The Burbea-Rao and Bhattacharyya centroids

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Means and centroids

In Euclidean geometry, centroid $c$ of a point set $\mathcal{P} = \{p_1, \ldots, p_n\}$:

Center of mass (also known as center of gravity):

$$c = \frac{1}{n} \sum_{i=1}^{n} p_i$$

Unique minimizer of average squared Euclidean distances

$$c = \arg \min_p \sum_{i=1}^{n} \frac{1}{n} \|p - p_i\|^2.$$

Two major ways to define means:

- by axiomatization, or
- by optimization (means defined by distances or penalty functions)
Means by axiomatization

Axioms for mean function $M(x_1, x_2)$:

- **Reflexivity.** $M(x, x) = x$,
- **Symmetry.** $M(x_1, x_2) = M(x_2, x_1)$,
- **Continuity and strict monotonicity.** $M(\cdot, \cdot)$ continuous and $M(x_1, x_2) < M(x'_1, x_2)$ for $x_1 < x'_1$, and
- **Anonymity.**

\[
M(M(x_{11}, x_{12}), M(x_{21}, x_{22})) = M(M(x_{11}, x_{21}), M(x_{12}, x_{22}))
\]

Yields unique function $f$ (up to an additive constant):

\[
M(x_1, x_2) = f^{-1}\left(\frac{f(x_1) + f(x_2)}{2}\right) \overset{\text{equal}}{=} M_f(x_1, x_2)
\]

$f$: continuous, strictly monotonous and increasing function.

(1930: Kolmogorov, Nagumo, + Aczél 1966)
Means by axiomatization: Quasi-arithmetic means

- arithmetic mean \( \frac{x_1 + x_2}{2} \leftarrow f(x) = x \)
- geometric mean \( \sqrt{x_1 x_2} \leftarrow f(x) = \log x \)
- harmonic mean \( \frac{2}{\frac{1}{x_1} + \frac{1}{x_2}} \leftarrow f(x) = \frac{1}{x} \)

Arithmetic barycenter on the \( f \)-representation \((y = f(x))\) :

\[
M_f(x_1, \ldots, x_n; w_1, \ldots, w_n) = f^{-1} \left( \sum_{i=1}^{n} w_i f(x_i) = \bar{x} \right)
\]

\[
f(\bar{x}) = \sum_{i=1}^{n} w_i f(x_i)
\]

\[
\bar{y} = \sum_{i=1}^{n} w_i y_i
\]
Dominance and interness of means

Dominance property:

\[ M_f(x_1, \ldots, x_n; w_1, \ldots, w_n) < M_g(x_1, \ldots, x_n; w_1, \ldots, w_n), \]

if and only if \( g \) dominates \( f \): \( \forall x, g(x) > f(x) \).

Interness property:

\[ \min(x_1, \ldots, x_n) \leq M_f(x_1, \ldots, x_n) \leq \max(x_1, \ldots, x_n), \]

limit cases \( p \to \pm \infty \) of power means for \( f(x) = x^p, p \in \mathbb{R}_* \).

\[ M_p(x_1, \ldots, x_n) = (\sum_{i=1}^{n} w_i x_i^p)^{\frac{1}{p}} \]

<table>
<thead>
<tr>
<th>name of power mean</th>
<th>value of ( p )</th>
<th>formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>maximum</td>
<td>( \to +\infty )</td>
<td>( \max_i x_i )</td>
</tr>
<tr>
<td>quadratic mean (root mean square)</td>
<td>2</td>
<td>( \sqrt{\sum_i w_i x_i^2} )</td>
</tr>
<tr>
<td>arithmetic mean</td>
<td>1</td>
<td>( \sum_i w_i x_i )</td>
</tr>
<tr>
<td>geometric mean</td>
<td>( \to 0 )</td>
<td>( \prod_i x_i^{w_i} )</td>
</tr>
<tr>
<td>harmonic mean</td>
<td>( \to -1 )</td>
<td>( \frac{1}{\sum_i \frac{w_i}{x_i}} )</td>
</tr>
<tr>
<td>minimum</td>
<td>( \to -\infty )</td>
<td>( \min_i x_i )</td>
</tr>
</tbody>
</table>

also called Hölder means.
Means by optimization

\[(\text{OPT}) : \min_x \sum_{i=1}^{n} w_i d(x, p_i) = \min_x L(x; \mathcal{P}, d),\]

Entropic means (Ben-Tal et al., 1989)

\[I_f(x, p) = p f \left( \frac{x}{p} \right),\]

\(f(\cdot)\): strictly convex differentiable function with \(f(1) = 0\) and \(f'(1) = 0\).

entropic means: linear scale-invariant (homogeneous degree 1):

\[M(\lambda p_1, ..., \lambda p_n; I_f) = \lambda M(p_1, ..., p_n; I_f)\]
Bregman means

\[ B_F(x, p) = F(x) - F(p) - (x - p)F'(p), \]

\( F(\cdot) \): strictly convex and differentiable function.

(OPT) is convex \(\rightarrow\) admits a unique minimizer:

\[ M(p_1, \ldots, p_n; B_F) = M_{F'}(p_1, \ldots, p_n) = F'^{-1} \left( \sum_{i=1}^{n} w_i F'(p_i) \right) \]

quasi-arithmetic mean for \( F' \), the derivative of \( F \).

Since \( d(x, p) \neq d(p, x) \), define a right-sided centroid \( M' \)

\[ (\text{OPT}') : \min_x \sum_{i=1}^{n} w_i d(p_i, x), \]
Visualizing Bregman divergences

\[ B_F(p, q) = F(p) - F(q) - \langle p - q, \nabla F(q) \rangle, \]

Kullback-Leibler \((F(x) = x \log x)\): \(KL(p, q) = \sum_{i=1}^{d} p^{(i)} \log \frac{p^{(i)}}{q^{(i)}}\)

Squared Euclidean \(L_2^2\) \((F(x) = x^2)\):
\[ L_2^2(p, q) = \sum_{i=1}^{d} (p^{(i)} - q^{(i)})^2 = \|p - q\|^2 \]
Information-theoretic sided means

Reference duality

- $f$-divergences

$$I_f(x, p) = I_{f^*}(p, x),$$

for $f^*(x) = x f(1/x)$.

*Any $f$-divergence can be symmetrized and stay in the class*

- Bregman divergences

$$B_F(x, p) = B_{F^*}(F'(p), F'(x))$$

for $F^*(\cdot)$ the Legendre convex conjugate ($F^{*'} = (F')^{-1}$)

*Only the squared Mahalanobis distances are symmetric Bregman divergences*
Separable divergence and means as projections

Separable divergence:

\[ d(x, p) = \sum_{i=1}^{d} d_i(x^{(i)}, p^{(i)}) , \]

with \( x^{(i)} \) denoting the \( i \)-th coordinate, and \( d_i \)'s univariate divergences. Typical non separable divergence: squared Mahalanobis distance (or other matrix trace divergences)

\[ d(x, p) = (x - p)^T Q (x - p) \]

View means of separable divergence as a projection

\[ \text{(PROJ)} : \inf_{u \in U} d(u, p) \]

with \( u_1 = \ldots = u_{d \times n} > 0 \), and \( p \) the \( (n \times d) \)-dimensional point obtained by stacking the \( d \) coordinates of each of the \( n \) points.
Burbea-Rao divergences

Based on Jensen’s inequality for a convex function $F$:

$$d(x, p) = \frac{F(x) + F(p)}{2} - F\left(\frac{x + p}{2}\right) \equiv BR_F(x, p) \geq 0.$$ 

strictly convex function $F(\cdot)$.

$$BR_F(p, q) = \sum_{i=1}^{d} BR_F(p^{(i)}, q^{(i)})$$

Includes the special case of Jensen-Shannon divergence:

$$JS(p, q) = H\left(\frac{p + q}{2}\right) - \frac{H(p) + H(q)}{2}$$

$F(x) = -H(x)$, the negative Shannon entropy $H(x) = -x \log x$.

→ generators are convex and entropies are concave (negative generators)
Visualizing Burbea-Rao divergences

\[ (p, F(p)) \]

\[ \left( \frac{p+q}{2}, \frac{F(p)+F(q)}{2} \right) \]

\[ \text{BR}_F(p, q) \]

\[ \left( \frac{p+q}{2}, F\left( \frac{p+q}{2} \right) \right) \]

\[ (q, F(q)) \]
Burbea-Rao divergences: Squared Mahalanobis

\[ \text{BR}_F(p, q) = \frac{F(p) + F(q)}{2} - F\left(\frac{p + q}{2}\right) \]

\[ = \frac{2\langle Qp, p \rangle + 2\langle Qq, q \rangle - \langle Q(p + q), p + q \rangle}{4} \]

\[ = \frac{1}{4}(\langle Qp, p \rangle + \langle Qq, q \rangle - 2\langle Qp, q \rangle) \]

\[ = \frac{1}{4}\langle Q(p - q), p - q \rangle = \frac{1}{4}\|p - q\|_Q^2. \]

(Not a metric. square root of Jensen-Shannon is a metric but not the square roots of all Burbea-Rao divergences.)
Symmetrizing Bregman divergences

Jeffreys-Bregman divergences.

\[ S_F(p; q) = \frac{B_F(p, q) + B_F(q, p)}{2} \]
\[ = \frac{1}{2} \langle p - q, \nabla F(p) - \nabla F(q) \rangle, \]

Jensen-Bregman divergences (diversity index).

\[ J_F(p; q) = \frac{B_F\left(p, \frac{p+q}{2}\right) + B_F\left(q, \frac{p+q}{2}\right)}{2} \]
\[ = \frac{F(p) + F(q)}{2} - F\left(\frac{p + q}{2}\right) = BR_F(p, q) \]
Skew Burbea-Rao divergences

\[
\begin{align*}
\text{BR}_F^{(\alpha)} : \mathcal{X} \times \mathcal{X} & \to \mathbb{R}^+ \\
\text{BR}_F^{(\alpha)}(p, q) &= \alpha F(p) + (1 - \alpha) F(q) - F(\alpha p + (1 - \alpha) q)
\end{align*}
\]

Skew symmetrization of Bregman divergences:

\[
\begin{align*}
\text{BR}_F^{(\alpha)}(p, q) &= \alpha F(p) + (1 - \alpha) F(q) - F(\alpha p + (1 - \alpha) q) \\
&= \text{BR}_F^{(1-\alpha)}(q, p)
\end{align*}
\]

\[
\alpha B_F(p, \alpha p + (1 - \alpha) q) + (1 - \alpha) B_F(q, \alpha p + (1 - \alpha) q) \overset{\text{equal}}{=} \text{BR}_F^{(\alpha)}(p, q)
\]

= skew Jensen-Bregman divergences.
Bregman as asymptotic skewed Burbea-Rao

\[
B_F(p, q) = \lim_{\alpha \to 1} \frac{1}{1 - \alpha} BR_F^{(\alpha)}(p, q)
\]

\[
B_F(q, p) = \lim_{\alpha \to 0} \frac{1}{\alpha} BR_F^{(\alpha)}(p, q)
\]

**Proof:** \(F(\alpha p + (1 - \alpha)q) = F(p + (1 - \alpha)(q - p)) \simeq_{\alpha \to 1} F(p) + (1 - \alpha)(q - p) \nabla F(p)\) (Taylor)

\[
F(\alpha p + (1 - \alpha)q) - \alpha F(p) - (1 - \alpha) F(q) \simeq_{\alpha \to 1} (1 - \alpha) F(p) + (1 - \alpha)(q - p) \nabla F(p) - (1 - \alpha) F(q)
\]

\[
\simeq_{\alpha \to 1} (1 - \alpha) (F(p) - F(q) - (p - q) \nabla F(p))
\]

\[
\lim_{\alpha \to 1} BR_F^{(\alpha)}(p, q) = (1 - \alpha) B_F(p, q)
\]

For \(0 < \alpha < 1\), swap arguments by setting \(\alpha \to 1 - \alpha\):

\[
BR_F^{(\alpha)}(p, q) = BR_F^{(1 - \alpha)}(q, p)
\]
Burbea-Rao centroids

$$\text{OPT} : c = \arg \min_x \sum_{i=1}^n w_i \text{BR}_F^{(\alpha_i)}(x, p_i) = \arg \min_x L(x)$$

Wlog., equivalent to minimize

$$E(c) = \left( \sum_{i=1}^n w_i \alpha_i \right) F(c) - \sum_{i=1}^n w_i F(\alpha_i c + (1 - \alpha_i) p_i)$$

Sum $E = F + G$ of convex $F$ + concave $G$ function $\Rightarrow$ Convex-ConCave Procedure (CCCP, NIPS*01)

Start from arbitrary $c_0$, and iteratively update as:

$$\nabla F(c_{t+1}) = -\nabla G(c_t)$$

$\Rightarrow$ guaranteed convergence to a local minimum.
ConCave Convex Procedure (CCCP)

\[
\min_x E(x) = F(x) + G(x) \\
\nabla F(c_{t+1}) = -\nabla G(c_t)
\]
Iterative algorithm for Burbea-Rao centroids

Apply CCCP scheme

\[
\nabla F(c_{t+1}) = \frac{1}{\sum_{i=1}^{n} w_i \alpha_i} \sum_{i=1}^{n} w_i \alpha_i \nabla F(\alpha_i c_t + (1 - \alpha_i)p_i)
\]

\[
c_{t+1} = \nabla F^{-1} \left( \frac{1}{\sum_{i=1}^{n} w_i \alpha_i} \sum_{i=1}^{n} w_i \alpha_i \nabla F(\alpha_i c_t + (1 - \alpha_i)p_i) \right)
\]

Get arbitrarily fine approximations of the (skew) Burbea-Rao centroids and barycenters.
Special cases: Closed-form Burbea-Rao centroids

Consider $F(x) = \langle x, x \rangle$.

$$
\min E(x) = \frac{F(x)}{2} - \sum_{i=1}^{n} w_i F\left(\frac{p_i + x}{2}\right),
$$

$$
= \min \frac{\langle x, x \rangle}{2} - \frac{1}{4} \sum_{i=1}^{n} w_i \left( \langle x, x \rangle + 2\langle x, p_i \rangle + \langle p_i, p_i \rangle \right)
$$

The minimum obtained when $\nabla E(x) = 0$

$$
x = \bar{p} = \sum_{i=1}^{n} w_i p_i
$$

Extremal skew cases (for $\alpha \to 0$ or $\alpha \to 1$):

Bregman sided centroids in closed-forms: $\bar{x} = \sum_{i=1}^{n} w_i p_i$ (right-sided) or

$$
\bar{x} = (\nabla F)^{-1} \left( \sum_{i=1}^{n} w_i \nabla F(p_i) \right) \text{ (left-sided)}
$$

But usually only approximation using CCCP iterations.
Bhattacharyya coefficients/distances

Bhattacharyya coefficient and non-metric distance:

\[ C(p, q) = \int \sqrt{p(x)q(x)} \, dx, \quad 0 < C(p, q) \leq 1, \quad B(p, q) = -\ln C(p, q). \]

(coefficient is always strictly positive)

Hellinger metric

\[ H(p, q) = \sqrt{\frac{1}{2} \int (\sqrt{p(x)} - \sqrt{q(x)})^2 \, dx}, \]

such that \( 0 \leq H(p, q) \leq 1. \)

\[
H(p, q) = \sqrt{\frac{1}{2} \left( \int p(x) \, dx + \int q(x) \, dx - 2 \int \sqrt{p(x)q(x)} \, dx \right)} \\
= \sqrt{1 - C(p, q)}.
\]
Chernoff coefficients/\(\alpha\)-divergences

Skew Bhattacharyya divergences based on Chernoff \(\alpha\)-coefficients.

\[
B_\alpha(p, q) = -\ln \int_x p^\alpha(x)q^{1-\alpha}(x)dx = -\ln C_\alpha(p, q)
\]

\[
= -\ln \int_x q(x) \left( \frac{p(x)}{q(x)} \right)^\alpha dx
\]

\[
= -\ln E_q[L^\alpha(x)]
\]

Amari \(\alpha\)-divergence:

\[
D_\alpha(p||q) = \begin{cases} 
\frac{4}{1-\alpha^2} \left( 1 - \int p(x) \frac{1-\alpha}{2} q(x) \frac{1+\alpha}{2} dx \right), & \alpha \neq \pm 1, \\
\int p(x) \log \frac{p(x)}{q(x)} dx = KL(p, q), & \alpha = -1, \\
\int q(x) \log \frac{q(x)}{p(x)} dx = KL(q, p), & \alpha = 1,
\end{cases}
\]

\[
D_\alpha(p||q) = D_{-\alpha}(q||p)
\]

Remapping \(\alpha' = \frac{1-\alpha}{2} (\alpha = 1 - 2\alpha')\) to get Chernoff \(\alpha'\)-divergences
Exponential families in statistics

- Probability measure
  - Parametric
    - Exponential families
      - Univariate
        - Uniparameter
          - Binomial
          - Bernoulli
          - Poisson
          - Exponential
          - Rayleigh
          - Gamma
          - Beta
        - Bi-parameter
          - Gamma
          - Beta
          - Poisson
          - Exponential
          - Rayleigh
        - Multi-parameter
          - Gamma
          - Beta
          - Poisson
          - Exponential
          - Rayleigh
      - Multivariate
        - Gamma
        - Beta
        - Poisson
        - Exponential
        - Rayleigh
    - Non-exponential families
      - Uniform
      - Cauchy
      - Lévy skew α-stable
  - Non-parametric
Exponential families in statistics

Gaussian, Poisson, Bernoulli/multinomial, Gamma/Beta, etc.:

\[ p(x; \lambda) = p_F(x; \theta) = \exp \left( \langle t(x), \theta \rangle - F(\theta) + k(x) \right). \]

Example: Poisson distribution

\[ p(x; \lambda) = \frac{\lambda^x}{x!} \exp(-\lambda), \]

- the sufficient statistic \( t(x) = x \),
- \( \theta = \log \lambda \), the natural parameter,
- \( F(\theta) = \exp \theta \), the log-normalizer,
- and \( k(x) = -\log x! \) the carrier measure (with respect to the counting measure).
Gaussians as an exponential family

\[ p(x; \lambda) = p(x; \mu, \Sigma) = \frac{1}{2\pi \sqrt{\det \Sigma}} \exp \left( -\frac{(x - \mu)^T \Sigma^{-1} (x - \mu)}{2} \right) \]

- \( \theta = (\Sigma^{-1} \mu, \frac{1}{2} \Sigma^{-1}) \in \Theta = \mathbb{R}^d \times \mathbb{K}_{d \times d} \), with \( \mathbb{K}_{d \times d} \) cone of positive definite matrices,
- \( F(\theta) = \frac{1}{4} \text{tr}(\theta_2^{-1} \theta_1 \theta_1^T) - \frac{1}{2} \log \det \theta_2 + \frac{d}{2} \log \pi, \)
- \( t(x) = (x, -x^T x), \)
- \( k(x) = 0. \)

Inner product: composite, sum of a dot product and a matrix trace:

\[ \langle \theta, \theta' \rangle = \theta_1^T \theta_1' + \text{tr}(\theta_2^T \theta_2'). \]

The coordinate transformation \( \tau : \Lambda \rightarrow \Theta \) is given for \( \lambda = (\mu, \Sigma) \) by

\[ \tau(\lambda) = \left( \lambda_2^{-1} \lambda_1, \frac{1}{2} \lambda_2^{-1} \right), \quad \tau^{-1}(\theta) = \left( \frac{1}{2} \theta_2^{-1} \theta_1, \frac{1}{2} \theta_2^{-1} \right) \]
Bhattacharyya/Chernoff of exponential families

Equivalence with skew Burbea-Rao distances:

\[ B_\alpha(p_F(x; \theta_p), p_F(x; \theta_q)) = BR^{(\alpha)}_F(\theta_p, \theta_q) = \alpha F(\theta_p) + (1-\alpha) F(\theta_q) - F(\alpha \theta_p + (1-\alpha) \theta_q) \]

Proof: Chernoff coefficients \( C_\alpha(p, q) \) of members \( p = p_F(x; \theta_p) \) and \( q = p_F(x; \theta_q) \) of the same exponential family \( \mathcal{E}_F \):

\[
C_\alpha(p, q) = \int p^\alpha(x)q^{1-\alpha}(x)dx = \int p^{\alpha}_F(x; \theta_p)p^{1-\alpha}_F(x; \theta_q)dx
\]

\[
= \int \exp(\alpha(\langle x, \theta_p \rangle - F(\theta_p))) \times \exp((1-\alpha)(\langle x, \theta_q \rangle - F(\theta_q)))dx
\]

\[
= \int \exp(\langle x, \alpha \theta_p + (1-\alpha) \theta_q \rangle - (\alpha F(\theta_p) + (1-\alpha) F(\theta_q)))dx
\]

\[
= \exp(-(\alpha F(\theta_p) + (1-\alpha) F(\theta_q))) \times \int \exp(\langle x, \alpha \theta_p + (1-\alpha) \theta_q \rangle - F(\alpha \theta_p + (1-\alpha) \theta_q))dx
\]

\[
= \exp(F(\alpha \theta_p + (1-\alpha) \theta_q) - (\alpha F(\theta_p) + (1-\alpha) F(\theta_q)) \times \int \exp(\langle x, \alpha \theta_p + (1-\alpha) \theta_q \rangle - F(\alpha \theta_p + (1-\alpha) \theta_q))dx
\]

\[
= \exp(-BR^{(\alpha)}_F(\theta_p, \theta_q)) > 0. \text{ Coefficient is always strictly positive. For } \theta_p = \theta_q, \quad C_\alpha(\theta_p, \theta_q) = \exp -0 = 1 \text{ and } B_\alpha(\theta_p, \theta_q) = 0.
\]
\(\alpha\)-div./Kullback-Leibler \leftrightarrow \text{Burbea-Rao/Bregman}

Skew Bhattacharyya distances on members of the same exponential family is equivalent to skew Burbea-Rao divergences on the natural parameters (without swapping order).

\[ B_\alpha(p_F(x; \theta_p), p_F(x; \theta_q)) = BR_F^{(\alpha)}(\theta_p, \theta_q) \]

For \(\alpha = \pm 1\), Kullback-Leibler of exp. fam. = \textit{Bregman divergence} (limit as \(\alpha \to 1\) or \(\alpha \to 0\)).

\[
\begin{align*}
\text{KL}(p, q) & = \text{KL}(p_F(x; \theta_p), p_F(x; \theta_q)) \\
& = \lim_{\alpha' \to 1} D_{\alpha'}(p_F(x; \theta_p), p_F(x; \theta_q)) \\
& = \lim_{\alpha' \to 1} \frac{1}{\alpha'(1 - \alpha')} (1 - C_{\alpha'}(p_F(x; \theta_p), p_F(x; \theta_q))) \\
& \quad \text{since } \exp x \approx x \approx 0 1 + x \\
& = \lim_{\alpha' \to 1} \frac{1}{\alpha'(1 - \alpha')} \underbrace{BR_{\alpha'}^{(\alpha)}(\theta_p, \theta_q)}_{(1 - \alpha')B_F(\theta_q, \theta_p)} \\
& = \lim_{\alpha' \to 1} \frac{1}{\alpha'} B_F(\theta_q, \theta_p) = B_F(\theta_q, \theta_p)
\end{align*}
\]
## Closed-form Bhattacharyya distances for exp. fam.

<table>
<thead>
<tr>
<th>Exp. fam.</th>
<th>$F(\theta)$ (up to a constant)</th>
<th>Bhattacharyya/Burbea-Rao $BR_F(\lambda_p, \lambda_q) = BR_F(\tau(\lambda_p))$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Multinomial</td>
<td>$\log(1 + \sum_{i=1}^{d-1} \exp \theta_i)$</td>
<td>$- \ln \sum_{i=1}^{d} \sqrt{p_i q_i}$</td>
</tr>
<tr>
<td>Poisson</td>
<td>$\exp \theta$</td>
<td>$\frac{1}{2} (\sqrt{\mu_p} - \sqrt{\mu_q})^2$</td>
</tr>
<tr>
<td>Gaussian</td>
<td>$- \frac{\theta_1^2}{4\theta_2} + \frac{1}{2} \log(-\frac{\pi}{\theta_2})$</td>
<td>$\frac{1}{4} \frac{(\mu_p - \mu_q)^2}{\sigma_p^2 + \sigma_q^2} + \frac{1}{2} \ln \frac{\sigma_p^2 + \sigma_q^2}{2\sigma_p \sigma_q}$</td>
</tr>
<tr>
<td>Gaussian</td>
<td>$\frac{1}{4} \text{tr}(\Theta^{-1} \theta \theta^T) - \frac{1}{2} \log \det \Theta$</td>
<td>$\frac{1}{8} (\mu_p - \mu_q)^T \left( \frac{\Sigma_p + \Sigma_q}{2} \right)^{-1} (\mu_p - \mu_q) + \frac{1}{2} \ln \frac{\det \frac{\Sigma_p + \Sigma_q}{2}}{\det \Sigma_p \det \Sigma_q}$</td>
</tr>
</tbody>
</table>

Bhattacharyya, Burbea-Rao, Tsallis, Rényi, $\alpha-$, $\beta$-divergences are in closed forms for members of the same exponential family.
**Application: statistical mixtures**

cdefintion Gaussian mixture models (GMMs, MoGs: mixture of Gaussians):
Probabilistic modeling of data:

\[
\Pr(X = x) = \sum_{i=1}^{k} w_i \Pr(X = x | \mu_i, \Sigma_i) \quad \text{(with } \sum_i w_i = 1 \text{ and all } w_i \geq 0). \]

\[
\Pr(X = x | \mu, \Sigma) = \frac{1}{(2\pi)^{d/2} \sqrt{\det \Sigma}} \exp \left(-\frac{1}{2} (x - \mu)^T \Sigma^{-1} (x - \mu)\right).
\]

Similar to \(k\)-means, soft clustering wrt. to log-likelihood is minimized by the expectation-maximization (EM) algorithm [Dempster’77]
Application: Statistical images and Gaussians

Consider 5D Gaussian Mixture Models (GMMs) of color images (image=RGBxy point set)

Get open source Java(TM) jMEF library:
www.informationgeometry.org/MEF/
Hierarchical clustering of GMMs wrt. Bhattacharyya distance. Simplify the number of components of an initial GMM.

(a) source

(b) $k = 48$

(c) $k = 16$
Summary of results

- Skew Burbea-Rao divergences occur when
  - Symmetrizing skew Bregman divergences: Jensen-Bregman divergences
  - Bhattacharyya/Chernoff coefficients/distances of exponential families

- Apply ConCave-Convex procedure (CCCP) for computing Burbea-Rao centroids

- Skewed Burbea-Rao yields *in the limit* Bregman divergences

- Application: Hierarchical clustering of Gaussian mixtures

  (In arXiv:1004.5049, alternative tailored matrix method generalizing ICASSP 2000 but not so efficient as the general scheme)

www.informationgeometry.org/BurbeaRao/
References

"Bhattacharyya clustering with applications to mixture simplifications," ICPR 2010.


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