

Statistical Anatomical Models: how to compute with phenotypes?

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Microscopie journalière

Resumé par mois

Date	Visites		Pages		Echanges		M
	Visites	Pages	Visites	Pages	Visites	Pages	
1603	631	374	3347	302156	8998	15148	5
2263	777	490	5300	474449	13203	24101	79482
2146	703	379	3869	532358	10631	19693	79482
1988	672	301	3560	17290258	9342	19693	65262
214	732	353	4045	2129561	10970	20848	66539
473	438	5314	2129561	10970	23486	61643	91
851	289	3654	2133742	13167	21986	61643	91
305	470	6430	2362204	8982	14682	69442	91
607	207	2059	3123665	14106	25554	47393	129
62	352	5676	997934	6437	10405	91373	83
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345	5976	292592	10628	18969	65719	1089	1143
		36881186	130110	20116			



Microsite journalière

Resumé par mois

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607	207	2059	3123665	6437	10405	29313	47	1089	1089
354	352	5676	997934	10926	18819	58906	1089	1089	1089
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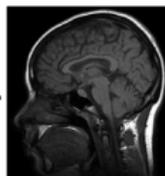
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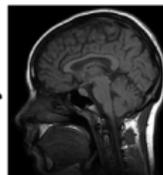
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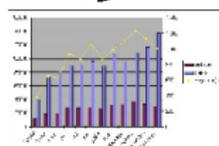
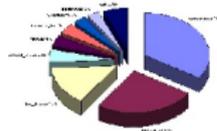


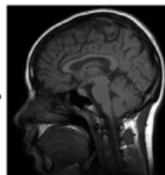
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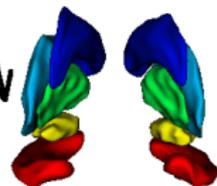


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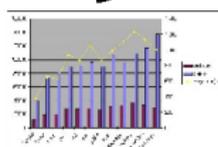
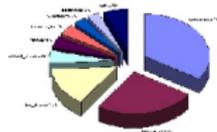


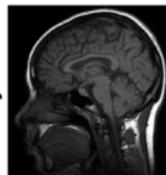
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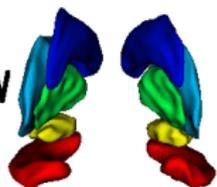
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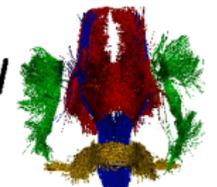




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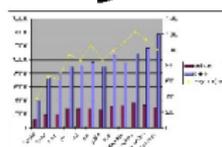
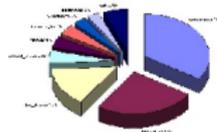


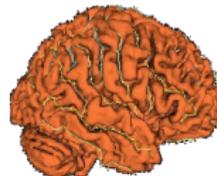
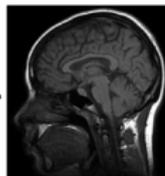
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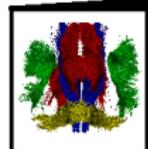
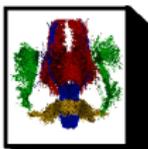
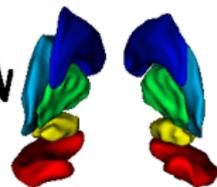
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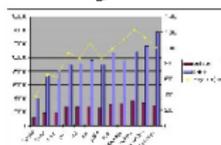
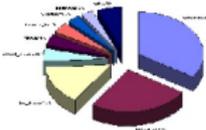




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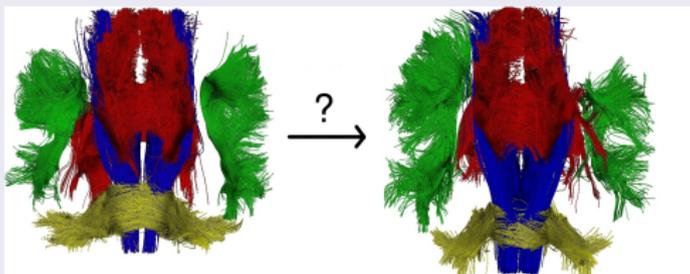
Vector data in \mathbb{R}^N :

- Hilbert space: $\|x - y\|^2$
- Gaussian variables: $y_j = \bar{x} + \varepsilon_j$, with $\varepsilon_j \sim \mathcal{N}(0, \Sigma)$

Anatomical data lie on a manifold:

- which manifold? which metric?
- how to compute first and second moment (mean and variance)?

Comparison of 2 anatomies



- “shape space” embedding [Kendall 1984]:

$$d(\mathcal{O}_1, \mathcal{O}_2) \quad (\text{or } \mathcal{O}_2 = \mathcal{O}_1 + \varepsilon)$$

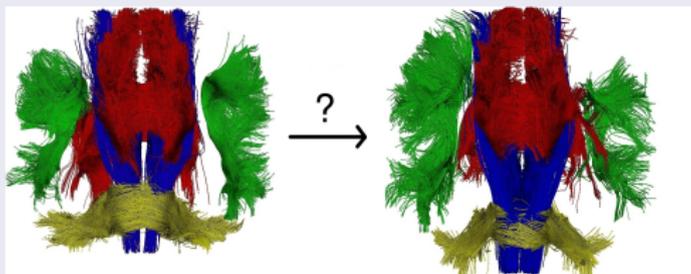
- invariance of the metric? shape ‘alignment’?
- deformation = nuisance factor

- measure of deformation [Grenander 1993]:

$$\hat{\phi} = \operatorname{argmin} d(\mathcal{O}_2, \phi.\mathcal{O}_1) \quad (\text{or } \mathcal{O}_2 = \hat{\phi}.\mathcal{O}_1 + \varepsilon)$$

- metric on objects derived from metric on deformations
- which deformations?
- residual as nuisance factor: overfitting

Comparison of 2 anatomies



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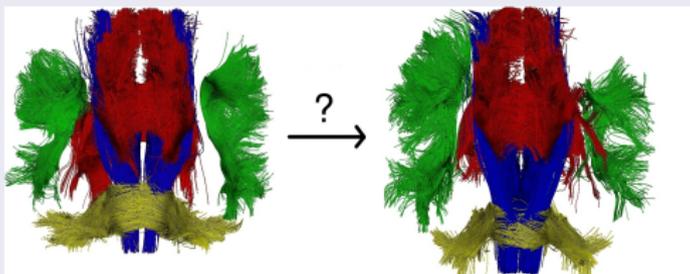
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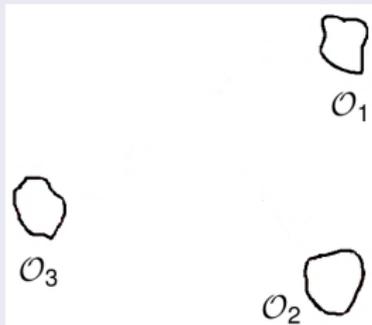
- Combined model:

$$\mathcal{O}_2 = \phi \cdot \mathcal{O}_1 + \varepsilon$$

- decomposition into geometry + residual
- all information taken into account
- which trade-off?

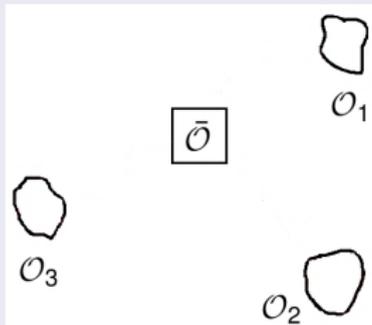
[For images, see Glasbey & Mardia JRSS'01, Allasonniere et al. JRSS'07]

Group-wise statistics



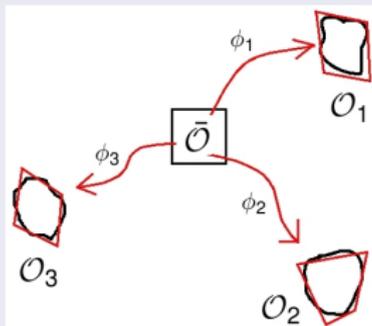
$$O_i = \phi_i \cdot \bar{O} + \varepsilon_i$$

Group-wise statistics



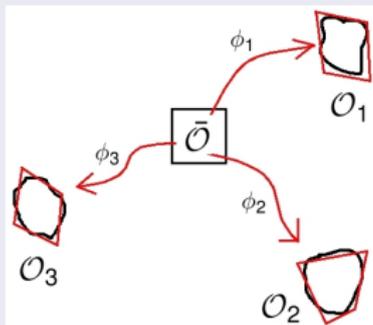
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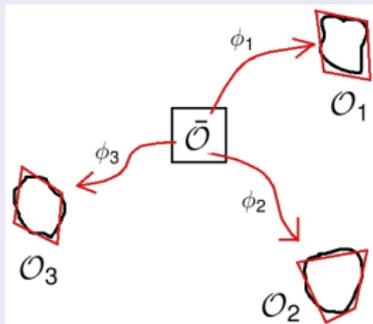
Group-wise statistics



$$O_i = \phi_i \cdot \bar{O} + \varepsilon_i$$

- deterministic template (\bar{O}): anatomical invariants
- random deformations (ϕ): geometrical variability
- random residuals (ε): "texture" variability

Group-wise statistics



$$\mathcal{O}_i = \phi_i \cdot \bar{\mathcal{O}} + \varepsilon_i$$

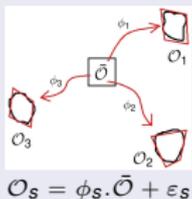
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Formal MAP derivation:

$$E(\bar{\mathcal{O}}, \phi_1, \dots, \phi_N) = \sum_{s=1}^N \frac{1}{2\sigma^2} \|\phi_s \cdot \bar{\mathcal{O}} - \mathcal{O}_s\|^2 + \mathbf{d}(\text{Id}, \phi_s)^2$$

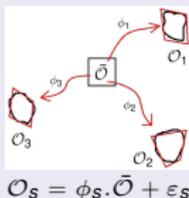
- Fréchet mean $\bar{\mathcal{O}}$ with the metric derived from diffeos
- Variance given by the distribution of
 - deformation parameters $\{\phi_s\}$
 - residuals $\varepsilon_s = \|\phi_s \cdot \bar{\mathcal{O}} - \mathcal{O}_s\|^2$

Group-wise statistics



$$E(\bar{\mathcal{O}}, \phi_1, \dots, \phi_N) = \sum_{i=s}^N \left\{ \frac{1}{2\sigma^2} \|\phi_s \cdot \bar{\mathcal{O}} - \mathcal{O}_s\|^2 + d(\text{Id}, \phi_s)^2 \right\}$$

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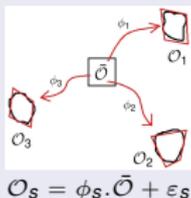
- Need to define:

- parameterization of diffeos (ϕ_s) + metric d
- deformation of anatomical objects $\phi_s \cdot \bar{\mathcal{O}}$
- norm between objects $\|\mathcal{O}_1 - \mathcal{O}_2\|^2$

- Then:

- differentiate E w.r.t. deformation parameters and template
- derive statistics on deformation parameters and residuals

Group-wise statistics



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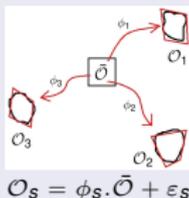
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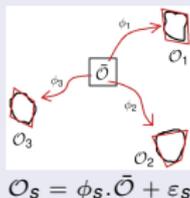
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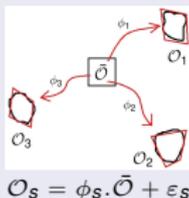
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... for any kind of data (image, surface mesh, curve, etc..)!

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Group-wise statistics



$$E(\bar{\mathcal{O}}, \phi_1, \dots, \phi_N) = \sum_{i=s}^N \left\{ \frac{1}{2\sigma^2} \|\phi_s \cdot \bar{\mathcal{O}} - \mathcal{O}_s\|^2 + d(\text{Id}, \phi_s)^2 \right\}$$

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Outline

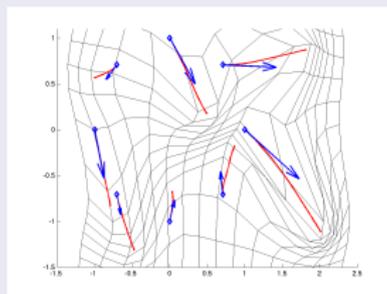
- 1 Sparse adaptive parameterization of diffeomorphisms
- 2 Atlas construction with landmark points
- 3 Atlas construction with currents
- 4 Atlas construction with images

Adaptive finite-dimensional subsets of diffeomorphisms

System of N self-interacting particles:

- control points $\{c_k\}$ with momenta $\{\alpha_k\}$
- Hamiltonian (kinetic energy):

$$H(\mathbf{c}, \boldsymbol{\alpha}) = \sum_{p=1}^{N_{cp}} \sum_{q=1}^{N_{cp}} \alpha_k^T K(c_k, c_p) \alpha_p$$



- Equations of motion: $\dot{c}_k = \partial H / \partial \alpha_k$, $\dot{\alpha}_k = -\partial H / \partial c_k$

$$\begin{cases} \frac{dc_k(t)}{dt} = \sum_{q=1}^N K(c_k(t), c_q(t)) \alpha_q(t) \\ \frac{d\alpha_k(t)}{dt} = -\sum_{q=1}^N \nabla K(c_k, c_q) \alpha_q(t)^T \alpha_k(t) \end{cases}$$

- Defines a dense diffeomorphic deformation:

$$v_t(x) = \sum_k K(x, c_k(t)) \alpha_k(t)$$

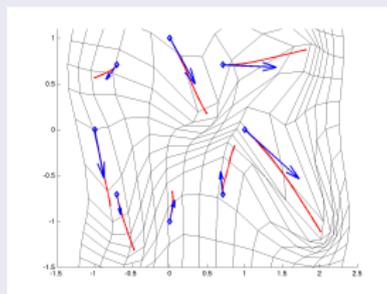
$$\frac{\phi_t(x)}{dt} = v_t(\phi_t(x)) \quad \phi_0(x) = x$$

Adaptive finite-dimensional subsets of diffeomorphisms

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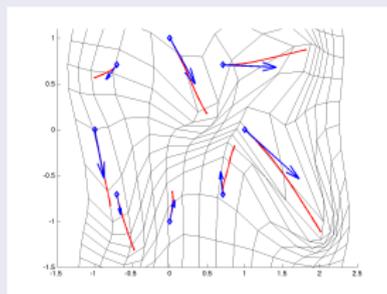
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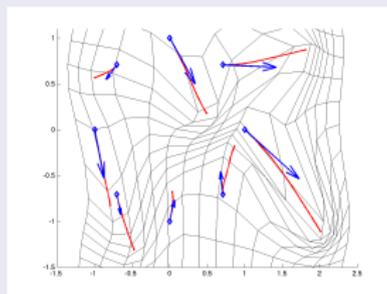
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Definition

The group of diffeomorphisms for a RKHS V :

$\mathcal{G}_V = \left\{ \phi_1; \frac{\partial \phi_t}{\partial t} = \mathbf{v}_t \circ \phi_t, \phi_0 = \text{id}, \text{ and } \mathbf{v}_t \in V \right\}$ is provided with

the metric: $d(\text{id}, \phi_1) = \int_0^1 \|\mathbf{v}_t\|^2 dt$ [Trounev'95, Dupuis et al.'98]

Theorem

The geodesics connecting two N -uples of distinct points \mathbf{c}_0 and \mathbf{c}_1 are such that:

- $\mathbf{v}_t(x) = \sum_k K(x, \mathbf{c}_k(t)) \alpha_k(t)$ (discrete support of velocity)
- $\|\mathbf{v}_t\|_V^2$ constant along the geodesic (energy conservation)

Since $\|\mathbf{v}_t\|_V^2 = H(\mathbf{c}, \alpha) = \sum_{p,q=1}^{N_{cp}} \alpha_p(t)^T K(\mathbf{c}_p(t), \mathbf{c}_q(t)) \alpha_q(t)$, control points and momenta are optimally transported along the geodesics according the Hamiltonian equations:

$\dot{\mathbf{c}}_k(t) = \partial H / \partial \alpha_k$ and $\dot{\alpha}_k(t) = -\partial H / \partial \mathbf{c}_k$

- for a *fixed* set of N control points:
 - diffeomorphisms parameterized by initial momenta $\{\alpha_k(0)\}$
 - geodesics: Log-map $\phi_1 \rightarrow \{\alpha_k(0)\}$
 - metric on the tangent-space at Id V : $\|v_0\|_V^2 = \alpha_0^T \mathbf{K}(\mathbf{c}_0, \mathbf{c}_0) \alpha_0$
- optimization of *the position* of control points:
 - optimize the choice of a deformation model of size $3N$
 - selection in a infinite-dimensional dictionary of deformation models
- optimization of *the number* of control points:
 - selection of the model complexity

Control-point parameterization of deformations not a new idea

(diffeo B-spline [Rueckert et al.], GRID [Grenander et al.]), however here:

- adaptive parameterization (number and position of CPs)
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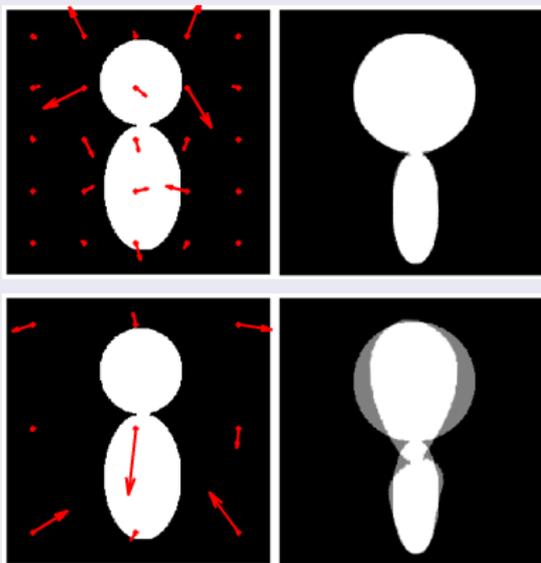
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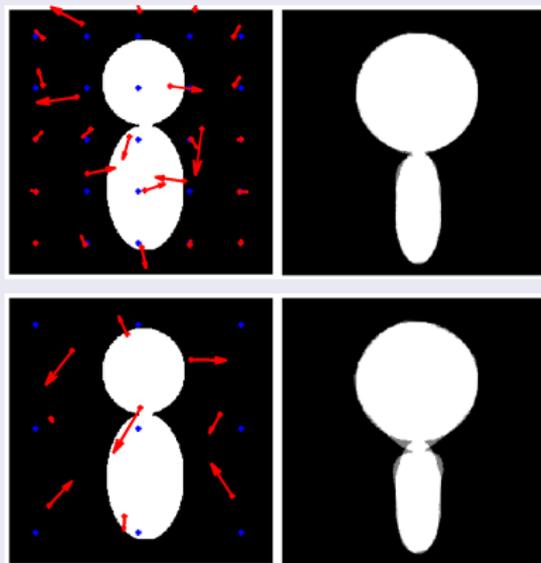
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Importance of optimizing Control-Points positions:

Fixed Positions

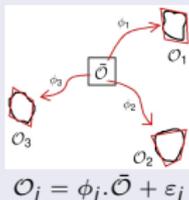


Updated Positions



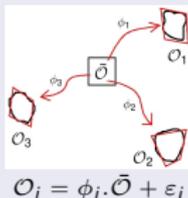
Optimizing positions of CP enables more compact encoding of the deformation

Group-wise statistics



$$E(\bar{\mathcal{O}}, \phi_1, \dots, \phi_N) = \sum_{s=1}^N \left\{ \frac{1}{2\sigma^2} \|\phi_s \cdot \bar{\mathcal{O}} - \mathcal{O}_s\|^2 + \mathbf{d}(\text{Id}, \phi_s)^2 \right\}$$

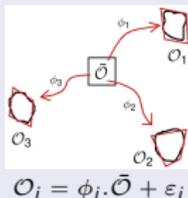
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- Hamiltonian equations: $\dot{\mathbf{S}}^s(t) = F(\mathbf{S}^s(t))$ with $\mathbf{S}^s(0) = \mathbf{S}_0^s$
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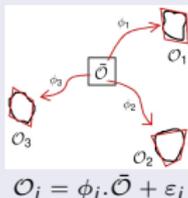
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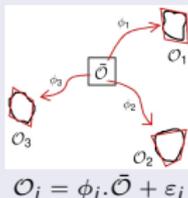
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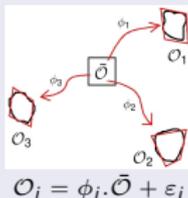
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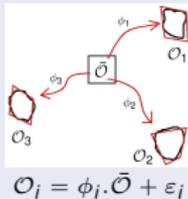
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Outline

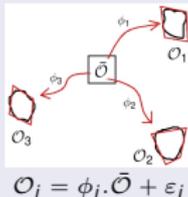
- 1 Sparse adaptive parameterization of diffeomorphisms
- 2 Atlas construction with landmark points**
- 3 Atlas construction with currents
- 4 Atlas construction with images

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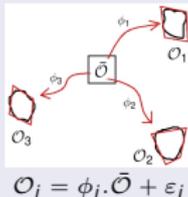
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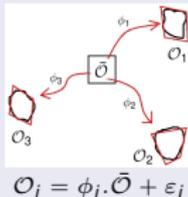
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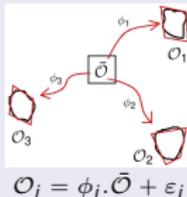
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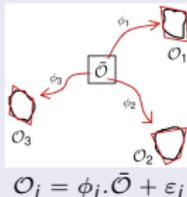
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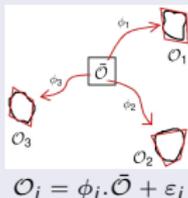
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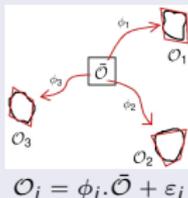
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$$\phi \cdot \mathcal{X}_0 = X(1) \text{ with } \begin{cases} \dot{X}(t) = G(\mathbf{S}(t), X(t)) \\ X(0) = \mathcal{X}_0 \end{cases}$$

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 $\nabla_{X(1)} \|X(1) - Y\|^2 = 2(X(1) - Y)$

Group-wise statistics



$$E(X_0, \mathbf{S}_0^1, \dots, \mathbf{S}_0^N) = \sum_{s=1}^N \left\{ \frac{1}{2\sigma^2} \|X_s(1) - Y_s\|^2 + L(\mathbf{S}_0^s) \right\}$$

- Landmark case: $\mathcal{O}_s = Y_s = \{y_{s,k}\}$

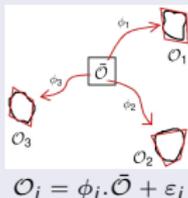
- Template of the form: $\bar{\mathcal{O}} = X_0 = \{x_{0,k}\}$
- $\phi \cdot X_0 = \{\phi(x_{0,k})\}$ solution of:

$$\dot{x}_k(t) = v_t(x_k(t)) = \sum_{k=1}^{N_{cp}} K(x_k(t), c_k(t)) \alpha_k(t) = G(\mathbf{S}(t), x_k(t))$$

$$\phi \cdot X_0 = X(1) \text{ with } \begin{cases} \dot{X}(t) = G(\mathbf{S}(t), X(t)) \\ X(0) = X_0 \end{cases}$$

- Sum of squared differences: $\|X_s(1) - Y_s\|_{\mathbb{R}^N}^2$,
 $\nabla_{X(1)} \|X(1) - Y\|^2 = 2(X(1) - Y)$

Group-wise statistics



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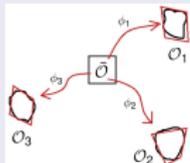
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Group-wise statistics

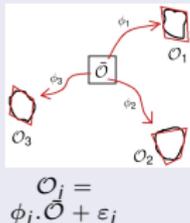


$$\phi_j \cdot O + \varepsilon_j$$

$$E(X_0, \mathbf{S}_0^1, \dots, \mathbf{S}_0^N) = \sum_{s=1}^N \left\{ \frac{1}{2\sigma^2} \|X_s(1) - Y_s\|^2 + L(\mathbf{S}_0^s) \right\}$$

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Group-wise statistics



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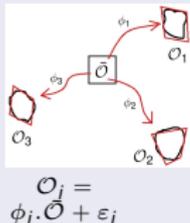
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$$\begin{aligned} \nabla_{\mathbf{S}_0^s} E &= \frac{1}{\sigma^2} \nabla_{\mathbf{S}_0^s} X_s(1)^T (X_s(1) - Y_s) + \nabla_{\mathbf{S}_0^s} L \\ \nabla_{X_0} E &= \frac{1}{\sigma^2} \nabla_{\mathbf{S}_0^s} X_s(1)^T (X_s(1) - Y_s) \end{aligned}$$

These gradients can be solved using linearized ODEs!!!

Group-wise statistics



$$E(X_0, \mathbf{S}_0^1, \dots, \mathbf{S}_0^N) = \sum_{s=1}^N \left\{ \frac{1}{2\sigma^2} \|X_s(1) - Y_s\|^2 + L(\mathbf{S}_0^s) \right\}$$

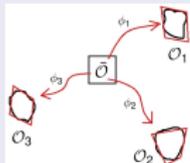
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Group-wise statistics



$$\phi_j \cdot \bar{O} + \varepsilon_j$$

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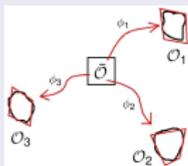
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Group-wise statistics

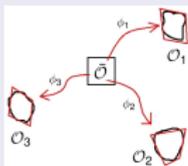


$$\mathcal{O}_j = \phi_j \cdot \bar{\mathcal{O}} + \varepsilon_j$$

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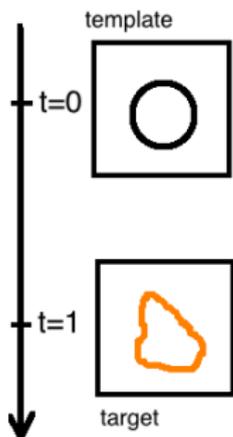
Group-wise statistics



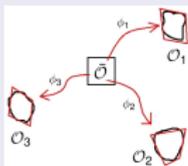
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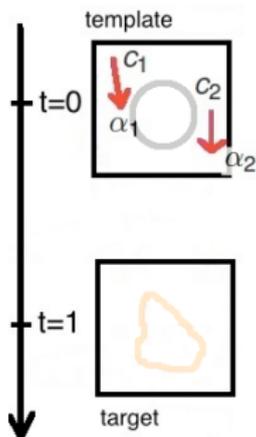
Group-wise statistics



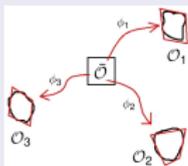
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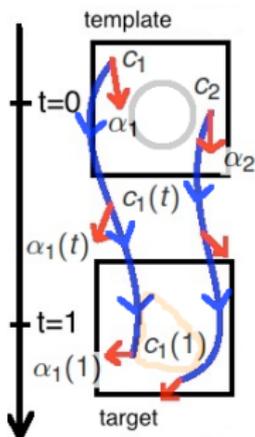
Group-wise statistics



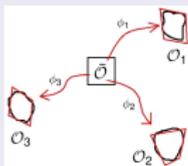
$$\phi_j = \phi_j \cdot \bar{\phi} + \varepsilon_j$$

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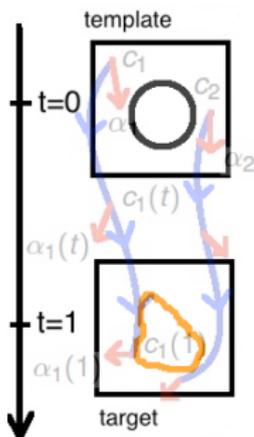
Group-wise statistics



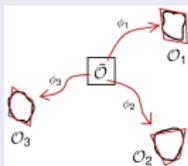
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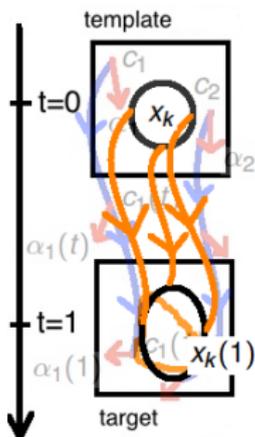
Group-wise statistics



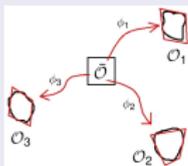
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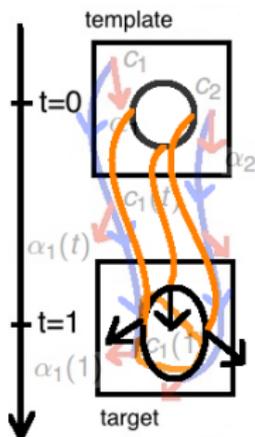
Group-wise statistics



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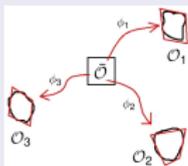
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$$\theta(1) = \frac{1}{\sigma^2} (X(1) - Y)$$

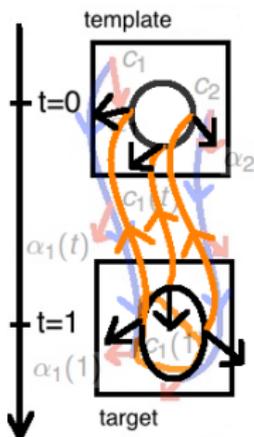
Group-wise statistics



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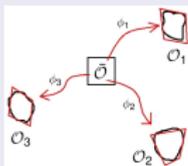
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$$\dot{\theta}(t) = \partial_{S(t)} G^T \theta(t), \quad \theta(1) = \frac{1}{\sigma^2} (X(1) - Y)$$

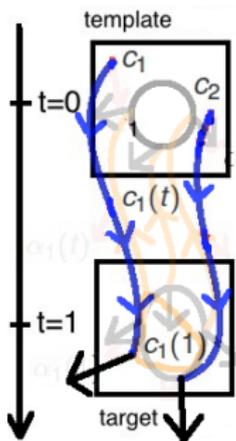
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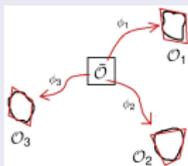
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$$\dot{\xi}(t) = \partial_{X(t)} G^T \theta(t) + dF^T \xi(t), \quad \xi(1) = 0$$

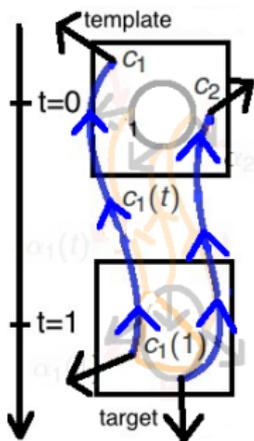
Group-wise statistics



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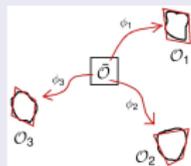
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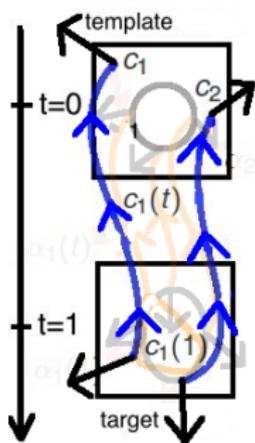
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$$\nabla_{S_0} E = \xi(0) \quad \nabla_{X_0} E = \theta(0)$$

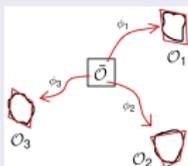
- Simultaneous optimization of template shape *and* registrations!
- at no additional cost!

demo

2 missing pieces:

- selection of control points
- currents

Group-wise statistics



$$\phi_j \cdot O + \varepsilon_j$$

$$E(X_0, \mathbf{S}_0^1, \dots, \mathbf{S}_0^N) = \sum_{s=1}^N \left\{ \frac{1}{2\sigma^2} \|X_s(1) - Y_s\|^2 + L(\mathbf{S}_0^s) \right\}$$

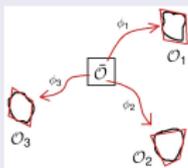
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Group-wise statistics



$$\mathcal{O}_j = \phi_j \cdot \bar{\mathcal{O}} + \varepsilon_j$$

$$E(X_0, \mathbf{S}_0^1, \dots, \mathbf{S}_0^N) = \sum_{s=1}^N \left\{ \frac{1}{2\sigma^2} \|X_s(1) - Y_s\|^2 + L(\mathbf{S}_0^s) + \gamma \sum_{k=1}^{N_{cp}} \|\alpha_{0,k}^s\| \right\}$$

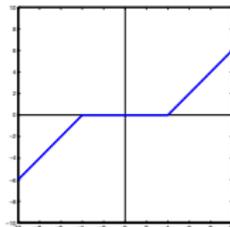
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$$\nabla_{\mathbf{S}_0} E = \xi(0) \quad \nabla_{X_0} E = \theta(0)$$

L^1 norm as a surrogate to model selection

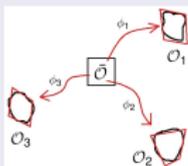
- Optimized using FISTA [Beck&Teboulle'09]

$$\alpha_0^{new} \leftarrow \left(\alpha_0^{old} - \tau \xi^\alpha(0) \right) ST_{\gamma\tau} \left(\frac{\alpha_0^{old} - \tau \xi^\alpha(0)}{\|\alpha_0^{old} - \tau \xi^\alpha(0)\|} \right)$$



- zero-out momenta small in magnitude

Group-wise statistics



$$O_j = \phi_j \cdot \bar{O} + \varepsilon_j$$

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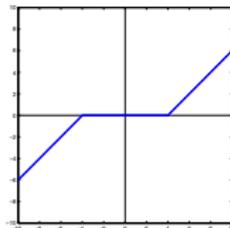
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L^1 norm as a surrogate to model selection

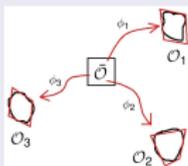
- Optimized using FISTA [Beck&Teboulle'09]

$$\alpha_0^{new} \leftarrow \left(\alpha_0^{old} - \tau \xi^\alpha(0) \right) \text{ST}_{\gamma\tau} \left(\frac{\alpha_0^{old} - \tau \xi^\alpha(0)}{\|\alpha_0^{old} - \tau \xi^\alpha(0)\|} \right)$$



- zero-out momenta small in magnitude

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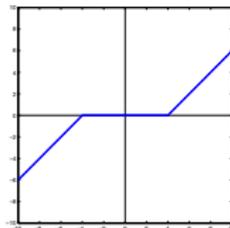
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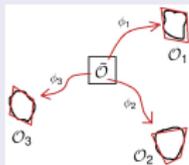


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Outline

- 1 Sparse adaptive parameterization of diffeomorphisms
- 2 Atlas construction with landmark points
- 3 Atlas construction with currents**
- 4 Atlas construction with images

Group-wise statistics



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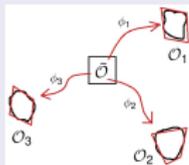
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- Landmark case:

- discrepancy measure: $\|X_s(1) - Y_s\|^2$
- driving force: $\nabla_{X_s(1)} \|X_s(1) - Y_s\|^2 = 2(X_s(1) - Y_s)$
- Needs point correspondence: hard to find (fiber bundles)
- Currents: an alternative for both curves and surfaces!

Group-wise statistics



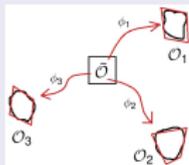
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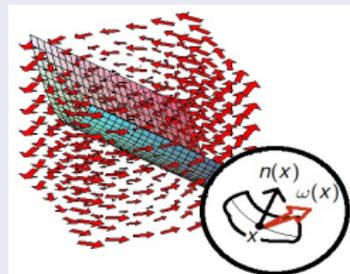
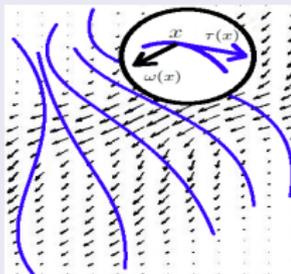
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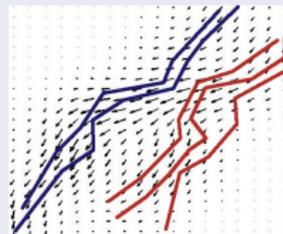
Shapes as currents: an object that integrates vector fields



$$F(\omega) = \sum_i \int_{F_i} \omega(x)^t \tau_i(x) dx \quad S(\omega) = \int_S \omega(x)^t n(x) d\sigma(x)$$

- Currents integrate vector fields: W (test space) $\longrightarrow \mathbb{R}$
- Vector space: addition = union, sign = orientation
- Distance **without point correspondence**:

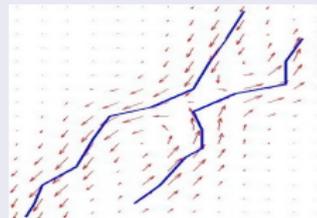
- $d(\mathcal{O}, \mathcal{O}') = \sup_{\|\omega\|_W \leq 1} |\mathcal{O}(\omega) - \mathcal{O}'(\omega)|$
- no point, no line correspondence
- robust to line interruption



[Vaillant and Glaunès IPMI'05, Glaunès PhD'06]

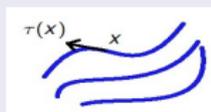
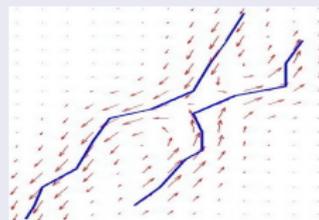
Numerical tools for currents

“Regularized” L^2 -metric: $W = L^2 * K$
 K Gaussian



Numerical tools for currents

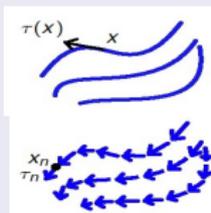
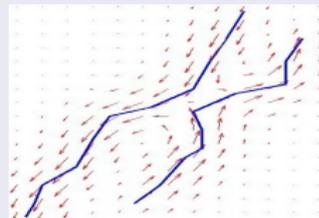
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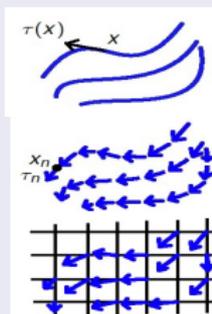
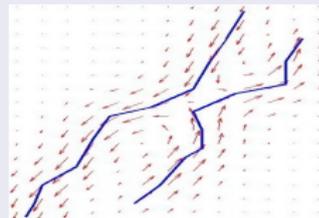


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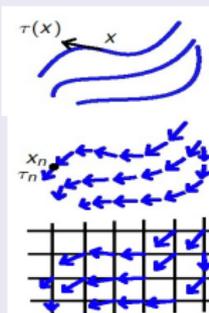
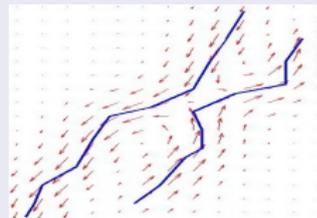
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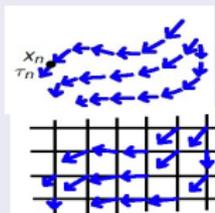
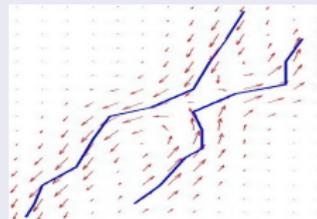
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	double sum	grid
complexity	$\mathcal{O}(N^2)$	$\mathcal{O}(N + N_{grid} \log(N_{grid}))$
approx. error	$\mathcal{O}(\max \tau_i)$	$\mathcal{O}(\Delta^2 / \lambda_W^2)$

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- discrepancy measure:

$$\|T_1 - T_2\|^2 = \|T_1\|^2 + \|T_2\|^2 - 2 \langle T_1, T_2 \rangle$$

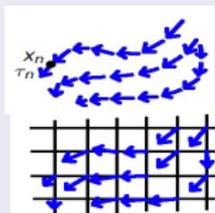
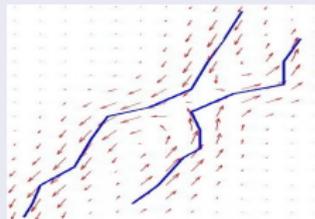
- driving force needs

$$\nabla_{x_k} \|T\|^2 = 2 \left(\sum_{i=1}^N \nabla_{x_k} K(x_k, x_i) \tau_i^T \right) \tau_k$$

- all computed with FFTs!

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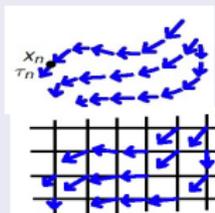
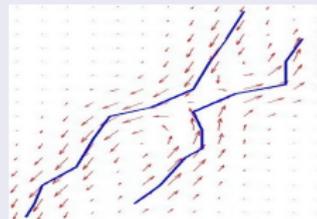
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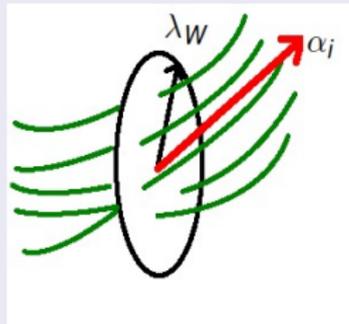
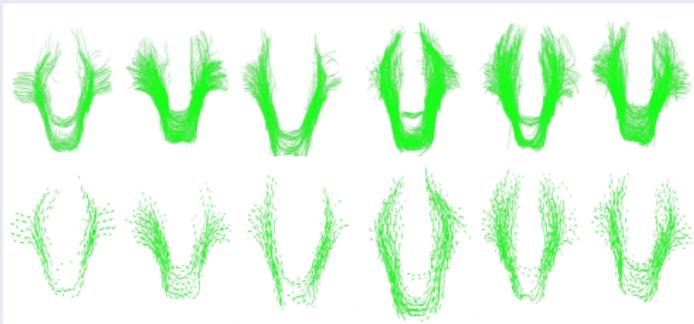
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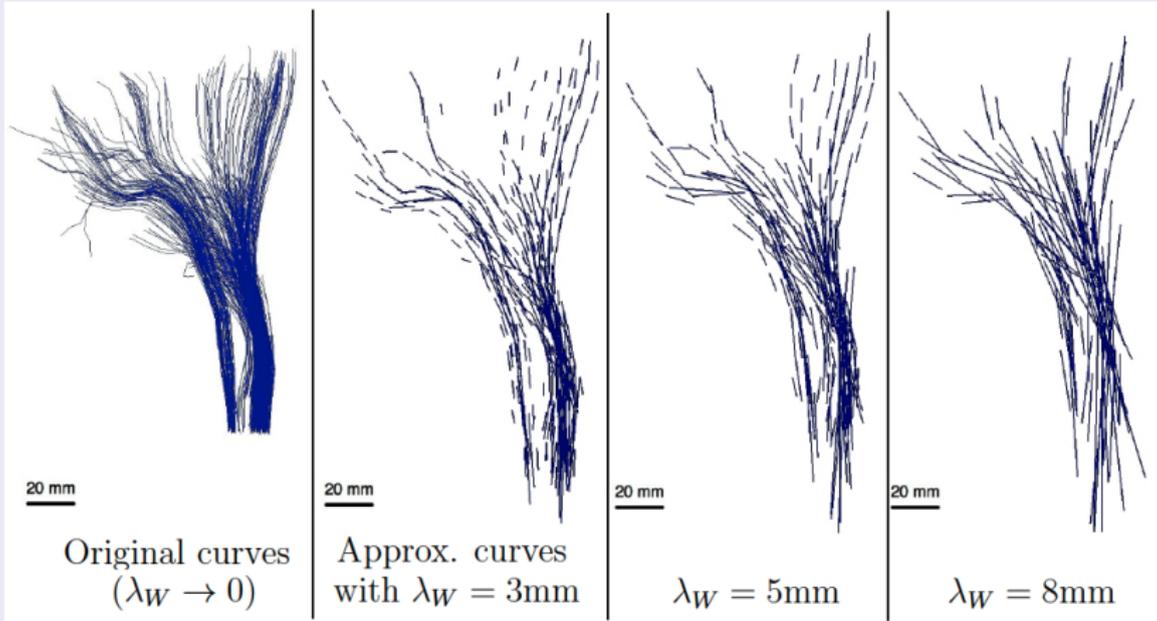


$\lambda_W = 3\text{mm}$, compression ratio: 85%, approx. error < 5% variance

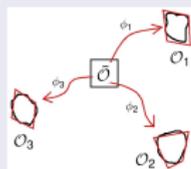
- Adjust the shape decomposition to the resolution
- via the search of adapted basis
- Iterative algorithm: matching pursuit
- proven to converge to the original currents

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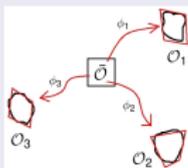
$$\dot{\theta}(t) = \partial_{\mathbf{S}(t)} \mathbf{G}^T \theta(t), \quad \theta(1) = \frac{1}{\sigma^2} (\mathbf{X}(1) - \mathbf{Y})$$

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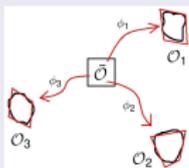
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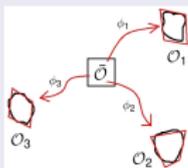
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$$\dot{\theta}(t) = \partial_{\mathbf{S}(t)} G^T \theta(t), \quad \theta(1) = \frac{1}{\sigma^2} \nabla_{\mathbf{X}_s(1)} \|\mathbf{X}_s(1) - \mathbf{Y}_s\|_{W^*}^2$$

$$\dot{\xi}(t) = \partial_{\mathbf{X}(t)} G^T \theta(t) + dF^T \xi(t), \quad \xi(1) = 0$$

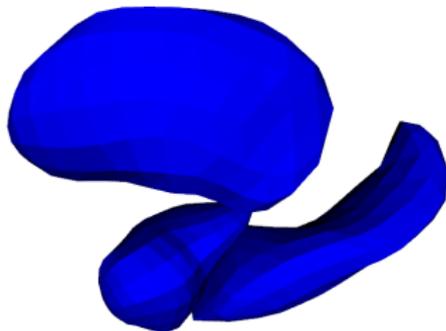
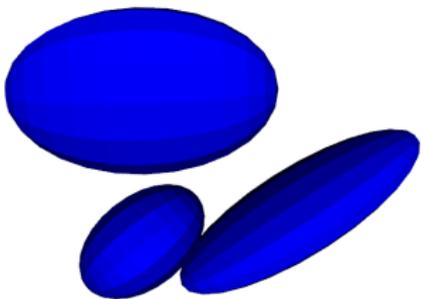
$$\boxed{\nabla_{\mathbf{S}_0} E = \xi(0) \quad \nabla_{\mathbf{X}_0} E = \theta(0)}$$

- Metric on currents to drive the atlas construction
- Only the positions of the vertices of the mesh are optimized
- The topology of the mesh is preserved during optimization!

Down's syndrome study



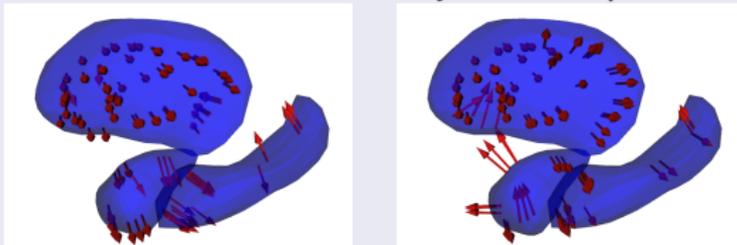
8 controls + 8 Down's syndrome patients



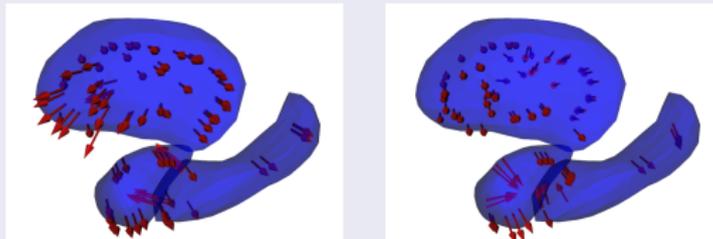
Down's syndrome study



8 controls + 8 Down's syndrome patients

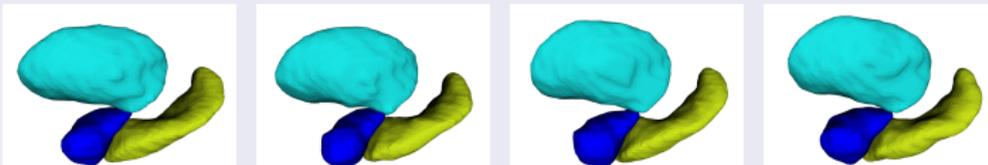


Deformation momenta to 2 Down's syndrome patients

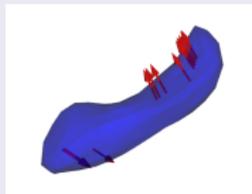
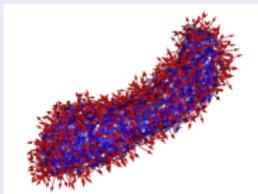
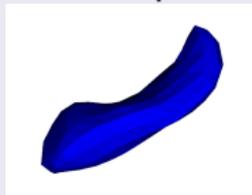


Deformation momenta to 2 control subjects

Down's syndrome study



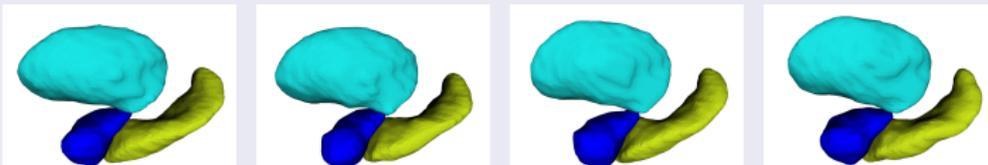
8 controls + 8 Down's syndrome patients



Durrleman Media 09

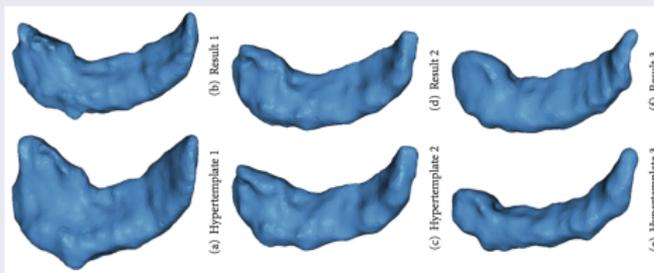
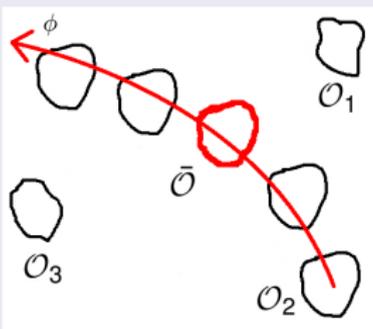
Our method

Down's syndrome study



8 controls + 8 Down's syndrome patients
 Comparison with Hypertemplate [Ma et al., 2010]

- constrained to be in the orbit of a given shape
- bias by the choice of the initial subject
- alternated minimization



Statistics on deformations

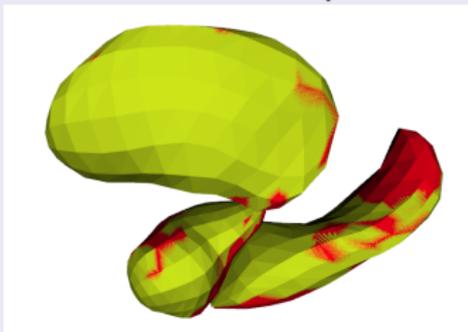
- Output of the atlas:
 - control points: $\mathbf{c} = \{c_k\}$
 - momenta for each subject: $\alpha^1, \dots, \alpha^N$ ($\alpha^i = \{\alpha_k^i\}$)
- Finite-dimensional RKHS with metric $\mathbf{K} = (K(c_i, c_j))_{i,j}$:
 $\langle \alpha^s, \alpha^u \rangle = \alpha^s \mathbf{K} \alpha^u$
- Fit a Gaussian distribution on the momenta set.
- Enables:
 - Principal Component Analysis (eigenmodes of $(\langle \alpha^i, \alpha^j \rangle)_{i,j}$)
 - Maximum-Likelihood Classification
 - Cross-validation

Down's syndrome study



8 controls + 8 Down's syndrome patients

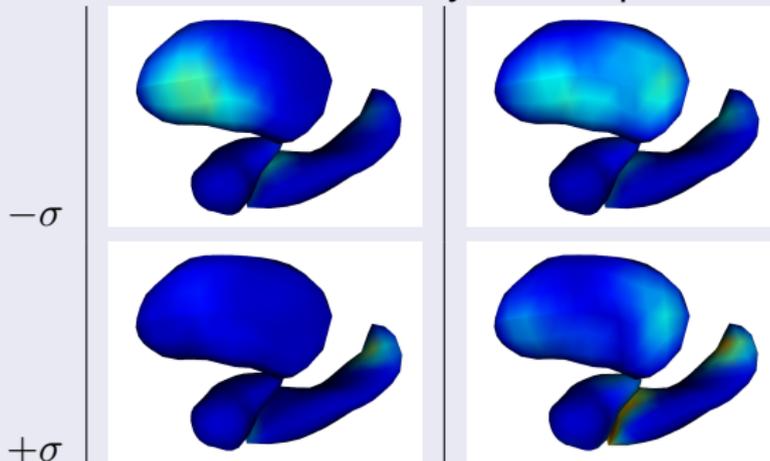
1 template for all, momenta pooled in 2 groups



Down's syndrome study



8 controls + 8 Down's syndrome patients



controls

patients

First deformation mode

Down's syndrome study



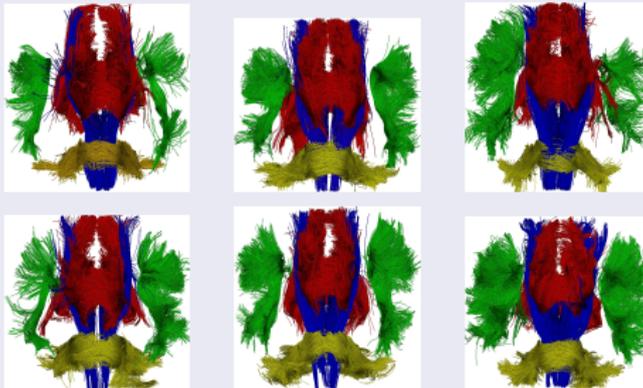
8 controls + 8 Down's syndrome patients

Classification based on initial momenta (leave-2-out):

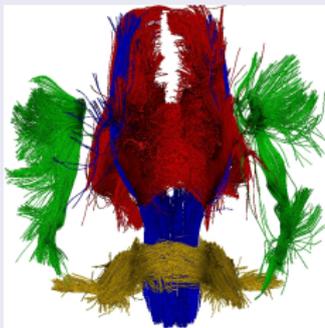
- False Positive Ratio: 3.1%
- True Positive Ratio: 87.5%

Statistical analysis of fiber tracts

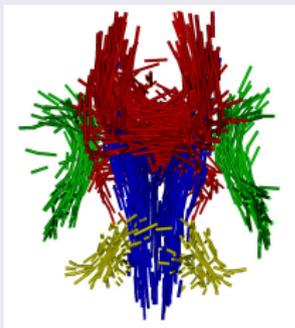
6 subjects - 5 tracts per subjects



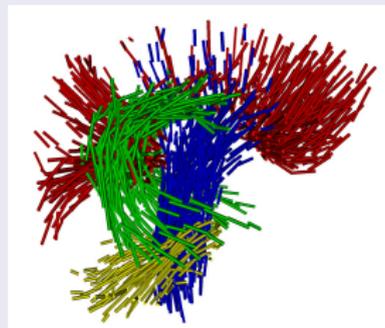
Statistical analysis of fiber tracts



(a) One subject
among 6



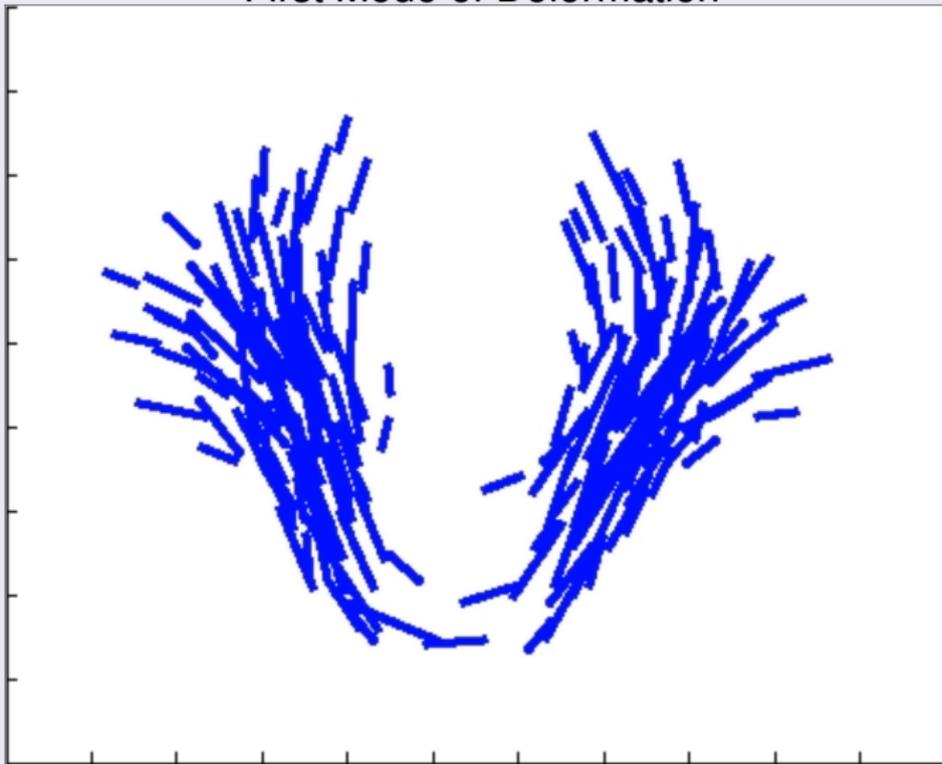
(b) template
(occipital view)



(c) template
(lateral view)

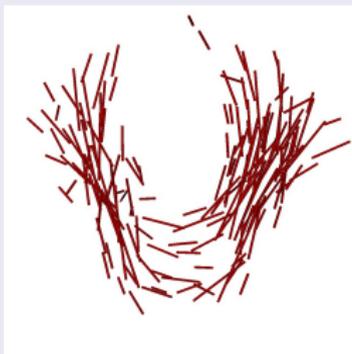
Statistical analysis of fiber tracts

First Mode of Deformation

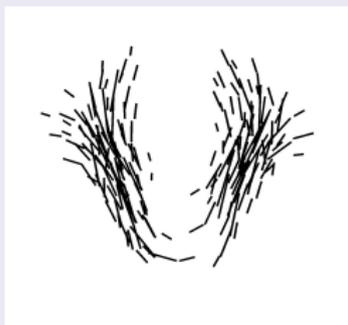


Statistical analysis of fiber tracts

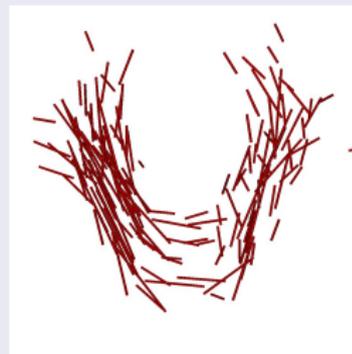
First "texture" Mode



texture mode at $-\sigma$
 $\bar{B} - m_\epsilon$



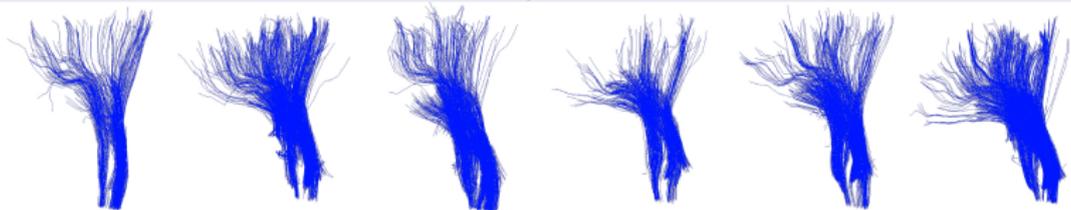
template
 \bar{B}



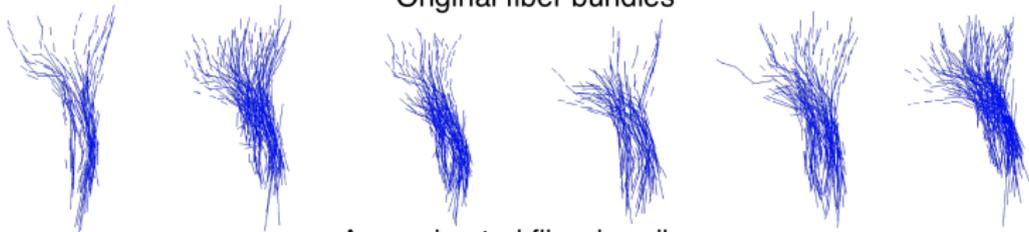
texture mode at $+\sigma$
 $\bar{B} + m_\epsilon$

Statistical analysis of fiber tracts

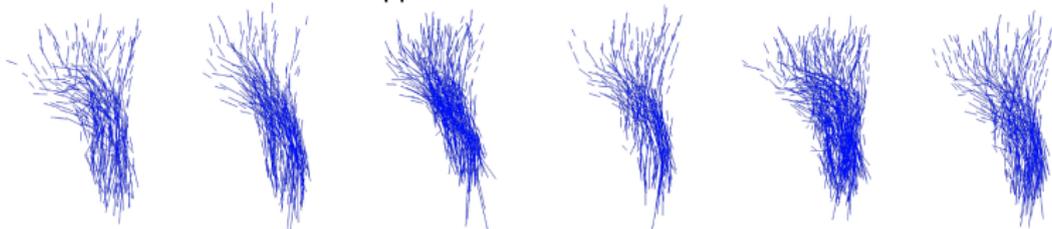
Simulation of synthetic bundles



Original fiber bundles



Approximated fiber bundles $\lambda_W = 5\text{mm}$

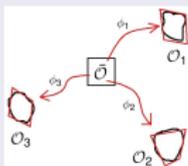


Six instances of simulated bundles

Outline

- 1 Sparse adaptive parameterization of diffeomorphisms
- 2 Atlas construction with landmark points
- 3 Atlas construction with currents
- 4 Atlas construction with images**

Group-wise statistics

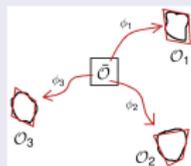


$$\mathcal{O}_j = \phi_j \cdot \bar{\mathcal{O}} + \varepsilon_j$$

$$E(X_0, \mathbf{S}_0^1, \dots, \mathbf{S}_0^N) = \sum_{s=1}^N \left\{ \frac{1}{2\sigma^2} \|X_s(1) - Y_s\|^2 + L(\mathbf{S}_0^s) + \gamma \sum_{k=1}^{N_{cp}} \|\alpha_{0,k}^s\| \right\}$$

$$s.t. \begin{cases} \dot{\mathbf{S}}_s(t) = F(\mathbf{S}_s(t)) \\ \mathbf{S}_s(0) = \mathbf{S}_0^s \end{cases} \quad \begin{cases} \dot{X}_s(t) = G(\mathbf{S}_s(t), X_s(t)) \\ X_s(0) = X_0 \end{cases}$$

Group-wise statistics



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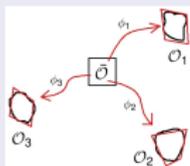
- Landmark case: $\mathcal{O}_s = Y_s = \{y_{s,k}\}$
 - Template of the form: $\bar{\mathcal{O}} = X_0 = \{x_{0,k}\}$
 - $\phi \cdot X_0 = \{\phi(x_{0,k})\}$ solution of:

$$\dot{x}_k(t) = v_t(x_k(t)) = \sum_{k=1}^{N_{cp}} K(x_k(t), c_k(t)) \alpha_k(t) = G(\mathbf{S}(t), x_k(t))$$

$$\phi \cdot X_0 = X(1) \text{ with } \begin{cases} \dot{X}(t) = G(\mathbf{S}(t), X(t)) \\ X(0) = X_0 \end{cases}$$

- Sum of squared differences: $\|X_s(1) - Y_s\|_{\mathbb{R}^N}^2$,
 $\nabla_{X(1)} \|X(1) - Y\|^2 = 2(X(1) - Y)$

Group-wise statistics



$$\mathcal{O}_j = \phi_j \cdot \bar{\mathcal{O}} + \varepsilon_j$$

$$E(X_0, \mathbf{S}_0^1, \dots, \mathbf{S}_0^N) = \sum_{s=1}^N \left\{ \frac{1}{2\sigma^2} \|X_s(1) - Y_s\|^2 + L(\mathbf{S}_0^s) + \gamma \sum_{k=1}^{N_{cp}} \|\alpha_{0,k}^s\| \right\}$$

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- **Landmark case:** $\mathcal{O}_s = Y_s = \{y_{s,k}\}$

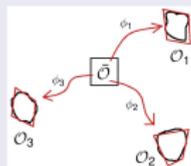
- Template of the form: $\bar{\mathcal{O}} = X_0 = \{x_{0,k}\}$
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Group-wise statistics



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- **Image case: $\mathcal{O}_s = \mathbf{I}_s$ arrays, domain discretization $\{y_k\}$**

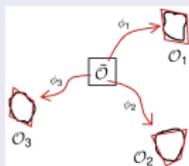
- Template of the form: $\bar{\mathcal{O}} = \mathbf{X}_0 = \{x_{0,k}\}$
- $\phi \cdot \mathbf{X}_0 = \{\phi(x_{0,k})\}$ solution of:

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 $\nabla_{\mathbf{X}(1)} \|\mathbf{X}(1) - \mathbf{Y}\|^2 = 2(\mathbf{X}(1) - \mathbf{Y})$

Group-wise statistics



$$\mathcal{O}_j = \phi_j \cdot \bar{\mathcal{O}} + \varepsilon_j$$

$$E(\mathbf{X}_0, \mathbf{S}_0^1, \dots, \mathbf{S}_0^N) = \sum_{s=1}^N \left\{ \frac{1}{2\sigma^2} \|\mathbf{X}_s(1) - I_s\|^2 + L(\mathbf{S}_0^s) + \gamma \sum_{k=1}^{N_{cp}} \|\alpha_{0,k}^s\| \right\}$$

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- Image case: $\mathcal{O}_s = I_s$ arrays, domain discretization $\{y_k\}$

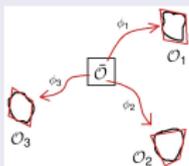
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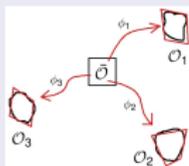
- Template of the form: $\bar{\mathcal{O}} = I_0$, an image
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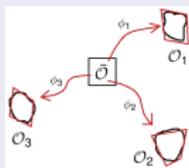
- Image case: $\mathcal{O}_s = I_s$ arrays, domain discretization $\{y_k\}$
 - Template of the form: $\bar{\mathcal{O}} = I_0$, an image
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Group-wise statistics



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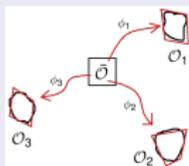
- Template of the form: $\bar{\mathcal{O}} = I_0$, an image
- $\phi \cdot I_0 = I_0 \circ \phi^{-1}$ with $\phi^{-1}(y_k)$ solution of:

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Group-wise statistics



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$$E(I_0, \mathbf{S}_0^1, \dots, \mathbf{S}_0^N) = \sum_{s=1}^N \left\{ \frac{1}{2\sigma^2} \|X_s(1) - I_s\|^2 + L(\mathbf{S}_0^s) + \gamma \sum_{k=1}^{N_{cp}} \|\alpha_{0,k}^s\| \right\}$$

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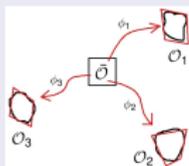
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Group-wise statistics



$$\mathcal{O}_j = \phi_j \cdot \bar{\mathcal{O}} + \varepsilon_j$$

$$E(I_0, \mathbf{S}_0^1, \dots, \mathbf{S}_0^N) = \sum_{s=1}^N \left\{ \frac{1}{2\sigma^2} \|X_s(1) - I_s\|^2 + L(\mathbf{S}_0^s) + \gamma \sum_{k=1}^{N_{cp}} \|\alpha_{0,k}^s\| \right\}$$

$$s.t. \begin{cases} \dot{\mathbf{S}}_s(t) = F(\mathbf{S}_s(t)) \\ \mathbf{S}_s(0) = \mathbf{S}_0^s \end{cases} \quad \begin{cases} \dot{X}_s(t) = G(\mathbf{S}_s(t), X_s(t)) \\ X_s(0) = X_0 \end{cases}$$

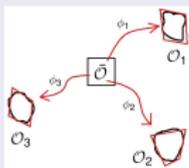
- Image case: $\mathcal{O}_s = I_s$ arrays, domain discretization $\{y_k\}$
 - Template of the form: $\bar{\mathcal{O}} = I_0$, an image
 - $\phi \cdot I_0 = I_0 \circ \phi^{-1}$ with $\phi^{-1}(y_k)$ solution of:

$$\dot{y}_k(t) = -v_t(y_k(t)) = - \sum_{k=1}^{N_{cp}} K(y_k(t), c_k(t)) \alpha_k(t)$$

$$\phi \cdot X_0 = X(1) \text{ with } \begin{cases} \dot{X}(t) = G(\mathbf{S}(t), X(t)) \\ X(0) = X_0 \end{cases}$$

- Sum of squared differences: $\|X_s(1) - Y_s\|_{\mathbb{R}^N}^2$,
 $\nabla_{X(1)} \|X(1) - Y\|^2 = 2(X(1) - Y)$

Group-wise statistics



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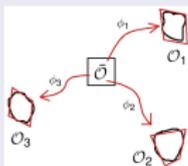
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Group-wise statistics



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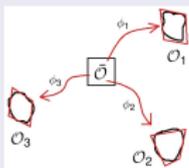
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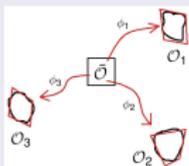
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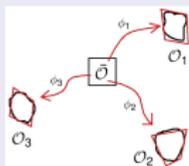
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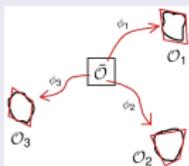
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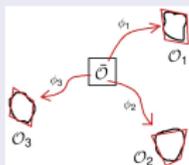
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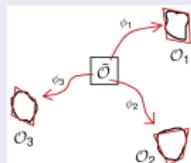
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- Sum of squared differences: $\|I_0(Y_s(0)) - I_s\|_{L^2}^2$,

$$\nabla_{Y_s(0)} \|I_0(Y_s(0)) - I_s\|^2 = 2(I_0(Y_s(0)) - I_s) \nabla_{Y_s(0)} I_0$$

Group-wise statistics

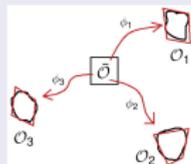


$$O_j = \phi_j \cdot O_0 + \varepsilon_j$$

$$E(I_0, \mathbf{S}_0^1, \dots, \mathbf{S}_0^N) = \sum_{s=1}^N \left\{ \frac{1}{2\sigma^2} \|I_0(Y_s(0)) - I_s\|^2 + L(\mathbf{S}_0^s) + \gamma \sum_{k=1}^{N_{cp}} \left\| \alpha_{0,k}^s \right\| \right\}$$

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Group-wise statistics



$$O_j = \phi_j \cdot O + \varepsilon_j$$

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- w/o L^1 prior:

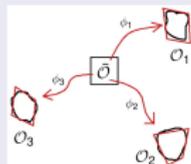
$$\dot{\eta}(t) = -\partial_{\mathbf{S}(t)} G^T \eta(t), \quad \eta(0) = \frac{1}{\sigma^2} (I_0(Y_s(0)) - I_s) \nabla I_0$$

$$\dot{\xi}(t) = -\partial_{Y(t)} G^T \eta(t) - dF^T \xi(t), \quad \xi(1) = 0$$

$$\nabla_{\mathbf{S}_0} E = \xi(0) + \nabla_{\mathbf{S}_0} L$$

- with L^1 prior: gradient soft-thresholded

Group-wise statistics



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$$E(I_0, \mathbf{S}_0^1, \dots, \mathbf{S}_0^N) = \sum_{s=1}^N \left\{ \frac{1}{2\sigma^2} \|I_0(Y_s(0)) - I_s\|^2 + L(\mathbf{S}_0^s) + \gamma \sum_{k=1}^{N_{cp}} \left\| \alpha_{0,k}^s \right\| \right\}$$

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- w/o L^1 prior:

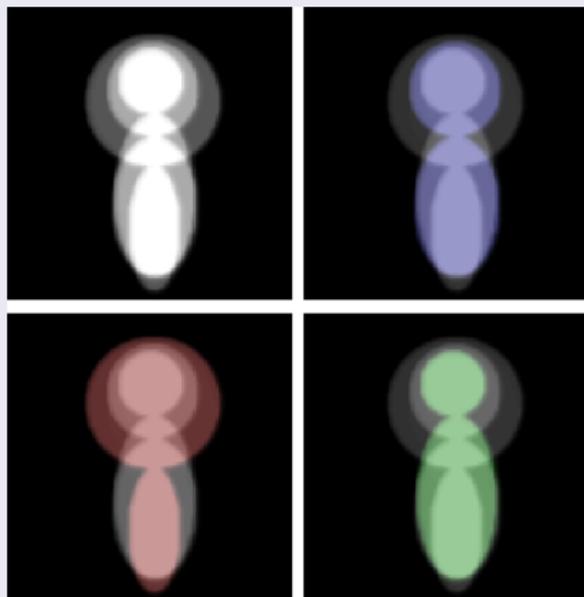
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- with L^1 prior: gradient soft-thresholded

Atlas construction: optimization

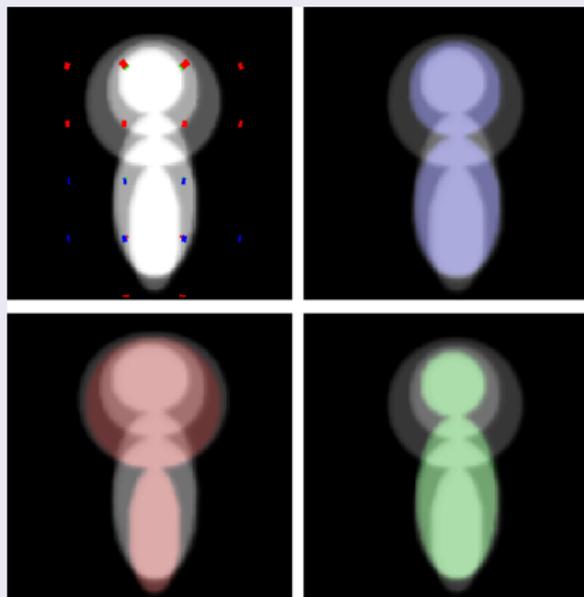


Single gradient descent:

- template image
- position of CP
- number of CP
- momenta

Image size=128², $\sigma_V = 25$, $\sigma^2 = 0.005$, $\gamma = 540$

Atlas construction: optimization

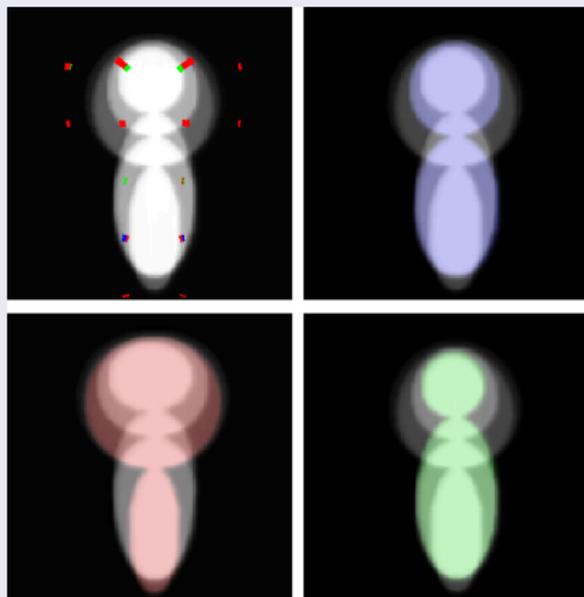


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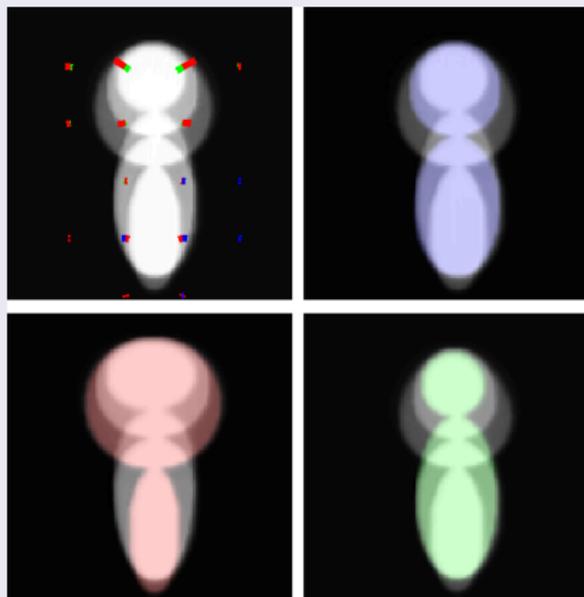


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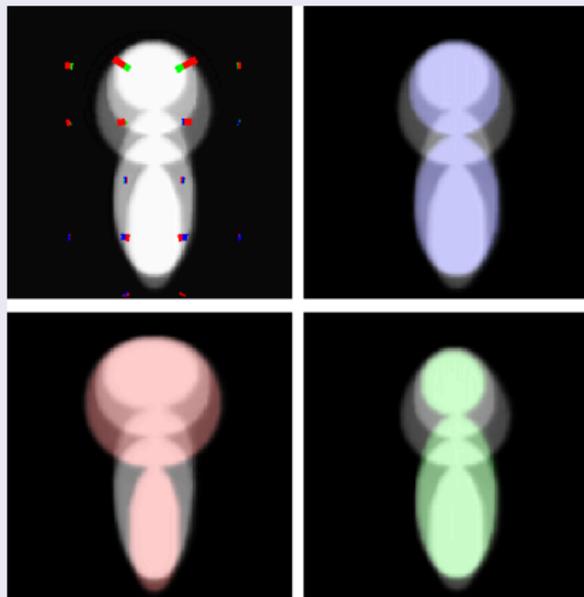


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Image size= 128^2 , $\sigma_V = 25$, $\sigma^2 = 0.005$, $\gamma = 540$

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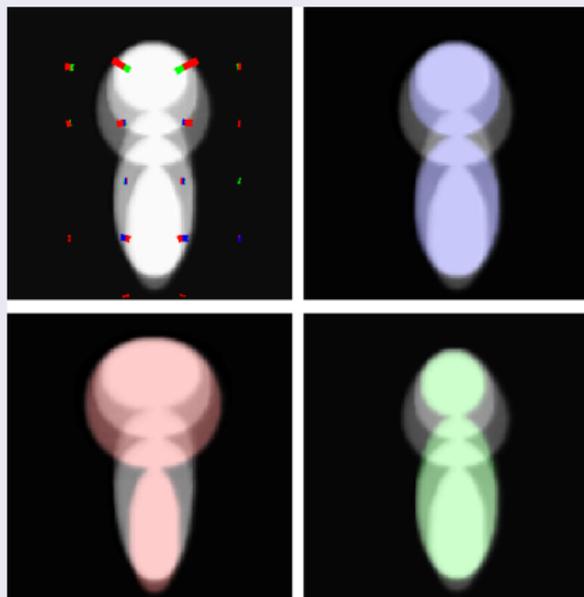


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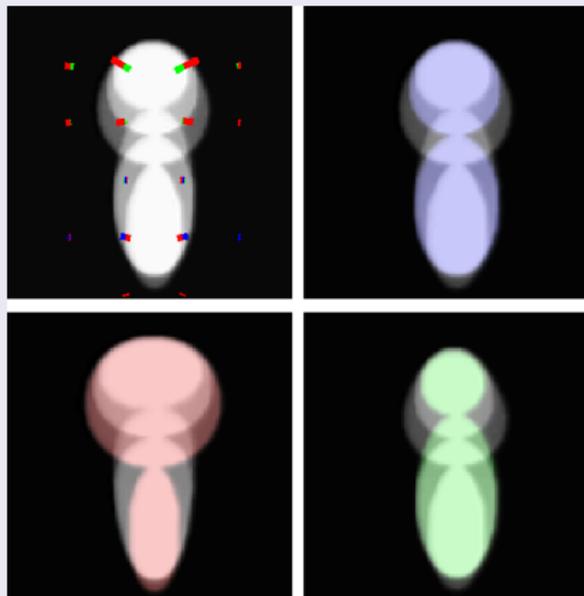


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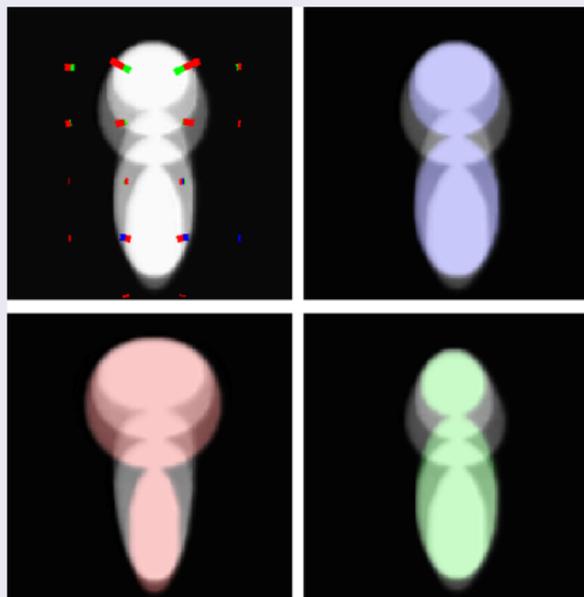


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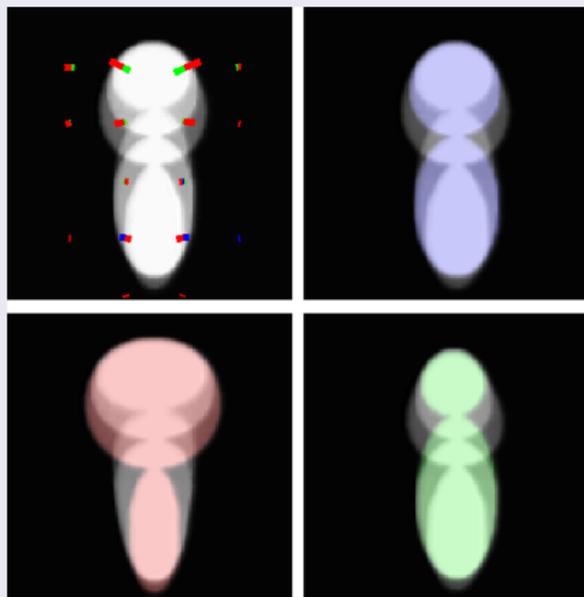


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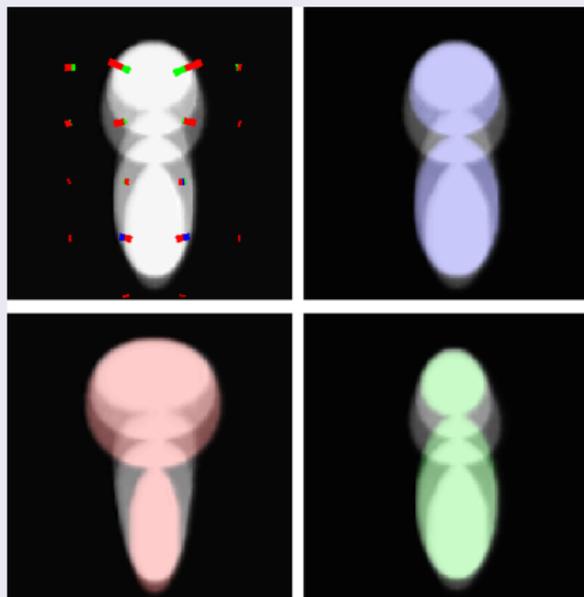


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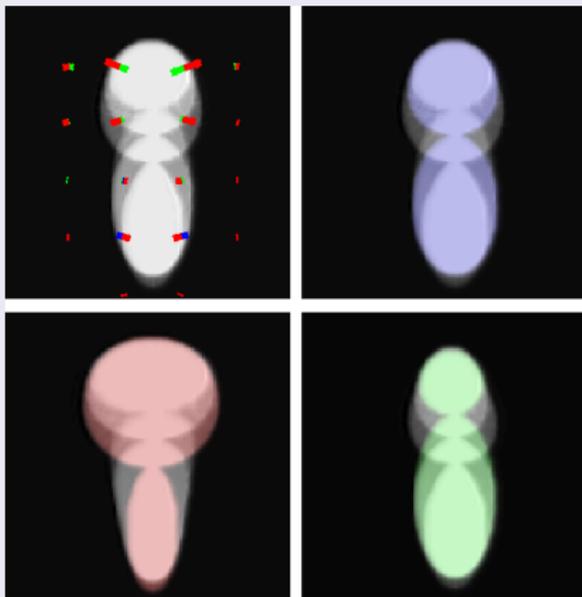


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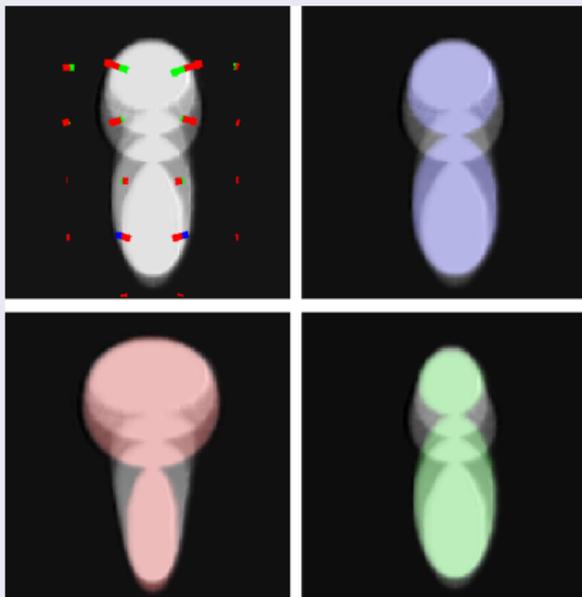


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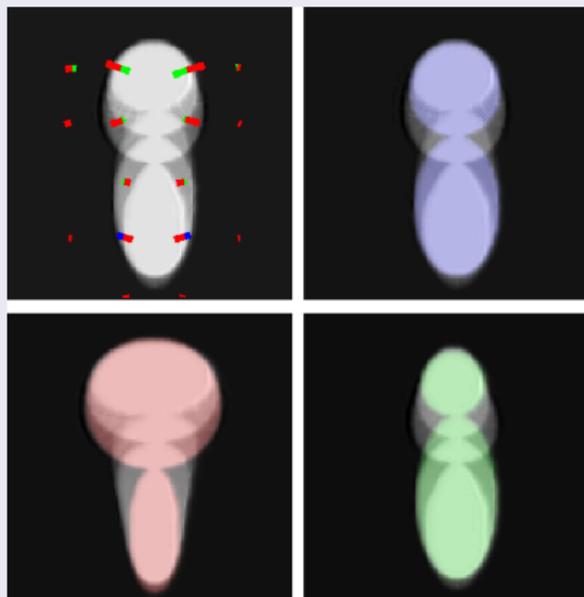


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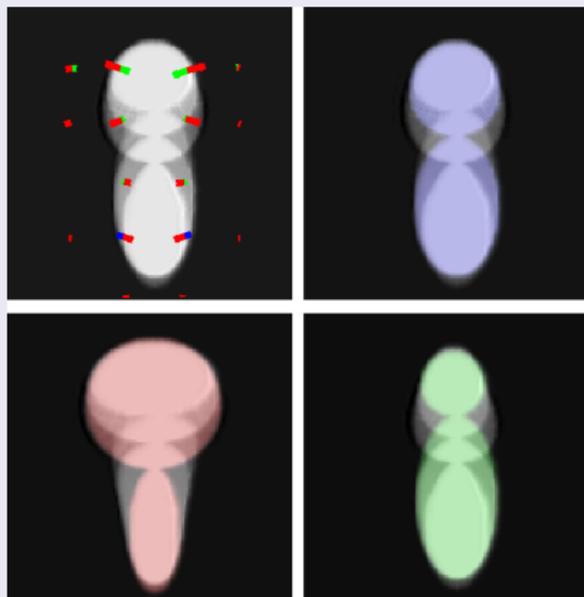


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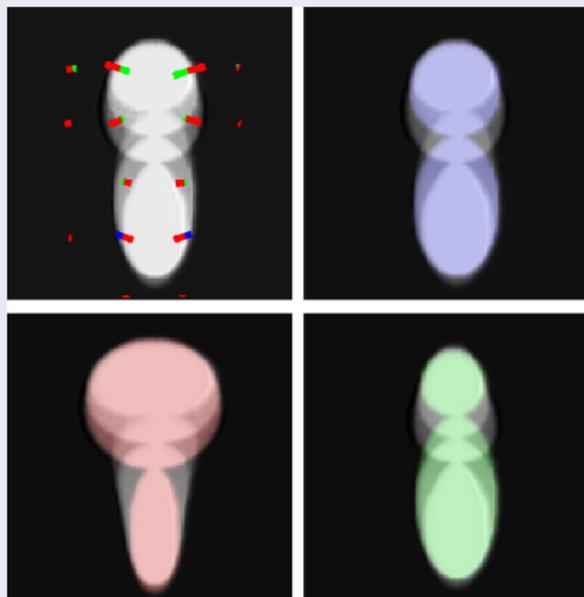


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Image size=128², $\sigma_V = 25$, $\sigma^2 = 0.005$, $\gamma = 540$

Atlas construction: optimization

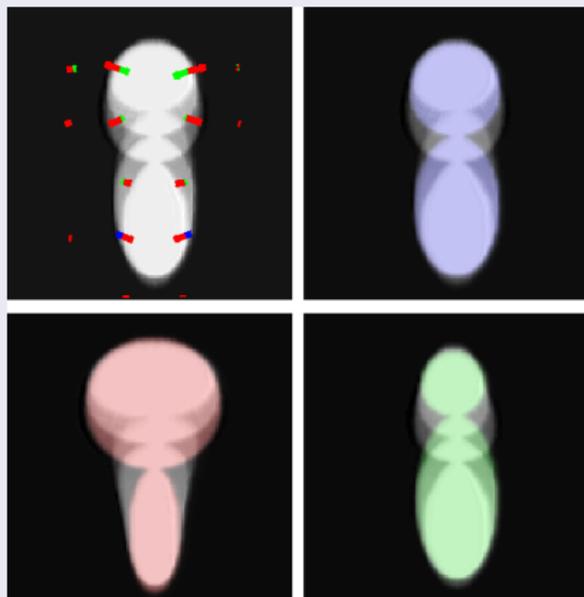


Single gradient descent:

- template image
- position of CP
- number of CP
- momenta

Image size= 128^2 , $\sigma_V = 25$, $\sigma^2 = 0.005$, $\gamma = 540$

Atlas construction: optimization

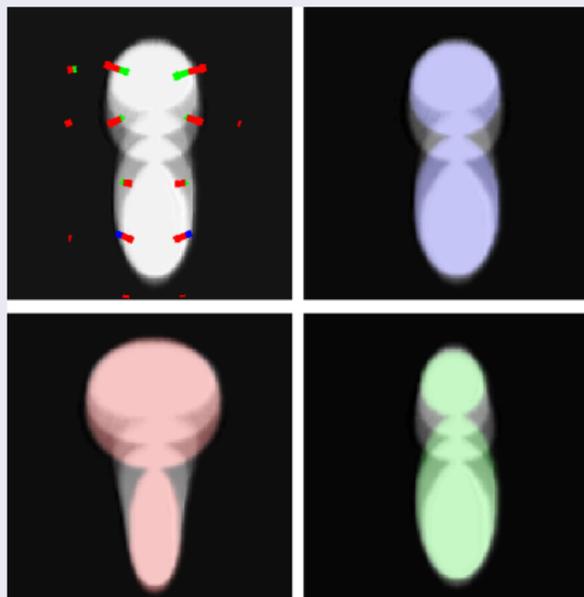


Single gradient descent:

- template image
- position of CP
- number of CP
- momenta

Image size= 128^2 , $\sigma_V = 25$, $\sigma^2 = 0.005$, $\gamma = 540$

Atlas construction: optimization

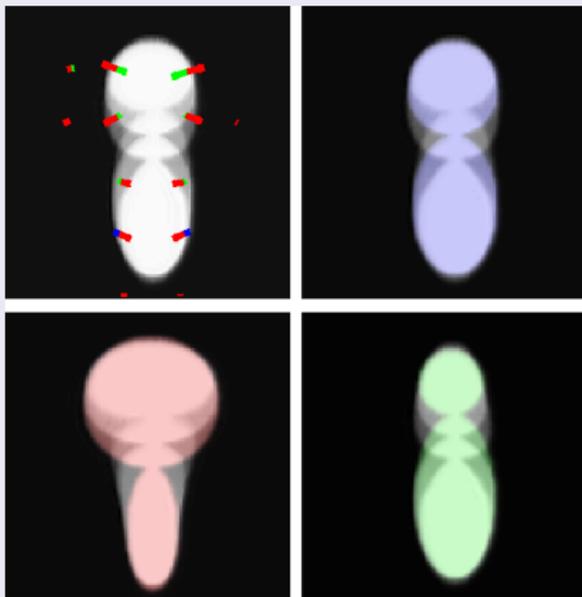


Single gradient descent:

- template image
- position of CP
- number of CP
- momenta

Image size= 128^2 , $\sigma_V = 25$, $\sigma^2 = 0.005$, $\gamma = 540$

Atlas construction: optimization

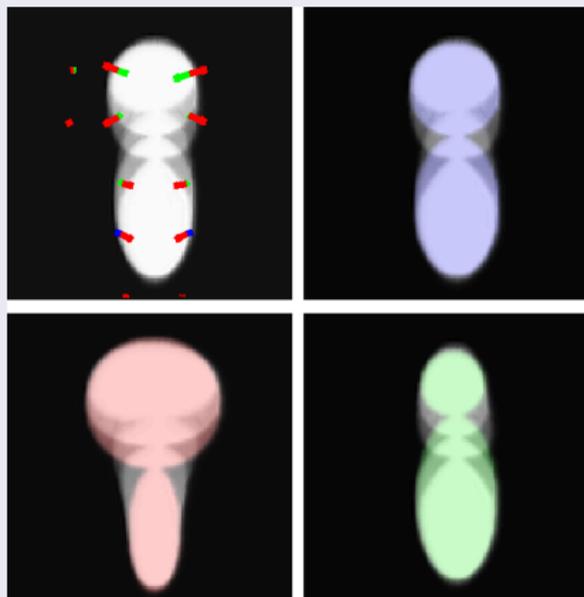


Single gradient descent:

- template image
- position of CP
- number of CP
- momenta

Image size= 128^2 , $\sigma_V = 25$, $\sigma^2 = 0.005$, $\gamma = 540$

Atlas construction: optimization

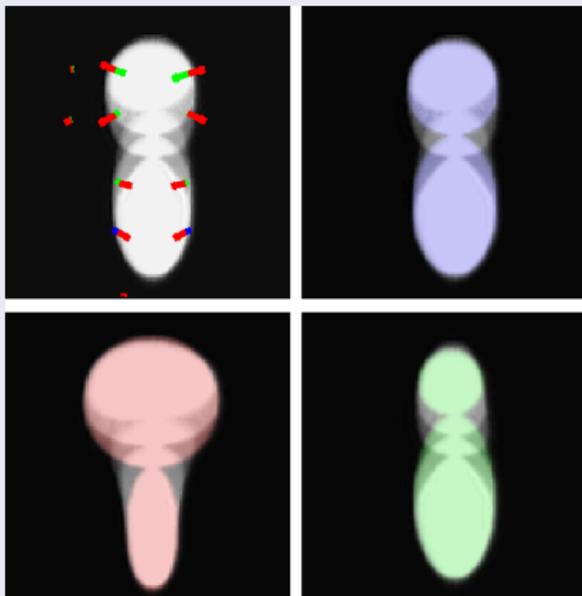


Single gradient descent:

- template image
- position of CP
- number of CP
- momenta

Image size= 128^2 , $\sigma_V = 25$, $\sigma^2 = 0.005$, $\gamma = 540$

Atlas construction: optimization

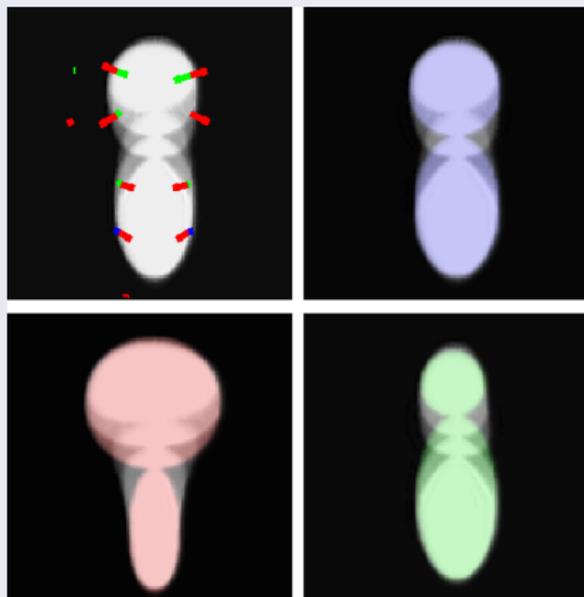


Single gradient descent:

- template image
- position of CP
- number of CP
- momenta

Image size= 128^2 , $\sigma_V = 25$, $\sigma^2 = 0.005$, $\gamma = 540$

Atlas construction: optimization

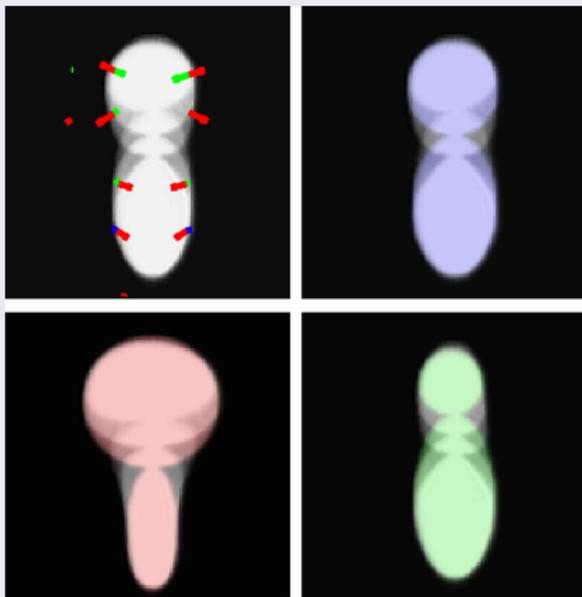


Single gradient descent:

- template image
- position of CP
- number of CP
- momenta

Image size=128², $\sigma_V = 25$, $\sigma^2 = 0.005$, $\gamma = 540$

Atlas construction: optimization

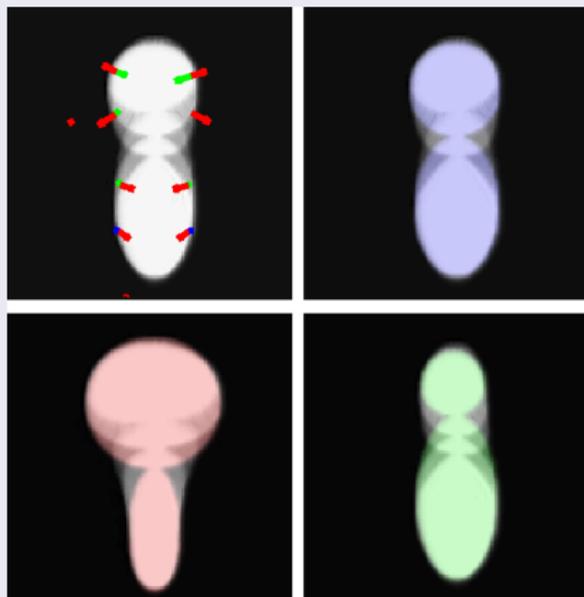


Single gradient descent:

- template image
- position of CP
- number of CP
- momenta

Image size=128², $\sigma_V = 25$, $\sigma^2 = 0.005$, $\gamma = 540$

Atlas construction: optimization

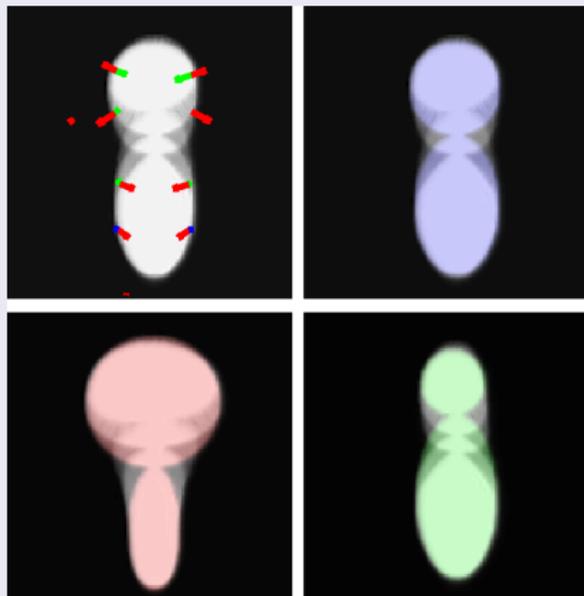


Single gradient descent:

- template image
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- momenta

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Atlas construction: optimization

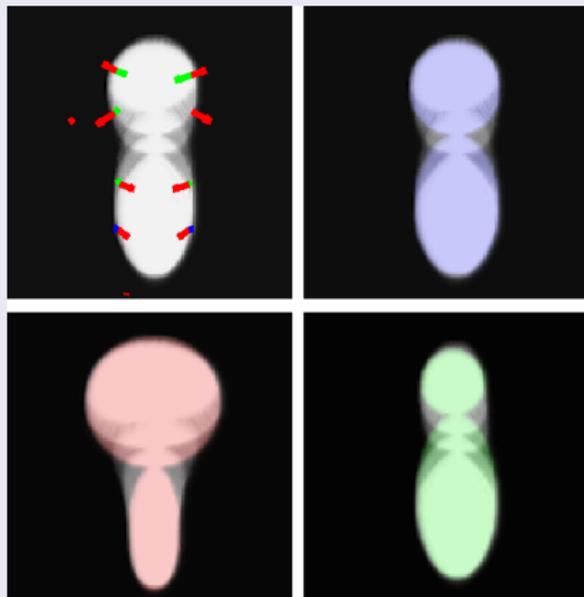


Single gradient descent:

- template image
- position of CP
- number of CP
- momenta

Image size=128², $\sigma_V = 25$, $\sigma^2 = 0.005$, $\gamma = 540$

Atlas construction: optimization

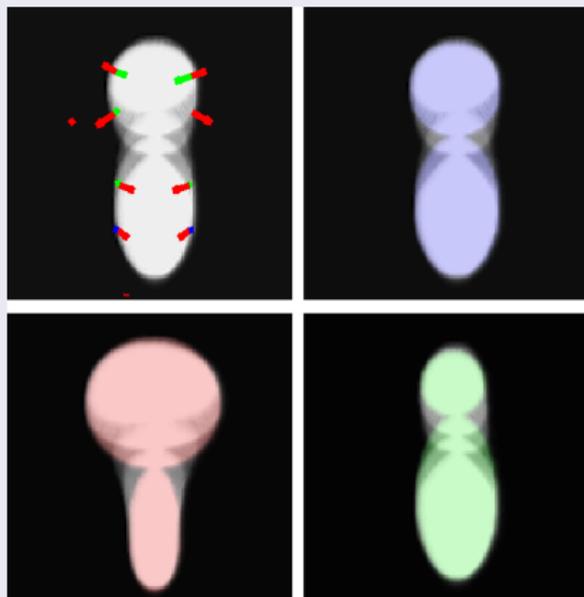


Single gradient descent:

- template image
- position of CP
- number of CP
- momenta

Image size=128², $\sigma_V = 25$, $\sigma^2 = 0.005$, $\gamma = 540$

Atlas construction: optimization

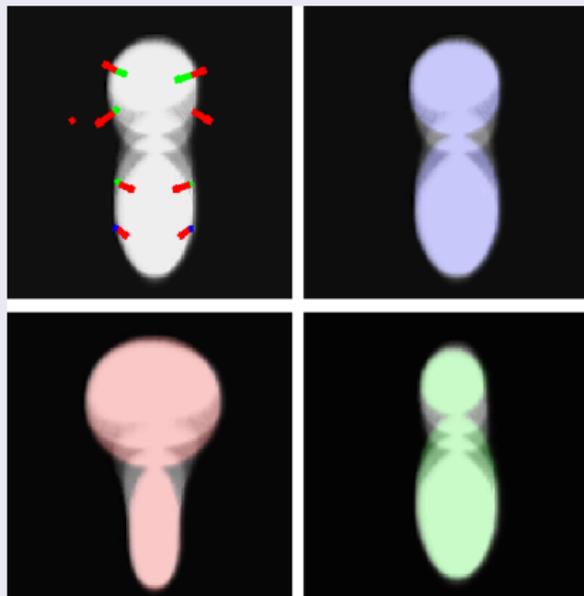


Single gradient descent:

- template image
- position of CP
- number of CP
- momenta

Image size= 128^2 , $\sigma_V = 25$, $\sigma^2 = 0.005$, $\gamma = 540$

Atlas construction: optimization

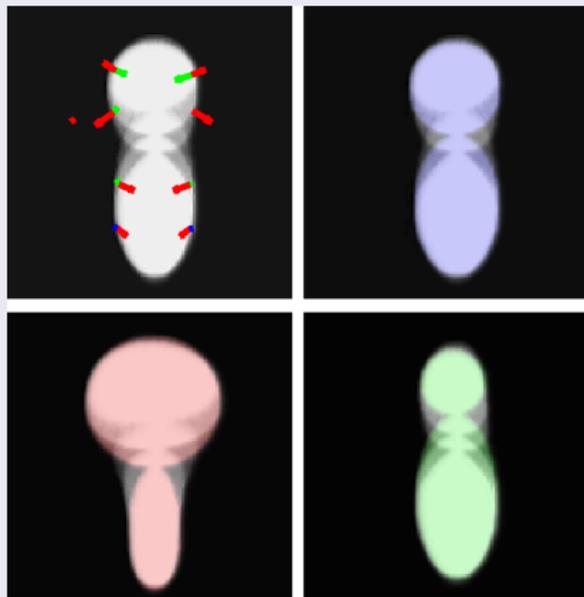


Single gradient descent:

- template image
- position of CP
- number of CP
- momenta

Image size= 128^2 , $\sigma_V = 25$, $\sigma^2 = 0.005$, $\gamma = 540$

Atlas construction: optimization

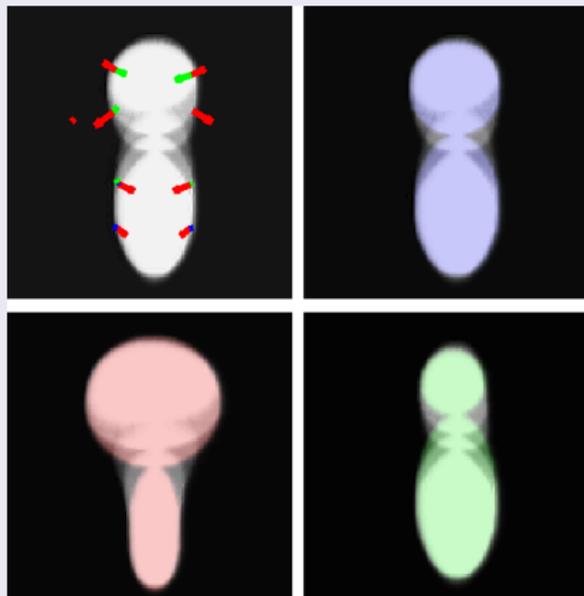


Single gradient descent:

- template image
- position of CP
- number of CP
- momenta

Image size=128², $\sigma_V = 25$, $\sigma^2 = 0.005$, $\gamma = 540$

Atlas construction: optimization

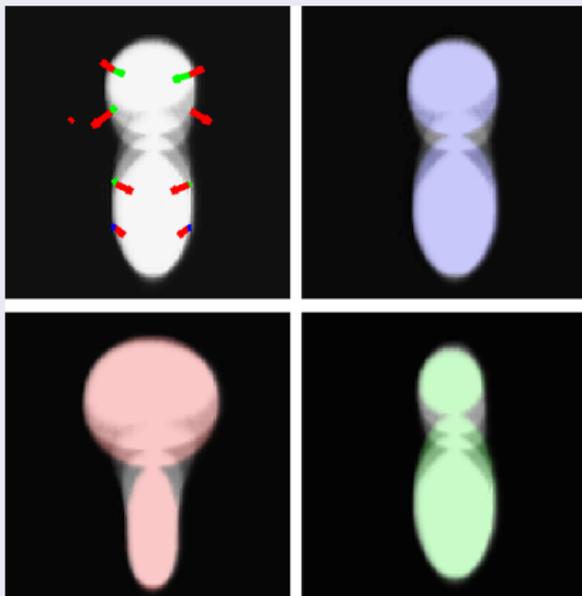


Single gradient descent:

- template image
- position of CP
- number of CP
- momenta

Image size=128², $\sigma_V = 25$, $\sigma^2 = 0.005$, $\gamma = 540$

Atlas construction: optimization

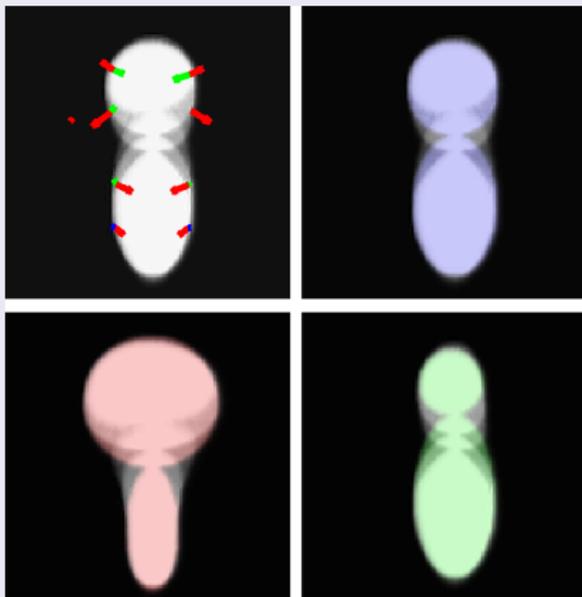


Single gradient descent:

- template image
- position of CP
- number of CP
- momenta

Image size= 128^2 , $\sigma_V = 25$, $\sigma^2 = 0.005$, $\gamma = 540$

Atlas construction: optimization

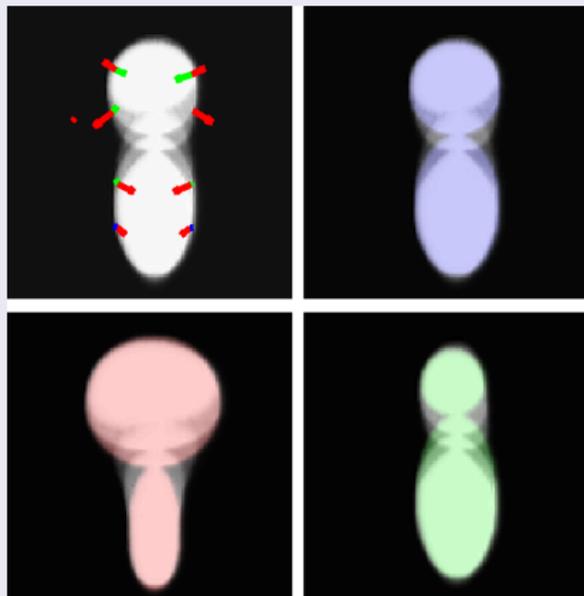


Single gradient descent:

- template image
- position of CP
- number of CP
- momenta

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Atlas construction: optimization

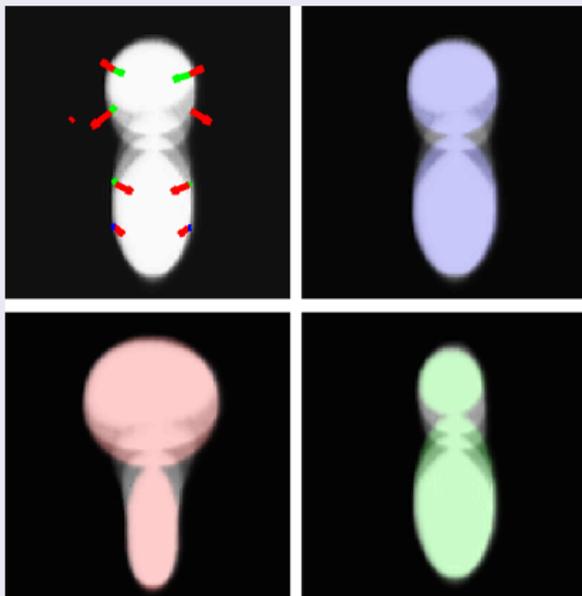


Single gradient descent:

- template image
- position of CP
- number of CP
- momenta

Image size= 128^2 , $\sigma_V = 25$, $\sigma^2 = 0.005$, $\gamma = 540$

Atlas construction: optimization

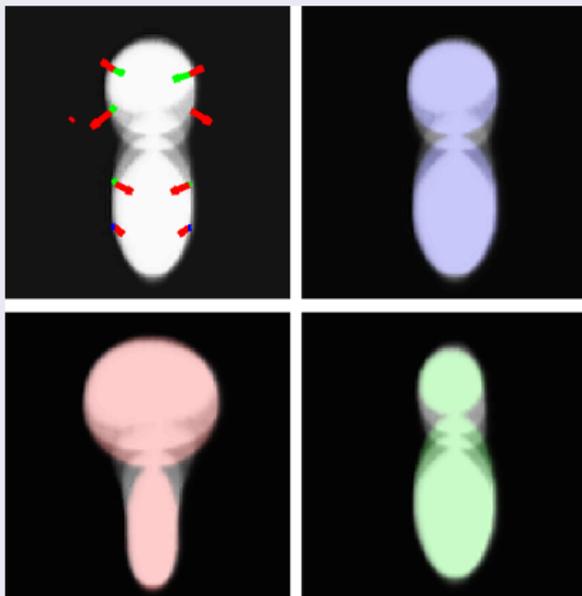


Single gradient descent:

- template image
- position of CP
- number of CP
- momenta

Image size= 128^2 , $\sigma_V = 25$, $\sigma^2 = 0.005$, $\gamma = 540$

Atlas construction: optimization

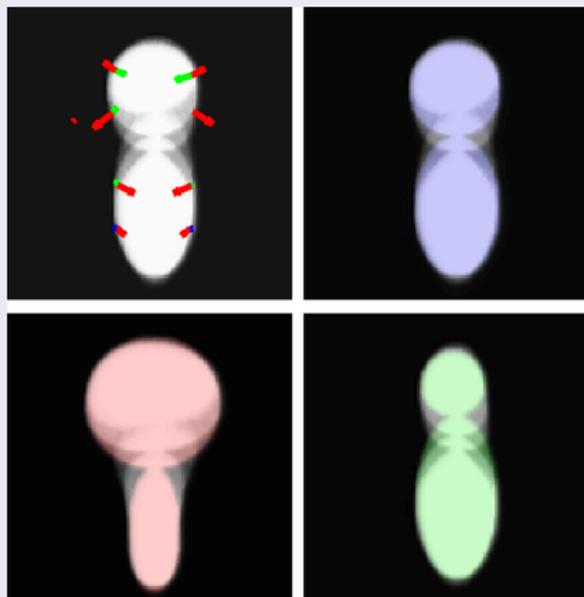


Single gradient descent:

- template image
- position of CP
- number of CP
- momenta

Image size=128², $\sigma_V = 25$, $\sigma^2 = 0.005$, $\gamma = 540$

Atlas construction: optimization

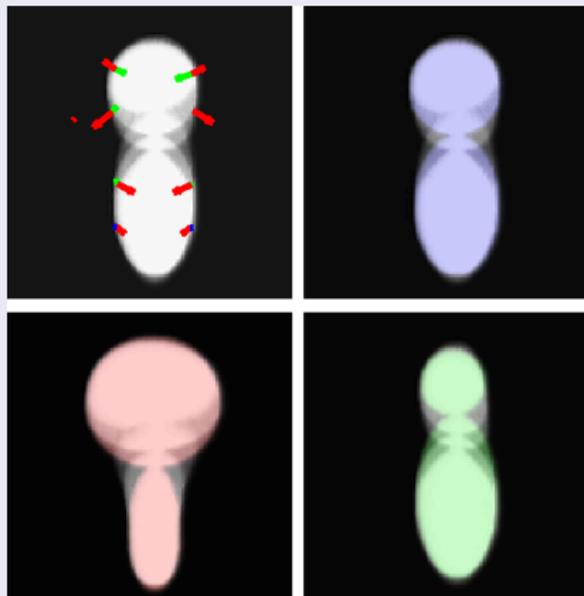


Single gradient descent:

- template image
- position of CP
- number of CP
- momenta

Image size=128², $\sigma_V = 25$, $\sigma^2 = 0.005$, $\gamma = 540$

Atlas construction: optimization

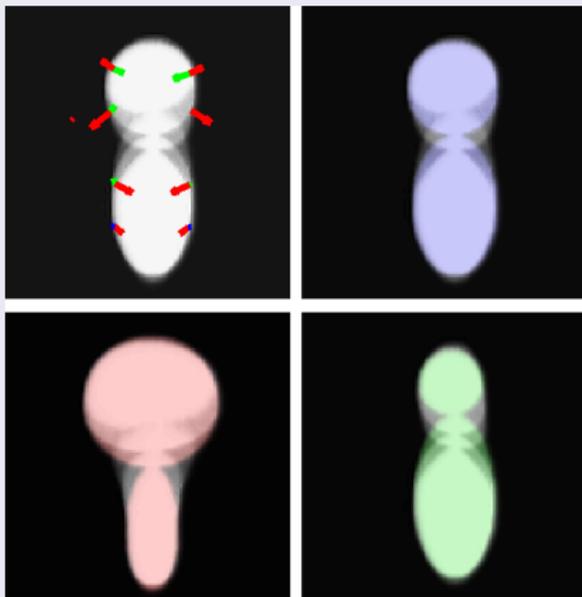


Single gradient descent:

- template image
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- number of CP
- momenta

Image size= 128^2 , $\sigma_V = 25$, $\sigma^2 = 0.005$, $\gamma = 540$

Atlas construction: optimization

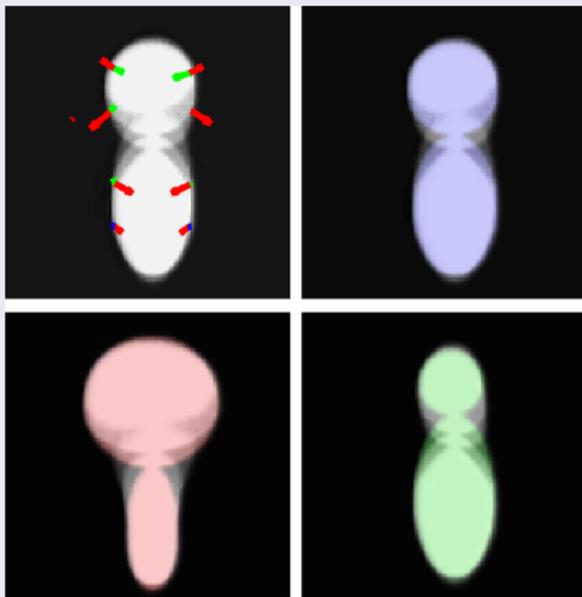


Single gradient descent:

- template image
- position of CP
- number of CP
- momenta

Image size= 128^2 , $\sigma_V = 25$, $\sigma^2 = 0.005$, $\gamma = 540$

Atlas construction: optimization

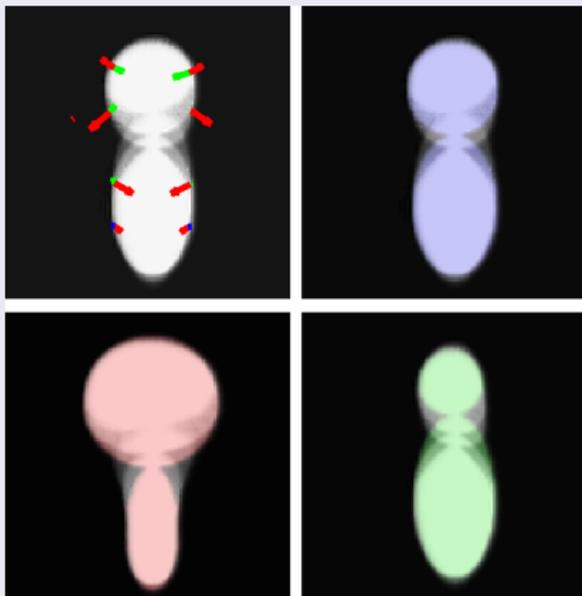


Single gradient descent:

- template image
- position of CP
- number of CP
- momenta

Image size=128², $\sigma_V = 25$, $\sigma^2 = 0.005$, $\gamma = 540$

Atlas construction: optimization

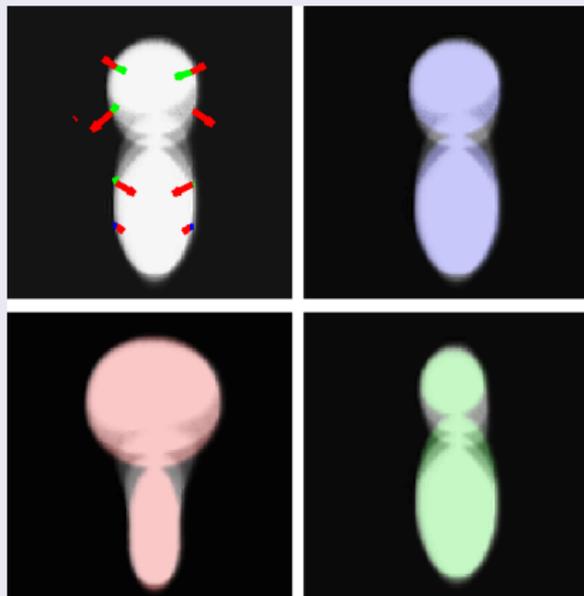


Single gradient descent:

- template image
- position of CP
- number of CP
- momenta

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Atlas construction: optimization

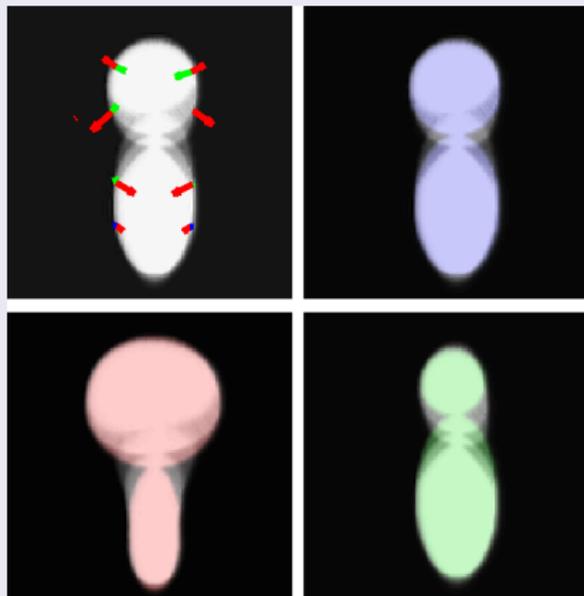


Single gradient descent:

- template image
- position of CP
- number of CP
- momenta

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Atlas construction: optimization

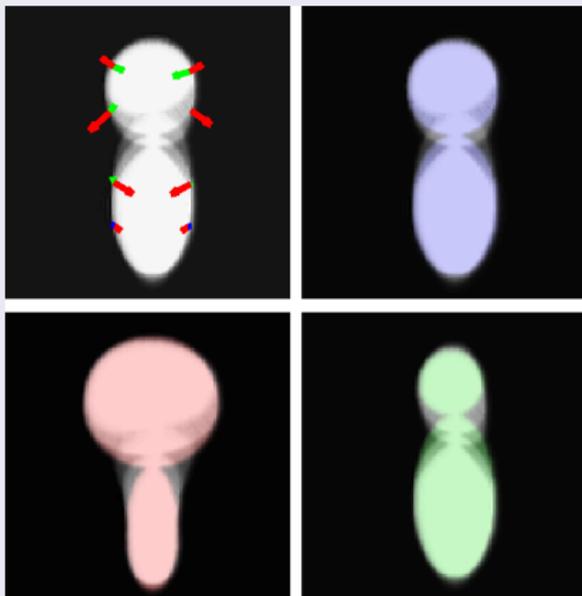


Single gradient descent:

- template image
- position of CP
- number of CP
- momenta

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Atlas construction: optimization

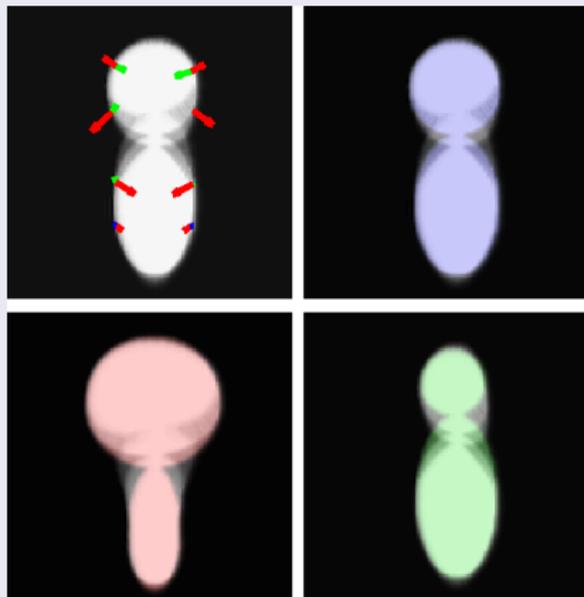


Single gradient descent:

- template image
- position of CP
- number of CP
- momenta

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Atlas construction: optimization

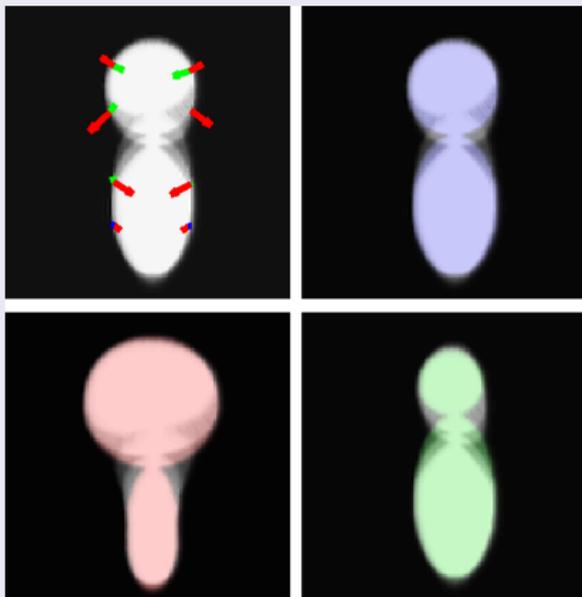


Single gradient descent:

- template image
- position of CP
- number of CP
- momenta

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Atlas construction: optimization

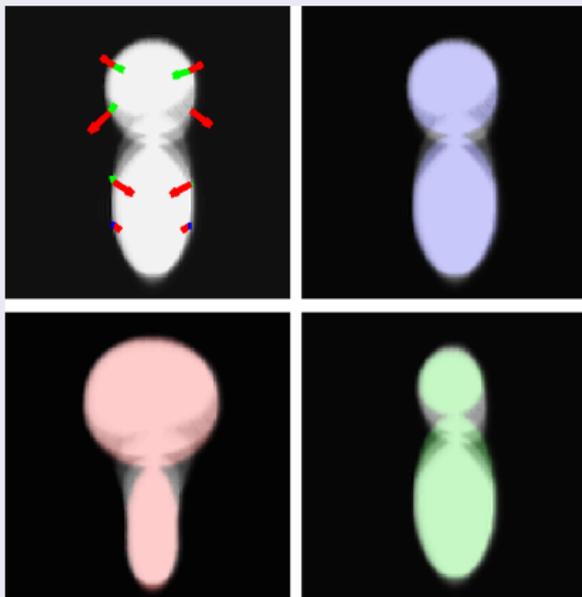


Single gradient descent:

- template image
- position of CP
- number of CP
- momenta

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Atlas construction: optimization

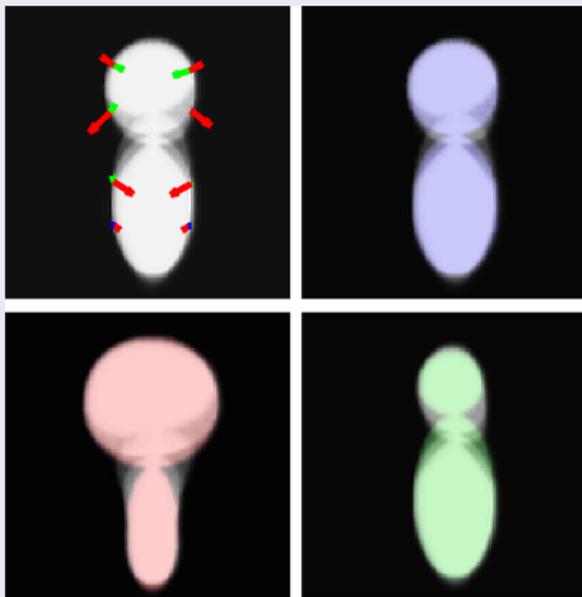


Single gradient descent:

- template image
- position of CP
- number of CP
- momenta

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Atlas construction: optimization

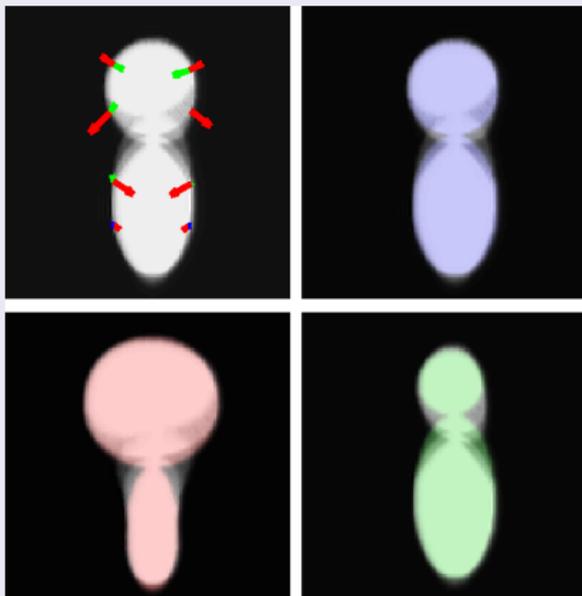


Single gradient descent:

- template image
- position of CP
- number of CP
- momenta

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Atlas construction: optimization

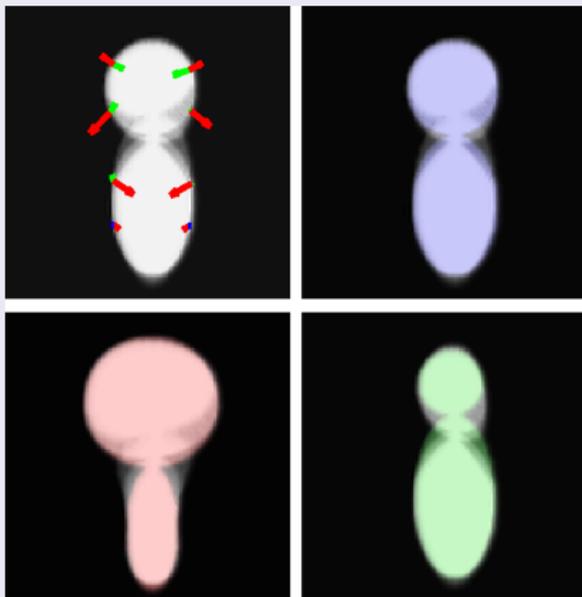


Single gradient descent:

- template image
- position of CP
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- momenta

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Atlas construction: optimization

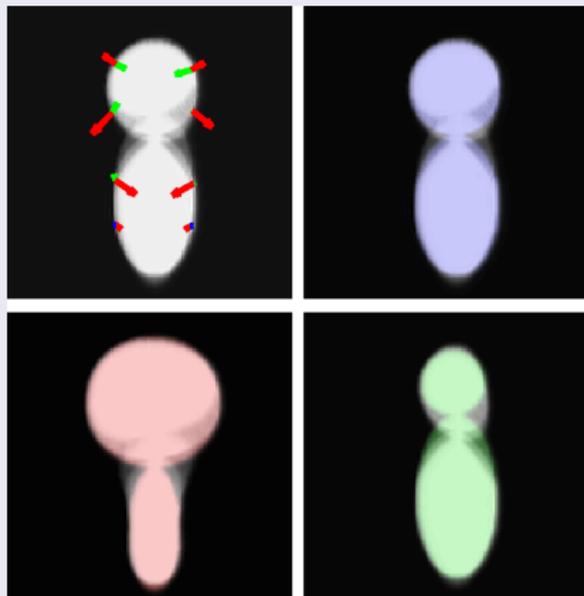


Single gradient descent:

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- position of CP
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Image size= 128^2 , $\sigma_V = 25$, $\sigma^2 = 0.005$, $\gamma = 540$

Atlas construction: optimization

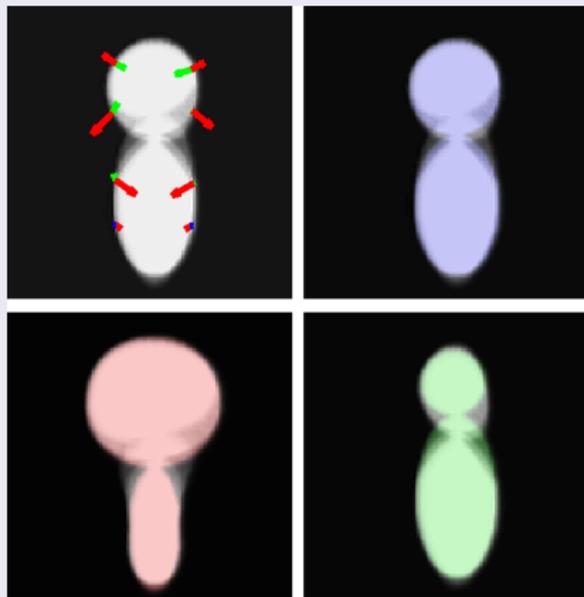


Single gradient descent:

- template image
- position of CP
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- momenta

Image size= 128^2 , $\sigma_V = 25$, $\sigma^2 = 0.005$, $\gamma = 540$

Atlas construction: optimization

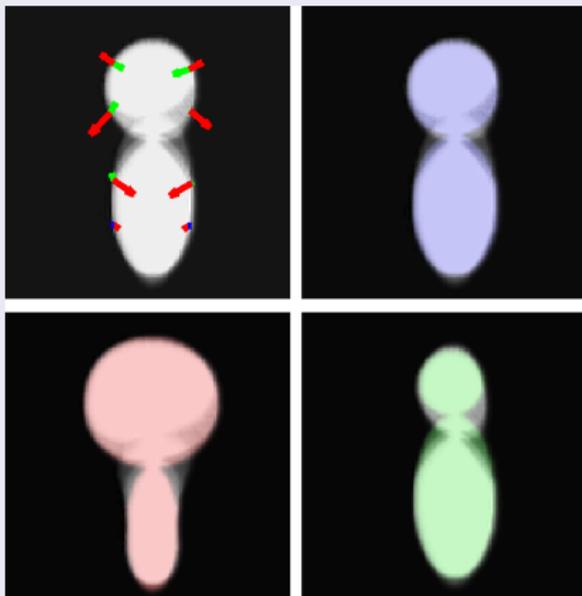


Single gradient descent:

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Atlas construction: optimization

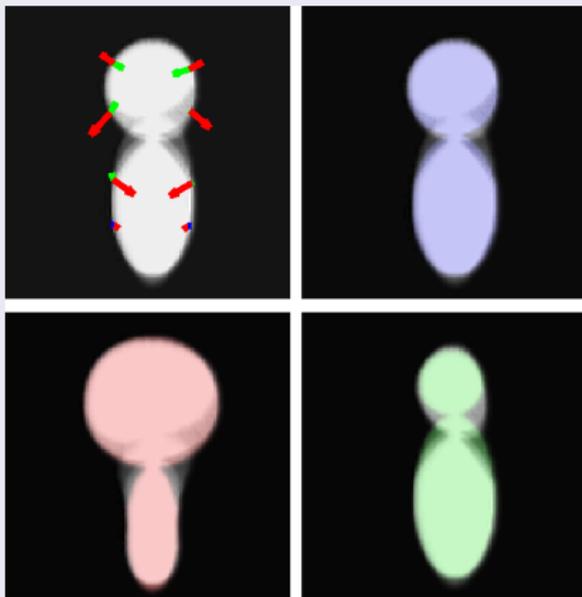


Single gradient descent:

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- position of CP
- number of CP
- momenta

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Atlas construction: optimization

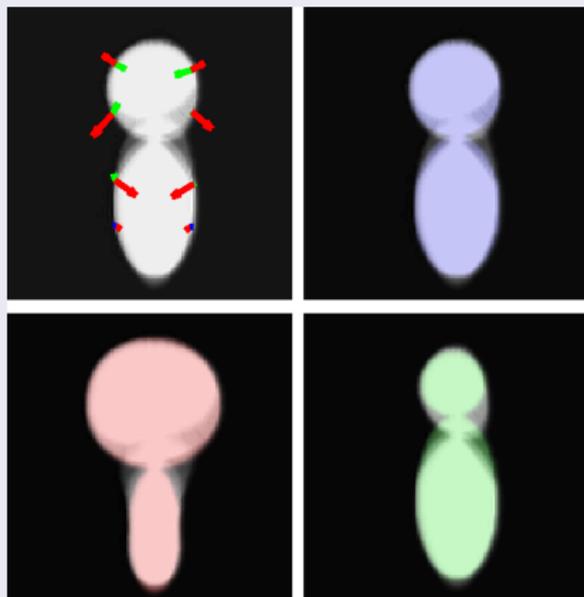


Single gradient descent:

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- position of CP
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- momenta

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Atlas construction: optimization

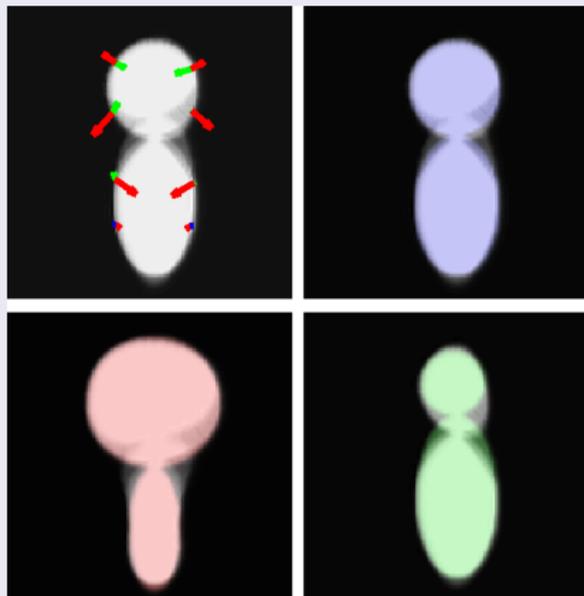


Single gradient descent:

- template image
- position of CP
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- momenta

Image size= 128^2 , $\sigma_V = 25$, $\sigma^2 = 0.005$, $\gamma = 540$

Atlas construction: optimization

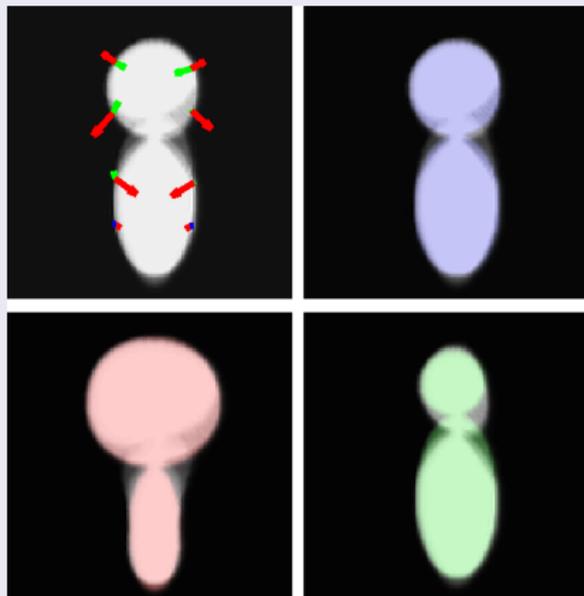


Single gradient descent:

- template image
- position of CP
- number of CP
- momenta

Image size=128², $\sigma_V = 25$, $\sigma^2 = 0.005$, $\gamma = 540$

Atlas construction: optimization

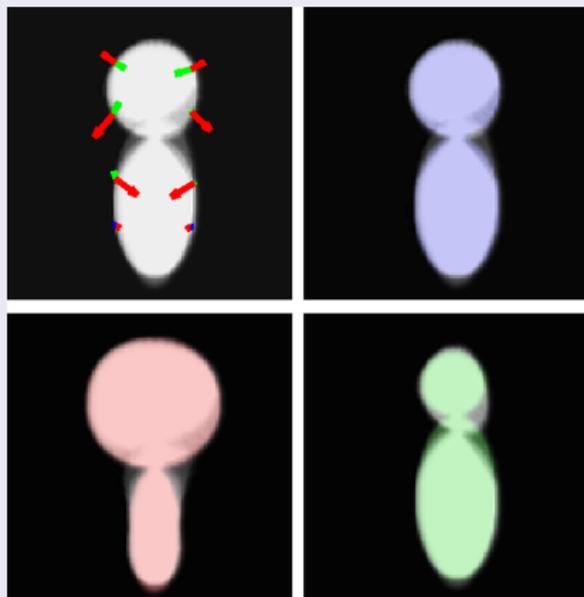


Single gradient descent:

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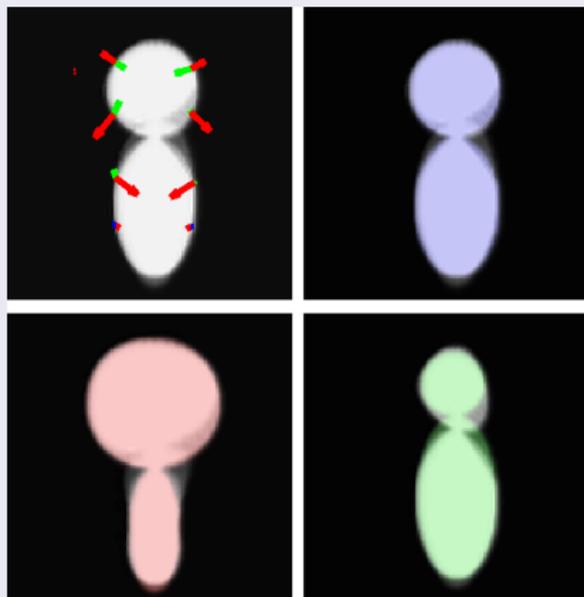


Single gradient descent:

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Image size=128², $\sigma_V = 25$, $\sigma^2 = 0.005$, $\gamma = 540$

Atlas construction: optimization

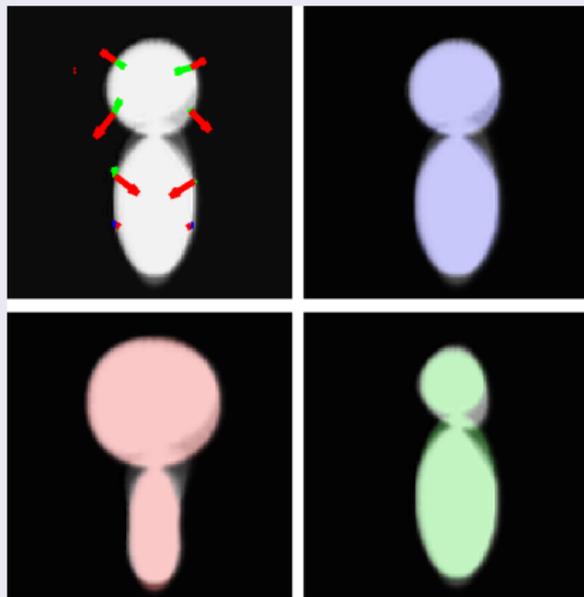


Single gradient descent:

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Atlas construction: optimization

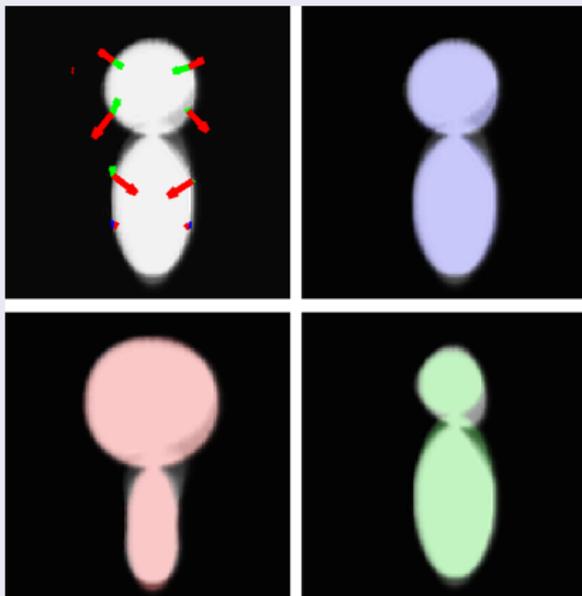


Single gradient descent:

- template image
- position of CP
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- momenta

Image size=128², $\sigma_V = 25$, $\sigma^2 = 0.005$, $\gamma = 540$

Atlas construction: optimization

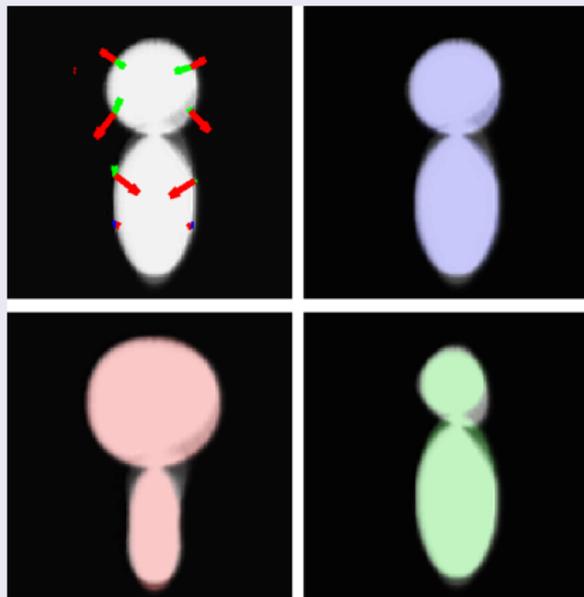


Single gradient descent:

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- position of CP
- number of CP
- momenta

Image size= 128^2 , $\sigma_V = 25$, $\sigma^2 = 0.005$, $\gamma = 540$

Atlas construction: optimization

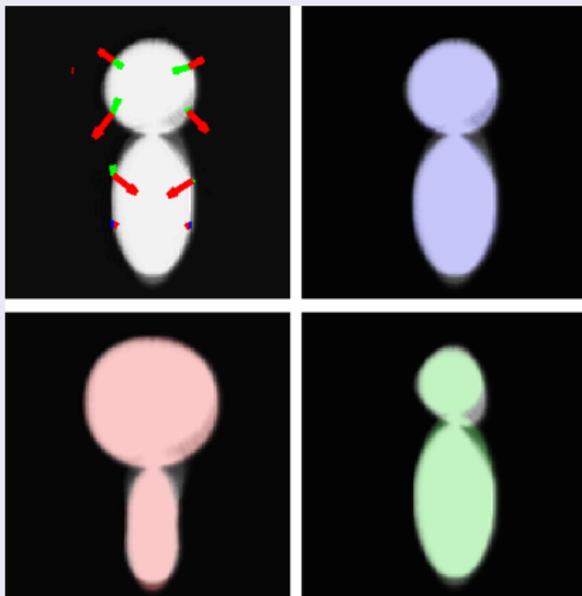


Single gradient descent:

- template image
- position of CP
- number of CP
- momenta

Image size=128², $\sigma_V = 25$, $\sigma^2 = 0.005$, $\gamma = 540$

Atlas construction: optimization

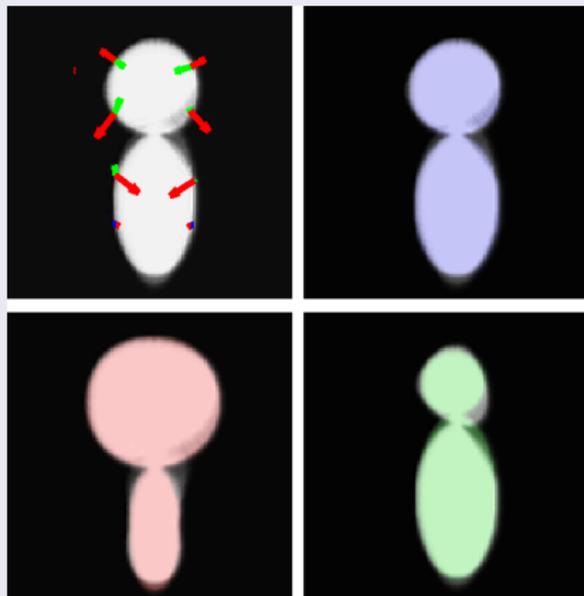


Single gradient descent:

- template image
- position of CP
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- momenta

Image size= 128^2 , $\sigma_V = 25$, $\sigma^2 = 0.005$, $\gamma = 540$

Atlas construction: optimization

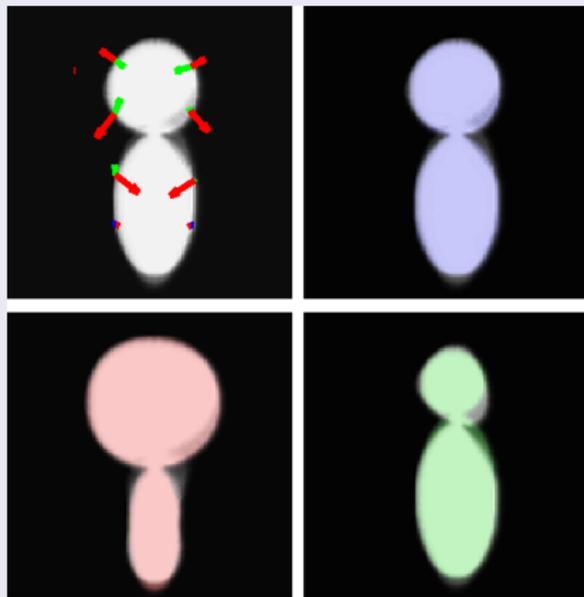


Single gradient descent:

- template image
- position of CP
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- momenta

Image size=128², $\sigma_V = 25$, $\sigma^2 = 0.005$, $\gamma = 540$

Atlas construction: optimization

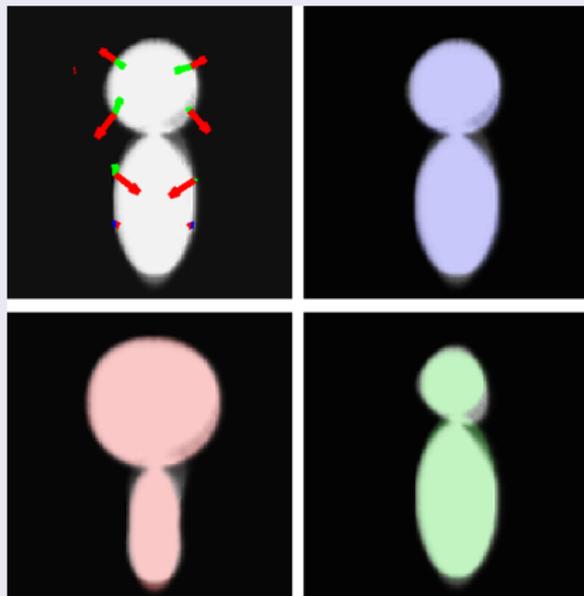


Single gradient descent:

- template image
- position of CP
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- momenta

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Atlas construction: optimization

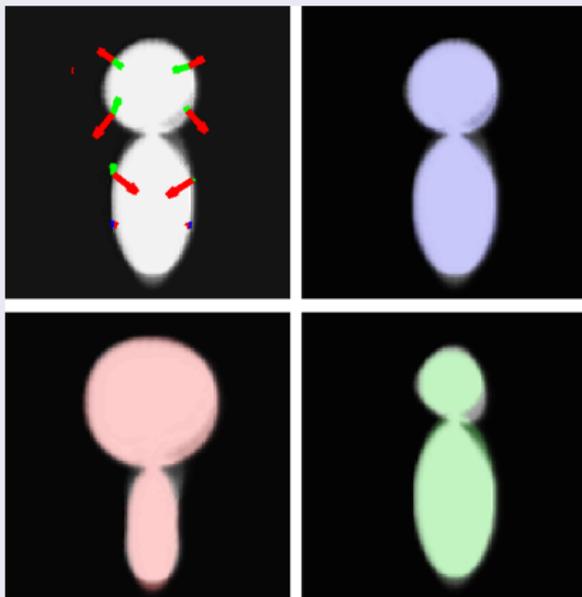


Single gradient descent:

- template image
- position of CP
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Image size=128², $\sigma_V = 25$, $\sigma^2 = 0.005$, $\gamma = 540$

Atlas construction: optimization

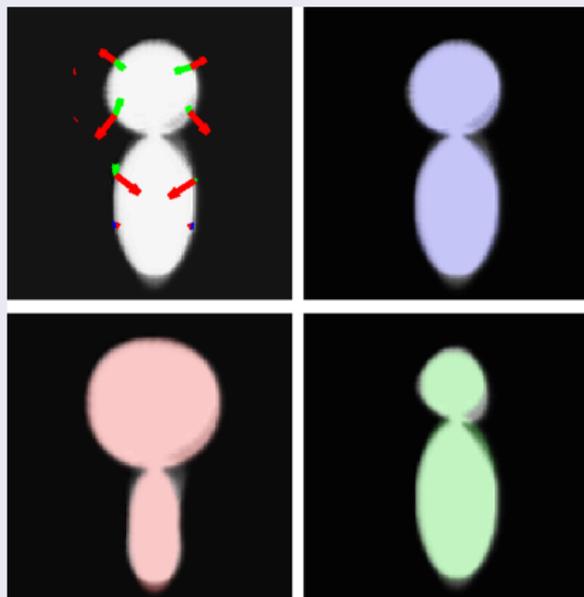


Single gradient descent:

- template image
- position of CP
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Atlas construction: optimization

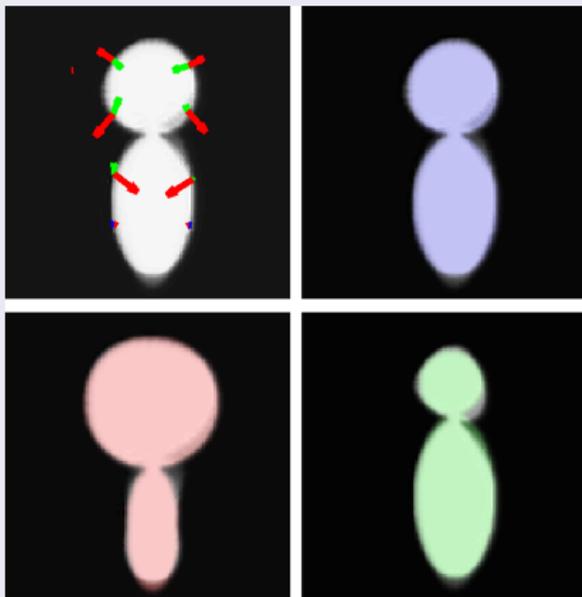


Single gradient descent:

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Image size= 128^2 , $\sigma_V = 25$, $\sigma^2 = 0.005$, $\gamma = 540$

Atlas construction: optimization

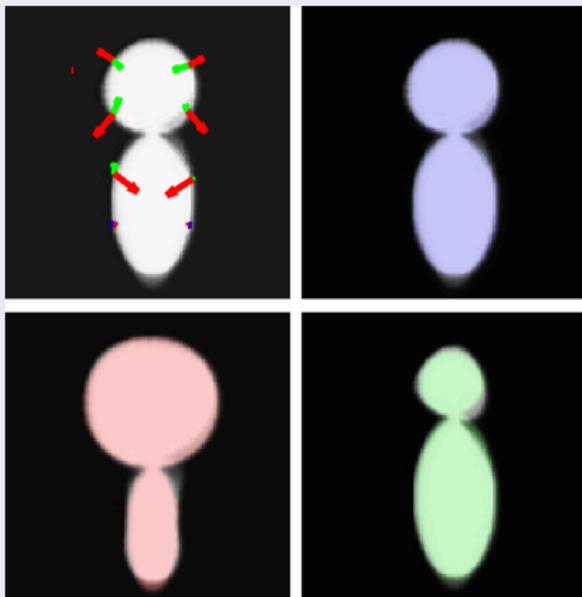


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Image size= 128^2 , $\sigma_V = 25$, $\sigma^2 = 0.005$, $\gamma = 540$

Atlas construction: optimization

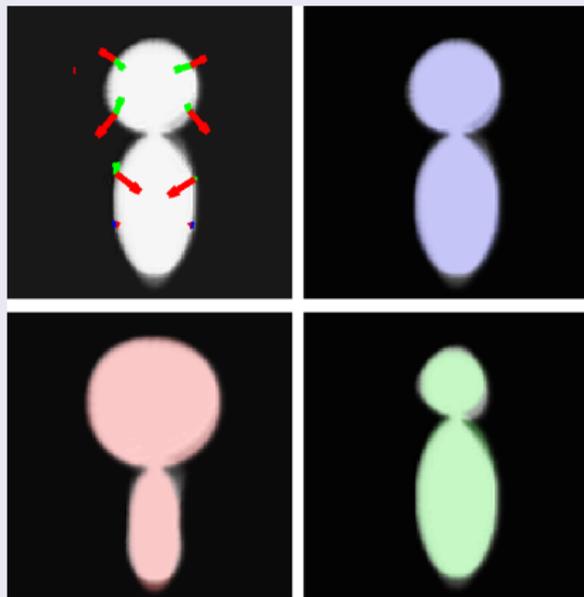


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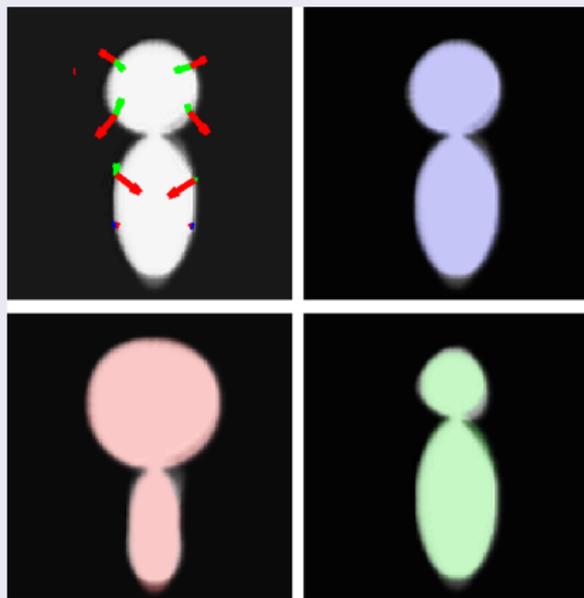


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Image size=128², $\sigma_V = 25$, $\sigma^2 = 0.005$, $\gamma = 540$

Atlas construction: optimization

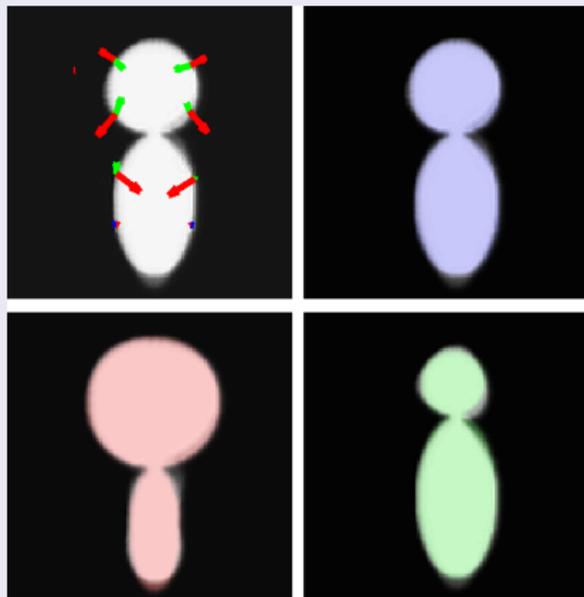


Single gradient descent:

- template image
- position of CP
- number of CP
- momenta

Image size= 128^2 , $\sigma_V = 25$, $\sigma^2 = 0.005$, $\gamma = 540$

Atlas construction: optimization

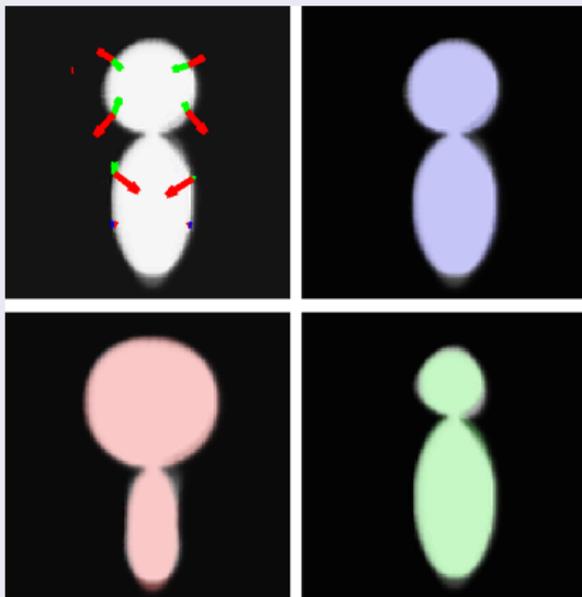


Single gradient descent:

- template image
- position of CP
- number of CP
- momenta

Image size= 128^2 , $\sigma_V = 25$, $\sigma^2 = 0.005$, $\gamma = 540$

Atlas construction: optimization

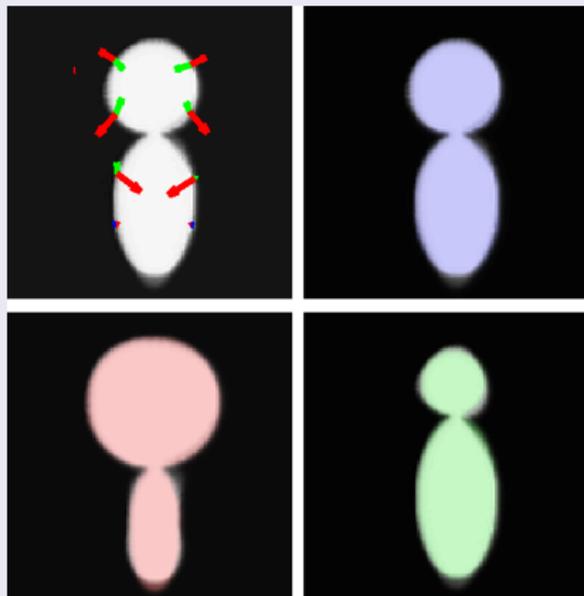


Single gradient descent:

- template image
- position of CP
- number of CP
- momenta

Image size= 128^2 , $\sigma_V = 25$, $\sigma^2 = 0.005$, $\gamma = 540$

Atlas construction: optimization

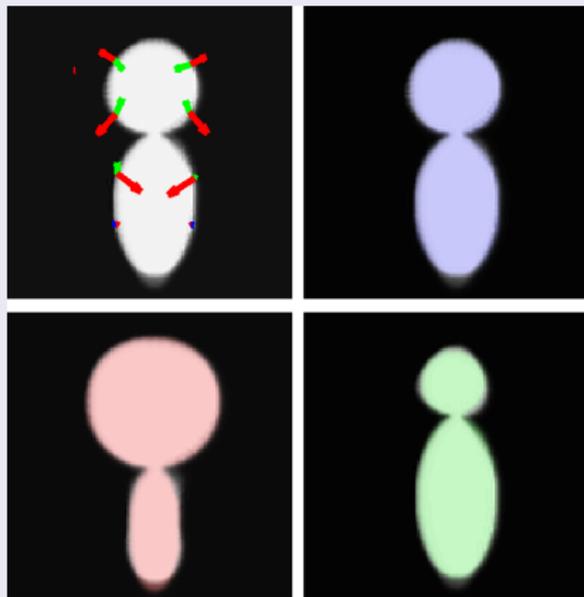


Single gradient descent:

- template image
- position of CP
- number of CP
- momenta

Image size= 128^2 , $\sigma_V = 25$, $\sigma^2 = 0.005$, $\gamma = 540$

Atlas construction: optimization

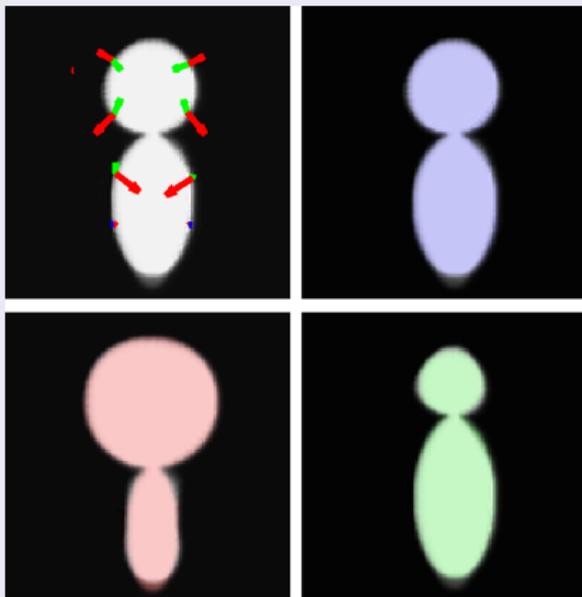


Single gradient descent:

- template image
- position of CP
- number of CP
- momenta

Image size=128², $\sigma_V = 25$, $\sigma^2 = 0.005$, $\gamma = 540$

Atlas construction: optimization

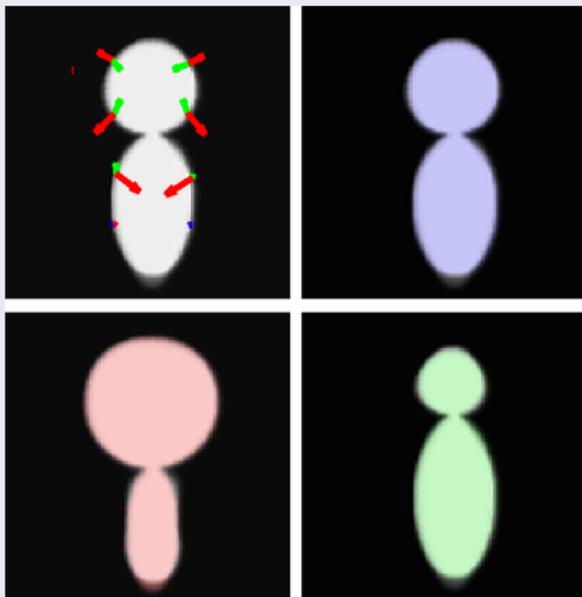


Single gradient descent:

- template image
- position of CP
- number of CP
- momenta

Image size= 128^2 , $\sigma_V = 25$, $\sigma^2 = 0.005$, $\gamma = 540$

Atlas construction: optimization

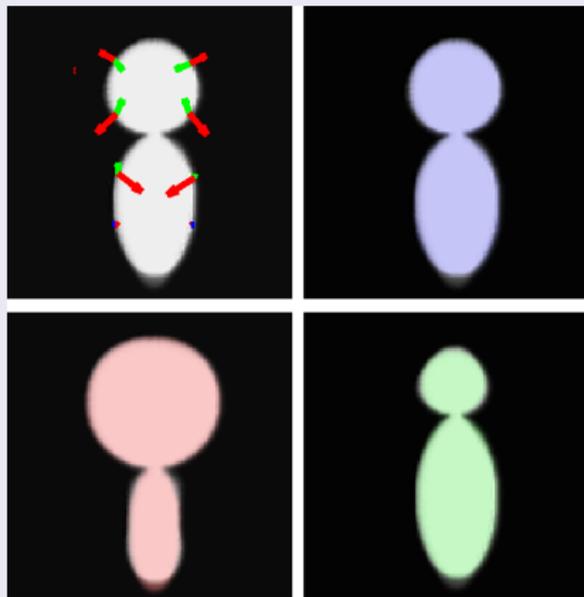


Single gradient descent:

- template image
- position of CP
- number of CP
- momenta

Image size= 128^2 , $\sigma_V = 25$, $\sigma^2 = 0.005$, $\gamma = 540$

Atlas construction: optimization

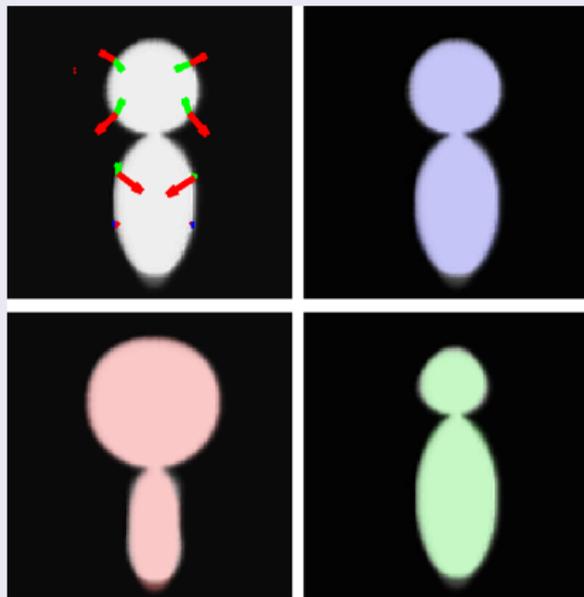


Single gradient descent:

- template image
- position of CP
- number of CP
- momenta

Image size=128², $\sigma_V = 25$, $\sigma^2 = 0.005$, $\gamma = 540$

Atlas construction: optimization

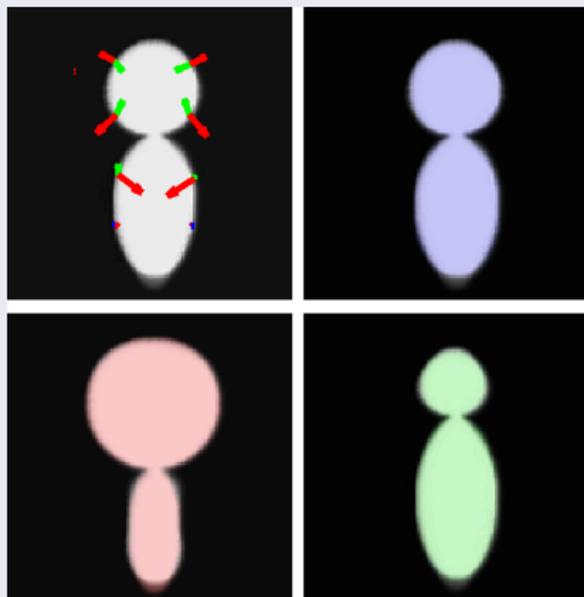


Single gradient descent:

- template image
- position of CP
- number of CP
- momenta

Image size=128², $\sigma_V = 25$, $\sigma^2 = 0.005$, $\gamma = 540$

Atlas construction: optimization

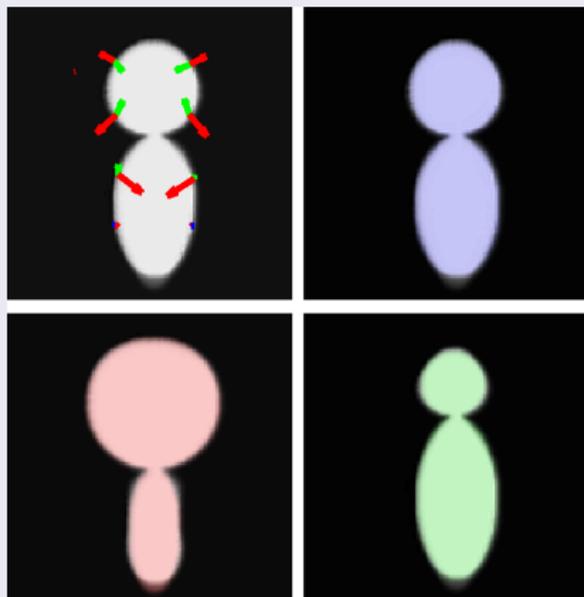


Single gradient descent:

- template image
- position of CP
- number of CP
- momenta

Image size= 128^2 , $\sigma_V = 25$, $\sigma^2 = 0.005$, $\gamma = 540$

Atlas construction: optimization

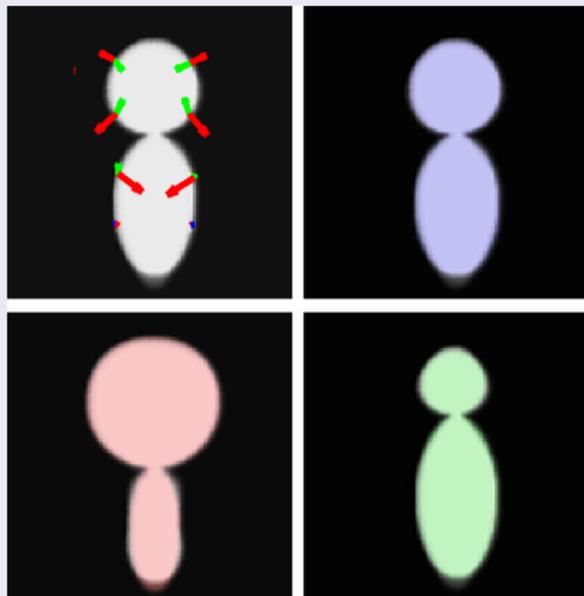


Single gradient descent:

- template image
- position of CP
- number of CP
- momenta

Image size=128², $\sigma_V = 25$, $\sigma^2 = 0.005$, $\gamma = 540$

Atlas construction: optimization

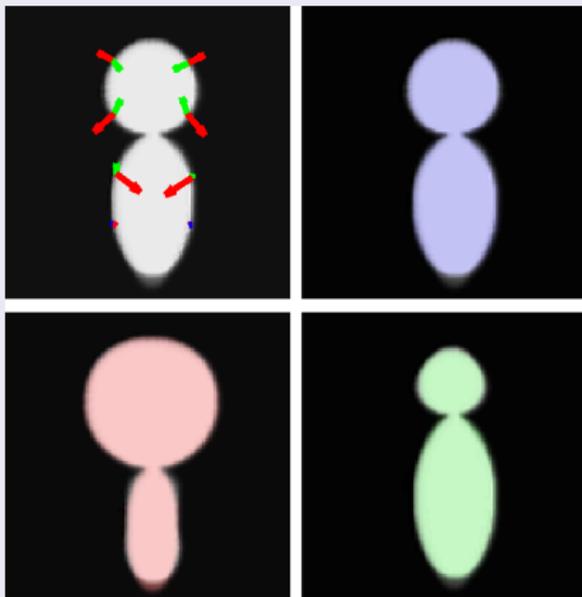


Single gradient descent:

- template image
- position of CP
- number of CP
- momenta

Image size= 128^2 , $\sigma_V = 25$, $\sigma^2 = 0.005$, $\gamma = 540$

Atlas construction: optimization

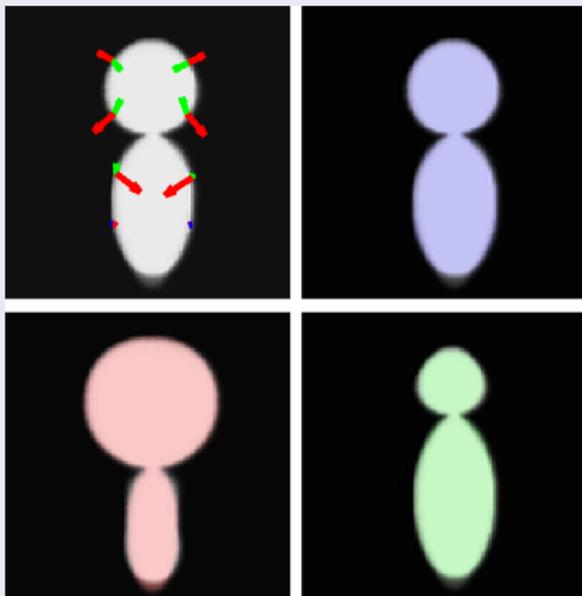


Single gradient descent:

- template image
- position of CP
- number of CP
- momenta

Image size= 128^2 , $\sigma_V = 25$, $\sigma^2 = 0.005$, $\gamma = 540$

Atlas construction: optimization

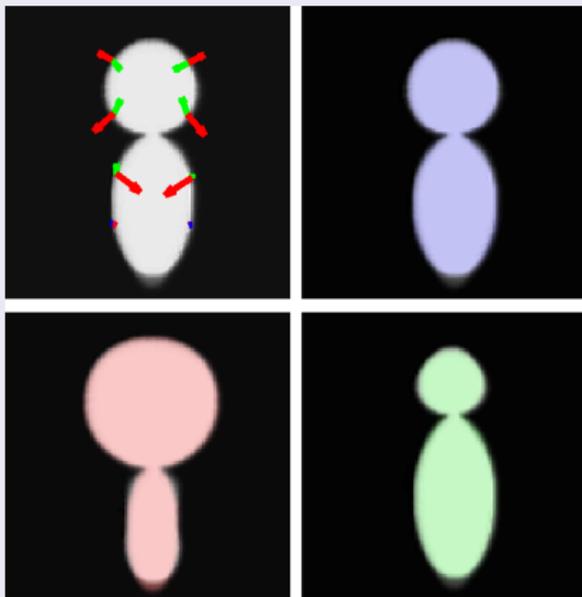


Single gradient descent:

- template image
- position of CP
- number of CP
- momenta

Image size= 128^2 , $\sigma_V = 25$, $\sigma^2 = 0.005$, $\gamma = 540$

Atlas construction: optimization

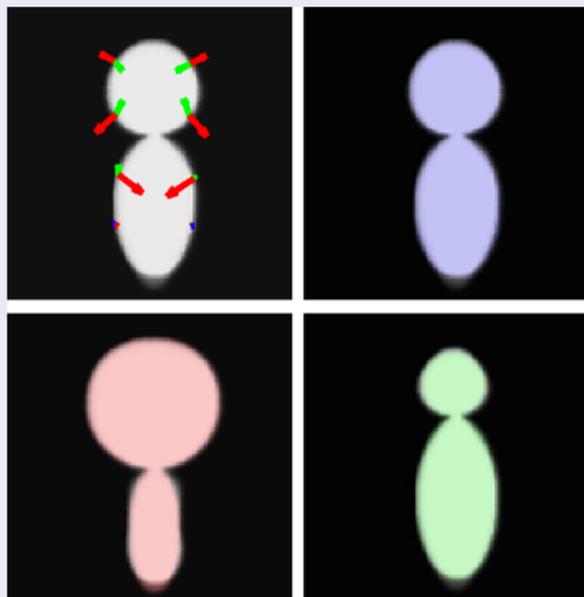


Single gradient descent:

- template image
- position of CP
- number of CP
- momenta

Image size= 128^2 , $\sigma_V = 25$, $\sigma^2 = 0.005$, $\gamma = 540$

Atlas construction: optimization

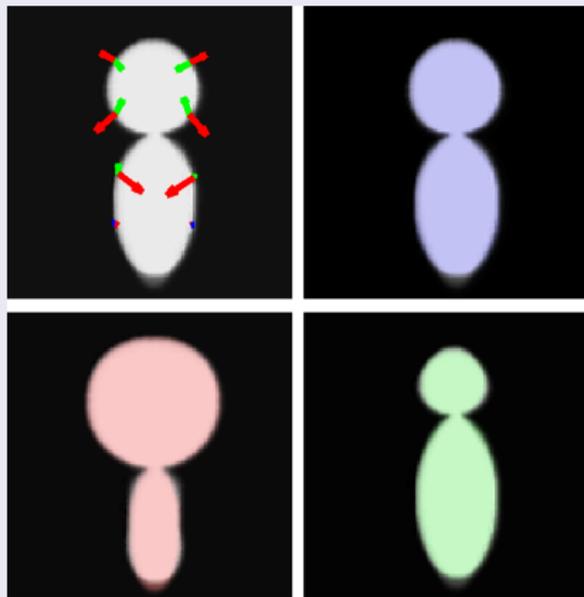


Single gradient descent:

- template image
- position of CP
- number of CP
- momenta

Image size=128², $\sigma_V = 25$, $\sigma^2 = 0.005$, $\gamma = 540$

Atlas construction: optimization

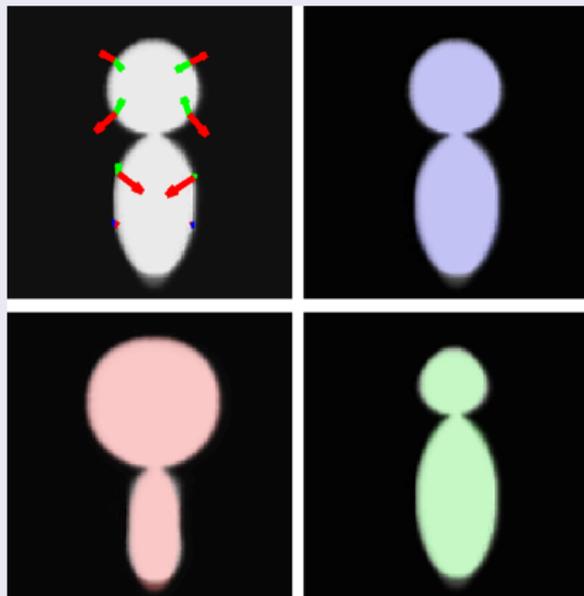


Single gradient descent:

- template image
- position of CP
- number of CP
- momenta

Image size=128², $\sigma_V = 25$, $\sigma^2 = 0.005$, $\gamma = 540$

Atlas construction: optimization

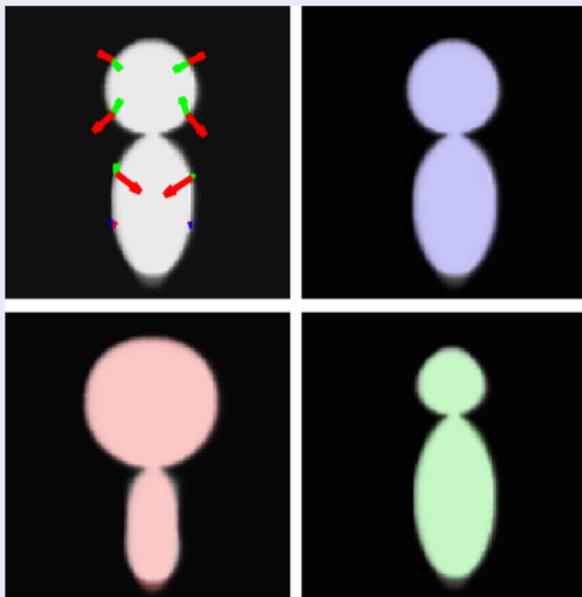


Single gradient descent:

- template image
- position of CP
- number of CP
- momenta

Image size= 128^2 , $\sigma_V = 25$, $\sigma^2 = 0.005$, $\gamma = 540$

Atlas construction: optimization

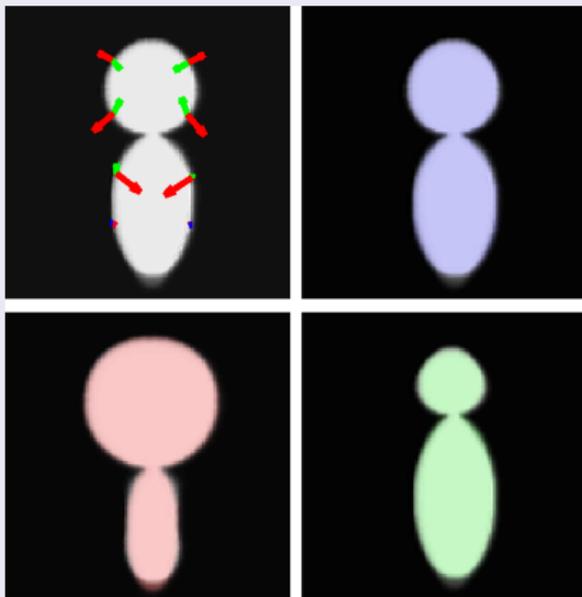


Single gradient descent:

- template image
- position of CP
- number of CP
- momenta

Image size= 128^2 , $\sigma_V = 25$, $\sigma^2 = 0.005$, $\gamma = 540$

Atlas construction: optimization

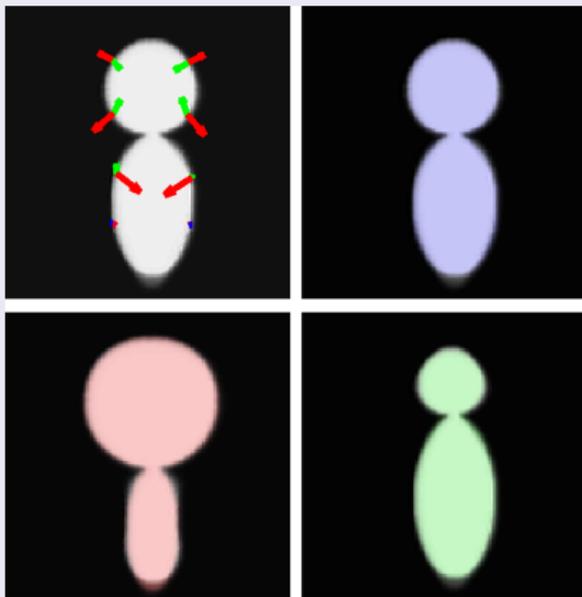


Single gradient descent:

- template image
- position of CP
- number of CP
- momenta

Image size= 128^2 , $\sigma_V = 25$, $\sigma^2 = 0.005$, $\gamma = 540$

Atlas construction: optimization

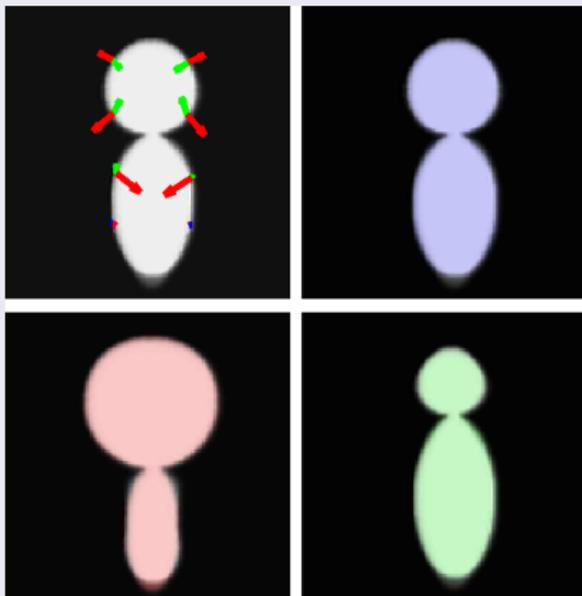


Single gradient descent:

- template image
- position of CP
- number of CP
- momenta

Image size= 128^2 , $\sigma_V = 25$, $\sigma^2 = 0.005$, $\gamma = 540$

Atlas construction: optimization

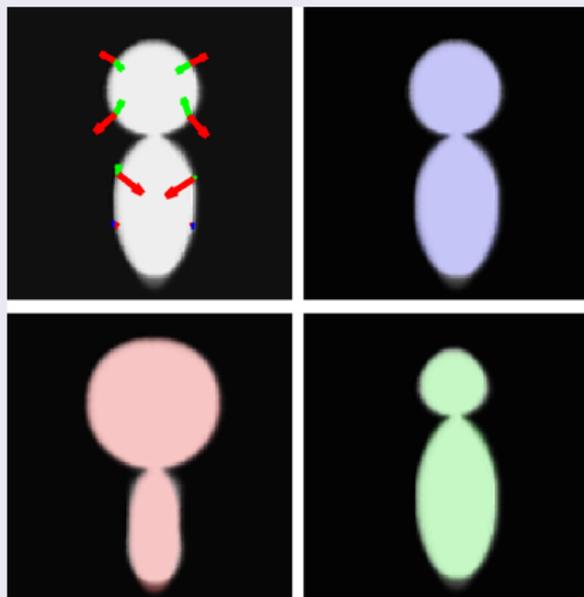


Single gradient descent:

- template image
- position of CP
- number of CP
- momenta

Image size= 128^2 , $\sigma_V = 25$, $\sigma^2 = 0.005$, $\gamma = 540$

Atlas construction: optimization

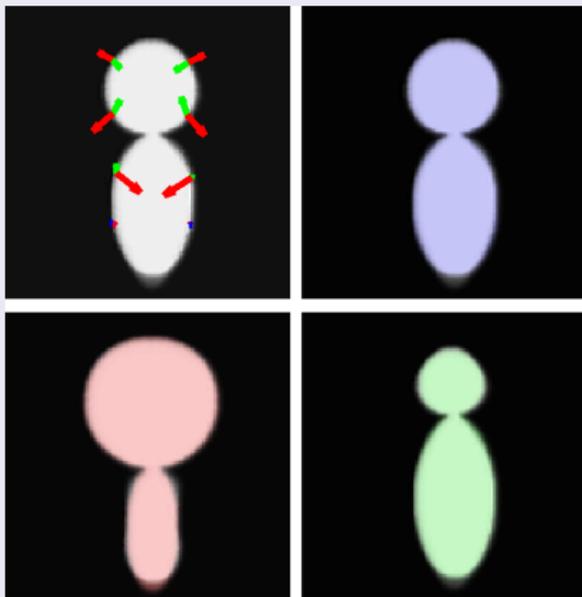


Single gradient descent:

- template image
- position of CP
- number of CP
- momenta

Image size= 128^2 , $\sigma_V = 25$, $\sigma^2 = 0.005$, $\gamma = 540$

Atlas construction: optimization

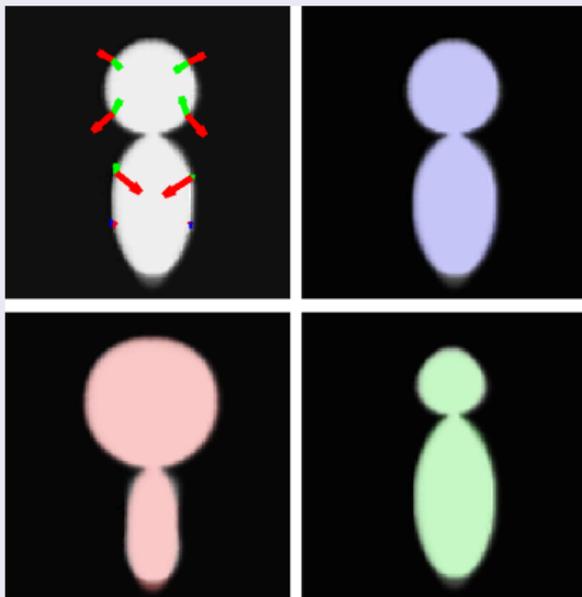


Single gradient descent:

- template image
- position of CP
- number of CP
- momenta

Image size= 128^2 , $\sigma_V = 25$, $\sigma^2 = 0.005$, $\gamma = 540$

Atlas construction: optimization

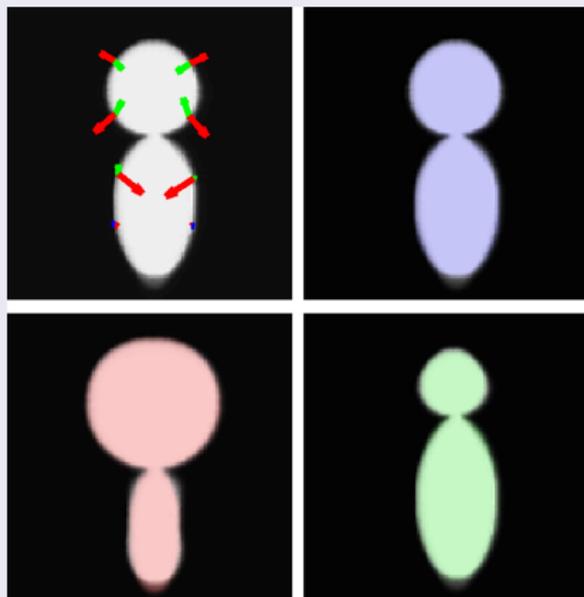


Single gradient descent:

- template image
- position of CP
- number of CP
- momenta

Image size= 128^2 , $\sigma_V = 25$, $\sigma^2 = 0.005$, $\gamma = 540$

Atlas construction: optimization

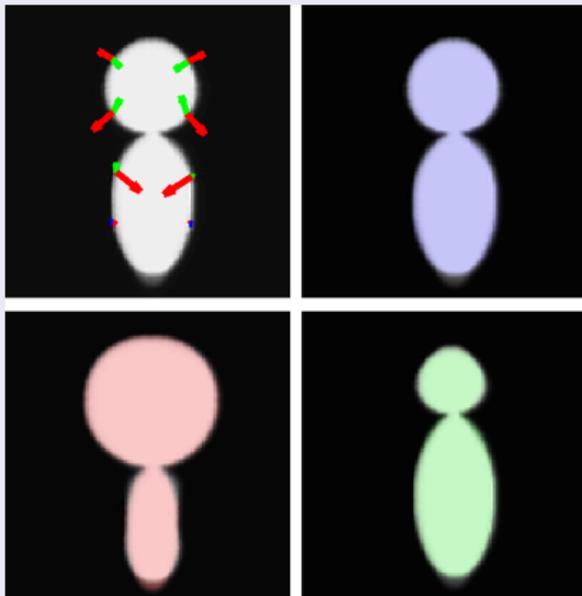


Single gradient descent:

- template image
- position of CP
- number of CP
- momenta

Image size= 128^2 , $\sigma_V = 25$, $\sigma^2 = 0.005$, $\gamma = 540$

Atlas construction: optimization



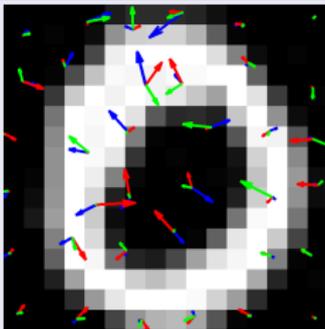
Single gradient descent:

- template image
- position of CP
- number of CP
- momenta

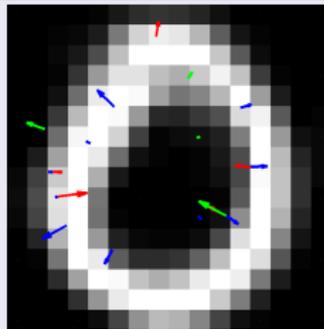
Image size= 128^2 , $\sigma_V = 25$, $\sigma^2 = 0.005$, $\gamma = 540$

8 estimated control points!

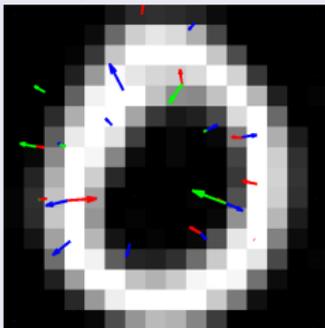
Atlas from the US postal database (20 training images)



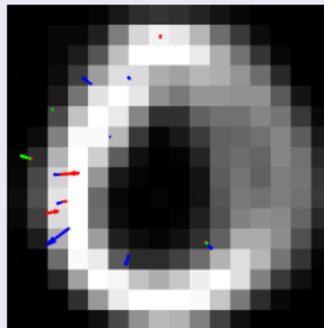
$\gamma = 0$; 36 active CPs



$\gamma = 700$; 21 active CPs

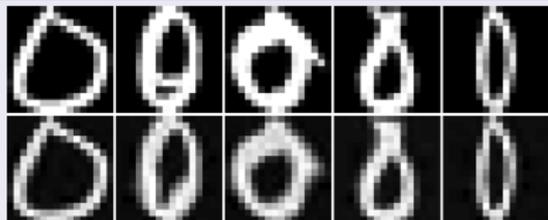


$\gamma = 400$; 27 active CPs

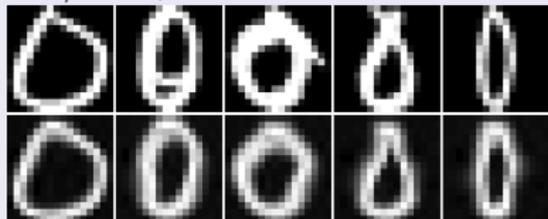


$\gamma = 800$; 13 active CPs

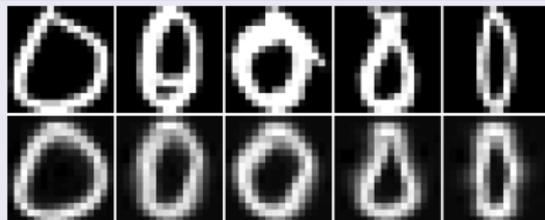
Atlas from the US postal database (20 training images)



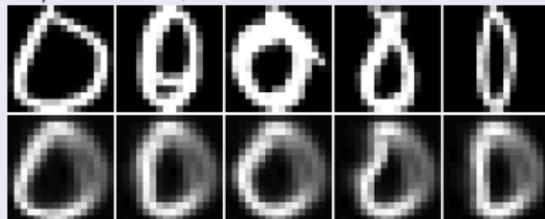
$\gamma = 0$; 36 active CPs



$\gamma = 400$; 27 active CPs

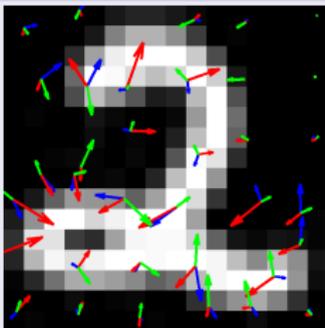


$\gamma = 700$; 21 active CPs

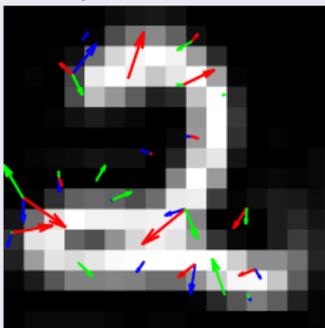


$\gamma = 800$; 13 active CPs

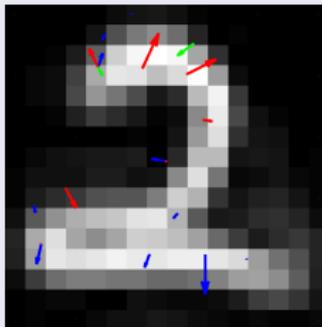
Atlas from the US postal database (20 training images)



$\gamma = 0$; 36 active CPs

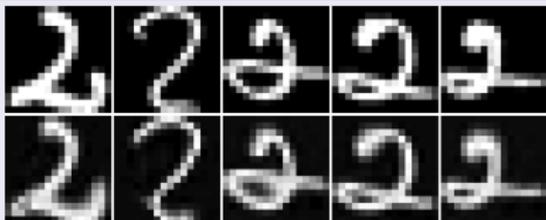


$\gamma = 200$; 27 active CPs

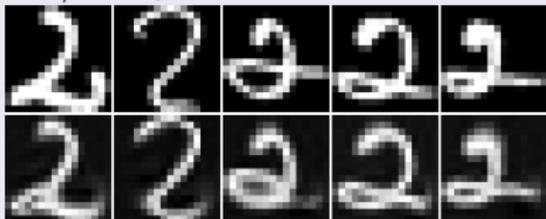


$\gamma = 400$; 18 active CPs

Atlas from the US postal database (20 training images)



$\gamma = 0$; 36 active CPs

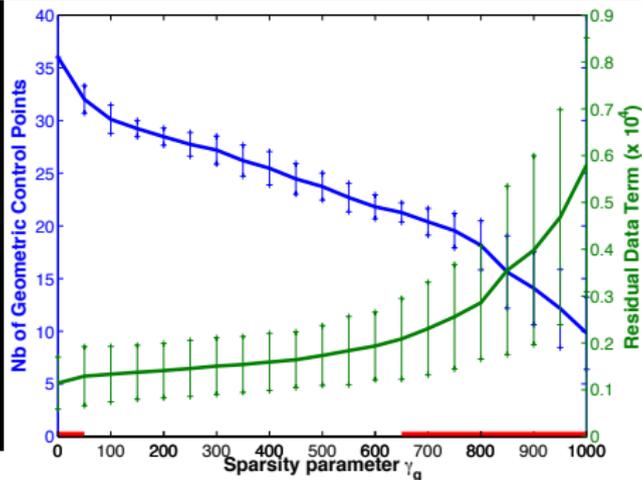
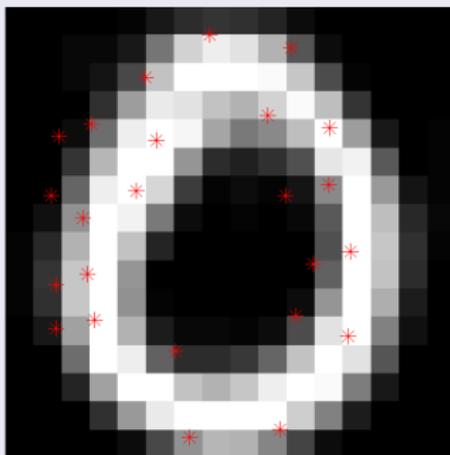


$\gamma = 200$; 27 active CPs

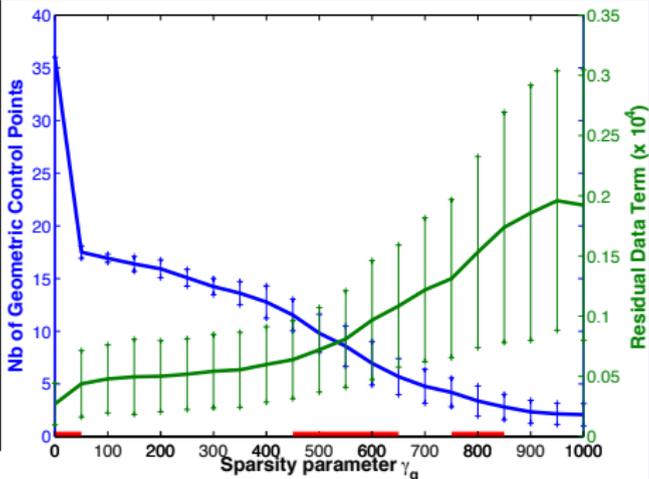
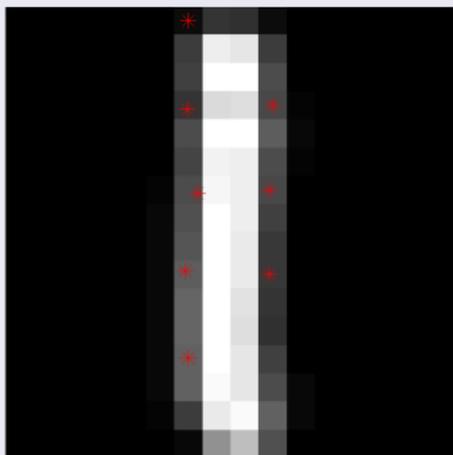


$\gamma = 400$; 18 active CPs

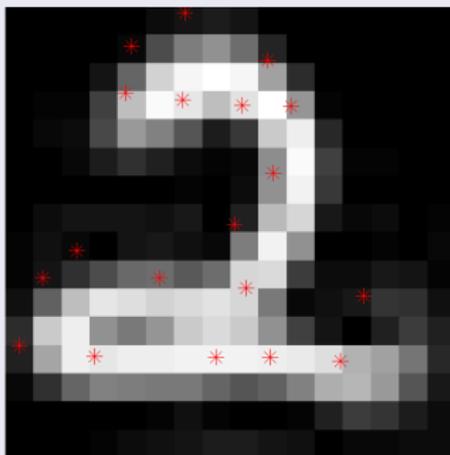
Atlas from the US postal database (20 training images)



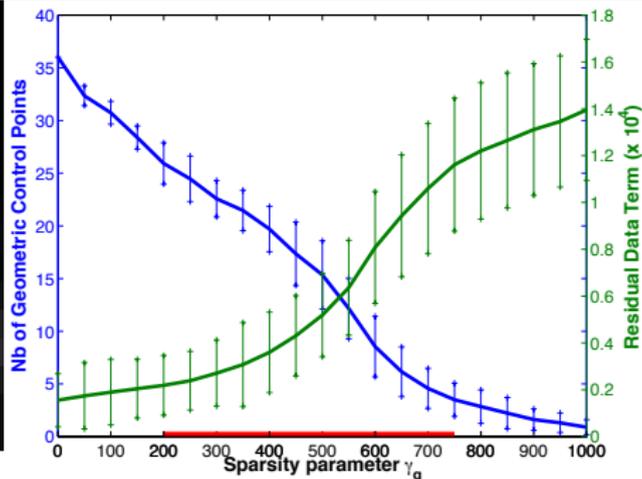
Atlas from the US postal database (20 training images)



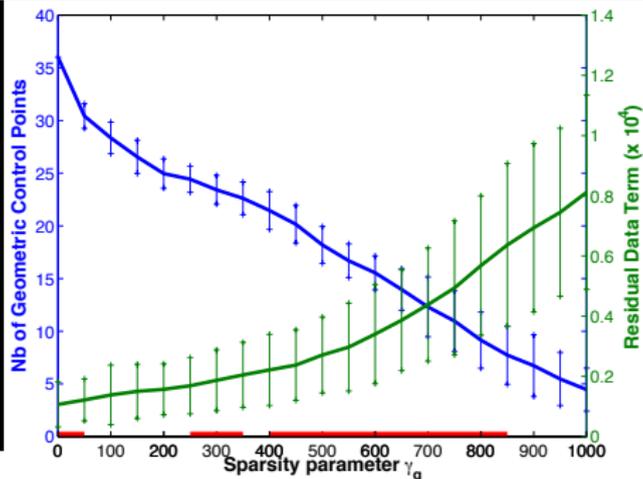
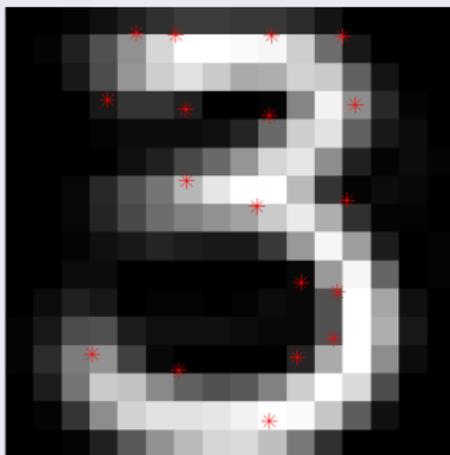
Atlas from the US postal database (20 training images)



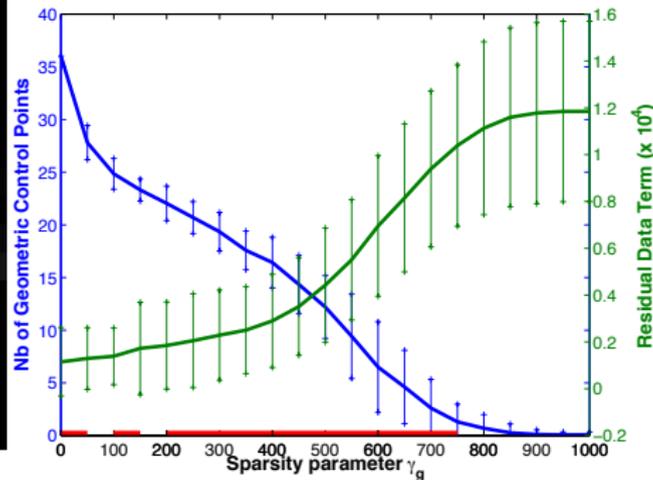
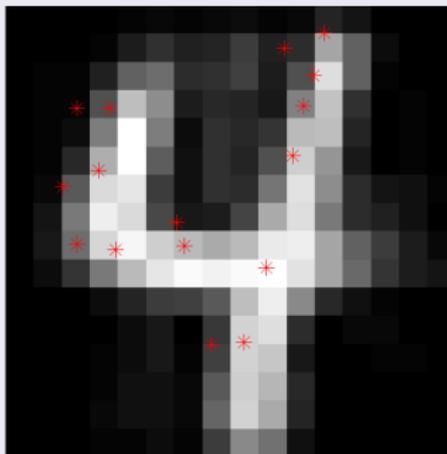
$\gamma_g = 350$



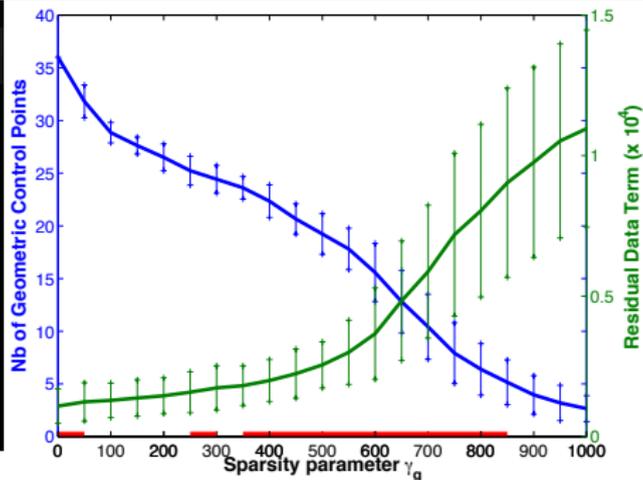
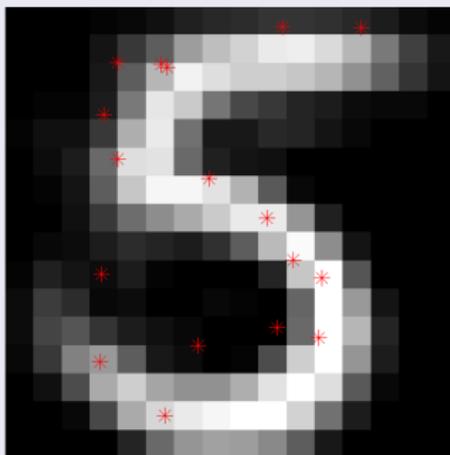
Atlas from the US postal database (20 training images)



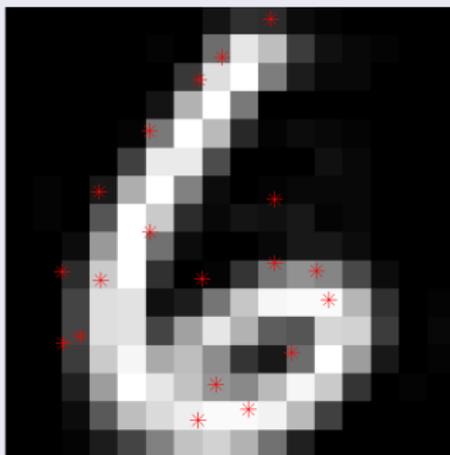
Atlas from the US postal database (20 training images)



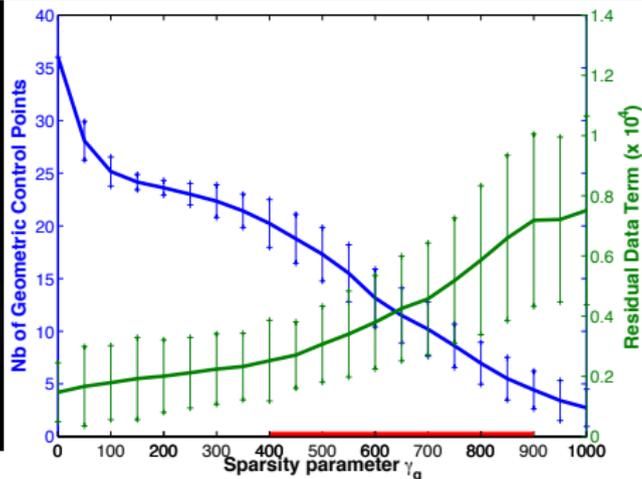
Atlas from the US postal database (20 training images)



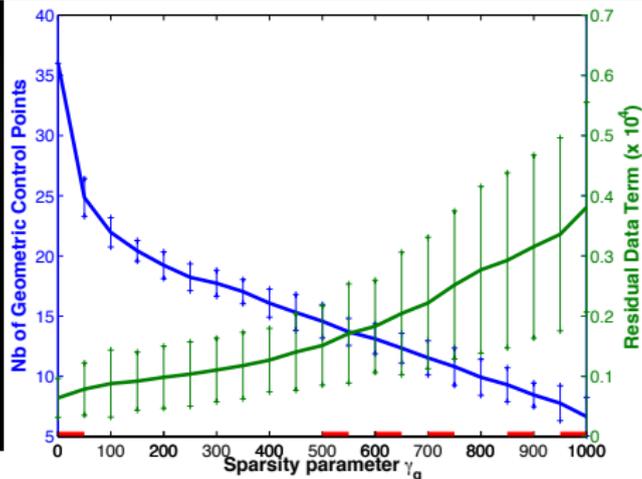
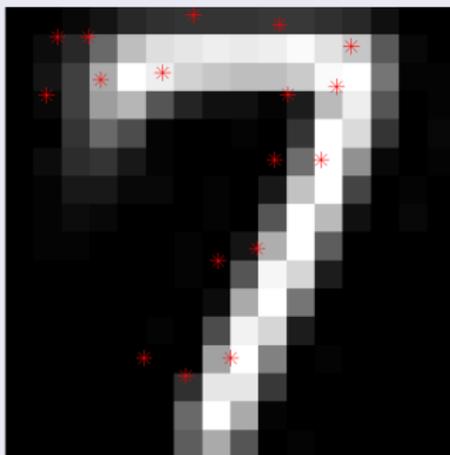
Atlas from the US postal database (20 training images)



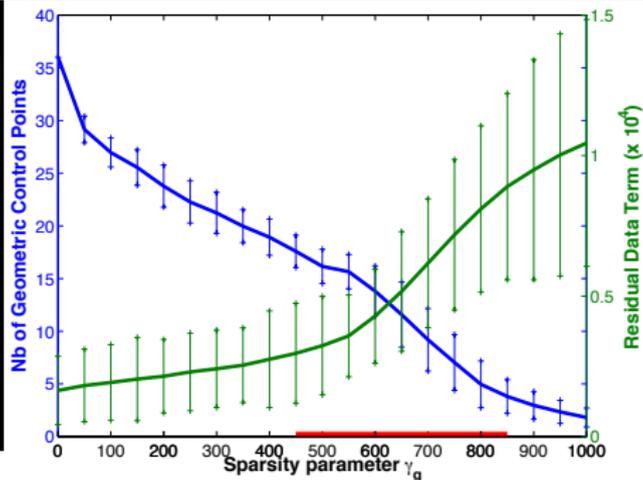
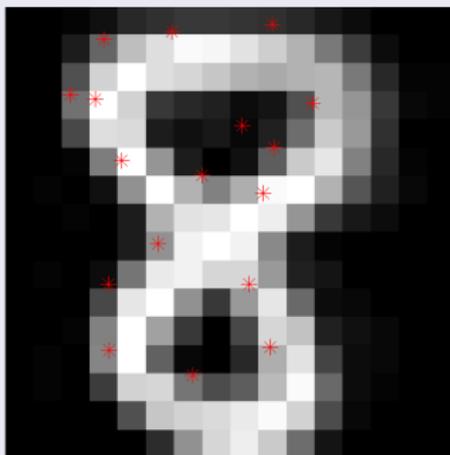
$\gamma_g = 450$



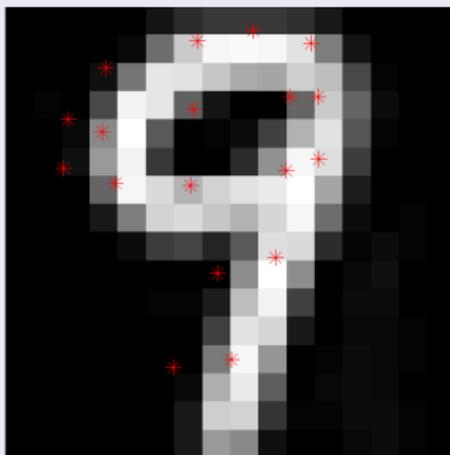
Atlas from the US postal database (20 training images)



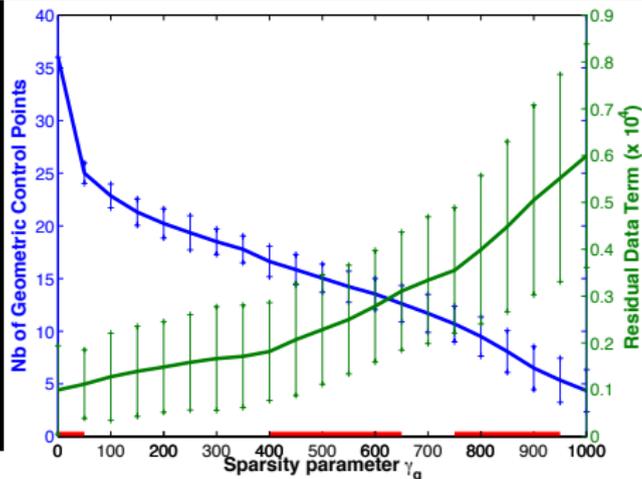
Atlas from the US postal database (20 training images)



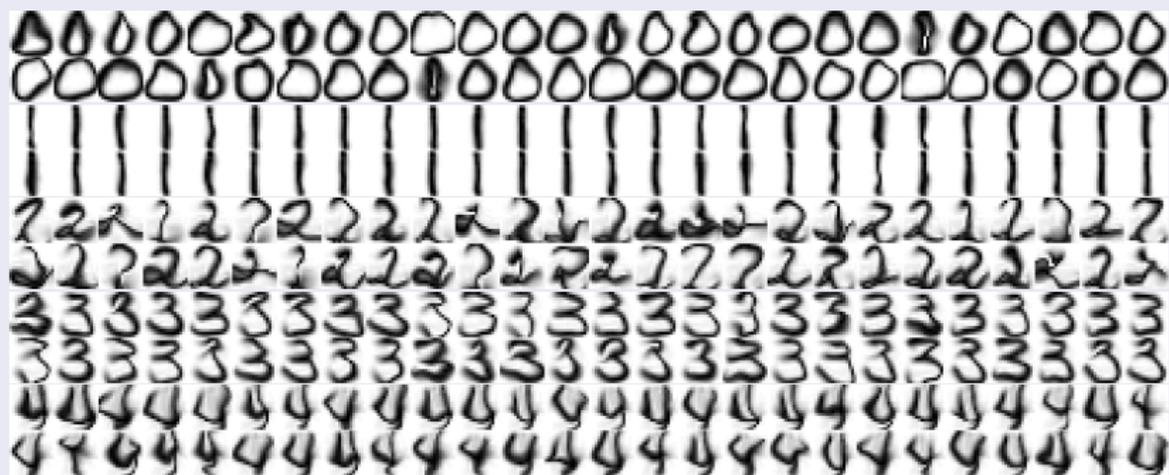
Atlas from the US postal database (20 training images)



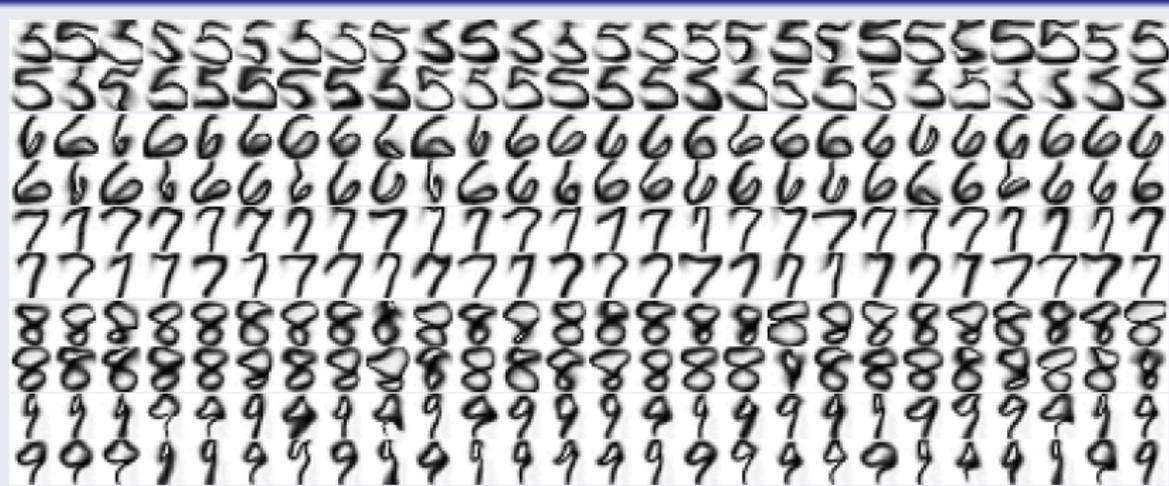
$\gamma_g = 400$



Atlas from the US postal database (20 training images)

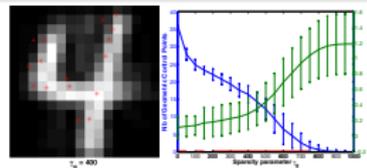
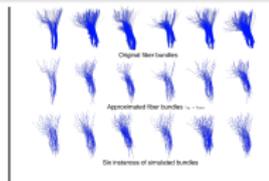
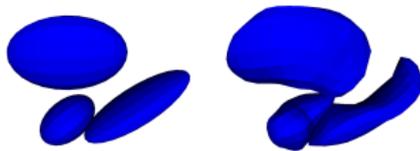


Atlas from the US postal database (20 training images)



Conclusion

- A generic method for atlas construction:
 - images, points, surface meshes, fiber bundles, etc..
 - control over the model complexity (template + variability)
 - single gradient descent for:
 - template estimation
 - best parameterization of shape variability
 - template-to-shape correspondence
- Useful for:
 - Characterization of atypical anatomical configuration
 - Classification, clustering
 - Regression against clinical or genetic variables



Conclusion

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- S. Allasonnière (X)
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