Statistical Anatomical Models: how to compute with phenotypes?

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COMMENT ?
Où ?


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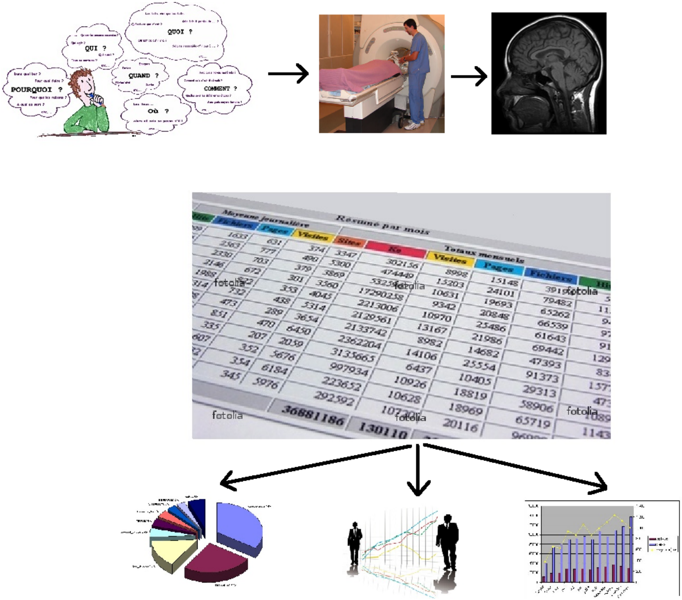
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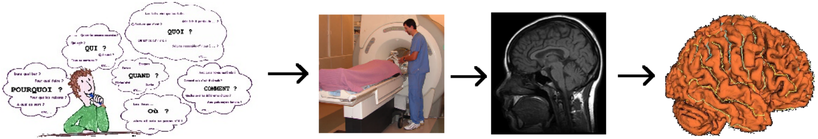
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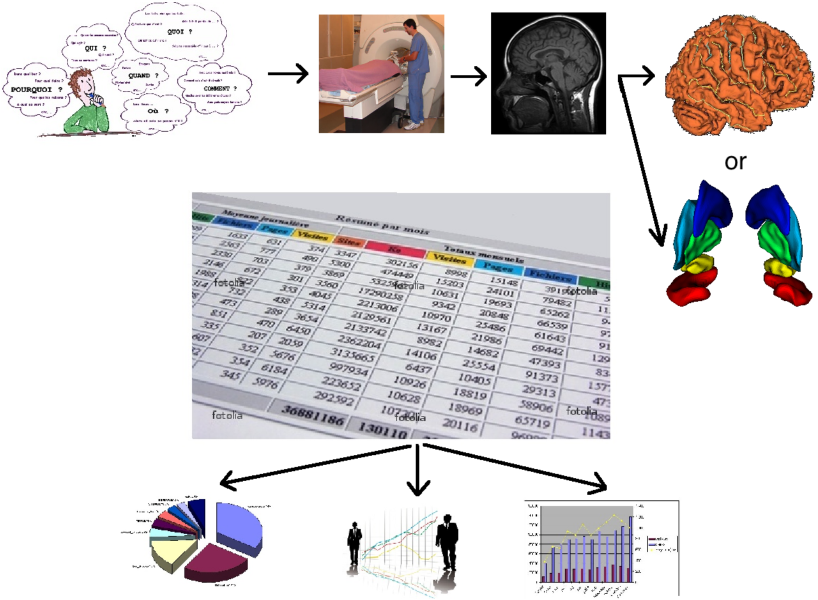
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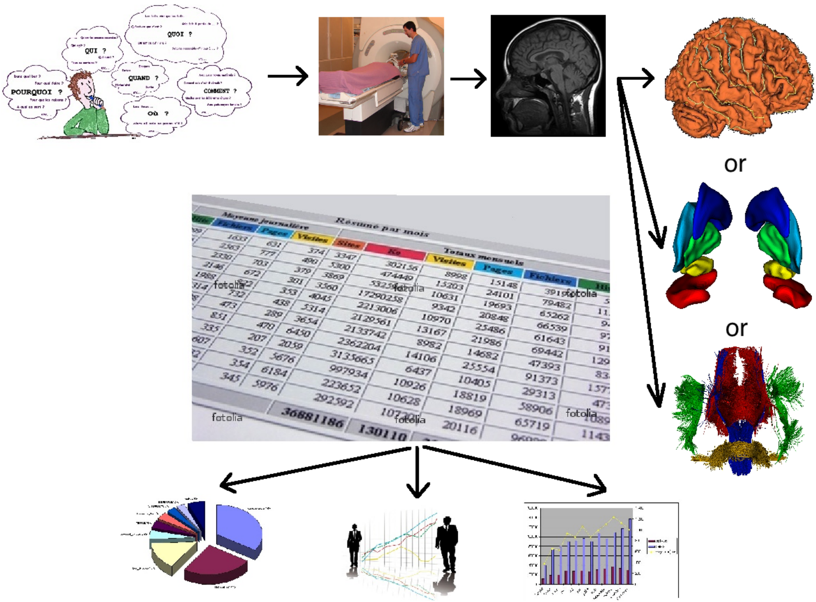
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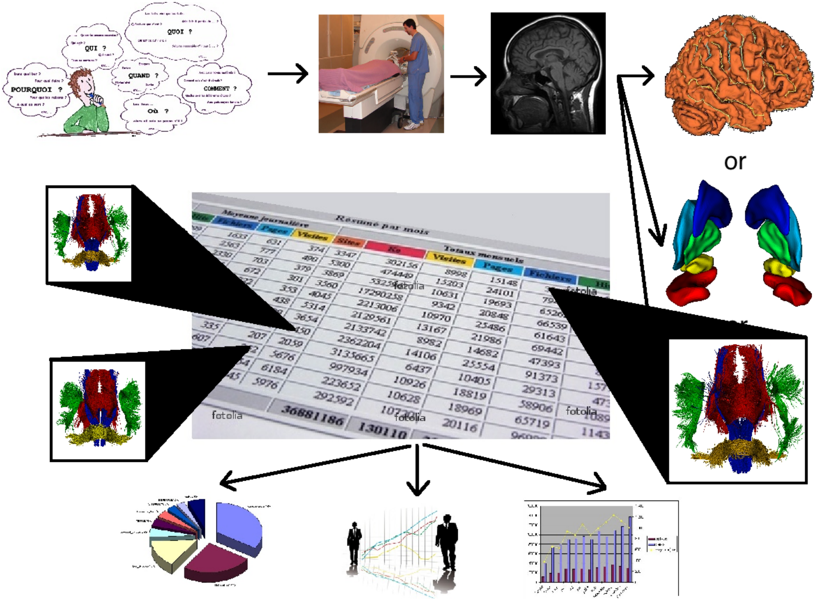
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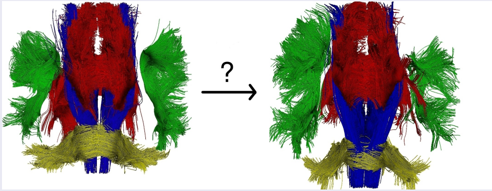
Vector data in $\mathbb{R}^N$:

- Hilbert space: $\|x - y\|^2$
- Gaussian variables: $y_i = \bar{x} + \varepsilon_i$, with $\varepsilon_i \sim \mathcal{N}(0, \Sigma)$

Anatomical data lie on a manifold:

- which manifold? which metric?
- how to compute first and second moment (mean and variance)?

Comparison of 2 anatomies



- "shape space" embedding [Kendall 1984]:

$$d(\mathcal{O}_1, \mathcal{O}_2) \qquad \text{(or } \mathcal{O}_2 = \mathcal{O}_1 + \varepsilon\text{)}$$

  - invariance of the metric? shape 'alignment'?
  - deformation = nuisance factor

- measure of deformation [Grenander 1993]:

$$\hat{\phi} = \operatorname{argmin} d(\mathcal{O}_2, \phi.\mathcal{O}_1) \qquad \text{(or } \mathcal{O}_2 = \hat{\phi}.\mathcal{O}_1 + \varepsilon\text{)}$$

  - metric on objects derived from metric on deformations
  - which deformations?
  - residual as nuisance factor: overfitting?

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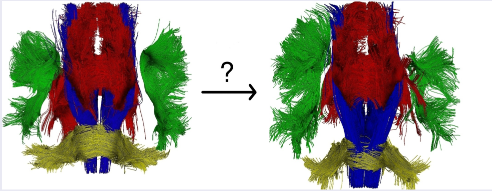

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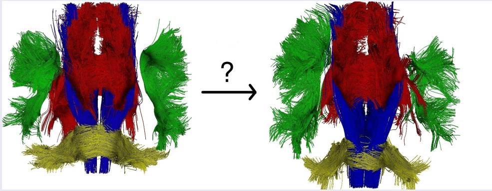


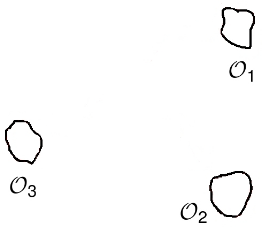
- Combined model:

$$\mathcal{O}_2 = \phi.\mathcal{O}_1 + \varepsilon$$

  - decomposition into geometry + residual
  - all information taken into account
  - which trade-off?

[For images, see Glasbey & Mardia JRSS'01, Allassonniere et al. JRSS'07]

Group-wise statistics



$$\mathcal{O}_i = \phi_i . \bar{\mathcal{O}} + \varepsilon_i$$

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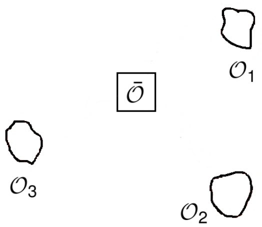

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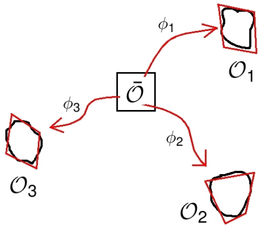

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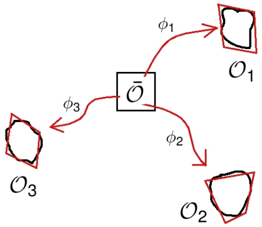

$$\mathcal{O}_i = \phi_i.\bar{\mathcal{O}} + \varepsilon_i$$

- deterministic template ($\bar{\mathcal{O}}$): anatomical invariants
- random deformations (ϕ): geometrical variability
- random residuals (ε): "texture" variability

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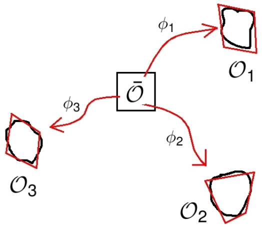

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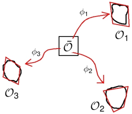
Formal MAP derivation:

$$E(\bar{\mathcal{O}}, \phi_1, \ldots, \phi_N) = \sum_{s=1}^{N} \frac{1}{2\sigma^2} \left\| \phi_s.\bar{\mathcal{O}} - \mathcal{O}_s \right\|^2 + \mathrm{d}(\mathrm{Id}, \phi_s)^2$$

- Fréchet mean $\bar{\mathcal{O}}$ with the metric derived from diffeos
- Variance given by the distribution of
  - deformation parameters $\{\phi_s\}$
  - residuals $\varepsilon_s = \left\| \phi_s.\bar{\mathcal{O}} - \mathcal{O}_s \right\|^2$

Group-wise statistics



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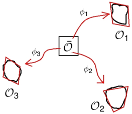

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- Need to define:
  - parameterization of diffeos (ϕ_s) + metric d
  - deformation of anatomical objects $\phi_s.\bar{\mathcal{O}}$
  - norm between objects $\|\mathcal{O}_1 - \mathcal{O}_2\|^2$
- Then:
  - differentiate E w.r.t. deformation parameters and template
  - derive statistics on deformation parameters and residuals

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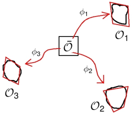

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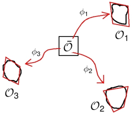

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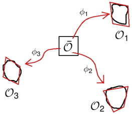

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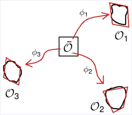

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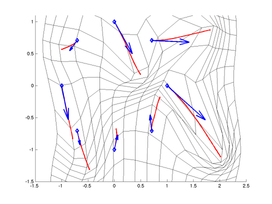
Outline

1. Sparse adaptive parameterization of diffeomorphisms
2. Atlas construction with landmark points
3. Atlas construction with currents
4. Atlas construction with images

Adaptive finite-dimensional subsets of diffeomorphisms

System of N self-interacting particles:

- control points $\{c_k\}$ with momenta $\{\alpha_k\}$



- Hamiltonian (kinetic energy):

$$H(\mathbf{c}, \alpha) = \sum_{p=1}^{N_{cp}} \sum_{q=1}^{N_{cp}} \alpha_k^t K(c_k, c_p) \alpha_p$$

- Equations of motion: $\dot{c}_k = \partial H / \partial \alpha_k$, $\dot{\alpha}_k = -\partial H / \partial c_k$

$$\begin{cases} \frac{dc_k(t)}{dt} = \sum_{q=1}^{N} K(c_k(t), c_q(t))\alpha_q(t) \\ \frac{d\alpha_k(t)}{dt} = -\sum_{q=1}^{N} \nabla K(c_k, c_q)\alpha_q(t)^T \alpha_k(t) \end{cases}$$

- Defines a dense diffeomorphic deformation:

$$v_t(x) = \sum_k K(x, c_k(t))\alpha_k(t)$$

$$\frac{\phi_t(x)}{dt} = v_t(\phi_t(x)) \qquad \phi_0(x) = x$$

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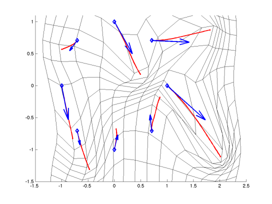

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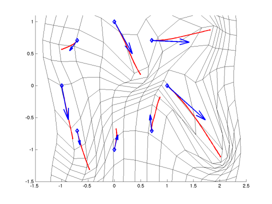

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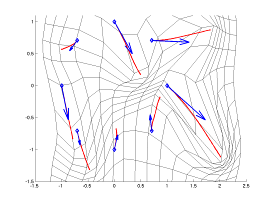

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Definition

The group of diffeomorphisms for a RKHS V:
$\mathcal{G}_V = \left\{ \phi_1 ; \frac{\partial \phi_t}{\partial t} = v_t \circ \phi_t, \phi_0 = \text{id}, \text{ and } v_t \in V \right\}$ is provided with the metric: $d(\text{id}, \phi_1) = \int_0^1 \|v_t\|^2 \, dt$ [Trouvé'95, Dupuis et al.'98]

Theorem

The geodesics connecting two N-uples of distinct points $\mathbf{c}_0$ *and* $\mathbf{c}_1$ *are such that:*

- $v_t(x) = \sum_k K(x, c_k(t))\alpha_k(t)$ *(discrete support of velocity)*
- $\|v_t\|_V^2$ *constant along the geodesic (energy conservation)*

Since $\|v_t\|_V^2 = H(\mathbf{c}, \boldsymbol{\alpha}) = \sum_{p,q=1}^{N_{cp}} \alpha_p(t)^T K(c_p(t), c_q(t))\alpha_q(t)$, *control points and momenta are optimally transported along the geodesics according the Hamiltonian equations:*
$\dot{c}_k(t) = \partial H / \partial \alpha_k$ *and* $\dot{\alpha}_k(t) = -\partial H / \partial c_k$

- for a *fixed* set of *N* control points:
  - diffeomorphisms parameterized by initial momenta $\{\alpha_k(0)\}$
  - geodesics: Log-map $\phi_1 \to \{\alpha_k(0)\}$
  - metric on the tangent-space at Id V: $\|v_0\|_V^2 = \alpha_0^T \mathbf{K}(\mathbf{c}_0, \mathbf{c}_0)\alpha_0$
- optimization of *the position* of control points:
  - optimize the choice of a deformation model of size 3*N*
  - selection in a infinite-dimensional dictionary of deformation models
- optimization of *the number* of control points:
  - selection of the model complexity

Control-point parameterization of deformations not a new idea (diffeo B-spline *[Rueckert et al.]*, GRID *[Grenander et al.]*), however here:

- adaptive parameterization (number and position of CPs)
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Control-point parameterization of deformations not a new idea (diffeo B-spline *[Rueckert et al.]*, GRID *[Grenander et al.]*), however here:

- adaptive parameterization (number and position of CPs)
- buit-in metric on deformations

- for a *fixed* set of N control points:
  - diffeomorphisms parameterized by initial momenta $\{\alpha_k(0)\}$
  - geodesics: Log-map $\phi_1 \to \{\alpha_k(0)\}$
  - metric on the tangent-space at Id V: $\|v_0\|_V^2 = \boldsymbol{\alpha}_0^T \mathbf{K}(\mathbf{c}_0, \mathbf{c}_0)\boldsymbol{\alpha}_0$
- optimization of *the position* of control points:
  - optimize the choice of a deformation model of size $3N$
  - selection in a infinite-dimensional dictionary of deformation models
- optimization of *the number* of control points:
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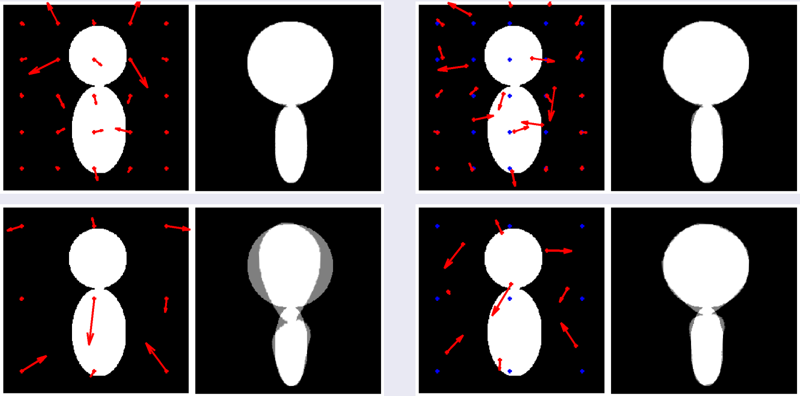
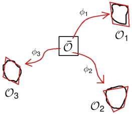
Importance of optimizing Control-Points positions:

Fixed Positions | *Updated* Positions



Optimizing positions of CP enables more compact encoding of the deformation

Group-wise statistics



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$$E(\bar{\mathcal{O}}, \phi_1, \dots, \phi_N) = \sum_{s=1}^{N} \left\{ \frac{1}{2\sigma^2} \left\| \phi_s.\bar{\mathcal{O}} - \mathcal{O}_s \right\|^2 + \mathrm{d}(\mathrm{Id}, \phi_s)^2 \right\}$$

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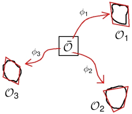

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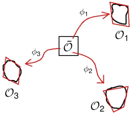

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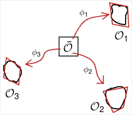

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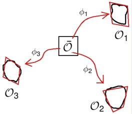

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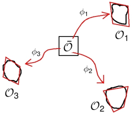

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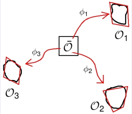
Outline

Group-wise statistics



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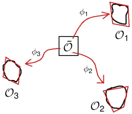

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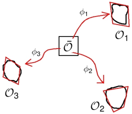

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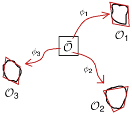

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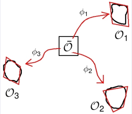

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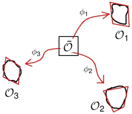

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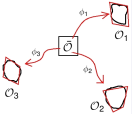

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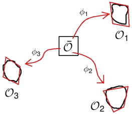

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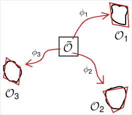

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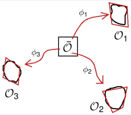

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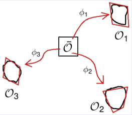

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These gradients can be solved using linearized ODEs!!!

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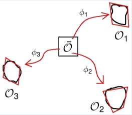

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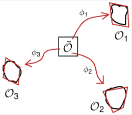

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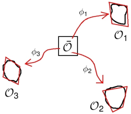
Group-wise statistics



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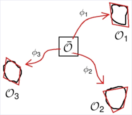

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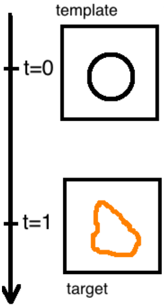

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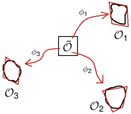

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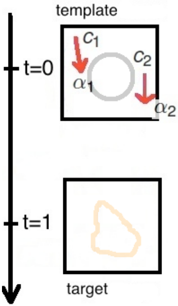

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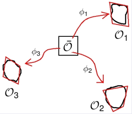

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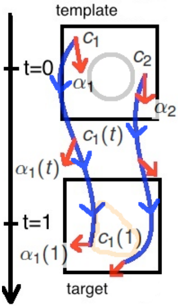

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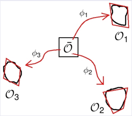

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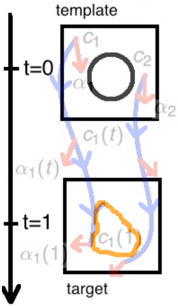

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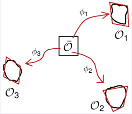

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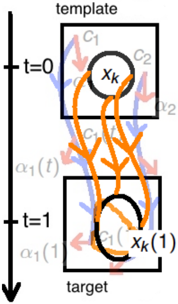

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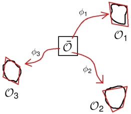

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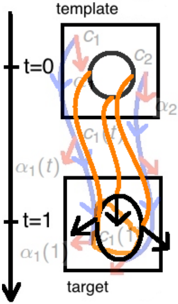
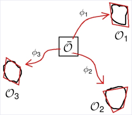


, $\theta(1) = \frac{1}{\sigma^2}(X(1) - Y)$

Group-wise statistics



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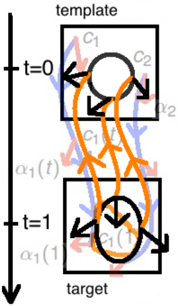

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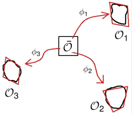
Group-wise statistics



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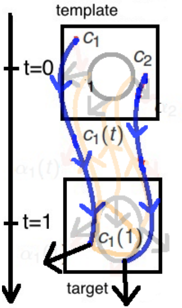

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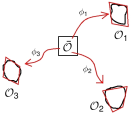
Group-wise statistics



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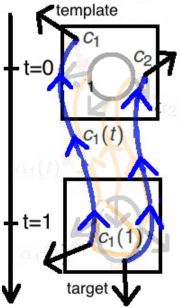

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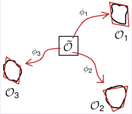

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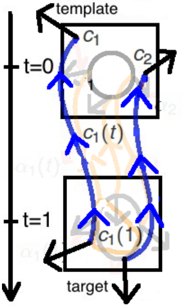

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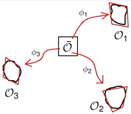
- Simultaneous optimization of template shape *and* registrations!
- at no additional cost!

demo

2 missing pieces:

- selection of control points
- currents

Group-wise statistics



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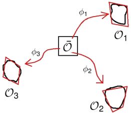

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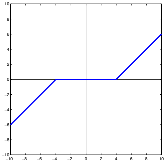
L^1 norm as a surrogate to model selection

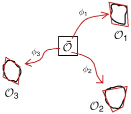
- Optimized using FISTA [Beck&Teboulle'09]

$$\alpha_0^{new} \leftarrow \left(\alpha_0^{old} - \tau\xi^\alpha(0)\right) ST_{\gamma\tau}\left(\frac{\alpha_0^{old} - \tau\xi^\alpha(0)}{\left\|\alpha_0^{old} - \tau\xi^\alpha(0)\right\|}\right)$$



- zero-out momenta small in magnitude

Group-wise statistics



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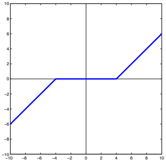

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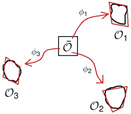

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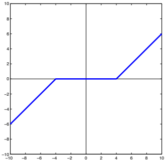

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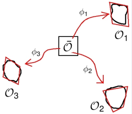
Outline

Group-wise statistics



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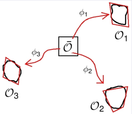
- Landmark case:
  - discrepancy measure: $\|X_s(1) - Y_s\|^2$
  - driving force: $\nabla_{X_s(1)} \|X_s(1) - Y_s\|^2 = 2(X_s(1) - Y_s)$
- Needs point correspondence: hard to find (fiber bundles)
- Currents: an alternative for both curves and surfaces!

Group-wise statistics



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- Landmark case:
  - discrepancy measure: $\|X_s(1) - Y_s\|^2$
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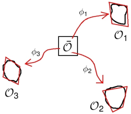
- Needs point correspondence: hard to find (fiber bundles)
- Currents: an alternative for both curves and surfaces!

Group-wise statistics



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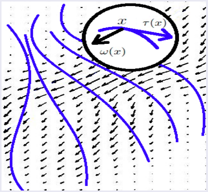
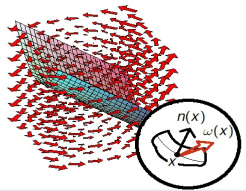
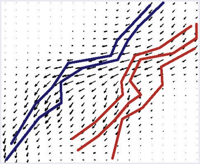
Shapes as currents: an object that integrates vector fields





$F(\omega) = \sum_i \int_{F_i} \omega(x)^t \tau_i(x) dx$ $\quad S(\omega) = \int_S \omega(x)^t n(x) d\sigma(x)$

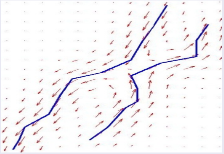
- Currents integrate vector fields: W (*test* space) $\longrightarrow \mathbb{R}$
- Vector space: addition = union, sign = orientation
- Distance **without point correspondence**:
  - $d(\mathcal{O},\mathcal{O}') = \sup_{\|\omega\|_W \leq 1} \left|\mathcal{O}(\omega) - \mathcal{O}'(\omega)\right|$
  - no point, no line correspondence
  - robust to line interruption



[Vaillant and Glaunès IPMI'05, Glaunès PhD'06]

Numerical tools for currents

"Regularized" L^2-metric: $W = L^2 * K$
K Gaussian



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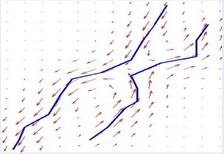

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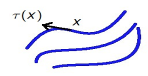

$$T = \sum_{i=1}^{\infty} \delta_{x_n}^{\tau_n} \qquad \|T\|^2 = \int_T \int_T \tau(x)^t K(x,y) \tau(y) dxdy$$

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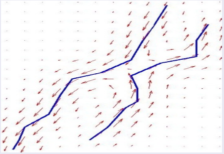

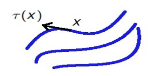

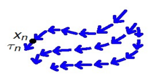

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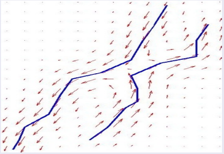

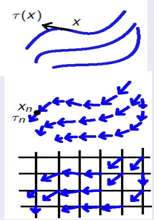

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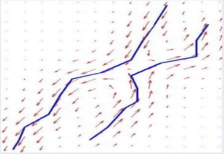

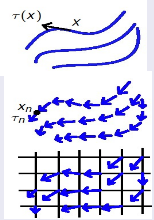

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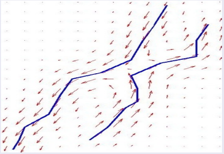
	double sum	grid
complexity	$\mathcal{O}(N^2)$	$\mathcal{O}(N + N_{grid} \log(N_{grid}))$
approx. error	$\mathcal{O}(\max \lvert \tau_i \rvert)$	$\mathcal{O}(\Delta^2 / \lambda_W^2)$

Numerical tools for currents



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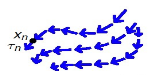

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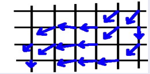

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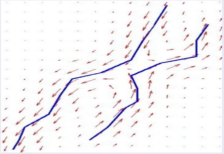
- all computed with FFTs!

Numerical tools for currents



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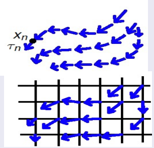

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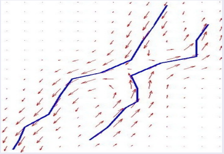

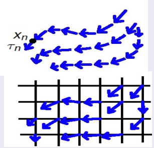

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$W = L^2 * K$, K Gaussian, Kernel bandwidth = resolution

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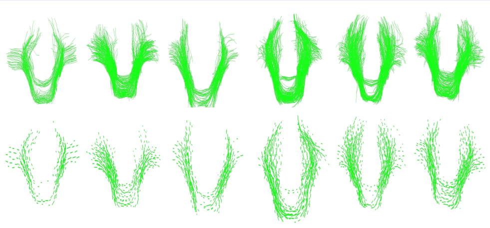

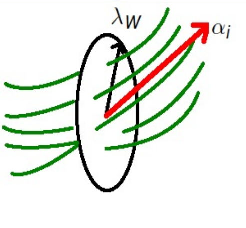

$\lambda_W = 3\text{mm}$, compression ratio: 85%, approx. error $<$ 5% variance

- Adjust the shape decomposition to the resolution
- via the search of adapted basis
- Iterative algorithm: matching pursuit
- proven to converge to the original currents

Approximation of currents at different resolutions

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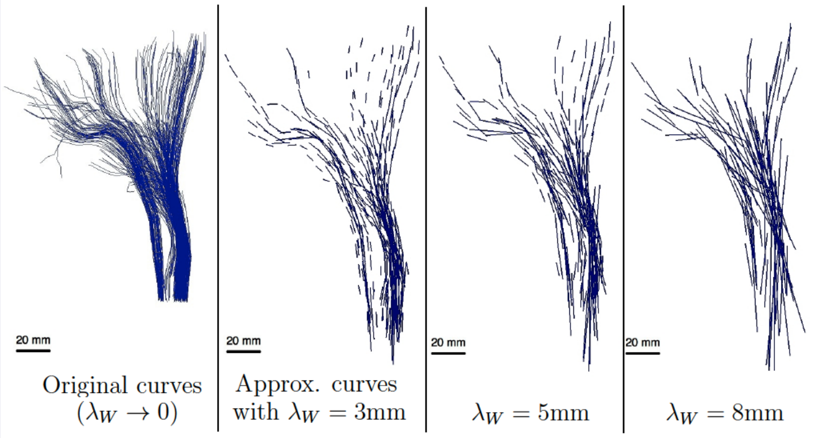
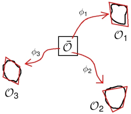


Group-wise statistics



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$$\boxed{\nabla_{S_0} E = \xi(0) \qquad \nabla_{X_0} E = \theta(0)}$$

- Metric on currents to drive the atlas construction
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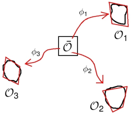

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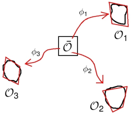

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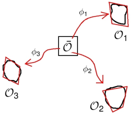

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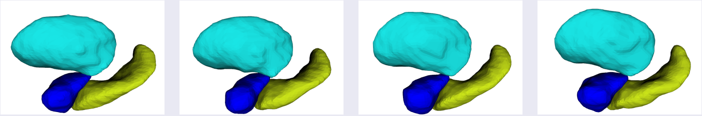
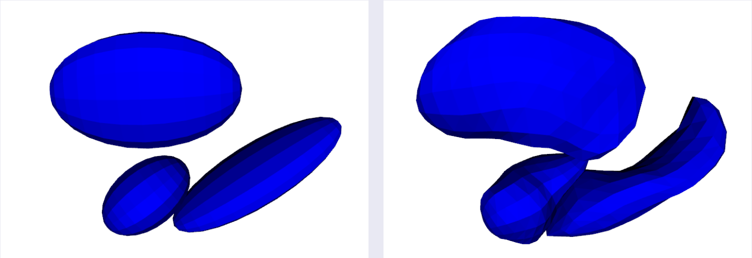
Down's syndrome study



8 controls + 8 Down's syndrome patients



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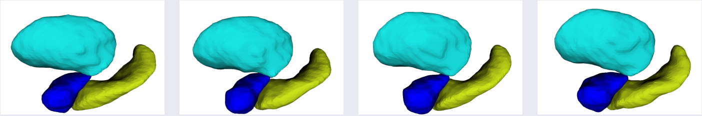

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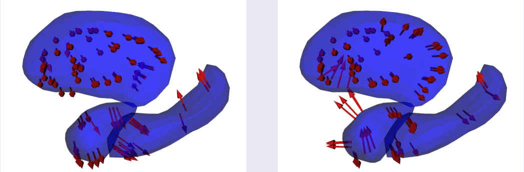
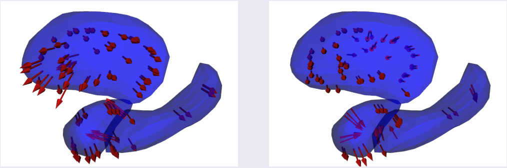
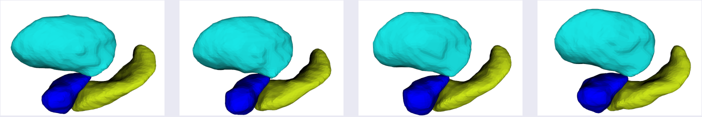


Deformation momenta to 2 Down's syndrome patients



Deformation momenta to 2 control subjects

Down's syndrome study



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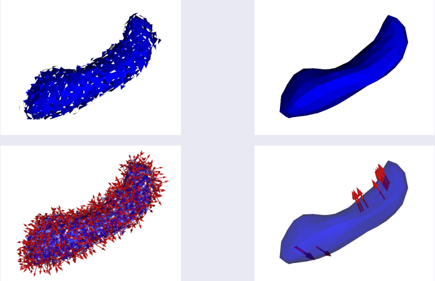
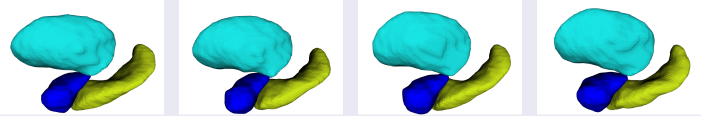


Durrleman Media 09 Our method

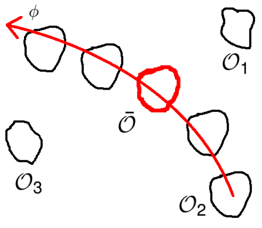
Down's syndrome study

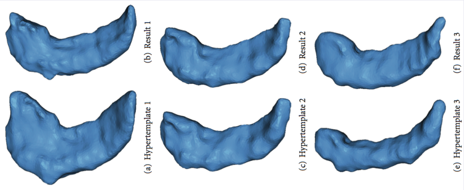


8 controls + 8 Down's syndrome patients

Comparison with Hypertemplate [Ma et al., 2010]

- constrained to be in the orbit of a given shape
- bias by the choice of the initial subject
- alternated minimization

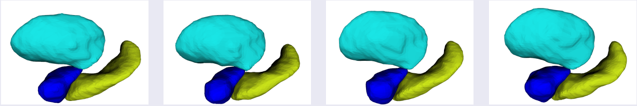




Statistics on deformations

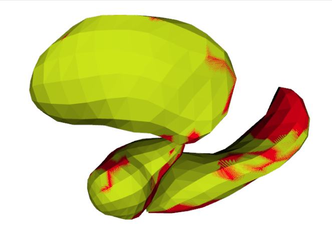
- Output of the atlas:
  - control points: $\mathbf{c} = \{c_k\}$
  - momenta for each subject: $\boldsymbol{\alpha}^1, \dots, \boldsymbol{\alpha}^N$ ($\boldsymbol{\alpha}^i = \{\alpha_k^i\}$)
- Finite-dimensional RKHS with metric $\mathbf{K} = \left(K(c_i, c_j)\right)_{i,j}$: $\langle \boldsymbol{\alpha}^s, \boldsymbol{\alpha}^u \rangle = \boldsymbol{\alpha}^s \mathbf{K} \boldsymbol{\alpha}^u$
- Fit a Gaussian distribution on the momenta set.
- Enables:
  - Principal Component Analysis (eigenmodes of $\left(\langle \boldsymbol{\alpha}^i, \boldsymbol{\alpha}^j \rangle\right)_{i,j}$)
  - Maximum-Likelihood Classification
  - Cross-validation

Down's syndrome study



8 controls + 8 Down's syndrome patients

1 template for all, momenta pooled in 2 groups



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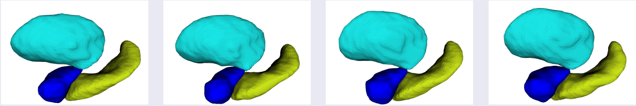

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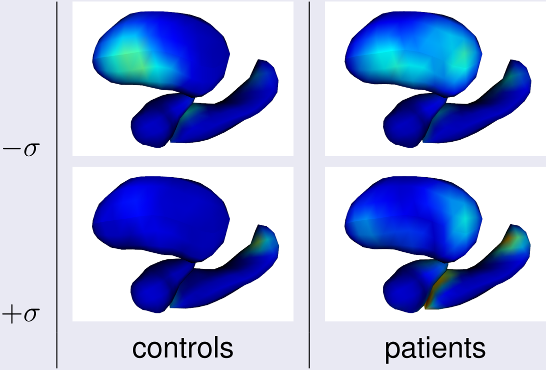
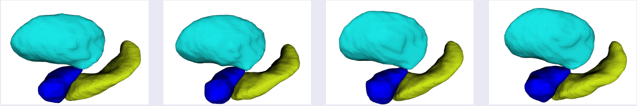


First deformation mode

Down's syndrome study



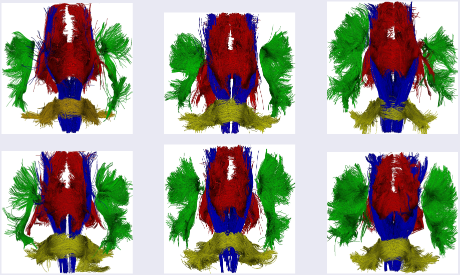
8 controls + 8 Down's syndrome patients

Classification based on initial momenta (leave-2-out):

- False Positive Ratio: 3.1%
- True Positive Ratio: 87.5%

Statistical analysis of fiber tracts

6 subjects - 5 tracts per subjects



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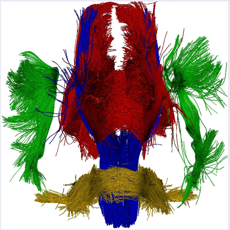
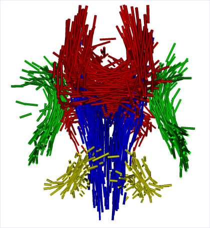
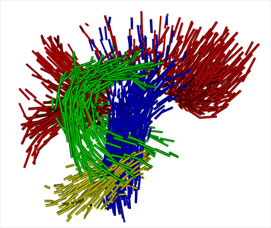


(a) One subject among 6

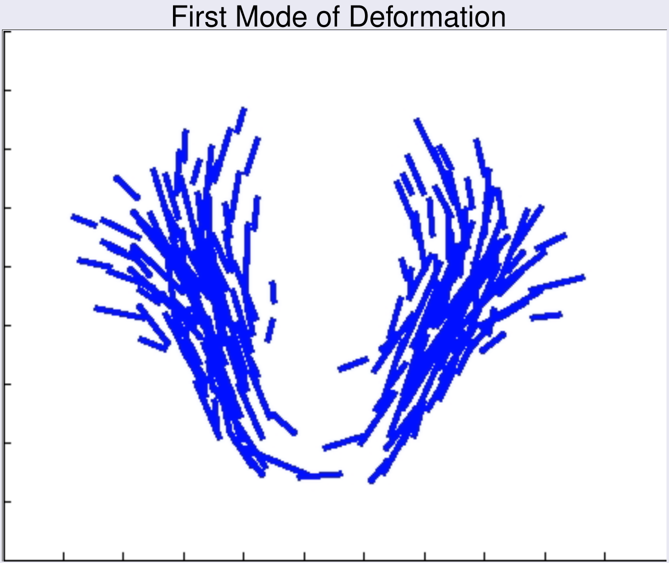


(b) template (occipital view)



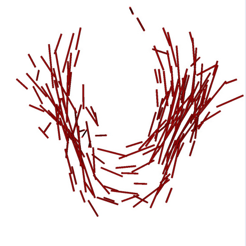
(c) template (lateral view)

Statistical analysis of fiber tracts

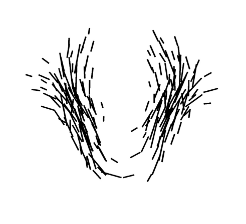


Statistical analysis of fiber tracts

First “texture” Mode



texture mode at $-\sigma$
$\bar{B} - m_\varepsilon$

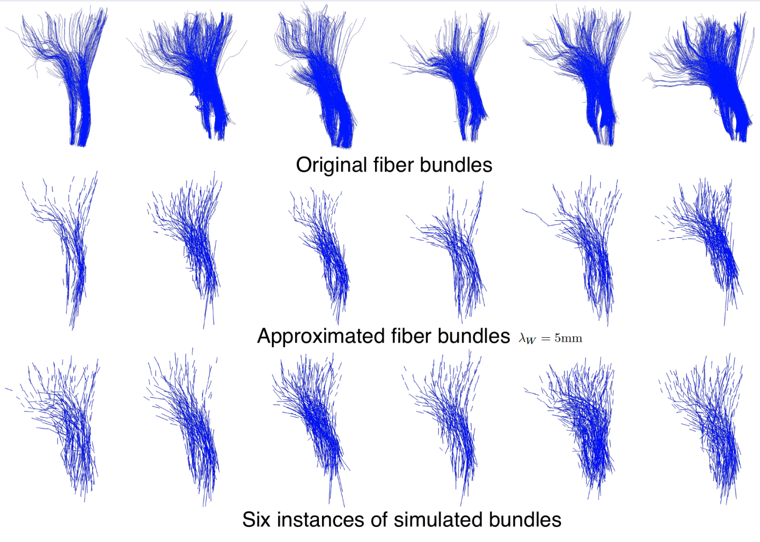


template
$\bar{B}$

texture mode at $+\sigma$
$\bar{B} + m_\varepsilon$

Statistical analysis of fiber tracts

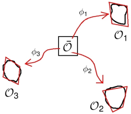
Simulation of synthetic bundles



Outline

1. Sparse adaptive parameterization of diffeomorphisms
2. Atlas construction with landmark points
3. Atlas construction with currents
4. Atlas construction with images

Group-wise statistics



$\mathcal{O}_i = \phi_i.\bar{\mathcal{O}} + \varepsilon_i$

$$E(X_0, \mathbf{S}_0^1, \ldots, \mathbf{S}_0^N) = \sum_{s=1}^N \left\{ \frac{1}{2\sigma^2} \|X_s(1) - Y_s\|^2 + L(\mathbf{S}_0^s) + \gamma \sum_{k=1}^{N_{cp}} \left\| \alpha_{0,k}^s \right\| \right\}$$

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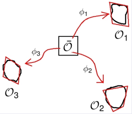

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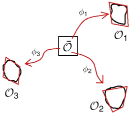

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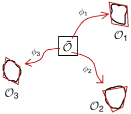

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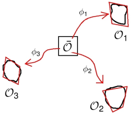

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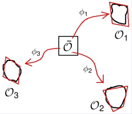

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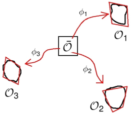

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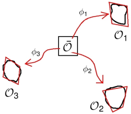

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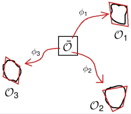

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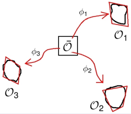

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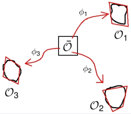
$$\phi.X_0 = X(1) \text{ with } \left\{ \begin{array}{l} \dot{X}(t) = G(\mathbf{S}(t), X(t)) \\ X(0) = X_0 \end{array} \right.$$

  - Sum of squared differences: $\|X_s(1) - Y_s\|^2_{\mathbb{R}^N}$, $\nabla_{X(1)} \|X(1) - Y\|^2 = 2(X(1) - Y)$

Group-wise statistics



$\mathcal{O}_i = \phi_i.\bar{\mathcal{O}} + \varepsilon_i$

$$E(I_0, \mathbf{S}_0^1, \ldots, \mathbf{S}_0^N) = \sum_{s=1}^{N} \left\{ \frac{1}{2\sigma^2} \|X_s(1) - I_s\|^2 + L(\mathbf{S}_0^s) + \gamma \sum_{k=1}^{N_{cp}} \left\| \alpha_{0,k}^s \right\| \right\}$$

$$s.t. \left\{ \begin{array}{l} \dot{\mathbf{S}}_s(t) = F(\mathbf{S}_s(t)) \\ \mathbf{S}_s(0) = \mathbf{S}_0^s \end{array} \right. \quad \left\{ \begin{array}{l} \dot{X}_s(t) = G(\mathbf{S}_s(t), X_s(t)) \\ X_s(0) = X_0 \end{array} \right.$$

- Image case: $\mathcal{O}_s = I_s$ arrays, domain discretization $\{y_k\}$
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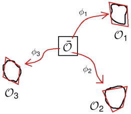
Group-wise statistics



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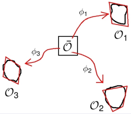
Group-wise statistics



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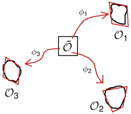

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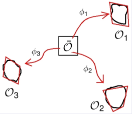

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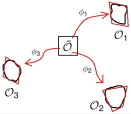
Group-wise statistics



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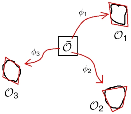
Group-wise statistics



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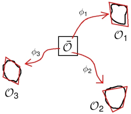
Group-wise statistics



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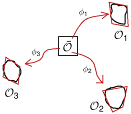
Group-wise statistics



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- w/o L^1 prior:

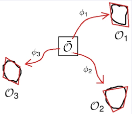
$$\dot{\eta}(t) = -\partial_{S(t)} G^T \eta(t), \quad \eta(0) = \frac{1}{\sigma^2}(I_0(Y_s(0)) - I_s)\nabla I_0$$

$$\dot{\xi}(t) = -\partial_{Y(t)} G^T \eta(t) - dF^T \xi(t), \quad \xi(1) = 0$$

$$\boxed{\nabla_{S_0} E = \xi(0) + \nabla_{S_0} L}$$

- with L^1 prior: gradient soft-thresholded

Group-wise statistics



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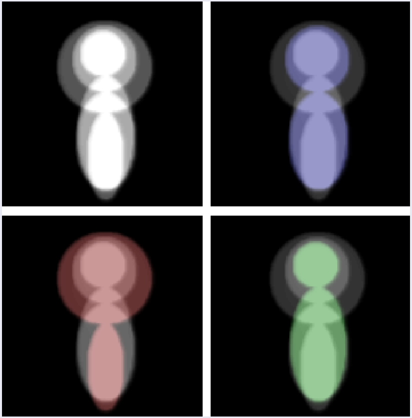
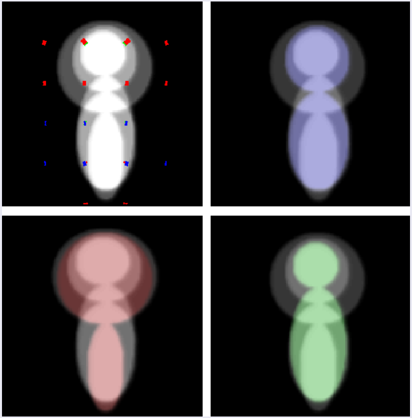
Atlas construction: optimization



Single gradient descent:
- template image
- position of CP
- number of CP
- momenta

Image size=128^2, $\sigma_V = 25$, $\sigma^2 = 0.005$, $\gamma = 540$

Atlas construction: optimization



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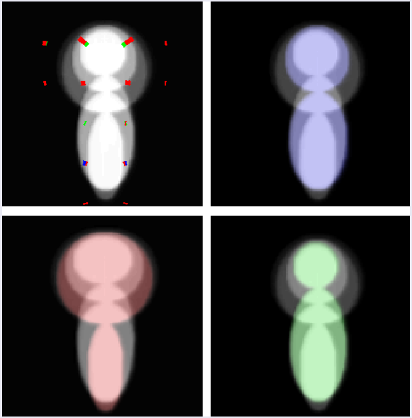

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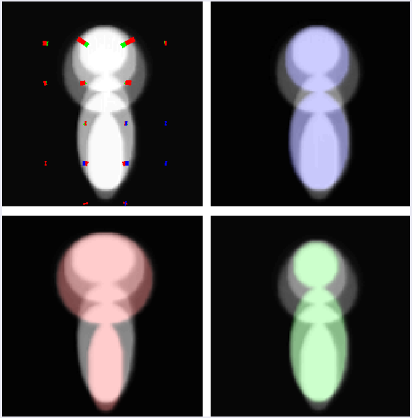

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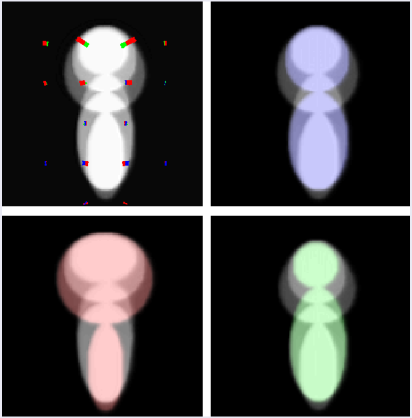

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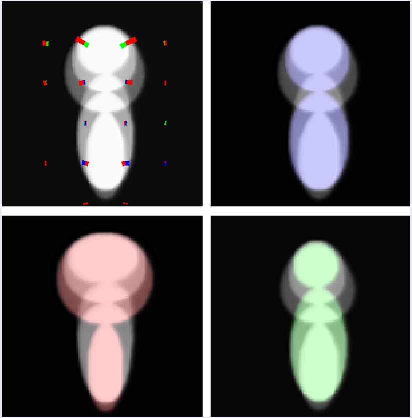

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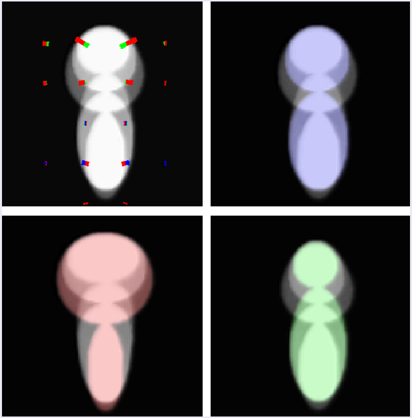

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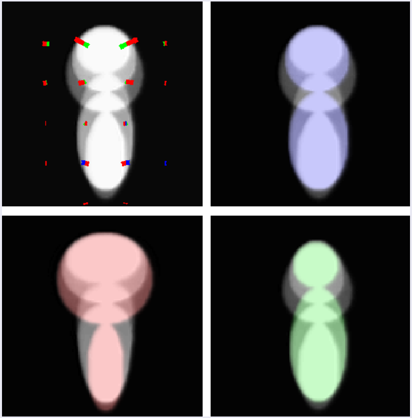

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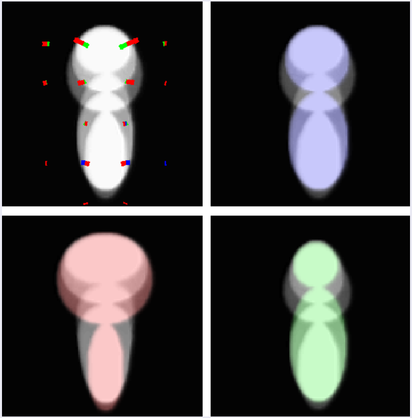

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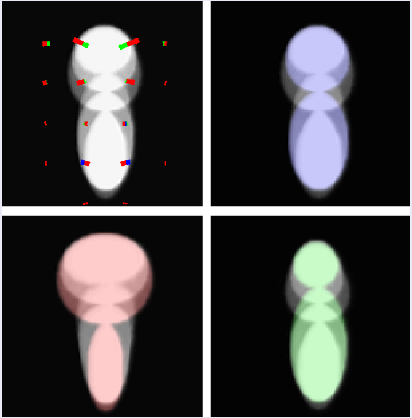

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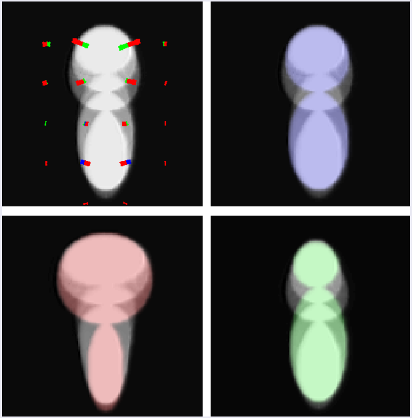

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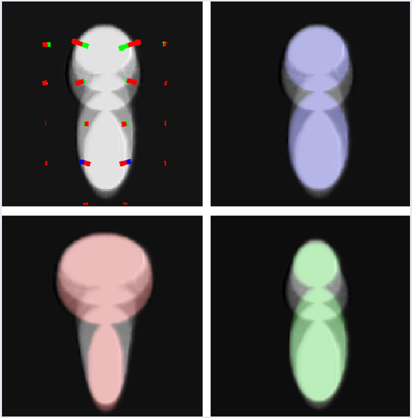

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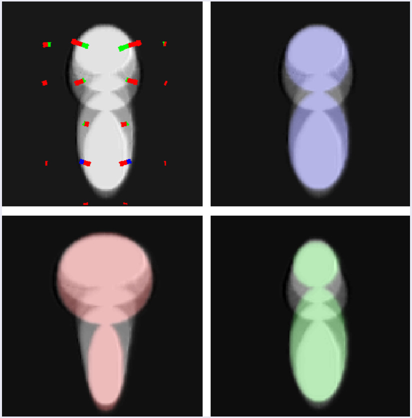
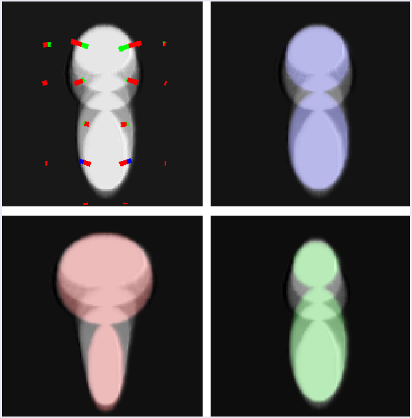


Single gradient descent:

- template image
- position of CP
- number of CP
- momenta

Image size=128^2, $\sigma_V = 25$, $\sigma^2 = 0.005$, $\gamma = 540$

Atlas construction: optimization



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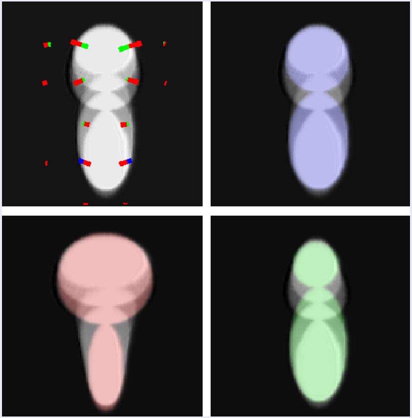

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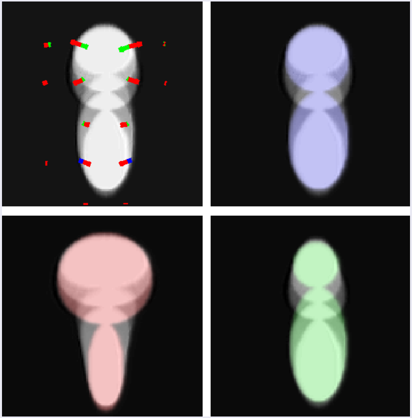

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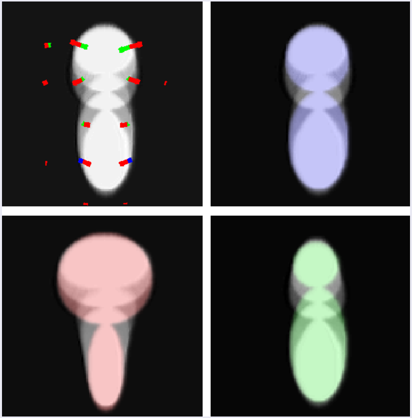

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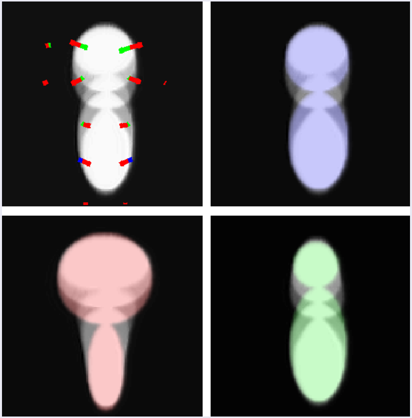

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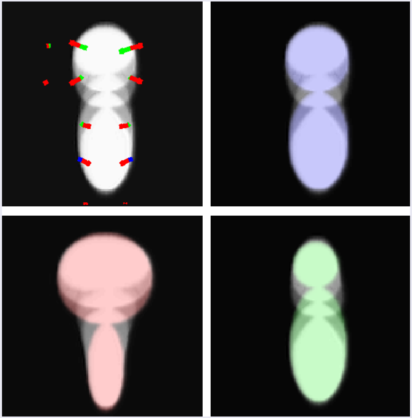

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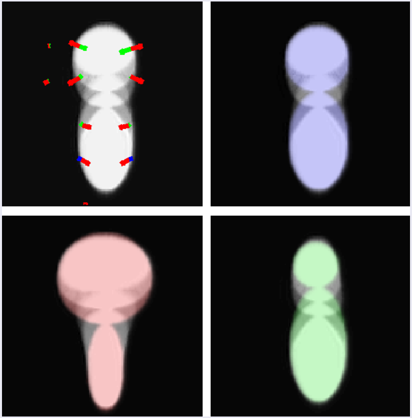

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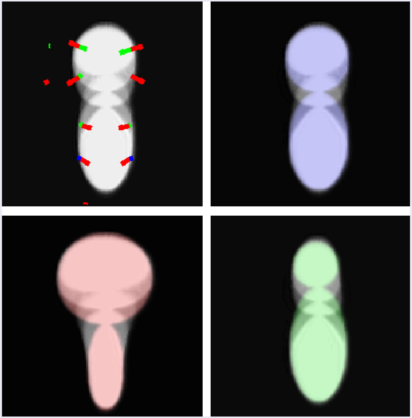

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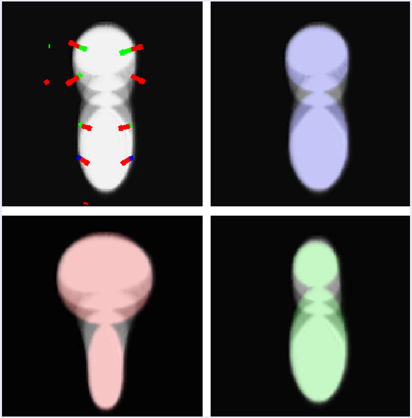

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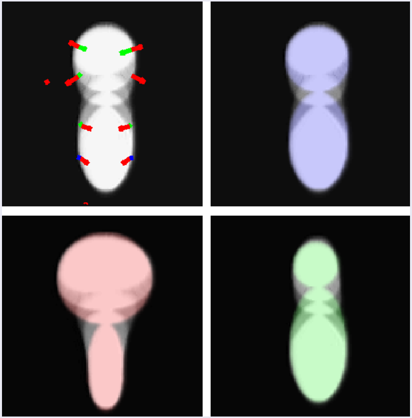

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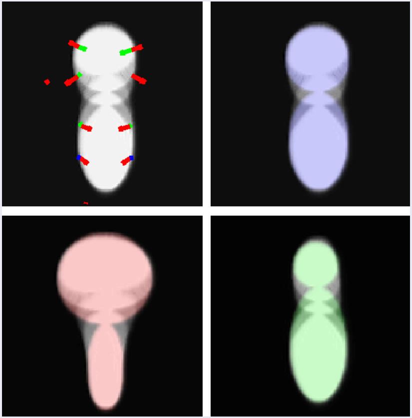

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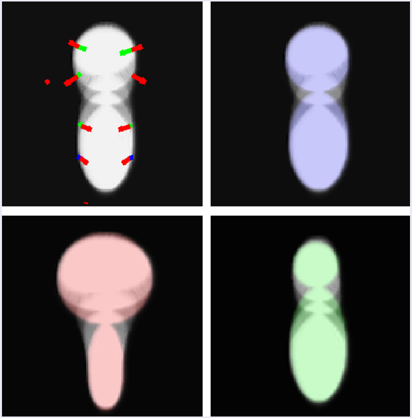

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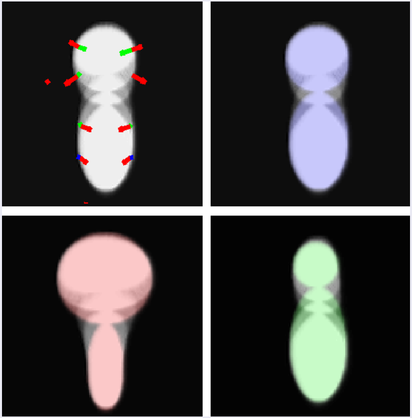

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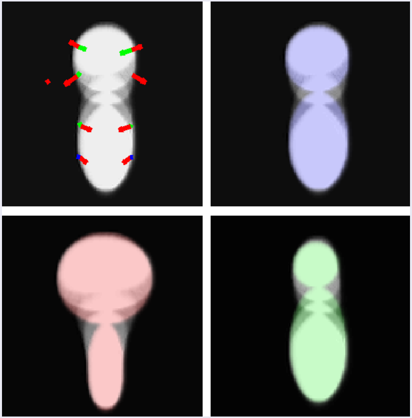

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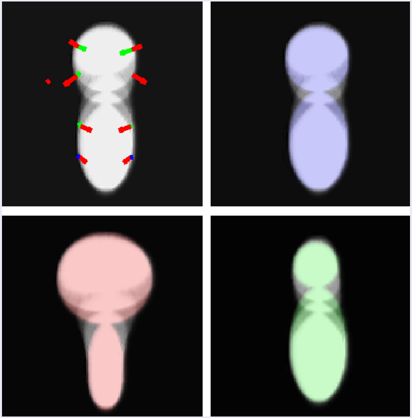

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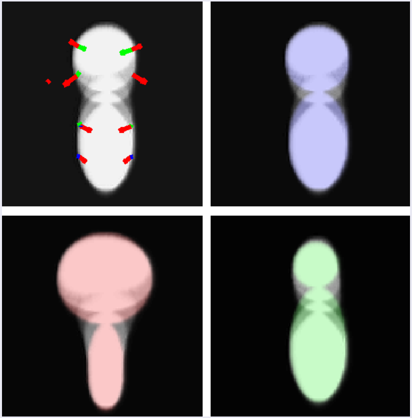

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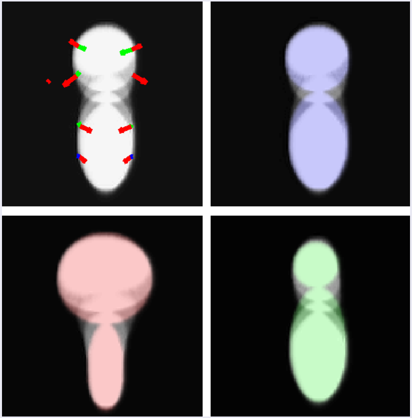

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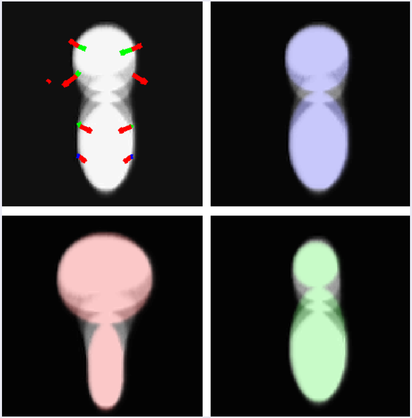

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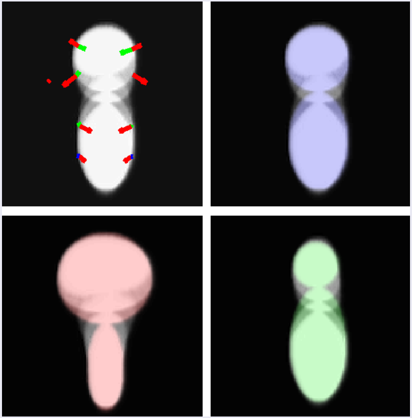

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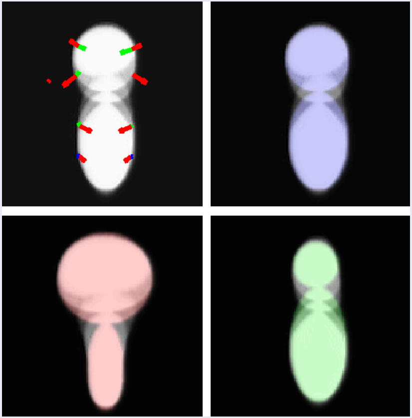

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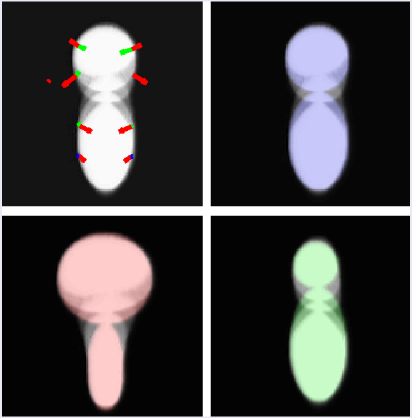

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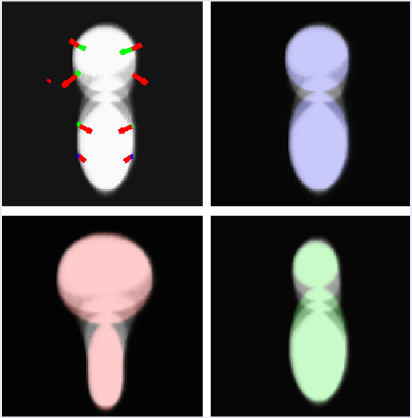

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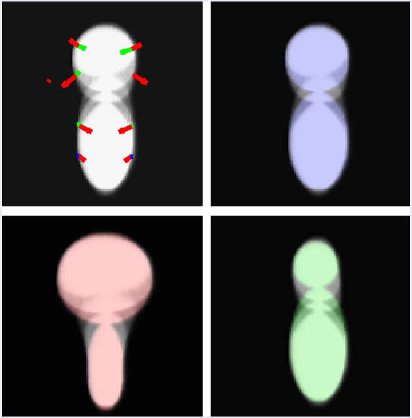

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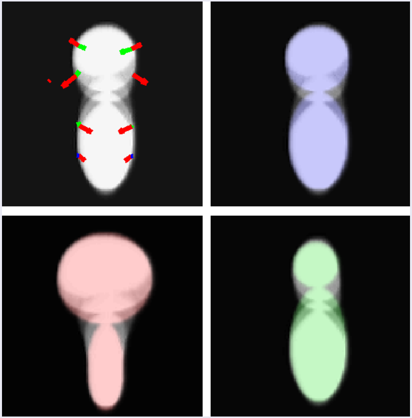

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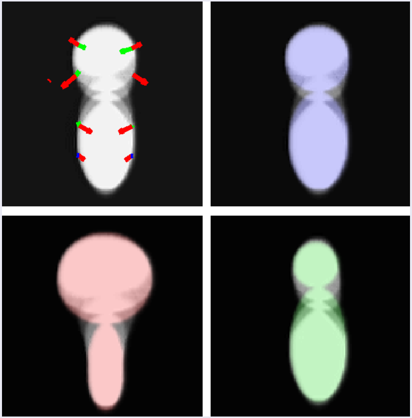

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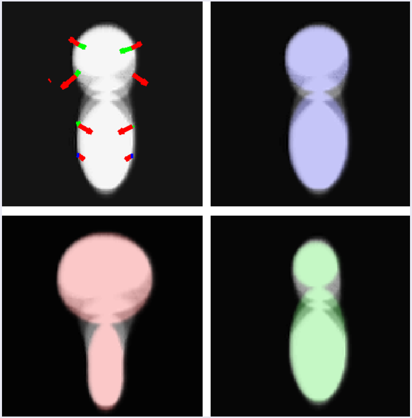

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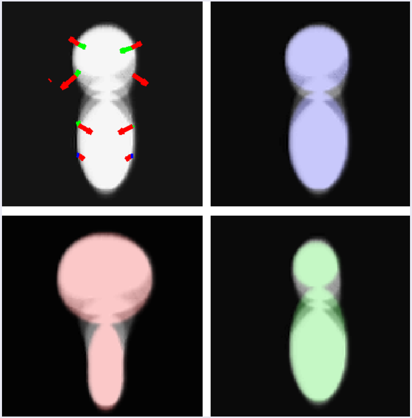

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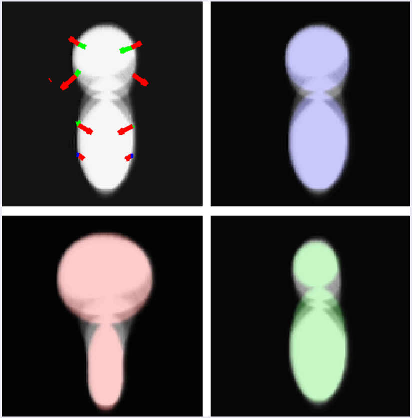

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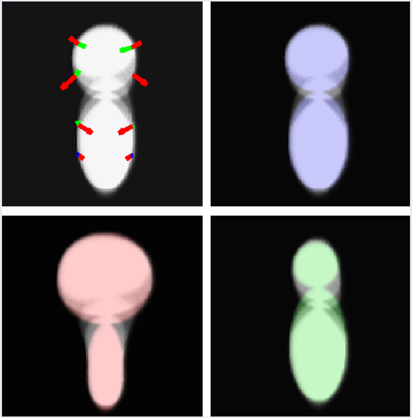

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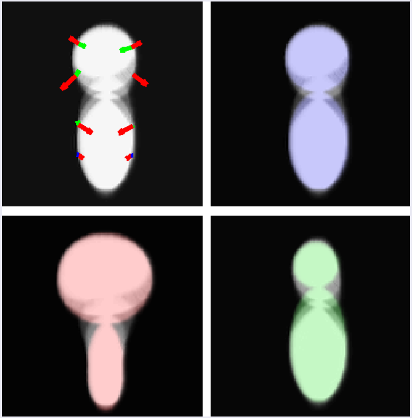

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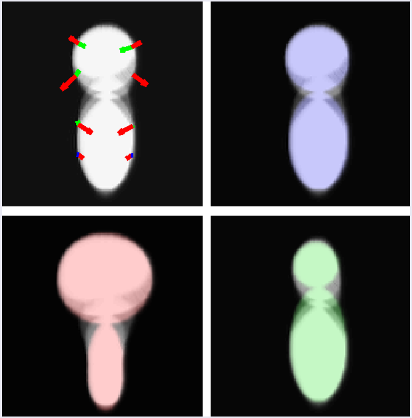

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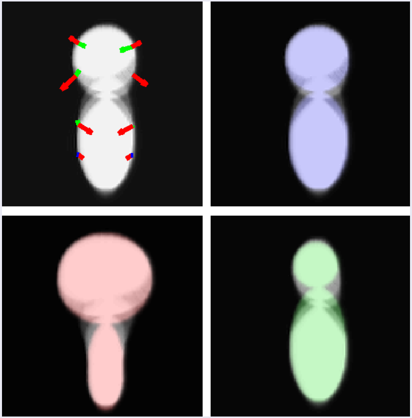

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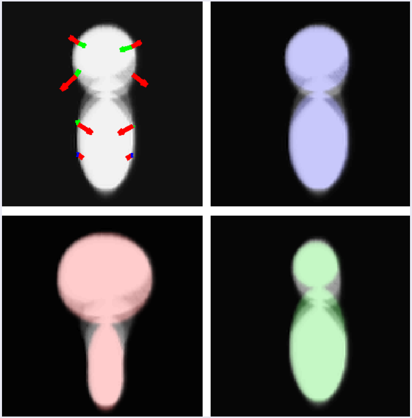

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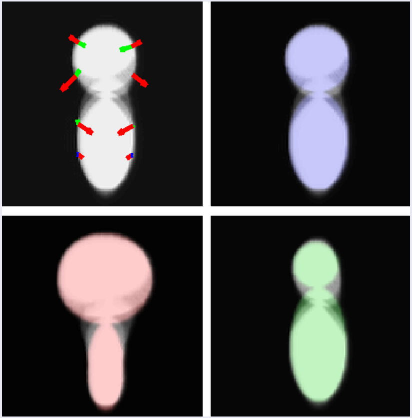

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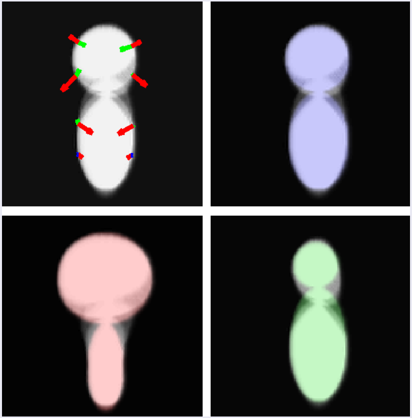

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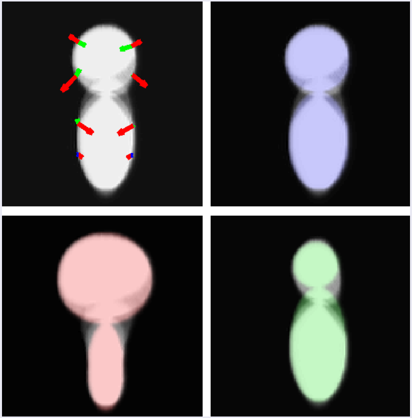

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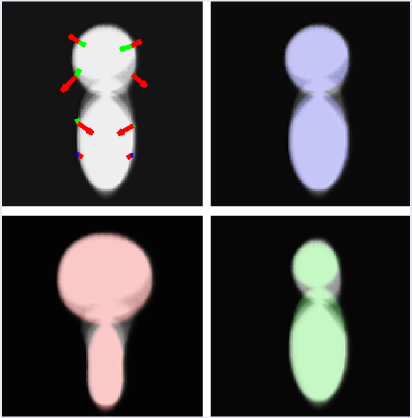

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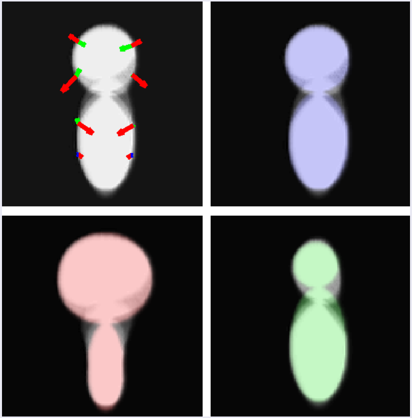

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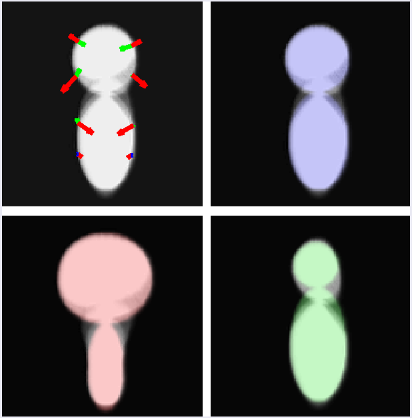

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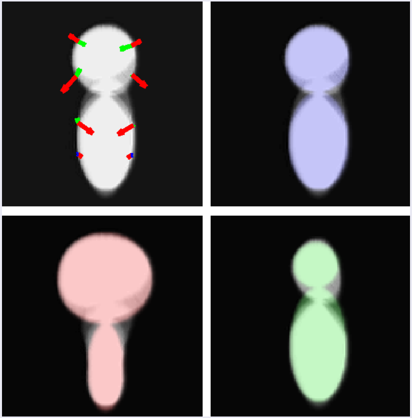

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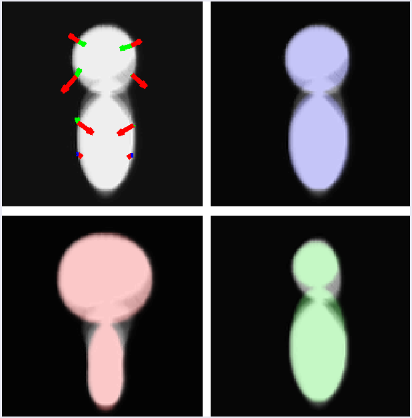

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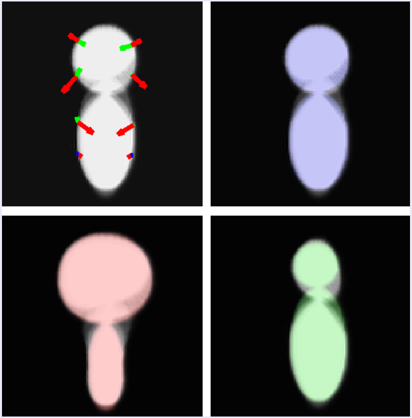

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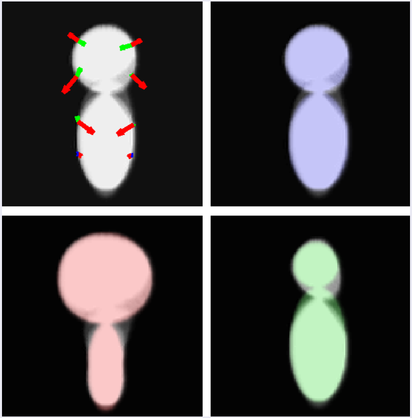

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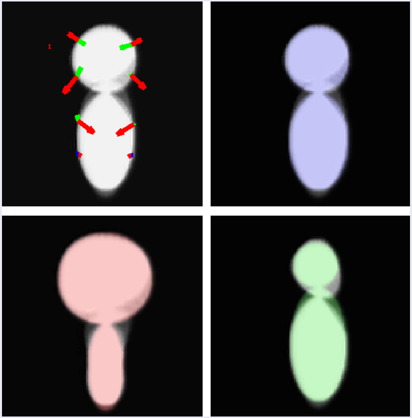

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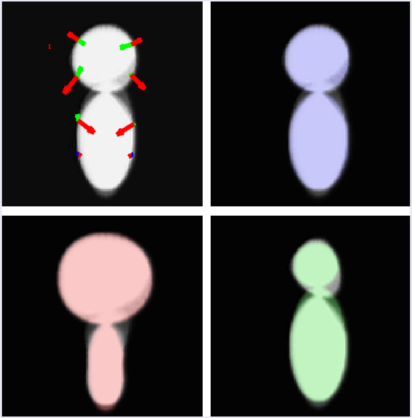

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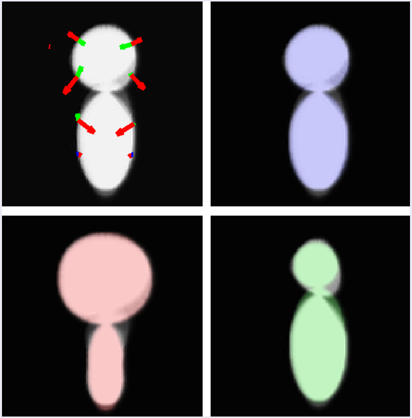

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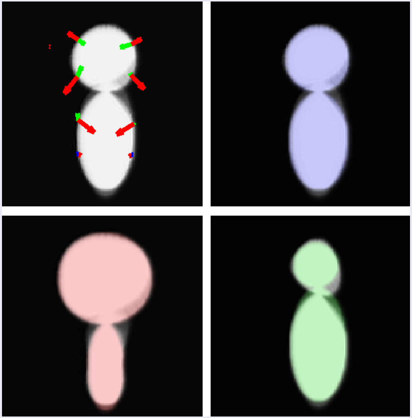

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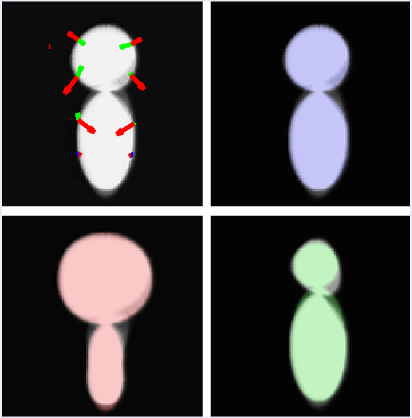
Image size=128^2, $\sigma_V = 25$, $\sigma^2 = 0.005$, $\gamma = 540$

Atlas construction: optimization

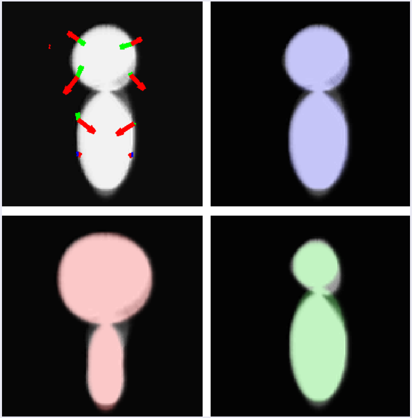


Single gradient descent:

- template image
- position of CP
- number of CP
- momenta

Image size=128^2, $\sigma_V = 25$, $\sigma^2 = 0.005$, $\gamma = 540$

Atlas construction: optimization



Single gradient descent:

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Image size=128^2, $\sigma_V = 25$, $\sigma^2 = 0.005$, $\gamma = 540$

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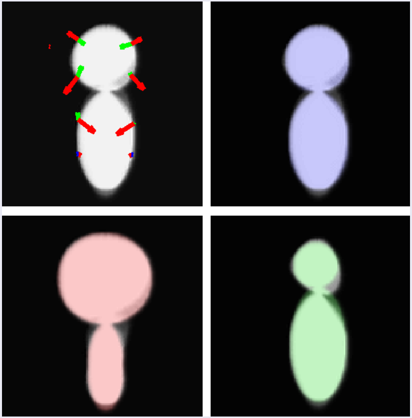

Single gradient descent:

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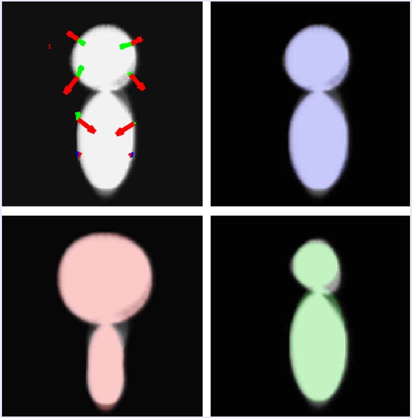
Image size=128^2, $\sigma_V = 25$, $\sigma^2 = 0.005$, $\gamma = 540$

Atlas construction: optimization



Single gradient descent:

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Image size=128^2, $\sigma_V = 25$, $\sigma^2 = 0.005$, $\gamma = 540$

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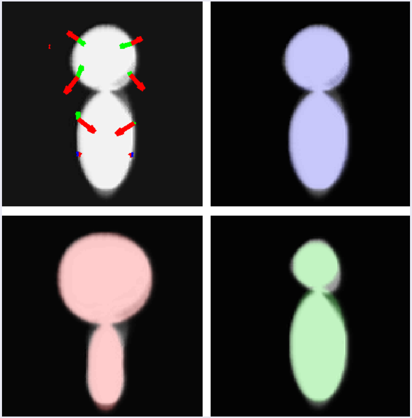

Single gradient descent:

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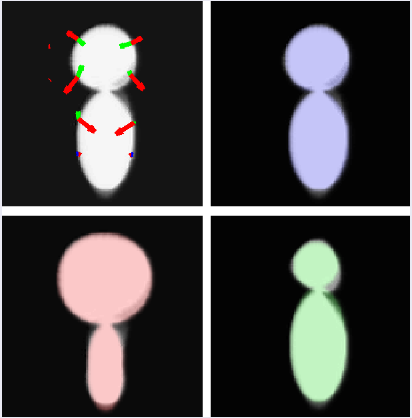
- position of CP
- number of CP
- momenta

Image size=128^2, $\sigma_V = 25$, $\sigma^2 = 0.005$, $\gamma = 540$

Atlas construction: optimization



Single gradient descent:

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- position of CP
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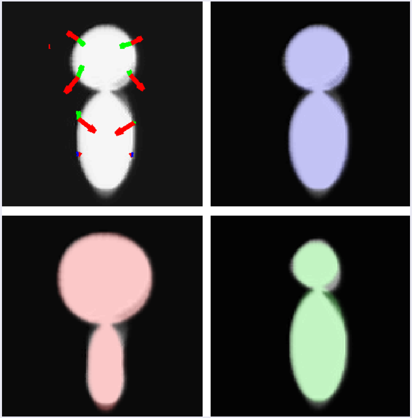
- momenta

Image size=128^2, $\sigma_V = 25$, $\sigma^2 = 0.005$, $\gamma = 540$

Atlas construction: optimization



Single gradient descent:

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- position of CP
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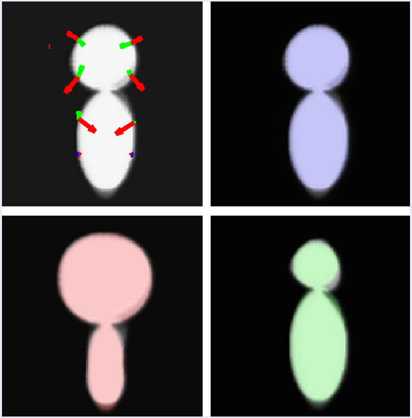
Image size=128^2, $\sigma_V = 25$, $\sigma^2 = 0.005$, $\gamma = 540$

Atlas construction: optimization



Single gradient descent:

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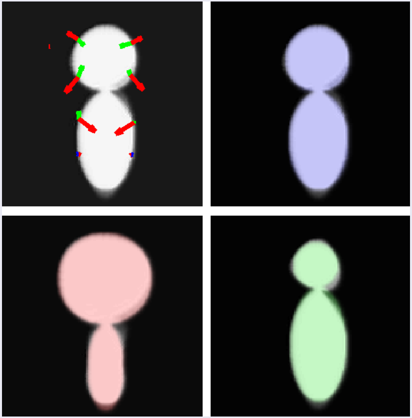
Image size=128^2, $\sigma_V = 25$, $\sigma^2 = 0.005$, $\gamma = 540$

Atlas construction: optimization



Single gradient descent:

- template image
- position of CP
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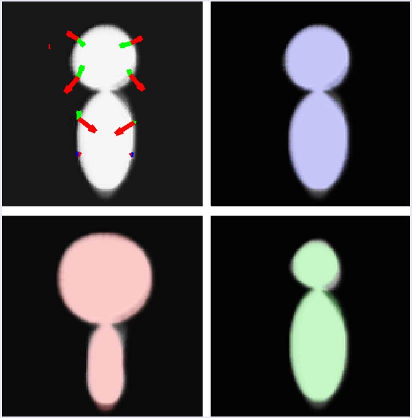
- momenta

Image size=128^2, $\sigma_V = 25$, $\sigma^2 = 0.005$, $\gamma = 540$

Atlas construction: optimization



Single gradient descent:

- template image
- position of CP
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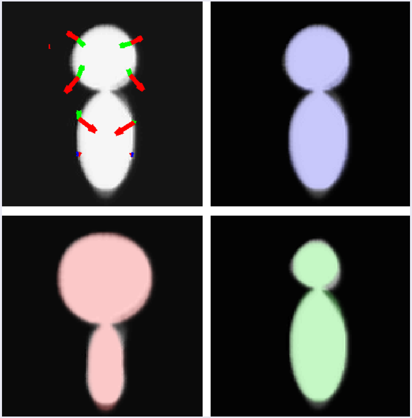
- momenta

Image size=128^2, $\sigma_V = 25$, $\sigma^2 = 0.005$, $\gamma = 540$

Atlas construction: optimization



Single gradient descent:

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- position of CP
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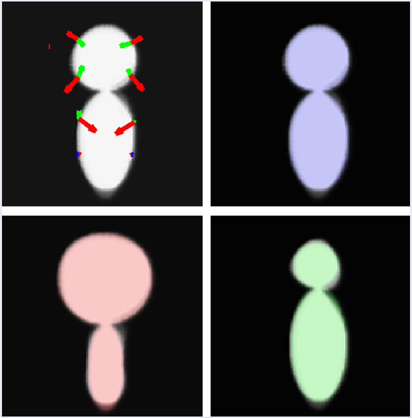

Single gradient descent:

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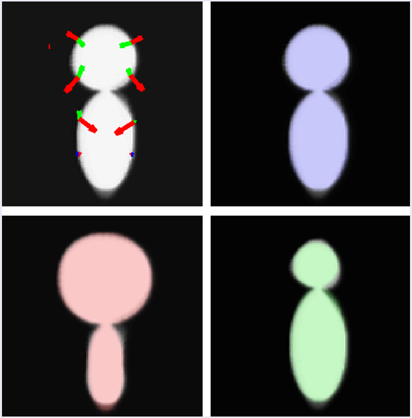

Single gradient descent:

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- position of CP
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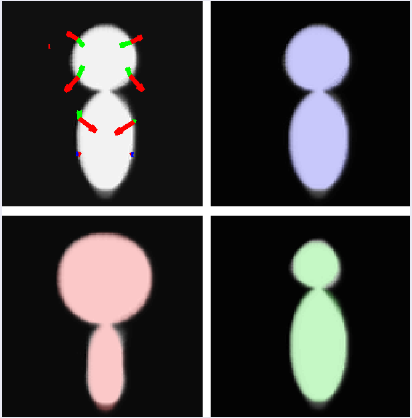
Image size=128^2, $\sigma_V = 25$, $\sigma^2 = 0.005$, $\gamma = 540$

Atlas construction: optimization

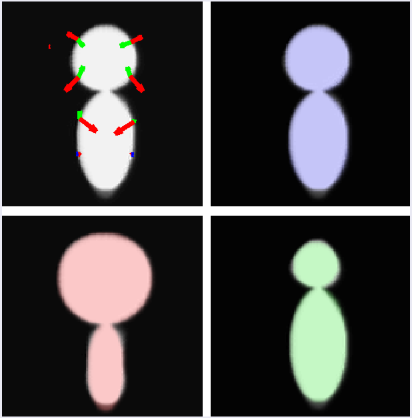


Single gradient descent:

- template image
- position of CP
- number of CP
- momenta

Image size=128^2, $\sigma_V = 25$, $\sigma^2 = 0.005$, $\gamma = 540$

Atlas construction: optimization



Single gradient descent:

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Image size=128^2, $\sigma_V = 25$, $\sigma^2 = 0.005$, $\gamma = 540$

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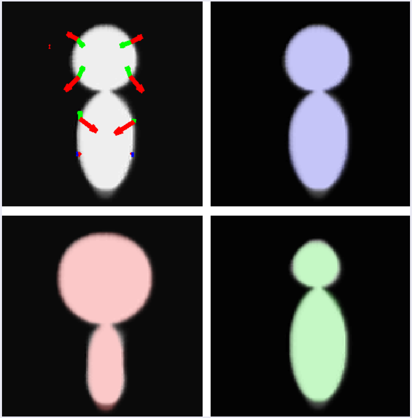

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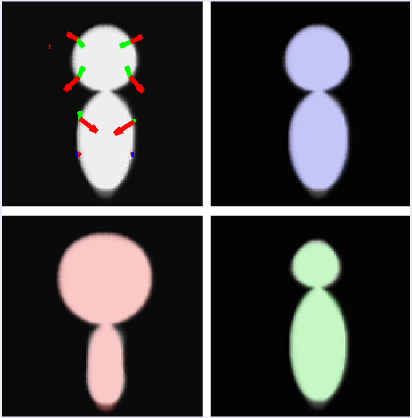

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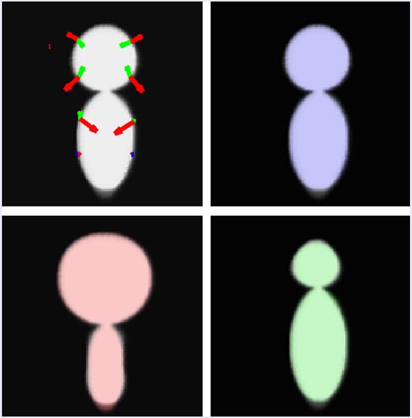

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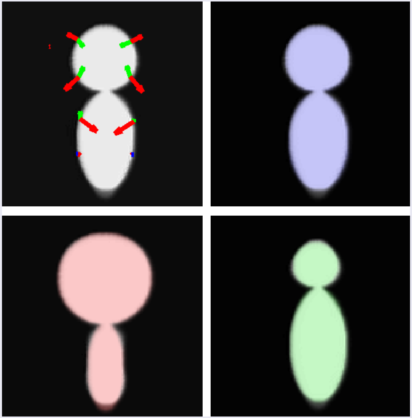

Single gradient descent:

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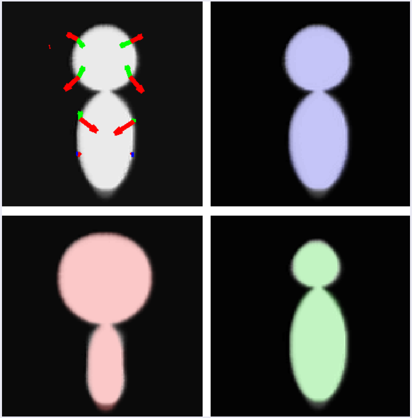

Single gradient descent:

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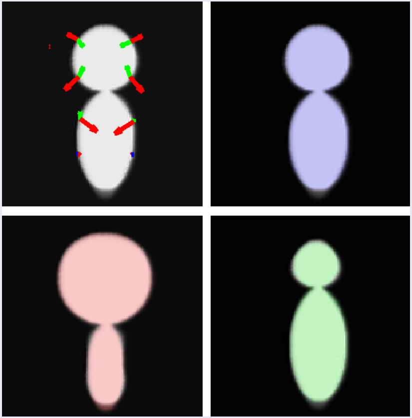

Single gradient descent:

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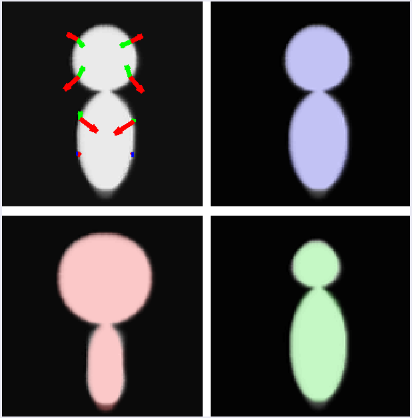

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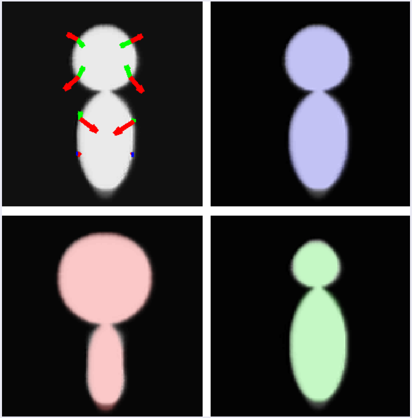

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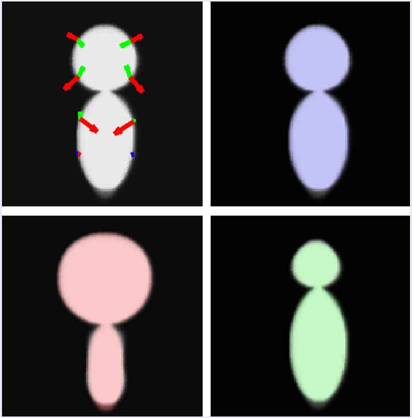

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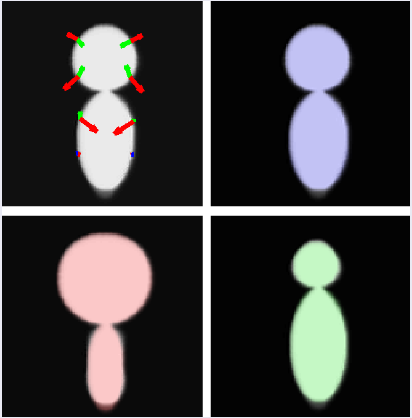

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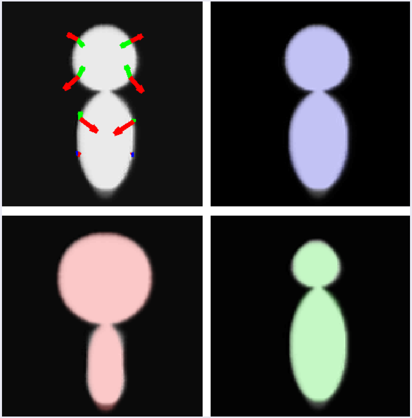

Single gradient descent:

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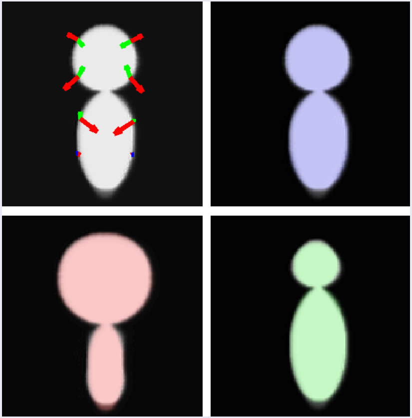

Single gradient descent:

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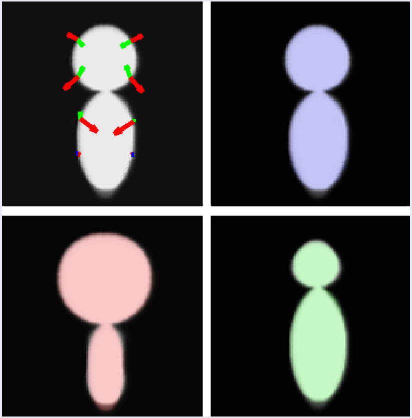

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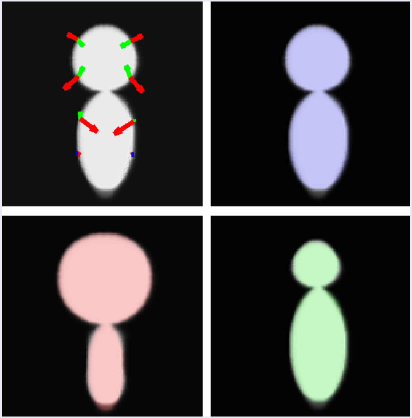

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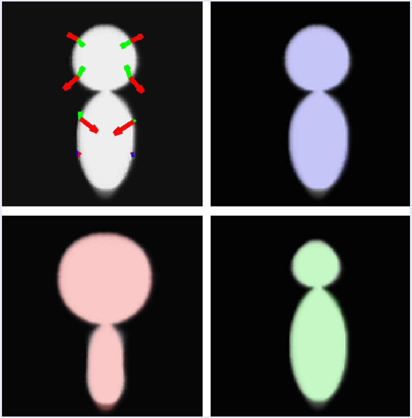

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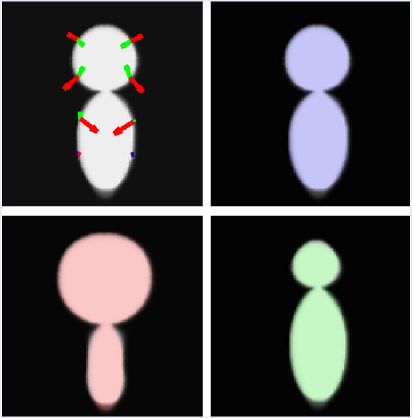

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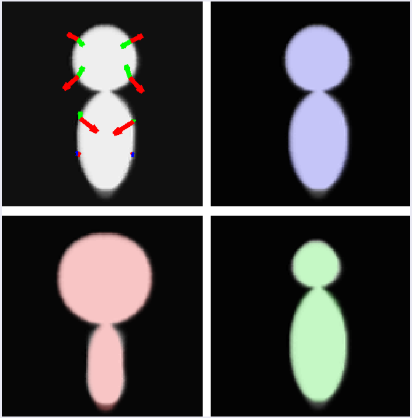

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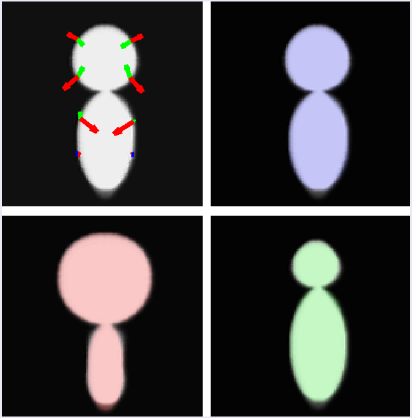

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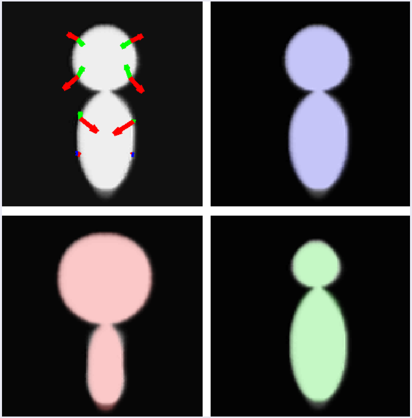

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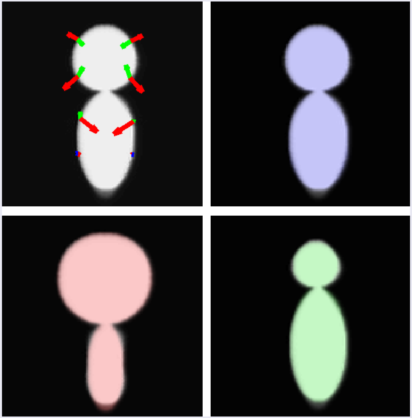

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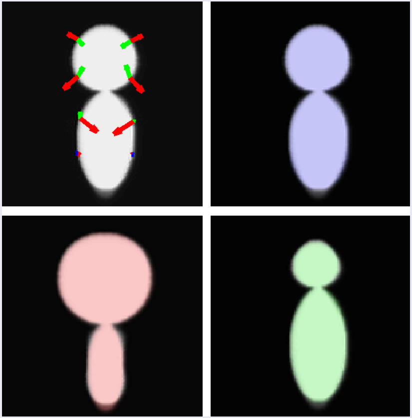

Single gradient descent:
- template image
- position of CP
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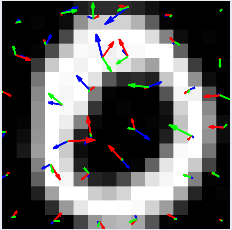
- momenta

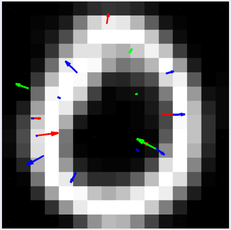
Image size=128^2, $\sigma_V = 25$, $\sigma^2 = 0.005$, $\gamma = 540$

8 estimated control points!

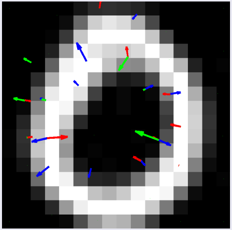
Atlas from the US postal database (20 training images)



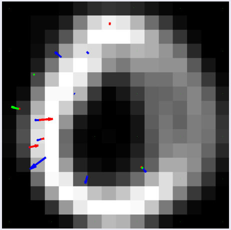
$\gamma = 0$; 36 active CPs



$\gamma = 700$; 21 active CPs

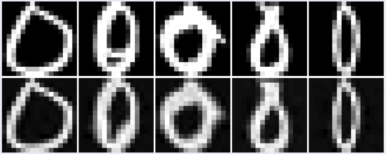


$\gamma = 400$; 27 active CPs

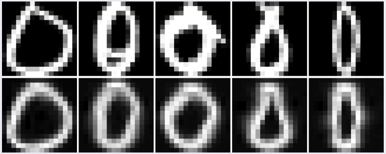


$\gamma = 800$; 13 active CPs

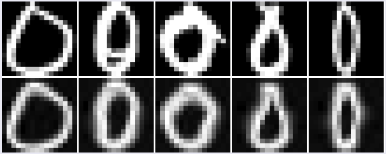
Atlas from the US postal database (20 training images)



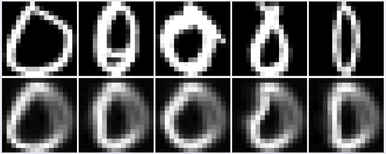
$\gamma = 0$; 36 active CPs



$\gamma = 700$; 21 active CPs

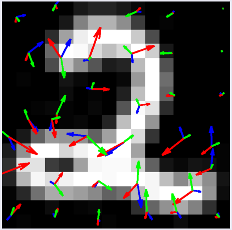


$\gamma = 400$; 27 active CPs

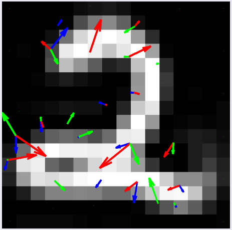


$\gamma = 800$; 13 active CPs

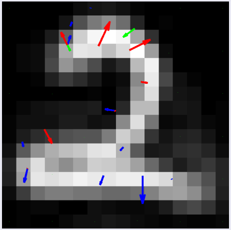
Atlas from the US postal database (20 training images)



$\gamma = 0$; 36 active CPs

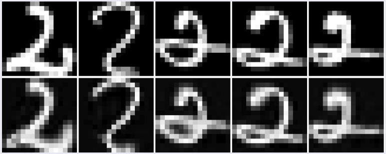


$\gamma = 200$; 27 active CPs

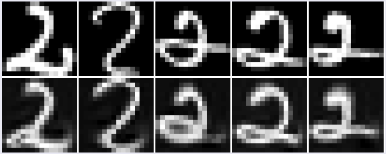


$\gamma = 400$; 18 active CPs

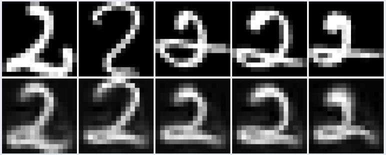
Atlas from the US postal database (20 training images)



$\gamma = 0$; 36 active CPs

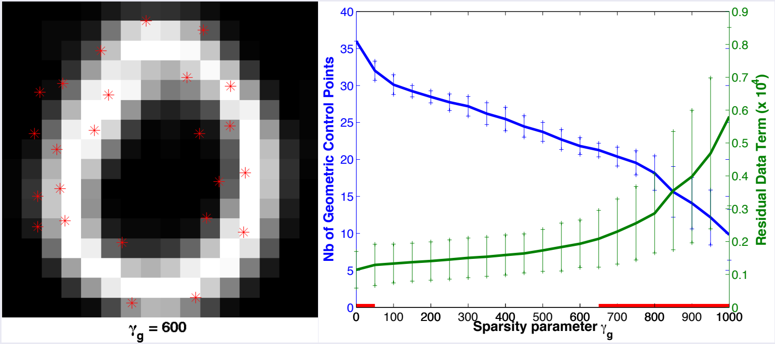


$\gamma = 200$; 27 active CPs

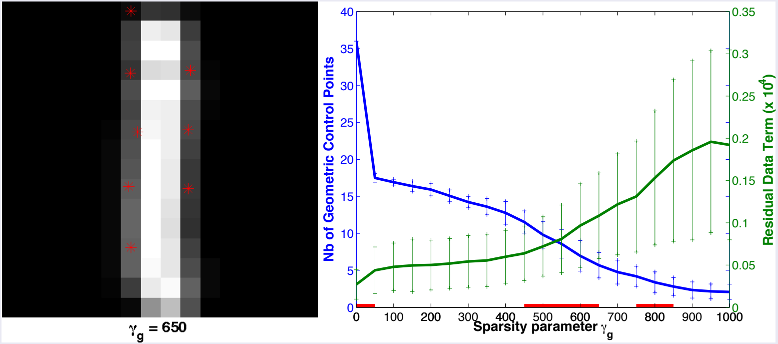


$\gamma = 400$; 18 active CPs

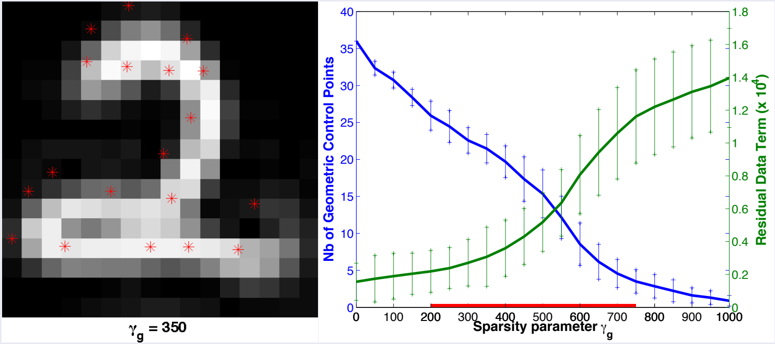
Atlas from the US postal database (20 training images)



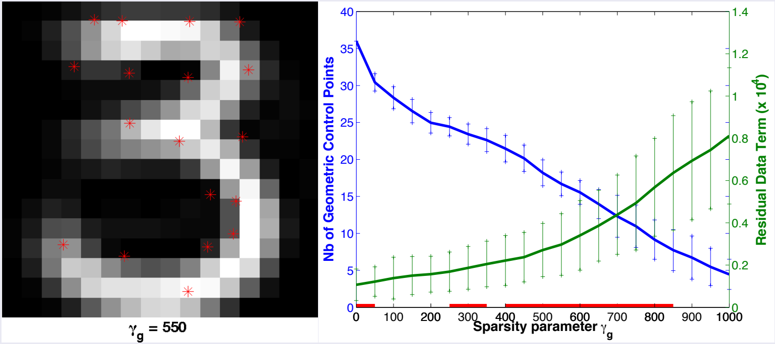
Atlas from the US postal database (20 training images)



Atlas from the US postal database (20 training images)



Atlas from the US postal database (20 training images)



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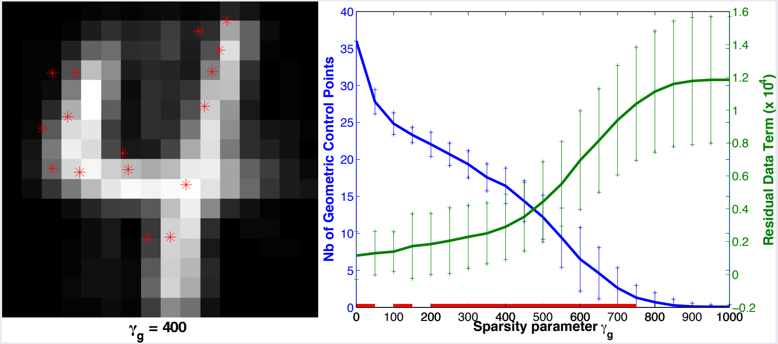

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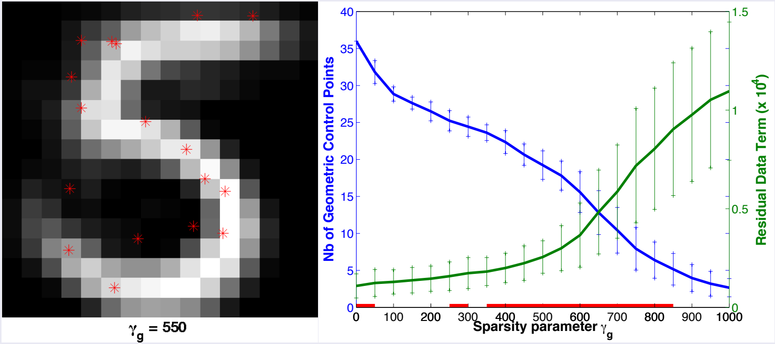

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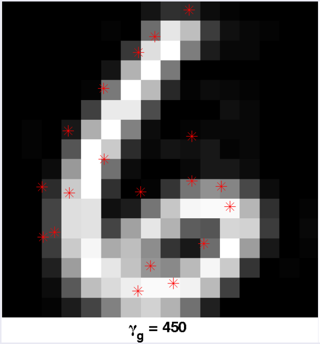

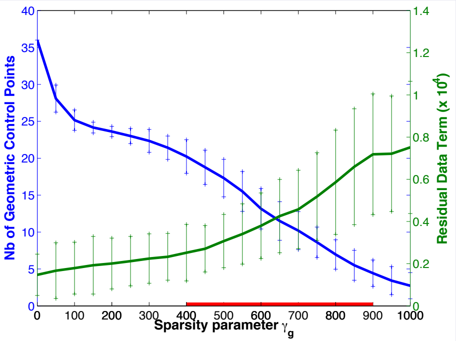

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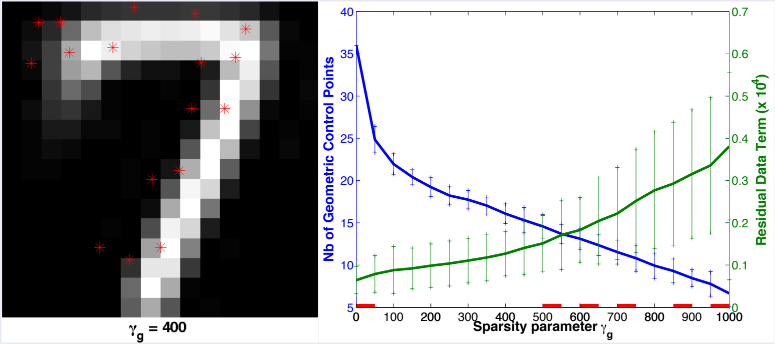

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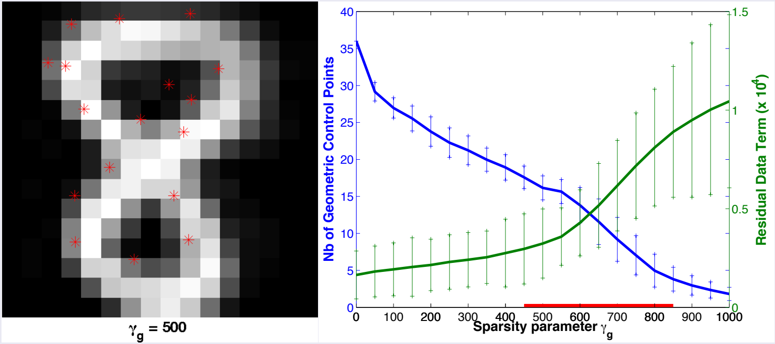

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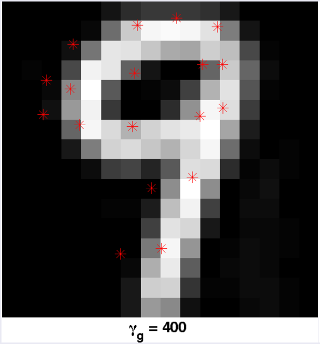

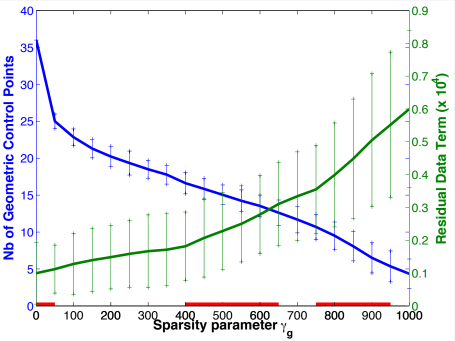

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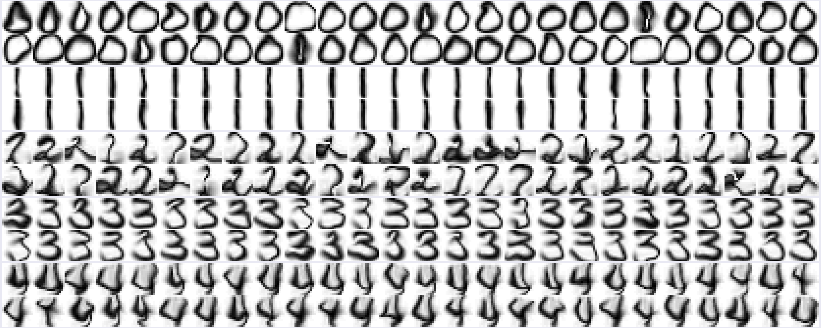

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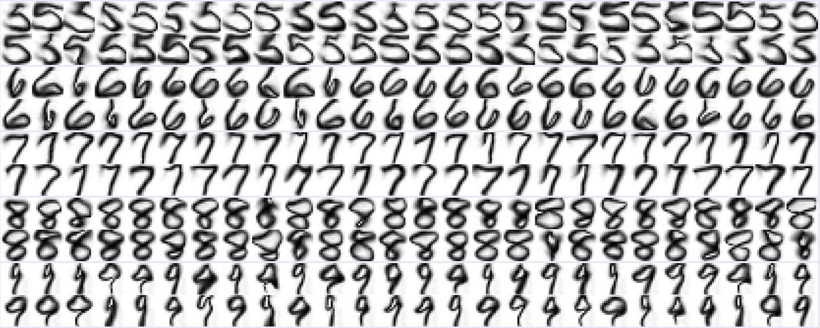
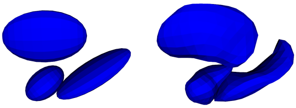
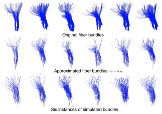
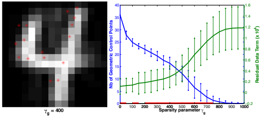


Conclusion

- A generic method for atlas construction:
  - images, points, surface meshes, fiber bundles, etc..
  - control over the model complexity (template + variability)
  - single gradient descent for:
    - template estimation
    - best parameterization of shape variability
    - template-to-shape correspondence
- Useful for:
  - Characterization of atypical anatomical configuration
  - Classification, clustering
  - Regression against clinical or genetic variables







Conclusion

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