Spatial Programming for Musical Transformations and Harmonization

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***Abstract*—This paper presents a spatial approach to build spaces of musical chords as simplicial complexes. The approach developed here enables the representation of a musical piece as an object evolving over time in this underlying space, leading to a *trajectory*. Well known spatial transformations can be applied to this trajectory as well as to the underlying space. These spatial transformations induce a corresponding musical transformation on the piece. Spaces and transformations are computed thanks to MGS, an experimental programming language dedicated to spatial computing.**

***Index Terms*—MGS; musical transformation ; harmonization; counterpoint; self-assembly; *Tonnetz*.**

I. Introduction

Musical objects and processes are frequently formalized with algebraic methods [1]. Such formalizations can sometimes be usefully represented by spatial structures. A well-known example is the *Tonnetz* (figure 1), a spatial organization of pitches illustrating the algebraic nature of triads (*i.e.*, *minor* and *major* 3-note chords) [2]. In [3] we have introduced an original method that extends and generalizes the approach of [4] for the building of pitch spaces using simplicial complexes [5]. This combinatorial structure is used to make explicit algebraic relations between notes and chords, as in *Tonnetze*, or more general relationships like co-occurrences.

In such spaces, a musical sequence is represented by a sub-complex evolving over time: a *trajectory*. It is then compelling to look at the musical effect of well known spatial transformations on a trajectory. In section IV we investigate geometrical transformations, as discrete translations and discrete central symmetries, leading to the well known operations of musical transpositions and inversions. Such discrete geometrical transformations can be generalized, leading to a new family of transformations with less known musical interpretation. Some audio examples are available online[1]. In section V, the problem of counterpoint is investigated from a spatial perspective. We propose for the first time to generate the additional voice such that the distance with other played notes in a particular underlying space is minimized. The underlying space is a parameter of the algorithm and by changing spaces, alternate (families of) solutions are generated.

II. A Short Introduction to MGS

MGS is an experimental domain specific language dedicated to spatial computing [6], [7]. MGS concepts are based on well

[1]see the web page: http://www.lacl.fr/~lbigo/scw13

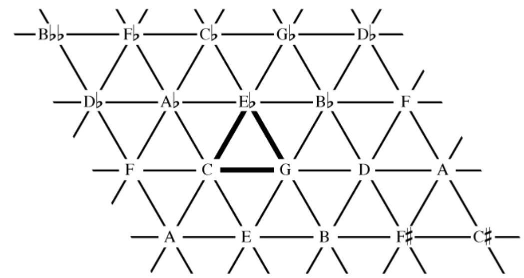

Figure 1. The original *Tonnetz*. Pitches are organized following the interval of fifth (horizontal axis), and the intervals of minor and major thirds (diagonal axis). Triangles represent minor and major triads.

established notions in algebraic topology [5] and uses of rules to compute declaratively spatial data structures. In MGS, all data structures are unified under the notion of *topological collection*. Simplicial complexes defined below are an example of topological collections. *Transformations* of topological collections are defined by rewriting rules [8] specifying replacement of sub-collections that can be recursively performed to build new spaces.

A *simplicial complex* is a space built by gluing together more elementary spaces called simplices. A p-*simplex* is the abstraction of an elementray space of dimension p and has exactly $p + 1$ faces in its border. A 0-simplex corresponds to a point, a 1-simplex corresponds to an edge, a 2-simplex is a triangle, *etc*. The geometric realization of a p-simplex is the convex hull of its $p + 1$ vertices as shown in Figure 3 for p-simplices with $p \in \{0, 1, 2, 3\}$.

For any natural integer n, the n-*skeleton* of a simplicial complex is defined by the set of faces of dimension n or less.

A simplicial complex can be built from a set of simplices by self-assembly, applying an accretive growing process [9]. The growth process is based on the identification of the simplices in the boundaries. This topological operation is not elementary and holds in all dimensions. Figure 2 illustrates the process. First, nodes E and G are merged. Then, the resulting edges $\{E, G\}$ are merged.

A simple way to compute the identification is to iteratively apply, until a fixed point is reached, the merge of topological cells that exactly have the same faces. The corresponding topological surgery can be expressed as a simple MGS transformation as follows:

```
transformation identification = {
  s1 s2 / (s1==s2 & faces(s1)==faces(s2))
  =>
  let c = new_cell (dim s1)
                   (faces s1)
                   (union (cofaces s1)
                          (cofaces s2))
  in  s1*c
}
```

The expression `new_cell` p f cf returns a new p-cell with faces f and cofaces cf. The rule specifies that two elements `s1` and `s2`, having the same label and the same faces in their boundaries, merge into a new element `c` (whose cofaces are the union of the cofaces of `s1` and `s2`) labeled by `s1` (which is also the label of `s2`).

In Fig. 2, the transformation `identification` is called twice. At the first application (from the left complex to the middle), vertices are identified. The two topological operations are made in parallel. At the second application (from the complex in the middle to the right), the two edges from E to G that share the same boundary, are merged. The cofaces of the resulting edge are the 2-simplices I_C and III_C corresponding to the union of the cofaces of the merged edges. Finally (on the right), no more merge operation can take place and the fixed point is reached.

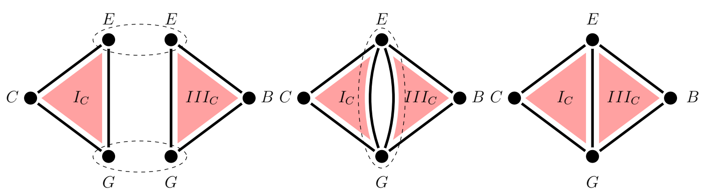

Figure 2. Identification of boundaries.

III. Chord Spaces

A. Building Chord Spaces

A chord space is an organization in space of a collection of musical chords. Such organizations are typically represented by graphs [10], or more recently by orbifolds [11]. Chords are generally represented in these spaces by vertices. A sequence of chords, which are included in the space, can thus be represented by a *trajectory*. A trajectory generalizes the notion of path to higher dimensional simplices and a trajectory is not necessarily connected.

We use a method presented in [12] to represent chords by simplices. A n-note chord, viewed as a set of n notes, is represented by a $(n-1)$-simplex. To simplify the presentation, we consider pitch classes instead of notes. This abstraction is customary in musical analysis and gathers all notes equivalent up to an octave under the same class. For example, the notes $C1$, $C2$, $C3$ …, all played by distinct keys on the piano, are grouped under the pitch class C.

In the simplicial representation of chord, a 0-simplex represents a single pitch class. More generally, a $n-1$-simplex represents a n-note chord, as illustrated on figure 3.

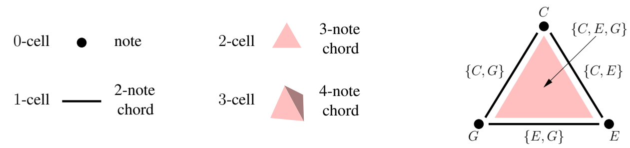

Figure 3. A chord represented as a simplex. The complex on the right corresponds to the 3-note chord C, E, G and all 2-note chords and notes included in it.

We build a *chord space simplicial complex* by representing each chord of a chord collection by a simplex, then by applying the self-assembly process described in section II. The application of this process to a collection of n-note chords gives rise to a $n-1$-dimensional simplicial complex. For example, the self-assembly process applied to the 24 major and minor triads (3-note chords) builds a toroidal simplicial complex. This complex extends the notion of *Tonnetz* developped in musical theory and illustrated on figure 1.

B. Chord Class Spaces

In this subsection, we present two particular types of chord spaces that will be used in next sections.

a) Chromatic Chord Class Spaces: Musical chords can be classified according different methods. One of the most popular classification spreads chords in 351 pitch class sets, called the *Forte Classes* [13]. Two pitch class sets belong to the same class if they are equivalent up to a transposition. We merge further chords equivalent up to transposition *and* inversion. The resulting classes can be obtained by listing orbits of the action of the dihedral group D_{12} on subsets of the cyclic group $\mathbb{Z}_{12}$ [14]. There are 224 such classes, we call chromatic chord classes. Chords belonging to a chromatic chord class share the same interval structure X: a sequence of intervals defined up to circular permutation and retrograde inversion. We note $\mathcal{C}(X)$ the simplicial complex representing the chromatic chord class associated with the interval structure X. In the chromatic system, the elements of X are elements of $\mathbb{Z}_{12}$.

b) Tonal Chord Class Spaces: A tonal chord class space is obtained by assembling chords sharing the same diatonic interval structure and including pitches of a particular tonality. If the scale of the tonality is heptatonic, (*i.e.*, the tonality includes seven pitch classes), the 16 spaces associated with the tonality can be obtained by enumerating the orbits of the action of the dihedral group D_7 on subsets of $\mathbb{Z}_7$. Such a space is noted $\mathcal{C}(X)$ where the elements of X belongs to $\mathbb{Z}_7$. For more details on these spaces, see [3].

C. Unfolding Chord Class Spaces

Chord class spaces of a same dimension can have different topologies. For example, $\mathcal{C}(3,4,5)$ and $\mathcal{C}(2,5,5)$ are both two-dimensional simplicial complexes but the first one has the shape of a torus and the second one has the shape of a strip [15]. However, chord class spaces of a given dimension can be *unfolded* in topologically equivalent infinite spaces. The unfolded representation is built as follows: an arbitrary

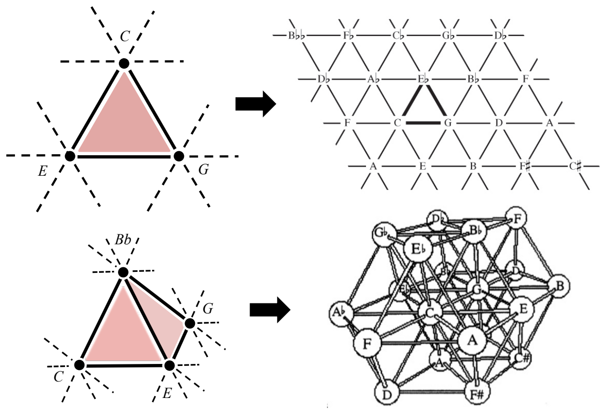

Figure 4. On the top, the unfolding process is applied to $\mathcal{C}(3,4,5)$ by extending C Major 1-simplices to infinite lines on the plane. At the bottom, unfolding process is applied to $\mathcal{C}(2,4,3,3)$ in the 3D space.

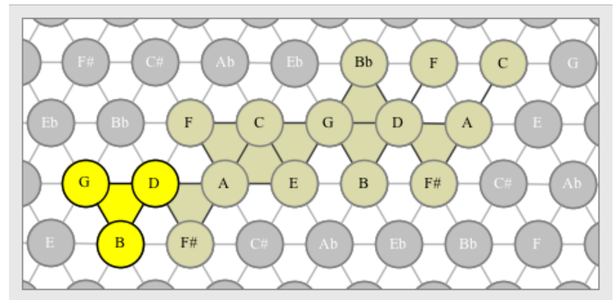

Figure 5. Path representing the first measures of J-S. Bach choral BWV 255. The chord class space used for the representation is obtained by unfolding $\mathcal{C}(3,4,5)$ which is the assembly of the 24 minor and major triads.

chord of the class is represented by the geometric realization of its simplex. Then, 1-simplices (*i.e.*, edges) are extended as infinite lines. The interval labelling an edge is assigned to the corresponding line and all its parallels. Pitch classes and chords are organized and repeated infinitely following the lines according their assigned intervals.

The major difference between a simplicial complex and its unfolded representation is that in the former, notes are represented once, and in the latter, by an infinite number of occurrences. Moreover, the associated 1-skeleton can be embedded in the euclidean space such that parallel 1-simplices (representing 2-note chords) relate to the same interval class. By considering 1-skeletons of the unfolded complexes representing major and minor triads (Figure 4), one gets the neo-Riemannian Tonnetz [2]. The 1-skeleton of the unfolded complex representing seventh and half-diminished seventh chords is equivalent to Gollin 3D Tonnetz [16]. These two complexes are chromatic chord class spaces.

Chord class spaces resulting from the assembly of n-note chords are unfolded as $(n-1)$-dimensional infinite spaces. From 2-note chords one gets an infinite line, from 3-note chords an infinite triangular tessellation. Note that n-simplices don't systematically tessellate the n-dimensional Euclidean space. For example, 2-simplices (triangles) tessellate the 2D plan but 3-simplices (tetrahedra) do not tessellate the 3D space. For this reason, the 3D unfolded representation of complexes as the one at the bottom right of the figure 4 contains some holes.

IV. Spatial Transformations and their Musical Interpretation

We focus on unfolded representations of chord class spaces resulting from the self assembly of 3-note chords. These unfoldings are infinite triangular tessellations which have the property to preserve local neighborhoods between elements. If two elements are neighbor in a folded space then they are neighbor its unfolded representation, and vice versa. Note that this property does not systematically hold at higher dimensions. The advantage to consider unfolded representations of 3-note chords is that it preserves the neighborhood of each simplex while enabling the specification of discrete counterparts of euclidean transformation. This is not easily achieved in the initial finite space.

A. *Representation of a Musical Sequence in a Chord Space*

As previously said, a note corresponds to an infinite number of possible locations in an unfolded chord space. To represent a musical sequence as a moving object in such a space, only one of those locations has to be chosen for each played note over time. The precise location of a note played at some date is chosen in order to minimize the distance with both previously and simultaneously played notes. These two criteria enable the representation of the sequence by a trajectory "as connected as possible". Figure 5 illustrates such a path in $\mathcal{C}(3,4,5)$.

B. *Spatial Transformations*

Now we have a spatial representation of a musical sequence, we can apply some spatial transformations to it and listen to the musical result. We consider two kinds of transformations:

- the first one applies a geometrical transformation *on the trajectory*, (*i.e.*, on the spatial object representing the sequence in a predefined space) as illustrated on figure 7;
- the second one applies transformations on *the underlying space* of the piece, that is, the triangular tessellation (figure 8). This is possible because all such transformations amount to change the labels of the underlying space.

Musical examples of different pieces, before and after transformations, are available in MIDI format at http://www.lacl.fr/~lbigo/scw13 .

1) Geometrical Transformations: The regularity of the triangular tessellation enables to specify a discrete counterpart of usual geometrical operations like translations or some rotations.

a) Translations: As previously mentioned, a direction in an unfolded space is associated with a constant interval. Then, a n-step translation of a path in a direction associated with the interval i reaches to a transposition of $n \times i$ on each

Figure 6. On the top, the first measures of the melody of the song *Hey Jude*. On the bottom, the same measures after three rotations in the complex $\mathcal{C}(1,2,4)$ related to F major tonality

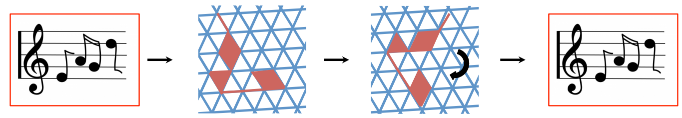

Figure 7. Rotation of a path in a triangular tessellation.

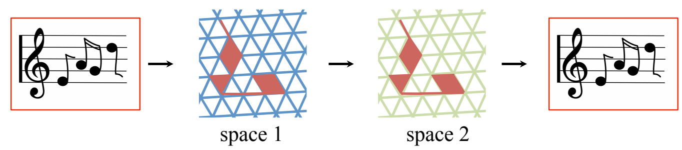

Figure 8. Transformation of the support space.

note of the sequence. This translation is thus interpreted as a transposition (if the chord space is chromatic) or as a modal transposition (if the chord space is tonal). Audio example 1 is the result of a one-step translation of the path representing Beethoven's piece *Für Elise* in $\mathcal{C}(3,4,5)$. The direction of the translation is associated with the interval of fourth (the left direction on figure 5). The result is the transposition of the whole piece a fourth higher. Example 2 is the beginning of Mozart's 16$^{\text{th}}$ sonata after a translation in $\mathcal{C}(1,2,4)$ related to C major tonality. Example 3 illustrates the same transformation on the song *Hey Jude* written by Paul McCartney, in $\mathcal{C}(1,2,4)$ related to F major tonality. This transformation corresponds to a modal transposition. The result is that the two original pieces switched from major mode to minor mode.

b) Rotations: Figure 7 illustrates a discrete $\pi/3$ rotation. Around a given vertex, five different rotations are possible in a triangular tessellation. This property is easily understandable by seeing that a note has six neighbors into six different directions. Thus, the motion to a note to one of his neighbors can be rotated five times around the starting note. Six rotations reach to an entire rotation around the center and is equivalent to identity. Three rotations are equivalent to a central symmetry.

This last operation is particularly interesting since it produces a trajectory having exactly the opposite direction from the original one. Each interval i being mapped to his opposite interval $-i$, this rotation is musically interpreted as an inversion (if the chord space is chromatic) or as an operation we could call a *modal inversion* (if the space is tonal).

Other rotations act as interval mappings depending on the properties of the chord space. Audio example 4 is the beginning of Mozart's 16$^{\text{th}}$ sonata after 3 rotations (*i.e.* central symmetry) in $\mathcal{C}(3,4,5)$. Examples 5 and 6 are the same sequence after respectively 2 and 3 rotations in $\mathcal{C}(1,2,4)$ related to C major tonality.

Examples 7, 8 and 9 result from the same operations on the song *Hey Jude*. Figure 6 compares the first measures of the melody of the song before and after the central symmetry in $\mathcal{C}(1,2,4)$ related to F major tonality.

2) Change of Space: This operation consists in changing the labels of the underlying space, which is a triangular tessellation, for the labels of another unfolded two-dimensional chord class space. Thanks to topological equivalence of the two unfolded representations, the label mapping between the two spaces is straightforward. In this operation, the trajectory representing the musical sequence stays "unchanged". Example 10 is the beginning of J.-S.Bach's choral BWV 256 after the initial support space $\mathcal{C}(3,4,5)$ is transformed into $\mathcal{C}(2,3,7)$. The transformation achieves a surprising use of the pentatonic scale, giving a particular color to the transformed sequence. Transforming a chromatic space into a tonal space will lead to a musical sequence including notes of a unique tonality. An atonal piece thus becomes tonal. Example 11 illustrates this phenomena with the atonal piece *Semi-Simple Variations for piano* of Milton Babbit: The piece is represented in $\mathcal{C}(1,4,7)$. Then, this complex is transformed in $\mathcal{C}(1,2,4)$ related to the D minor tonality. The transformation, maps each note of the piece to a note in the D minor tonality.

3) Musical Interpretation: Some of these transformations have a natural interpretation in music. For example, the translation in a chromatic scale corresponds to a transposition. Our spatial approach highlights many other transformations that are not systematically studied in music theory, like for instance the n-rotations (with $1 \leq n \leq 5$ and $n \neq 3$).

These transformations can be combined to generate new musical results. For example, one can apply successively a rotation, a translation and a change of space, enabling a huge set of recombinations to generate new material from an initial musical sequence. Notice that some of these operations are equivalent and produce the same musical result. For example, the central symmetry operation corresponds to the same musical inversion in all chromatic chord spaces. Note also that these transformations impact pitches only. However, the representation of a musical sequence in a space that does not include all the pitches (this is the case for tonal spaces), will induce a loss of some notes, thus impacting the rhythm of the sequence.

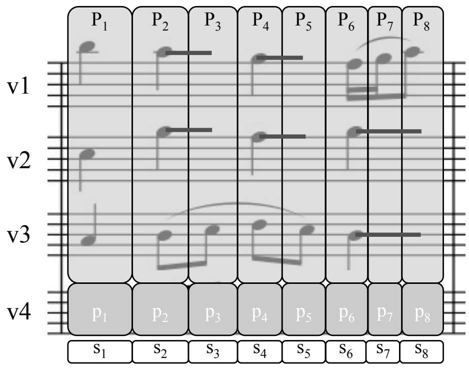

Figure 9. The generation of the voice $v4$ consists in adding to the pitch set P_t a pitch p_t (or a silence) for each segment s_t.

V. Spatial Counterpoint

Counterpoint consists in the writing of musical lines that are independent from each other but sound harmonious when played simultaneously. Numerous rules have been proposed to determine a note on a line according to previously played notes on the same line and simultaneously played notes of other lines. The way these notes are chosen determine to a large extent the musical style of the piece. Different sets of counterpoint rules have been proposed over time by music theorists. One of the most popular is probably the *Gradus Ad Parnassum* from Joseph Fux [17], used for composition by, among others, Haydn, Mozart, Beethoven and Schubert. This set of rules, published in 1725, still fascinates music theorists, and has been formalized relying on various frameworks, agebraic [1] or spatial [18].

We propose a method to translate some counterpoint rules, as the ones defined in Fux's *Gradus Ad Parnassum*, in chord spaces. The goal of this study is not to propose yet another more efficient and exhaustive method for counterpoint composition, but to show how the spatial approach can be used to express existing rules and can suggest some new rules for composition.

A. Segmentation

We focus on the generation of a melodic voice, which will be added to a pre-existing musical sequence.

First we divide the sequence in successive temporal segments. For each segment s_t, a pitch p_t or a silence has to be chosen and concatenated to the generated voice. If a same pitch is generated for two successive segments, the note can be hold or repeated. We use a simple segmentation process in this preliminary study: Each time a note is played or stopped in the pre-existing sequence, the previous segment stops and a new one starts. Figure 9 illustrates this process for the generation of a voice, in parallel with three others. The set P_t includes other voice's pitches sounding during the segment s_t. In this example, 8 pitches have to be determined to complete the fourth voice of this measure.

Note that this process only allows the generated voice to move simultaneously with an other pre-existing one. More sophisticated systems would typically allow new notes to be generated between pre-existing notes. The approach described here is constrained but sufficient for this preliminary study.

B. Translation of the Rules

Counterpoint rules can generally be classified in three categories:

- Vertical (or harmonic) rules: How p_t fits with pitches in P_t;
- Horizontal (or melodic) rules: How p_t fits with p_{t-1} (and sometimes with p_{t-2} or earlier);
- Transverse rules: How $\{P_t, p_t\}$ fits with $\{P_{t-1}, p_{t-1}\}$.

1) Vertical Rules: Vertical rules typically consist in promoting, avoiding or forbidding the formation of particular intervals or chords in $\{P_t, p_t\}$. We build a chord complex V corresponding to these rules by assembling simplices specifying the permitted intervals and chords. For example, a rule that allows the formation of minor and major chords is typically translated by the choice of the chord class complex $\mathcal{C}(3, 4, 5)$. If the rule forbids a particular interval, V contains no edge corresponding to this interval.

Once the complex V is assembled, we use a method presented in [3] to measure how the set of pitches defined by $\{P_t, p_t\}$ fits within this space. This method consists in measuring the compactness of the sub-complex made by the pitches of $\{P_t, p_t\}$ in V. For a given V, p_t is chosen in order to maximize this compactness. Informally, for a connected set S of simplices in a complex V, the compactness depends on the length of the paths in V between two arbitrary simplices of S.

An entire set of vertical rules rarely matches the structure of a particular complex, and a compromise needs generally to be done.

2) Horizontal Rules: Horizontal rules mostly specify allowed or forbidden intervals between p_{t-1} and p_t. Note that some complex rules can forbid some longer pitch sequences, for example p_t may depend also on p_{t-2}. In this preliminary study, we focus on rules concerning only the previous generated pitch p_{t-1}.

We build the complex H by assembling all the edges corresponding to allowed intervals. The resulting space is a one-dimensional complex, which is an undirected graph. The pitches p_t are successively determined by constructing a trajectory as connected as possible in H. Notice that a pitch transition in H is not oriented: for instance, if the notes F and G are neighbor in H, both transitions $F \rightarrow G$ and $G \rightarrow F$ are allowed.

3) Transverse Rules: A transverse rule consists in allowing or forbidding particular *n-pitch transitions*. A n-pitch transition consists in two consecutive sets of n pitches. Here is an example of a rule on 2-pitch transitions: If the pitches of two voices are separated by an interval of fifth during the segment s_{t-1}, they cannot be separated by this same interval during s_t. This rule is related to the *parallel fifth* rule, widely used during the baroque period.

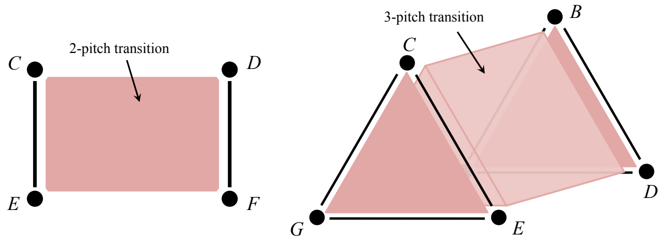

Figure 10. On the left, an allowed 2-pitch transition, between $\{C, E\}$ and $\{D, F\}$, represented by a square-shaped 2-cell. This 2-cell is not a simplex: it has four edges in its border while a 2-simplex has only three edges in its border. On the right, the 3-pitch transition between $\{C, E, G\}$ and $\{B, D, F\}$.

We represent an allowed n-pitch transition by a n-cell built as the *extrusion* of a $(n-1)$-simplex. Here, the extrusion is the product of an arbitrary simplex with a 1-simplex. The two sides of the extrusion correspond to the $(n-1)$-simplices respectively representing the pitch set $\{P_{t-1}, p_{t-1}\}$ and $\{P_t, p_t\}$. For example, a 2-transition is represented by a square-shaped cell (the extrusion of an edge). A 3-transition is represented by the extrusion of a triangle (figure 10). Notice that the resulting cell is not a simplicial cell, it is a a *simploid i.e.*, the product of two simplices.

As for horizontal spaces, this representation does not specify the direction of the transition. For example, the square cell on the left of figure 10 represents both transitions $\{C, E\} \rightarrow \{D, F\}$ and $\{D, F\} \rightarrow \{C, E\}$. To specify rules on directed transitions, n-cells representing n-transitions have to be oriented, in the same way that an edge (which is a 1-cell) can be oriented. An alternative approach would consist in updating the structure of H at each segment according to the played pitch set.

We build the space H by assembling the allowed n-pitch transitions. The resulting space is a *simploidal set*, a slight generalization of a simplicial complex [19].

All the notions we have presented on simplicial complexes lift immediately on simploidal sets. Thus, the pitch p_t is chosen so that the simploid spanned by $\{P_{t-1}, p_{t-1}\}$ and $\{P_t, p_t\}$ exists and maximizes the compactness in H.

4) Application: The respect of the rules by a potential pitch p_n is evaluated in each of the three spaces V, H and T. If no pitch is found, rules can be weakened by relaxing some constraint in one of the spaces, for instance by including some additional cells. An other possibility is to put a silence.

Some traditional rules cannot be easily represented solely by the structure of these complexes. However, we believe that the spatial approach can be an inspiration to propose new sets of rules for various kinds of music.

Moreover, the analyse of a set of musical pieces in a particular style can provide elements to design customized spaces to realize counterpoint in a similar style. For example, one can look for the complex in which a piece (or a set of pieces) is represented as compact as possible [3]. Using this complex for V is a good starting point to harmonize another piece in a similar style. Similar processes can be done to determine H and T.

VI. Conclusion

The starting point of this work is the abstract spatial representations of various musical objects defined in [4], [12], [3]. These representations have been unified using a simple self-assembly process, and further defined in MGS. They have already shown their usefulness, for instance for the computation of all-interval series [12].

In this paper we take a step further and we propose two spatial formulations of some non trivial compositional processes: the definition of a class of musical transformations that includes the transposition from a major to a minor mode and the spatial formulation of counterpoint rules, leading to a new algorithm to generate an additional voice.

The spatial framework works here as a powerful heuristic. In section IV we show that some straightforward spatial transformations have a well defined musical interpretation. The others, that is the straightforward spatial transformations that do not correspond to a well known chord or melodic transformation, suggest alternative musical transformations that are not easily expressed in the usual algebraic setting used in musicology. In section V, we demonstrate how counterpoint rules can be encoded on three cellular complexes that respectively represent the constraints on the notes that are played simultaneously, the possible successions of notes in a line, and the possible succession of chords in the sequence. Again, the spatial framework suggests some alternative rules, or new rule parametrization.

All the mechanisms described here have been implemented and the audio examples illustrating this work are accessible at the url http://www.lacl.fr/~lbigo/scw13 . The first results are very encouraging and open various perspectives. We mention two of them. In another direction, the research of an adapted space with a musical piece rarely accommodates with a unique complex. The comparison of how complexes fit with a piece over the time gives elements for an harmonic segmentation of the piece. A study of the successive most adapted complexes during a piece can be represented by another complex and gives interesting elements on composers practices.

The building and processing of abstract spaces appears to be a key issue for musical analysis and composition. We believe that the path taken in this paper can help to improve and to develop new tools.

Acknowledgment

The authors are very grateful to M. Andreatta, C. Agon and G. Assayag from the REPMUS team at IRCAM and to O. Michel from the LACL Lab. at University of Paris Est for endless fruitful discussions. This research is supported in part by the IRCAM and the University Paris Est-Créteil Val de Marne.

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