Group Homework 2

- Histogram Problem
  - Imagine a random variable X such that, $P_X(k) = \pi_k,\ k \in 1, \ldots, N$
  - Suppose we draw *n* independent observations from X and form a random vector $C = (C_1, \cdots, C_N)^T$ where C_k is the number of times where the observed value is *k*
  - **C** is then a *histogram* and has a *multinomial distribution:*

$$P_{C_1,\ldots,C_N}(c_1, \ldots, c_N) = \frac{n!}{\prod_{k=1}^{N} c_k!} \prod_{j=1}^{N} \pi_j^{c_j}$$

  - Note that $\pi = (\pi_1, \ldots, \pi_w)$ are probabilities and thus: $\pi_i \geq 0 \ , \ \sum \pi_i = 1$

1. Derive the ML estimate for parameters $\pi_k,\ k \in \{1, ..., N\}$
   - *hint:* If you know about lagrange multipliers, use them! Otherwise, keep in mind that minimizing for a function *f(a,b)* constraint to *a+b=1* is equivalent to minimizing for *f(a,1-a)*.

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2. Derive the MAP solution using Dirichlet priors:
   - One possible prior model over π_k is the *Dirichlet Distribution:*

$$P_{\Pi_1,\ldots,\Pi_N}(\pi_1,\ldots,\pi_N) = \frac{\Gamma(\sum_{j=1}^{N})u_j)}{\prod_{j=1}^{N}\Gamma(u_j)}\prod_{j=1}^{N}\pi_j^{u_j-1}$$

   - where *u* is the set of *hyper-parameters* (prior parameters to solve) and

$$\Gamma(x) = \int_O^\infty e^{-t}t^{x-1}dt$$

   is the Gamma function.
   - You should show that the posterior is equal to:

$$P_{\Pi|C}(\pi|c) = \frac{\Gamma(\sum_{j=1}^{W} c_j + u_j)}{\prod_{k=1}^{W}\Gamma(c_j+u_j)}\prod_{j=1}^{W}\pi_j^{c_j+u_j-1}$$

3. Compare the MAP estimator with that of ML in part (1). What is the role of this prior compared to ML?