Topology of wireless networks

L. Decreusefond

Institut Mines-Telecom

Also starring (by chronological order of appearance)

P. Martins, E. Ferraz, F. Yan, A. Vergne, I. Flint, N.K. Le

Séminaire Brillouin

Outline

History

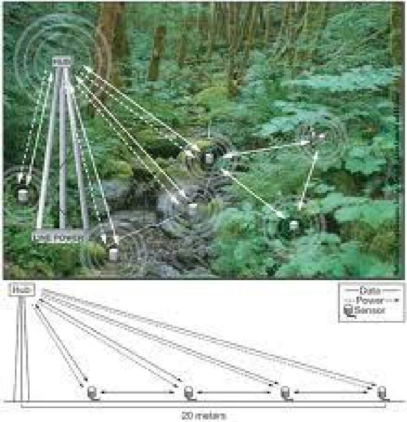
1870

1970

2000

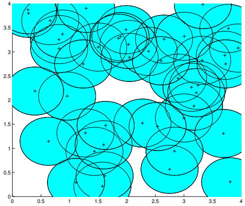
Sensors : ambient or pervasive computing

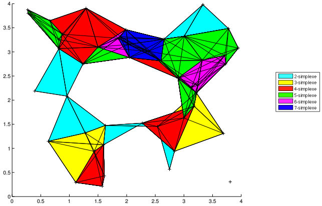
Applications : intelligent vehicle, agriculture, house, ...



Outline

Coverage

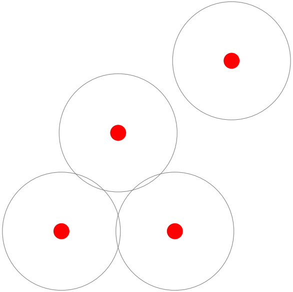



2-simplexe
3-simplexe
4-simplexe
5-simplexe
6-simplexe
7-simplexe

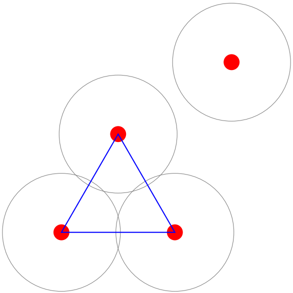
Mathematical framework

- Geometry leads to a combinatorial object
- Combinatorial object is equipped with a Linear algebra structure
- Coverage and connectivity reduce to compute the rank of a matrix
- Localisation of hole: reduces to the computation of a basis of a vector matrix, obtained by matrix reduction (as in Gauss algorithm).

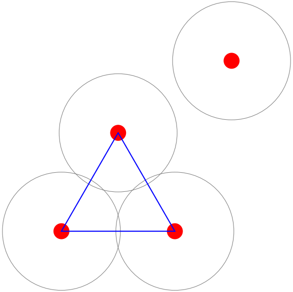
Cech complex



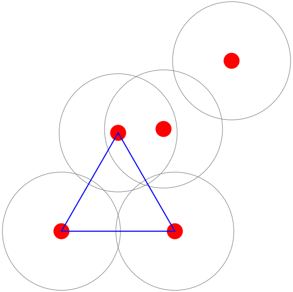
Cech complex



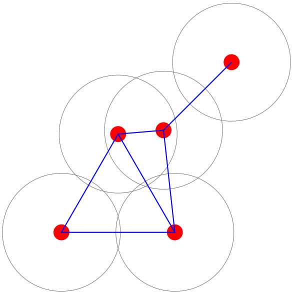
Cech complex



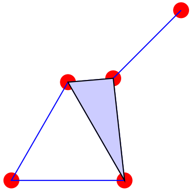
Cech complex



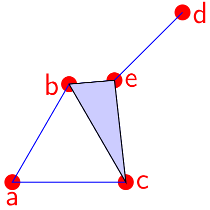
Cech complex



Cech complex



Cech complex



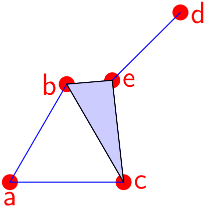
Vertices : a, b, c, d, e

Edges : ab, bc, ca, be, ec, ed

Triangles : bec

Tetrahedron : $\emptyset$

Cech complex



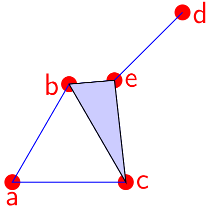
Vertices : a, b, c, d, e

Edges : ab, bc, ca, be, ec, ed

Triangles : bec

Tetrahedron : $\emptyset$

Cech complex



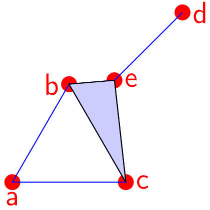
Vertices : a, b, c, d, e

Edges : ab, bc, ca, be, ec, ed

Triangles : bec

Tetrahedron : $\emptyset$

Cech complex



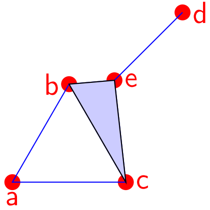
Vertices : a, b, c, d, e

Edges : ab, bc, ca, be, ec, ed

Triangles : bec

Tetrahedron : $\emptyset$

Cech complex



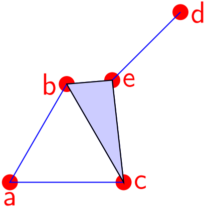
Vertices : a, b, c, d, e

Edges : ab, bc, ca, be, ec, ed

Triangles : bec

Tetrahedron : $\emptyset$

Cech complex



Vertices : $\{$ a, b, c, d, e $\} = \mathcal{C}_0$

Edges : $\{$ab, bc, ca, be, ec, ed $\} = \mathcal{C}_1$

Triangles : $\{$bec$\} = \mathcal{C}_2$

Tetrahedron : $\emptyset = \mathcal{C}_3$

◂ Comput. Rips

Cech complex

k-simplices

$$C_k = \bigcup \{[x_0, \cdots, x_{k-1}],\ x_i \in \omega, \cap_{i=0}^{k} B(x_i, \epsilon) \neq \emptyset\}$$

Nerve theorem

We can read some topological properties of $\bigcup_{x \in \omega} B(x, \epsilon)$ on $(C_k, k \geq 0)$

- Same nb of connected components
- Same nb of holes
- Same Euler characteristic

Boundary operator

Definition

$$\begin{aligned} \partial_k \ : \ C_k &\longrightarrow C_{k-1} \\ [v_0, \cdots, v_k] &\longmapsto \sum_{j=0}^{k} (-1)^j [v_0, \cdots, \hat{v}_j, \cdots] \end{aligned}$$

Boundary operator

Definition

$$\begin{aligned} \partial_k \ : \ C_k &\longrightarrow C_{k-1} \\ [v_0, \cdots, v_k] &\longmapsto \sum_{j=0}^{k} (-1)^j [v_0, \cdots, \hat{v}_j, \cdots] \end{aligned}$$

Example

$\partial_2(bec) = ec - bc + be$

Boundary operator

Definition

$$\partial_k : C_k \longrightarrow C_{k-1}$$
$$[v_0, \cdots, v_k] \longmapsto \sum_{j=0}^{k} (-1)^j [v_0, \cdots, \hat{v}_j, \cdots]$$

Example

$\partial_2(bec) = ec - bc + be$
$\partial_1 \partial_2(bec) = c - e - (c - b) + e - b = 0$

Theorem

$$\partial_k \circ \partial_{k+1} = 0$$

Consequence

$$\text{Im } \partial_{k+1} \subset \ker \partial_k$$

Definition

$$H_k = \ker \partial_k / \text{Im} \partial_{k+1} \text{ and } \beta_k = \dim \ker \partial_k - \text{range } \partial_{k+1}$$

Interpretation : The magic

- β_0 : number of connected components
- β_1 : number of holes
- β_2 : number of voids
- to be continued

Example

$$\partial_0 \equiv 0,\ \partial_1 = \begin{pmatrix} -1 & 0 & 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{pmatrix}$$

Nb of connected components

dim ker $\partial_0 = 5$, range $\partial_1 = 4$ hence $\beta_0 = 1$

Number of holes

$$\partial_2 = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

Nb of holes

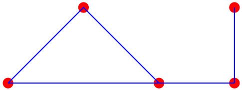
dim ker$\partial_1 = 2,$ range $\partial_2 = 1$ hence $\beta_1 = 1$

Polygons=cycles

$$\beta_1 = \text{ Nb of independent polygons} - \text{ Nb of independent triangles.}$$

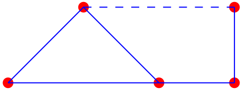
Polygons=cycles

$$\beta_1 = \text{ Nb of independent polygons} - \text{ Nb of independent triangles.}$$



Polygons=cycles

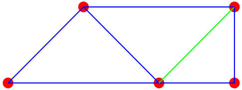
$$\beta_1 = \text{ Nb of independent polygons} - \text{ Nb of independent triangles.}$$



$$\beta_1 = 2 - 1 = 1.$$

Polygons=cycles

$$\beta_1 = \text{ Nb of independent polygons} - \text{ Nb of independent triangles.}$$



$$\beta_1 = 2 - 2 = 0.$$

Euler characteristic

Definition

$$\chi = \sum_{j=0}^{d} (-1)^j \beta_j$$

Discrete Morse inequality

$$-|\mathcal{C}_{k-1}| + |\mathcal{C}_k| - |\mathcal{C}_{k+1}| \leq \beta_k \leq |\mathcal{C}_k|$$

Euler characteristic

Definition

$$\chi = \sum_{j=0}^{d} (-1)^j \beta_j = \sum_{j=0}^{\infty} (-1)^j \left| \mathcal{C}_k \right|$$

Discrete Morse inequality

$$-\left|\mathcal{C}_{k-1}\right| + \left|\mathcal{C}_k\right| - \left|\mathcal{C}_{k+1}\right| \leq \beta_k \leq \left|\mathcal{C}_k\right|$$

Alternative complex

Cech complex

$$[v_0, \cdots, v_k] \in \mathcal{C}_k \iff \cap_{j=0}^{k} B(x_j, \epsilon) \neq \emptyset$$

Rips-Vietoris complex

$$[v_0, \cdots, v_k] \in \mathcal{R}_k \iff B(x_j, \epsilon) \cap B(x_k, \epsilon) \neq \emptyset$$

k simplex $=$ clique of $k+1$ points

Difference RV vs Cech

For the l^∞ distance

RV=Cech

Euclidean norm : false negative

Rips complex may miss some holes

Difference RV vs Cech

For the l^{∞} distance

RV=Cech

Euclidean norm : false negative

Rips complex may miss some holes

Cech vs Rips

$$\mathcal{R}_{\epsilon'}(\mathcal{V}) \subset \check{\mathrm{C}}_{\epsilon}(\mathcal{V}) \subset \mathcal{R}_{2\epsilon}(\mathcal{V}) \qquad \text{whenever} \qquad \frac{\epsilon}{\epsilon'} \geq \sqrt{\frac{d}{2(d+1)}}$$

Euclidean distance (D.-Feng-Martins)

- Coverage radius R_S
- Communication radius $R_C = \gamma R_S$

Upper-bound of the error

Theorem ($\sqrt{3} \leq \gamma \leq 2$)

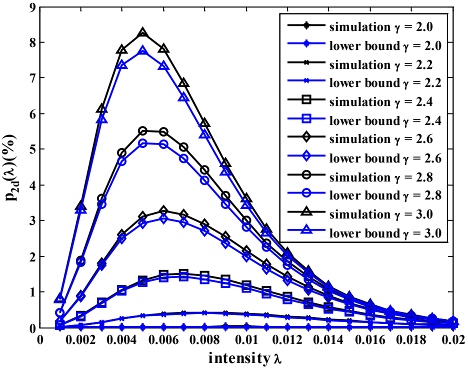
$$p_{2dl}(\lambda) = 2\pi\lambda^2 \int_{R_s}^{R_c/\sqrt{3}} r_0\, dr_0 \int_{\varphi_l(r_0)}^{\varphi_u(r_0)} d\varphi_1 \int_{r_0}^{R_1(r_0,\varphi_1)} e^{-\lambda\pi r_0^2} \times e^{-\lambda|S^+(r_0,\varphi_1)|}\left(1 - e^{-\lambda|S^-(r_0,r_1,\varphi_1)|}\right) r_1\, dr_1 \quad (1)$$

where

$$\varphi_l(r_0) = 2\arccos(R_c/(2r_0)),\ \varphi_u(r_0) = 2\arcsin(R_c/(2r_0)) - 2\arccos(R_c/(2r_0))$$

$$R_1(r_0,\varphi_1) = \min\Big(\sqrt{R_c^2 - r_0^2\sin^2\varphi_1} - r_0\cos\varphi_1$$

$$\sqrt{R_c^2 - r_0^2\sin^2(\varphi_1+\varphi_l(r_0))} + r_0\cos(\varphi_1+\varphi_l(r_0))\Big)$$



Probability to miss a hole using $\mathcal{R}_{R_s}$ and $\mathcal{R}_{R_c}$

Goals and related works

- Evaluate Betti nb and Euler charac. in some random settings
- Penrose : Asymptotics of $\mathbf{E}[|\mathcal{C}_k|^m]$ for Euclidian-RG Rips complex on the whole space ($m = 1, 2$)
- Kähle : Asymptotics of $\mathbf{E}[\beta_k]$ for Euclidian-RG Cech complex (deterministic number of points) and ER

Goals and related works

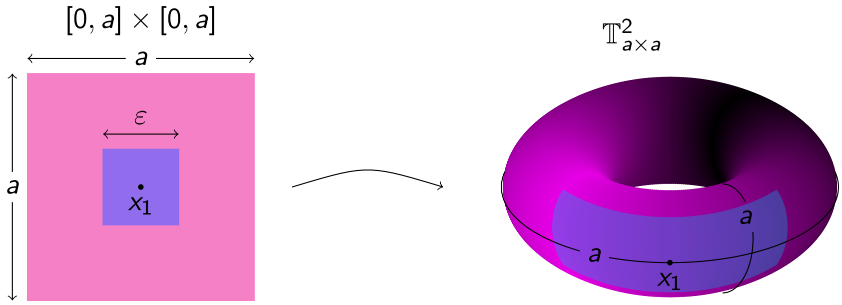
- Evaluate Betti nb and Euler charac. in some random settings
- Penrose : Asymptotics of $\mathbf{E}[|\mathcal{C}_k|^m]$ for Euclidian-RG Rips complex on the whole space ($m = 1, 2$)
- Kähle : Asymptotics of $\mathbf{E}[\beta_k]$ for Euclidian-RG Cech complex (deterministic number of points) and ER

Our results

Exact expressions of all moments of $|\mathcal{C}_k|$ and χ in any dimension for RG complex on a torus for the l^∞ norm

Outline

Random setting



Euler characteristic

- d=1 : $\{\chi = 0 \cap \beta_0 \neq 0\} \Leftrightarrow$ { circle is covered }
- d=2 : $\{\chi = 0 \cap \beta_0 \neq \beta_1\} \Leftrightarrow$ { domain is covered }
- d=3 : $\{\chi = 0 \cap \beta_0 + \beta_2 \neq \beta_1\} \Leftrightarrow$ { space is covered }

Euler characteristic (D.-Ferraz-Randriam-Vergne)

Euler characteristic

$$\mathbf{E}[\chi] = -\frac{\lambda e^{-\theta\, a^d}}{\theta} B_d(-\theta\, a^d) \text{ where } \theta = \lambda \left(\frac{2\epsilon}{a}\right)^d.$$

where B_d is the d-th Bell polynomial

$$B_d(x) = \left\{ {d \atop 1} \right\} x + \left\{ {d \atop 2} \right\} x^2 + \ldots + \left\{ {d \atop d} \right\} x^d$$

k simplices

The key remark

$$|\mathcal{C}_k| = \int h(x_1, \cdots, x_k) d\omega^{(k)}(x_1, \cdots, x_k)$$

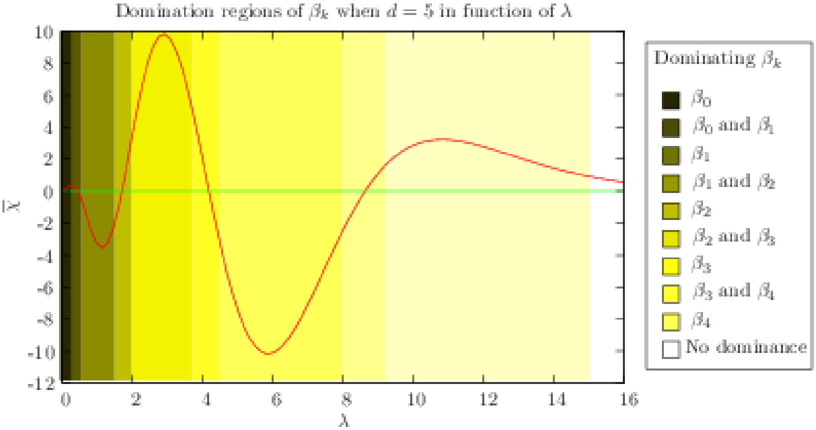
where

$$h(x_1, \cdots, x_k) \triangleq \frac{1}{k!} \prod_{i \neq j} \mathbf{1}_{\{\|x_i - x_j\| < \epsilon\}}$$

First moments

$$\mathbf{E}[|\mathcal{C}_k|] = \lambda a^d \frac{(k+1)^d}{(k+1)!} (a^d \theta)^k$$

Dimension 5


Domination regions of β_k when $d = 5$ in function of λ
10
8
6
4
2
0
-2
-4
-6
-8
-10
-12
$\bar{\chi}$
0
2
4
6
8
10
12
14
16
λ
Dominating β_k
β_0
β_0 and β_1
β_1
β_1 and β_2
β_2
β_2 and β_3
β_3
β_3 and β_4
β_4
No dominance

Depoissonization

k simplices

$$\mathbf{E}[|\mathcal{C}_k| \mid |\mathcal{C}_0| = n] = \binom{n}{k+1}(k+1)^d \left(\frac{2\epsilon}{a}\right)^{dk}$$

Euler characteristic

$$\mathbf{E}[\chi \mid |\mathcal{C}_0| = n] = \sum_{k=0}^{n} \binom{n}{k+1}(-1)^k(k+1)^d \left(\frac{2\epsilon}{a}\right)^{dk}$$

Second order moments

$$\operatorname{Cov}(|\mathcal{C}_k|, |\mathcal{C}_l|) = \left(\frac{1}{2\epsilon}\right)^d \sum_{i=0}^{l-1} \frac{1}{i!(k-l+i)!(l-i)!}\theta^{k+i} \times \left(k+i+2\frac{i(k-l+i)}{l-i+1}\right)^d .$$

Second order moments

$$\mathrm{Cov}(|\mathcal{C}_k|, |\mathcal{C}_l|) = \left(\frac{1}{2\epsilon}\right)^d \sum_{i=0}^{l-1} \frac{1}{i!(k-l+i)!(l-i)!}\theta^{k+i}$$
$$\times \left(k+i+2\frac{i(k-l+i)}{l-i+1}\right)^d.$$

Tools

- Chaos decomposition of $|\mathcal{C}_k|$
- Chaos multiplication formula

Poisson chaos

Theorem

$$I_n(f^{\otimes n}) = \left(\int f(x_1)\ (d\omega(x_1) - \lambda\ dx_1) \right)^n$$

$$= \int \dots \int \prod_{j=1}^{k} f(x_j) \otimes_{j=1}^{n} (\ d\omega(x_j) - \lambda dx_j)$$

$$= \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k}$$

$$\times \int \prod_{j=1}^{k} f(x_j)\ d\omega^{(k)}(x_1, \cdots, x_k)\ (\int f(x)\lambda\ dx)^{n-k}$$

Chaos

Extension

- Any symmetric $f(x_1, \cdots, x_k)$ is the limit of

$$g_n = \sum_i \alpha_{n,i} \prod_{j=1}^{k} f_{n,i}(x_j)$$

- $I_k(f)$ is defined by linearity
- For f non symmetric, let

$$f^s(x_1, \cdots, x_k) = \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} f(x_{\sigma(1)}, \cdots, x_{\sigma(k)})$$

and

$$I_n(f) := I_n(f^s)$$

Chaos multiplication formula

Theorem

$$I_i(f_i)I_j(f_j) = \sum_{s=0}^{2(i\wedge j)} I_{i+j-s}\left(\sum_{s\le 2t\le 2(s\wedge i\wedge j)} t! \binom{i}{t}\binom{j}{t}\binom{t}{s-t} f_i \circ_t^{s-t} f_j \right)$$

Euler characteristic

Euler characteristic

$$\mathrm{var}[\chi] = \left(\frac{a}{2\epsilon}\right)^d \sum_{n=1}^{\infty} c_n^d\, \theta^n,$$

where

$$c_n^d = \sum_{j=\lceil (n+1)/2 \rceil}^{n} \left[2 \sum_{i=n-j+1}^{j} \frac{(-1)^{i+j}}{(n-j)!(n-i)!(i+j-n)!} \left(n + \frac{2(n-i)(n-j)}{1+i+j-n}\right)^d - \frac{1}{(n-j)!^2 (2j-n)!} \left(n + \frac{2(n-j)^2}{1+2j-n}\right)^d \right].$$

Euler characteristic

Euler characteristic

$$\mathrm{var}[\chi] = \left(\frac{a}{2\epsilon}\right)^d \sum_{n=1}^{\infty} c_n^d\, \theta^n,$$

where

$$c_n^d = \sum_{j=\lceil (n+1)/2 \rceil}^{n} \left[2 \sum_{i=n-j+1}^{j} \frac{(-1)^{i+j}}{(n-j)!(n-i)!(i+j-n)!} \left(n + \frac{2(n-i)(n-j)}{1+i+j-n} \right)^d - \frac{1}{(n-j)!^2 (2j-n)!} \left(n + \frac{2(n-j)^2}{1+2j-n} \right)^d \right].$$

In dimension 1,

$$\mathrm{Var}(\chi) = \left(\theta e^{-\theta} - 2\theta^2 e^{-2\theta} \right)$$

Asymptotic results

If $\lambda \to \infty$, $\beta_i(\omega) \xrightarrow{p.s.} \beta_i(\mathbb{T}^d) = \binom{d}{i}$.

Limit theorems

CLT for Euler characteristic

$$\text{distance}_{TV}\left(\frac{\chi - \mathbf{E}[\chi]}{\sqrt{V_\chi}},\ \mathfrak{N}(0,1)\right) \le \frac{c}{\sqrt{\lambda}}.$$

Limit theorems

CLT for Euler characteristic

$$\text{distance}_{TV}\left(\frac{\chi-\mathbf{E}[\chi]}{\sqrt{V_\chi}},\ \mathfrak{N}(0,1)\right)\le\frac{c}{\sqrt{\lambda}}.$$

Method

- Stein method
- Malliavin calculus for Poisson process

Concentration inequality

- Discrete gradient $D_x F(\omega) = F(\omega \cup \{x\}) - F(\omega)$
- $D_x \beta_0 \in \{1,\, 0,\, -1,\, -2,\, -3\}$

Concentration inequality

- Discrete gradient $D_x F(\omega) = F(\omega \cup \{x\}) - F(\omega)$
- $D_x \beta_0 \in \{1, 0, -1, -2, -3\}$

Concentration inequality

- Discrete gradient $D_x F(\omega) = F(\omega \cup \{x\}) - F(\omega)$
- $D_x \beta_0 \in \{1, 0, -1, -2, -3\}$

$c > \mathbf{E}[\beta_0]$

$$P(\beta_0 \geq c) \leq \exp\left[-\frac{c - \mathbf{E}[\beta_0]}{6} \log\left(1 + \frac{c - \mathbf{E}[\beta_0]}{3\lambda}\right)\right]$$

Complexity

An important remark

- Construction of the complex is exponential (worst case)

Complexity

An important remark

- Construction of the complex is exponential (worst case)
- Computations of Betti numbers is polynomial

Further application (D.-Martins-Vergne)

Green networking

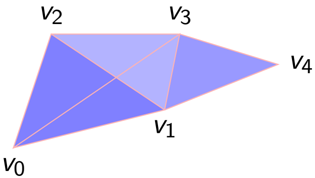
Switch off some sensors keeping the coverage

Height of an edge

Rank of the highest simplex it belongs to

Index of a vertex

Infimum of the height of its adjacent edges

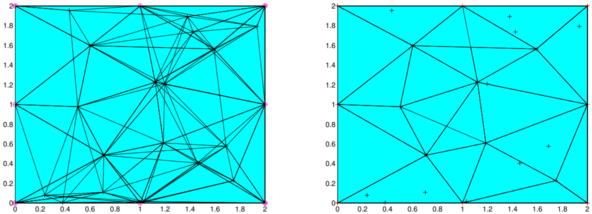


$$D[v_0, v_1, v_2] = D[v_0, v_1, v_3] = D[v_0, v_2, v_3] = D[v_1, v_2, v_3] = 3$$

$$D[v_1, v_3, v_4] = 2$$

$$I[v_0] = I[v_2] = 3 \text{ and } I[v_1] = I[v_3] = I[v_4] = 2$$

Example



- Complexity C bounded by 2^H

Complexity

$\theta_n = (r_n/a)^d$

$$\theta'_k = \frac{k^{\frac{1+\eta-d}{k-1}}}{n^{\frac{k}{k-1}}},\ \theta_k = \frac{k^{-\frac{1+\eta+d}{k-1}}}{n^{\frac{k}{k-1}}}$$

$$\theta_n \in [\theta'_k,\ \theta_k] \Longrightarrow C \xrightarrow{n\to\infty} k$$

Other regimes

Theorem (Critical: $n\theta_n \to 1$)

$C = O(n^3 \ln n)$.

Other regimes

Theorem (Critical: $n\theta_n \to 1$)

$C = O(n^3 \ln n)$.

Theorem (Super-critiqual: $n\theta_n \to \infty$)

$$C_n = O(2^n n^3)$$

Improvement

Energy saving

- Path loss

$$\text{Received Power} = K \, \frac{\text{Emitted power}}{\text{distance}(E, R)^{\gamma}}$$

Improvement

Energy saving

- Path loss

$$\text{Received Power} = K \, \frac{\text{Emitted power}}{\text{distance}(E, R)^{\gamma}}$$

- Covering radius such that

$$\text{Received Power} \geq \text{Threshold}$$

Improvement

Energy saving

- Path loss

$$\text{Received Power} = K \, \frac{\text{Emitted power}}{\text{distance}(E, R)^{\gamma}}$$

- Covering radius such that

$$\text{Received Power} \geq \text{Threshold}$$

- Covering radius proportional to $(\text{Emitted power})^{1/\gamma}$

Power saving algorithm I

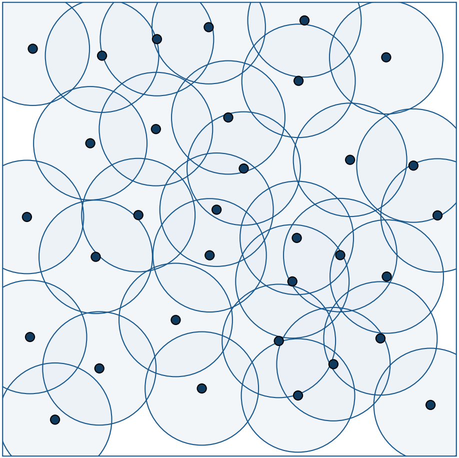
get collection of cells $\mathbb{C}$ and corresponding vertice $\mathbb{V}$;
build Čech complex for $\mathbb{C}$;
compute Betti number β_0^*, β_1^* and index $\hat{i}_v$ for each $v \in \mathbb{V}$;
flag fence, critical cells as not reducible;
flag cells whose index ≤ 2 as not reducible;
$\hat{i}_{\max} = \max\{\hat{i}_v | v \in \mathbb{V}\}$;
while exist a reducible cell **do**
  $\mathbb{C}^* \leftarrow$ collection of cells whose index $= \hat{i}_{\max}$
  c is a cell $\in \mathbb{C}^*$ whose biggest radius;
  $R_{\text{old}} \leftarrow R_c$;
  if $R_c - \Delta R_c \geq R_{c,\min}$ **then**
    $R_c \leftarrow R_c - \Delta R_c$;
  else
    turn off cell c;

Power saving algorithm II

end if
build Čech complex for $\mathbb{C}$ and compute β_0, β_1;
compute index for cell c;
if $\beta_0 \neq \beta_0^*$ or $\beta_0 \neq \beta_0^*$ or $\hat{i}_c < 2$ **then**
$R_c = R_c + \Delta R_c$;
set cell c is not reducible and set index of c to -1;
end if
compute index for every cell $\in \mathbb{C}$;
$\hat{i}_{\max} = \max\{\hat{i}_v | v \in \mathbb{V}\}$;
end while

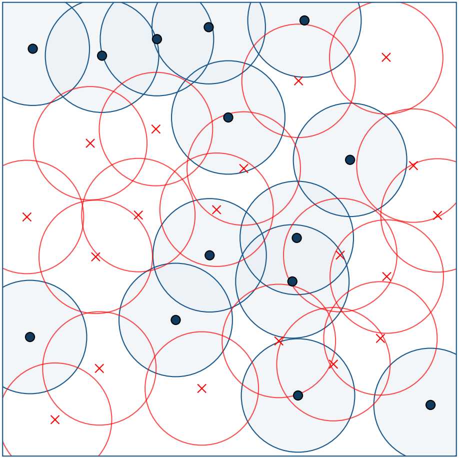
Around 60% of power saving

Damaged wireless network



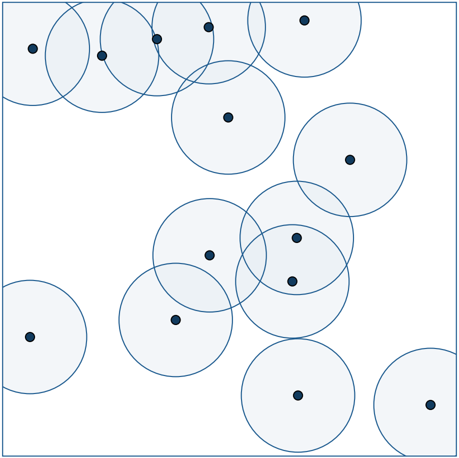
- Quality of Service
  - Coverage

Damaged wireless network



- Quality of Service
  - Coverage
- Disaster
  - Damaged nodes
  - Coverage holes
  - Several connected components

Damaged wireless network



- Quality of Service
  - Coverage
- Disaster
  - Damaged nodes
  - Coverage holes
  - Several connected components

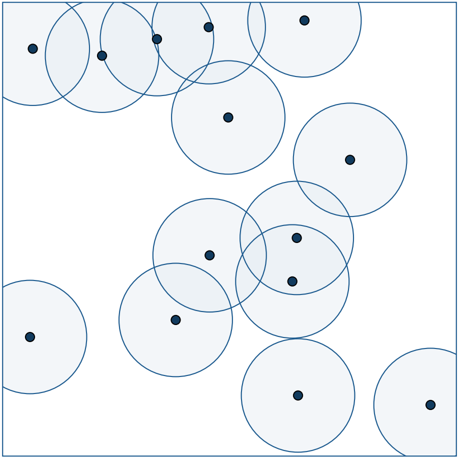
Recovery

- Addition of temporary nodes
  - Not in the same positions as the previous ones
  - Just enough to repair the network

Questions

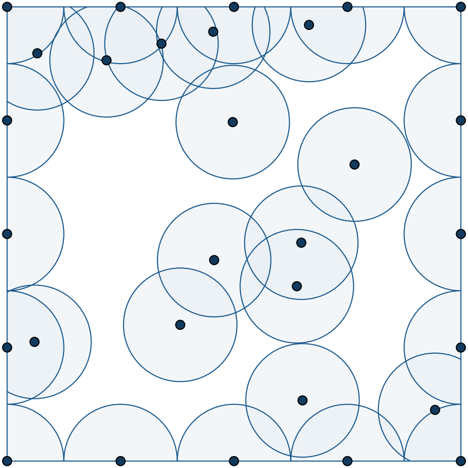
- How do we know that the network is repaired?
- Where do we put the new nodes?

Damaged wireless network



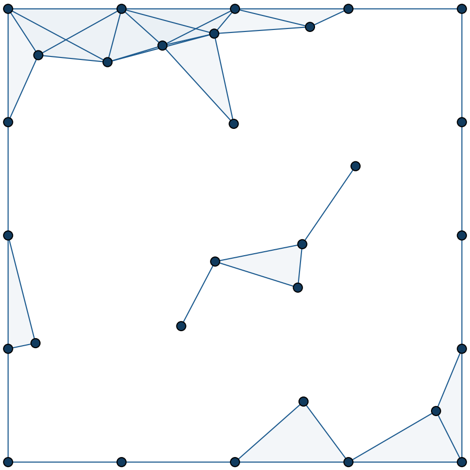
- Addition of virtual boundary nodes
  - To define the area to cover

Damaged wireless network



- Addition of virtual boundary nodes
  - To define the area to cover

Damaged wireless network



- Addition of virtual boundary nodes
  - To define the area to cover
- Computation of the coverage complex
  - $\beta_0 = 2$
  - $\beta_1 = 1$

Where do we put the new nodes?

Constraints

- Add enough nodes to repair the network
- Not too much

Last step

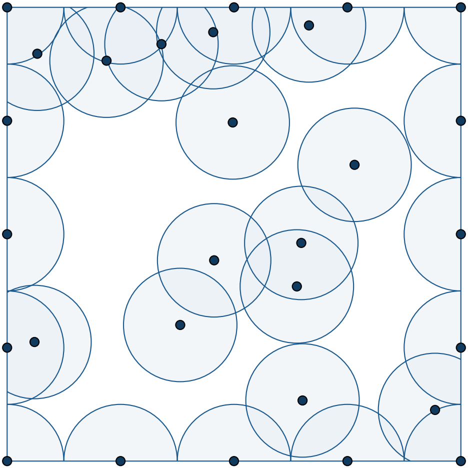
Remove superfluous nodes

Where do we put the new nodes?

Virtual nodes addition methods

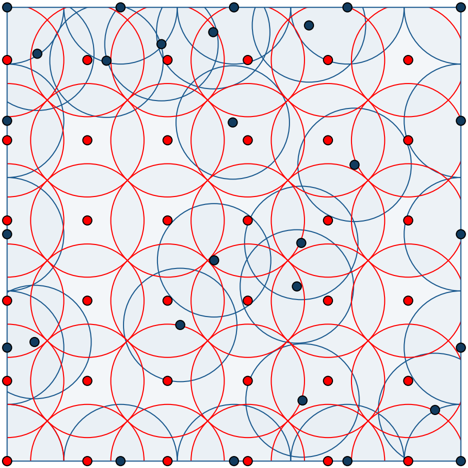
- Grid
- Uniform
- Determinantal

Classic addition methods



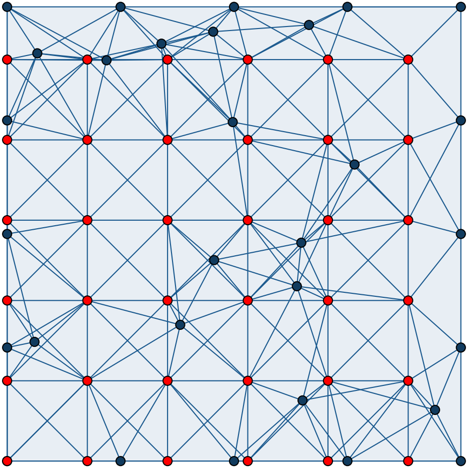
- Grid method
  - Deterministic number of nodes
  - Deterministic positions

Classic addition methods



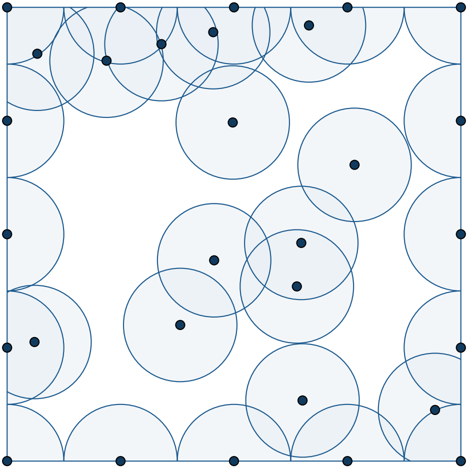
- Grid method
  - Deterministic number of nodes
  - Deterministic positions

Classic addition methods



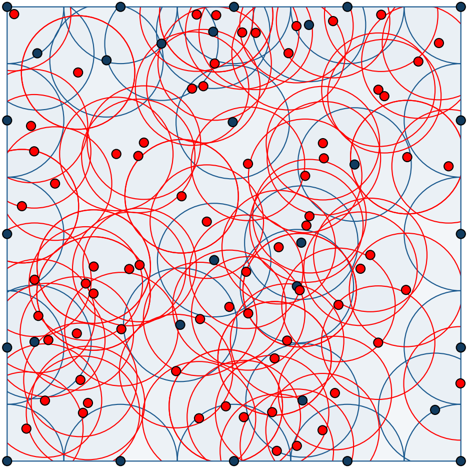
- Grid method
  - Deterministic number of nodes
  - Deterministic positions

Classic addition methods



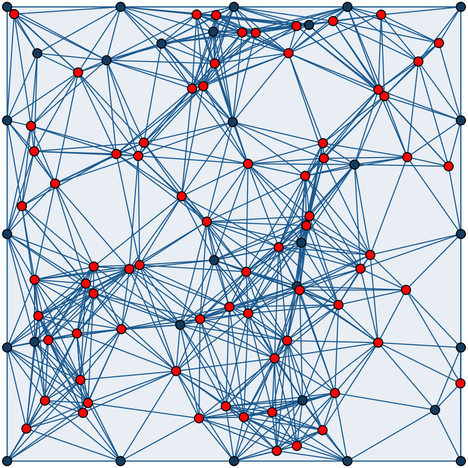
- Uniform method
  - Add nodes until the square is covered
  - Random positions following a uniform law on the square

Classic addition methods



- Uniform method
  - Add nodes until the square is covered
  - Random positions following a uniform law on the square

Classic addition methods



- Uniform method
  - Add nodes until the square is covered
  - Random positions following a uniform law on the square

Ginibre point process

Ginibre point process

- Let $K(x,y)=\sum_{k=1}^{\infty} B_k\phi_k(x)\overline{\phi_k(y)}$, where $B_k, k=1,2,\ldots,$ are k independent Bernoulli variables and $\phi_k(x)=\frac{1}{\sqrt{\pi k!}}e^{\frac{-|x|^2}{2}}x^k$ for $x\in\mathbb{C}$ and $k\in\mathbb{N}$
- The Ginibre point process is the point process with correlation function given by

$$\rho_n(x_1,\ldots,x_n)=\det(K(x_i,x_j)_{1\le i,j\le n})$$

$$E[\xi(K)(\xi(K)-1)\ldots(\xi(K)-n+1)]$$
$$=\int_{K^n}\rho_n(x_1,\ldots,x_n)dx_1\ldots dx_n$$

Ginibre point process

Ginibre point process

- Let $K(x,y) = \sum_{k=1}^{\infty} B_k \phi_k(x)\overline{\phi_k(y)}$, where $B_k, k = 1, 2, \ldots,$ are k independent Bernoulli variables and $\phi_k(x) = \frac{1}{\sqrt{\pi k!}} e^{\frac{-|x|^2}{2}} x^k$ for $x \in \mathbb{C}$ and $k \in \mathbb{N}$
- The Ginibre point process is the point process with correlation function given by

$$\rho_n(x_1, \ldots, x_n) = \det(K(x_i, x_j)_{1 \leq i,j \leq n})$$

$$E[\xi(K)(\xi(K)-1)\ldots(\xi(K)-n+1)] = \int_{K^n} \rho_n(x_1, \ldots, x_n) dx_1 \ldots dx_n$$

Repulsion

Definition (Papangelou intensity)

$$c(x, \omega) = P(x + \omega \subset \xi \,|\, \omega \subset \xi)$$

Repulsion

Definition (Papangelou intensity)

$$c(x,\,\omega) = P(x+\omega \subset \xi \,|\, \omega \subset \xi)$$

Theorem (For a Ginibre process)

$$c(x,\,\{x_1,\cdots,x_n\}) = \frac{\det J(\{x_1,\cdots,x_n,x\})}{\det J(\{x_1,\cdots,x_n\})}$$

where

$$J(x,y) = \sum_{n\geq 1} K^{\circ(n)}(x,y)$$

$$K^{\circ(n)}(x,y) = \int K^{\circ(n-1)}(x,z)K(z,y)\;dz$$

Repulsion

Definition (Papangelou intensity)

$$c(x,\,\omega) = P(x + \omega \subset \xi \,|\, \omega \subset \xi)$$

Theorem (For a Ginibre process)

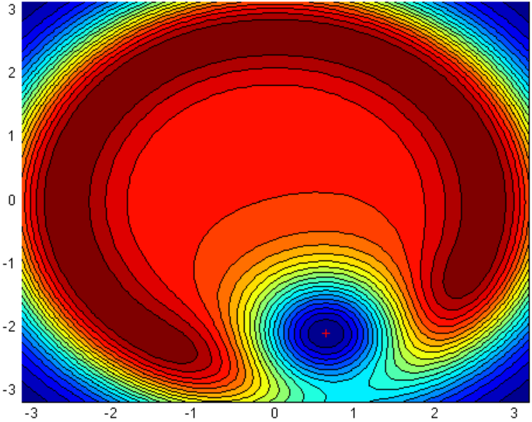
$$c(x,\,\{x_1, \cdots, x_n\}) = \frac{\det J(\{x_1, \cdots, x_n, x\})}{\det J(\{x_1, \cdots, x_n\})}$$

where

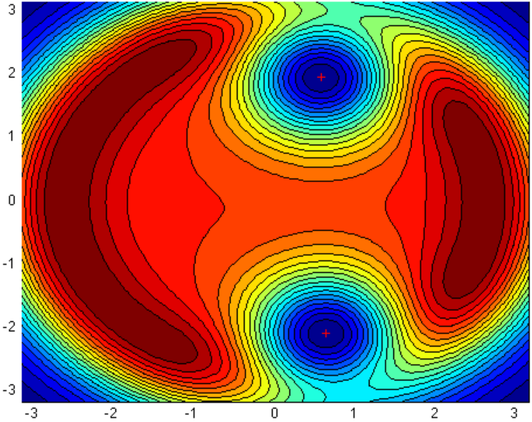
$$J(x, y) = \sum_{n \geq 1} K^{\circ(n)}(x, y)$$

$$K^{\circ(n)}(x, y) = \int K^{\circ(n-1)}(x, z) K(z, y)\; dz$$

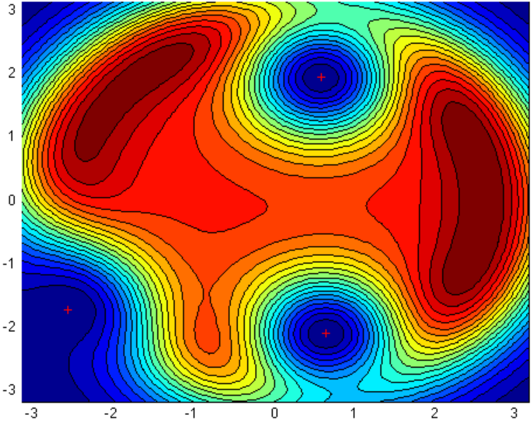
Repulsion (in picture)



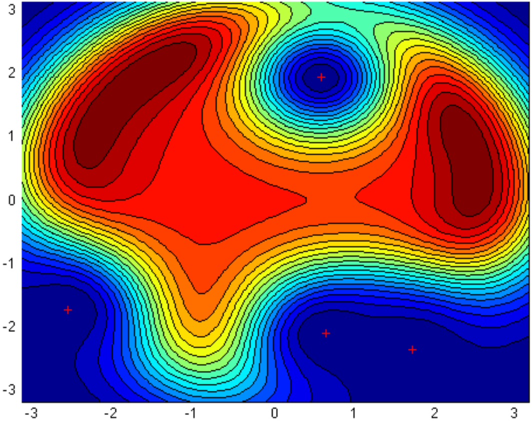
Repulsion (in picture)



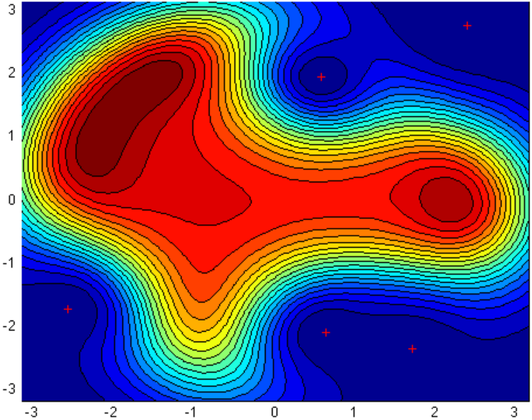
Repulsion (in picture)



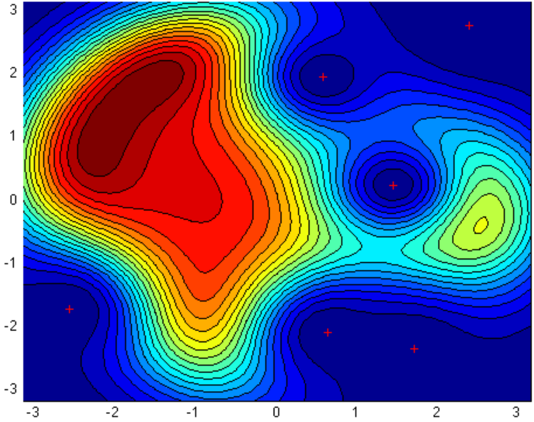
Repulsion (in picture)



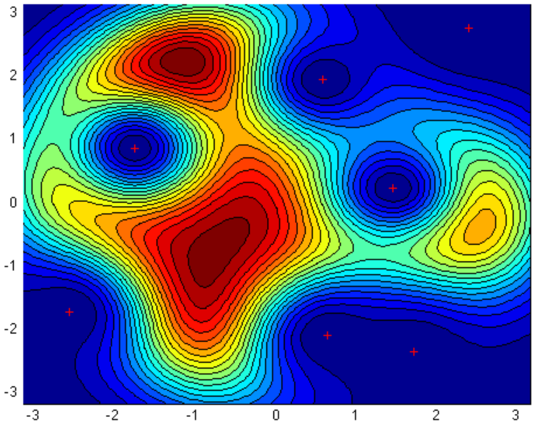
Repulsion (in picture)



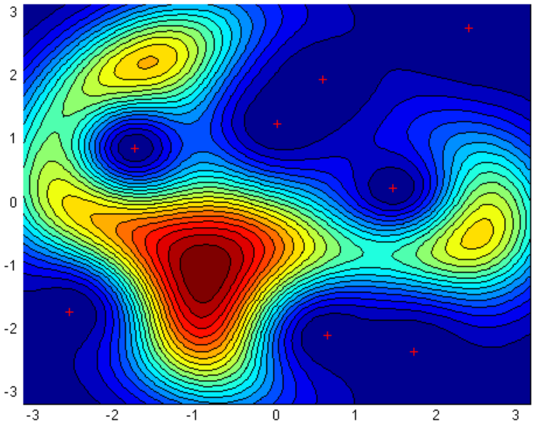
Repulsion (in picture)



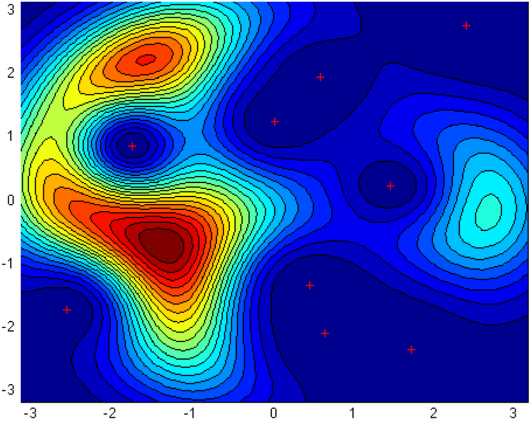
Repulsion (in picture)



Repulsion (in picture)



Repulsion (in picture)



Ginibre determinantal point process

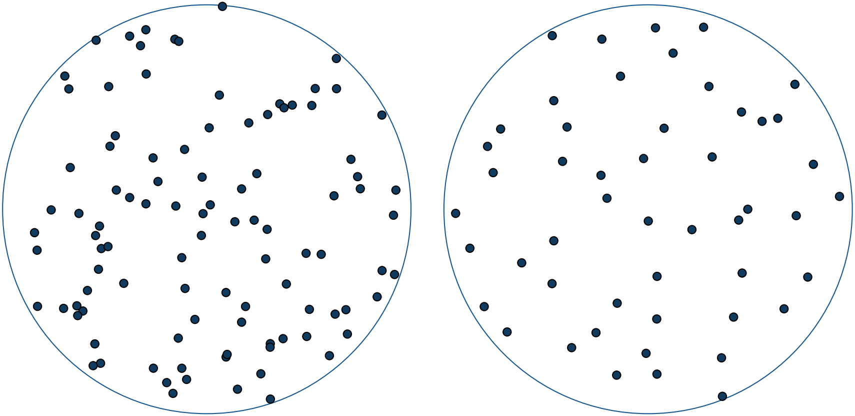
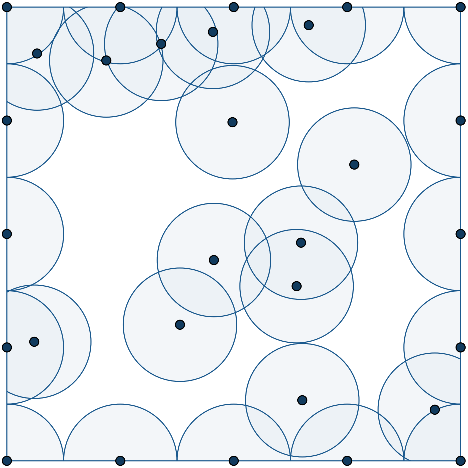


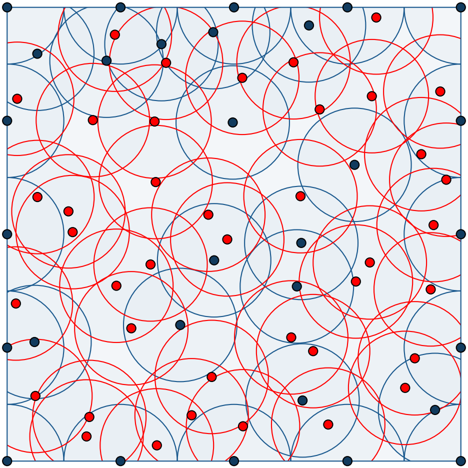
Figure : Poisson point process vs Ginibre determinantal point process

Determinantal addition method



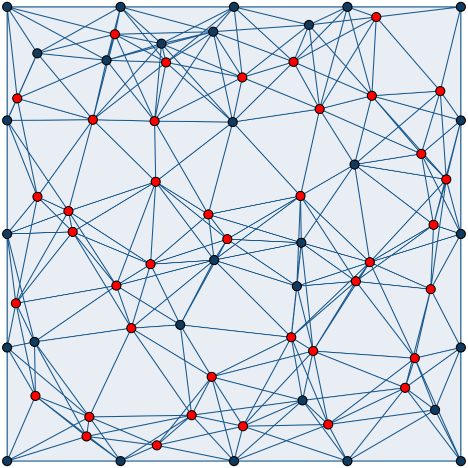
- Determinantal method
  - Add nodes until the square is covered
  - Random positions using a Ginibre point process

Determinantal addition method



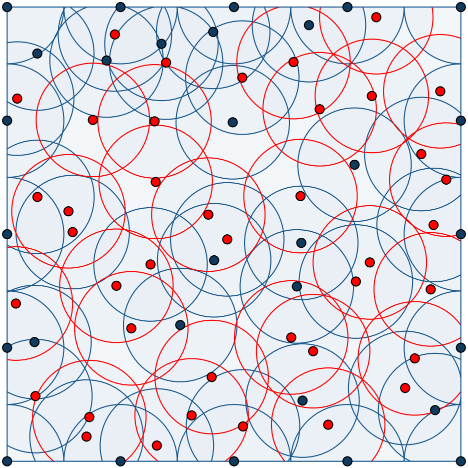
- Determinantal method
  - Add nodes until the square is covered
  - Random positions using a Ginibre point process

Determinantal addition method



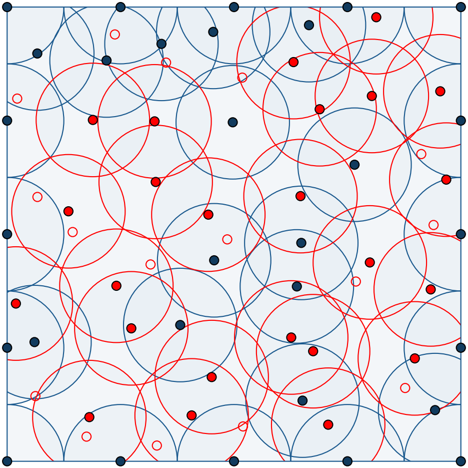
- Determinantal method
  - Add nodes until the square is covered
  - Random positions using a Ginibre point process

Final configuration (with determinantal addition method)



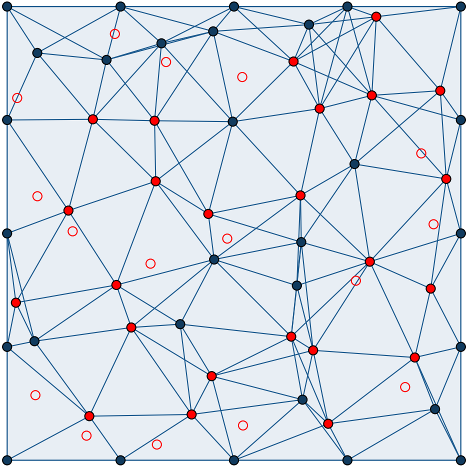
- Reduction algorithm
  - Removal of superfluous added nodes
  - Optimized number of added nodes

Final configuration (with determinantal addition method)



- Reduction algorithm
  - Removal of superfluous added nodes
  - Optimized number of added nodes

Final configuration (with determinantal addition method)



- Reduction algorithm
  - Removal of superfluous added nodes
  - Optimized number of added nodes

Comparison between addition methods

- Different scenarios depending on the percentage of area still covered after the disaster
- Square side of 1
- Coverage radius of 0.25
- Mean number of added vertices:

% of area initially covered	20%	40%	60%	80%
Grid method	9.00	9.00	9.00	9.00
Uniform method	32.51	29.34	24.64	15.63
Determinantal method	16.00	14.62	12.36	7.79

Comparison with a greedy algorithm

- Greedy algorithm
  - Lays nodes along a grid
  - Adds nodes from the furthest to the nearest
  - Until the furthest is already covered
- Similar to our algorithm with the grid addition method
- Mean final number of added vertices:

% of area initially covered	20%	40%	60%	80%
Greedy algorithm	3.69	3.30	2.84	1.83
Homology algorithm	4.42	3.87	2.97	1.78

Robustness

% of area initially covered	20%	40%	60%	80%
Greedy algorithm	0.68	0.67	0.48	0.35
Homology algorithm	0.58	0.52	0.36	0.26

Table : Mean number of holes after a Gaussian perturbation

% of area initially covered	20%	40%	60%	80%
Greedy algorithm	40.7%	45.2%	58.8%	68.9%
Homology algorithm	54.0%	58.0%	68.8%	76.1%

Table : Probability that there is no hole after a Gaussian perturbation

Conclusion

- Disaster recovery algorithm
  - patches damaged wireless network
  - adds enough virtual nodes to repair the network
  - runs a reduction algorithm on the virtual added nodes
- Simplicial homology representation
  - to compute the connectivity and the coverage of a wireless network
- Ginibre determinantal point process
  - to place nodes where they are needed

Bibliography I

[1] L. Decreusefond, E. Ferraz, H. Randriam and A. Vergne. Simplicial Homology of Random Configurations. In *Advances in Applied Probability*, 46(2), 2014. hal-00578955

[2] L. Decreusefond, I. Flint, and A. Vergne. Efficient simulation of the Ginibre process. hal-00869259, to appear in Adv. in Applied Prob.

[3] A. Vergne, L. Decreusefond, and P. Martins. Reduction algorithm for simplicial complexes. In *INFOCOM, 2013 Proceedings IEEE*, pages 475–479, 2013. hal-00864303

Bibliography II

[4] A. Vergne, I. Flint, L. Decreusefond, and P. Martins. Disaster Recovery in Wireless Networks: A Homology-Based Algorithm. In *International Conference on Telecommunications 2014 (ICT2014)*, May 2014. Best paper award. hal-00800520

[5] A. Vergne. Topologie algébrique appliquée aux réseaux de capteurs, PhD (2013) Prix de thèse Futur et Ruptures, 2014