Non-Standard Multiset

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Gamma and beyond

• Gamma considers seriously multiset (rewriting) for programming
• However, sometimes even multisets lack of structures
• Hence:
  – Structured Gamma
  – HOCL
  – negative (abelian group) and infinite multiplicities
  – MGS
  – ... ?
• Gamma is a unconventional language but based on conventional multiset. Can we parallel set theory: Non well founded multisets?
From hydromel to hyperset (according T. Forster)

- Hydromel is made of chouchen and chufere
- Chouchen is made of pure hydromel
- Chufere is made of hydromel and chouchen

⇒ Hydromel is made of hydromel (right!)
But how distinguishing between hydromel and chouchen?

\[
\begin{align*}
\text{hydromel} &= \{ \text{chouchen}, \text{chufere} \} \\
\text{chouchen} &= \{ \text{hydromel} \} \\
\text{chufere} &= \{ \text{hydromel}, \text{chouchen} \}
\end{align*}
\]
Hyerset (= non-well-founded set)

- A set $b$ is a hyperset if there exists an infinite descending sequence
  \[ \ldots a_{n+1} \in a_n \in a_{n+1} \in \ldots \in a_{n+1} \in b \quad (\text{illfounded}) \]
- $J = \{ J \}$
- Standard set theory (ZFC): every set is well-founded
- (FA) $\Rightarrow x \neq [x, y]$ and $y \neq [x, y]$ for any $x$ and $y$

C struct with pointers

```c
struct C = {
    C* x;
    int y;
}
C X; X.x = &c; X.y = 1;

type ('a, 'b) pair = Pair of 'a * 'b;; let rec x = Pair(x, 1);;
```
Non Standard Multiset  (following Louis Kauffman for hyperset)

- Non standard Multiset (NSM) as
  - Words
  - Planar subsets
  - Graphs

- Tools from
  - Language theory
  - Topology
  - Diagrams

to investigate NSM
and check that there is no dangers
Words on parenthesis and the nesting of sets

• A finite word $E$ on $\{\{,\}\}$ is well-formed iff
  
  – $E$ is empty.
  
  – $E = \{F\}G$ where $F$ and $G$ are well-formed

• A finite ordered multiset is an expression
  
  \[ S = \{T\} \]
  
  where $T$ is well-formed
  thus $T = A_1 A_2 \ldots A_n$ where the $A_i$ are the elements of $S$, are finite ordered multiset

• Finite multisets are the equivalence classes generated by
  $XY = YX$ where $X$ and $Y$ are well formed

• Example:
  
  $S = \{\{\}\} \{\{\}\}\}$ multiset with 2 elements $\{\} \ et \ \{\{\}\}\}$
  $X = \{\{\}\} \{\} \{\} \} \}$ (three times the same element)
Trees and boxes

- Multisets can be represented by trees
- Multisets can be represented by boxes
  (you can move and stretch the boxes but not cross them)
Forms and Non Standard Multisets (NSM)

- **Forms** are (eventually infinite) collections of rectangles such that two rectangles are either disjoints or one included into the other.

- **NSMs** are (eventually infinite) collections of rectangles:
  - there is one outermost rectangle $R$
  - the elements inside $R$ are disjoint unions of elements of NSM

- **NSM** are framed forms: $\text{NSM} = \{ \text{Form} \}$
- The simplest example : $J = \{ J \}$

\[
J = \{ \{ \{ \{ \{ \{ \{ \ldots \} \} \} \} \} \} \} \}
\]
Two NSM (form) are equal if you can superpose them (= if they are homeomorph in the plane)
NSM defined by a set of recursive equations

\[ A = \{ \{ \} \} B \] 
\[ B = \{ A \} \]

\[ A = \{ \{ \} \} B \]
= \{ \{ \} \} \{ A \}
= \{ \{ \} \} \{ \{ \} \} \{ A \}
= \ldots \]
Recursive notation

• \( J = \{J\} \)

• \( A = \{B\} \) and \( B = \{A\} \) thus \( A = \{\{A\}\} \)

• \( F = \{\{F\}\} F \)
Number of divisions of a Form

• The number $[X]_n$ of divisions of a form $X$ at depth $n$
  
  $[XY]_n = [X]_n + [Y]_n$
  
  $[{\{X}\}}]_n = [X]_{n-1}$

• For $F = \{ \{F\} F\}$:

  $[F]_n = [{\{F\}}]_{n-1} + [F]_{n-1}$
  
  $= [F]_{n-2} + [F]_{n-1}$
NSM defined by a finite set of recursive equations
= directed graph (à la Aczel)
They are more NSM than finite directed graphs
Do we avoid the Russel Paradox?

- We do not refer to the set of all multisets
- An axiomatic definition of NSM will enforce hereditarily constructions. Here, this is achieved by putting in the plane already pictured NSMs.
- NSM are limits of well-founded multisets (NSS à la Aczel are less than the limits of well-founded sets)
- We can defines the Russel set of a multiset $M$
  \[ r(M) = \{ x \in M \mid x \notin x \} \]
- ZFC: $r(M) = M$
  This is not necessarily true for NSM

<table>
<thead>
<tr>
<th>$M$</th>
<th>$r(M)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$J = {J}$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$b = {1, b}$</td>
<td>${1}$</td>
</tr>
<tr>
<td>$b = {0, {1, b}}$</td>
<td>$b = {0, {1, b}}$</td>
</tr>
</tbody>
</table>
From Form to Boolean Algebra...

• From NSM to SET: 
  add the equivalence $XX = X$

• From form to (almost) Boolean algebra
  – add $\{\{\}\} = \emptyset$
  – Example: $\{\{\}\}\{\} = \{\}$
  – Interpret
    • $\{X\}$ as the negation of $X$
    • $XY$ as the disjunction “$X$ or $Y$”
    • $\{\{X\}\{Y\}\}$ as the conjunction “$X$ and $Y$”
    • $\{\}$ as true and $\{\{\}\}$ as false
  – They are more values than just true and false: the infinite forms
  – Infinite forms give solution to equation like
    $P = \neg(P)$ that is $P = \{P\}$
    (very similar to the extension of $\mathbb{R}$ to $\mathbb{C}$ to give solution to $x^2 = -1$)
From Form to Knot...

• add the equivalence $XX = \_$.  
  (cancellation of pair instead of condensation)
Reidemeister Moves

- Move 1: enables self-membership
- Move 2: pairs of elements disappear. So: $X = \{C\} = \{X \times C\}$ thus, look membership only in the reduced knot
- Move 3: does not change memberships
Ribon twist
Reidemeister Moves

• Move 1: enables self-membership
• Move 2: pairs of elements disappear. So: \( X = \{C\} = \{X \times C\} \) thus, look membership only in the reduced knot
• Move 3: does not change memberships
For instance...

Borromean rings fall apart upon the removal of any one of the triplet

\[
\begin{align*}
  a & \in b \\
  b & \in c \\
  c & \in a
\end{align*}
\]

\[
\begin{align*}
  a & = \{c\} \\
  b & = \{a\} \\
  c & = \{b\}
\end{align*}
\]
Space, intrinsically
Graphs, intrinsically

• Graph $G = (V, E)$ avec $E \subseteq V \times V$

• This definition is \textit{extrinsic}

$V$ pre-exist to the graph.

– What we want is vertices that are only the organization between them, as co-existence, not as pre-existence.

– In addition, vertices have a position only relatively to the others vertices, not an absolute position (Leibniz vs. Newton)

• My motivation come from biological development where the organism build its own space
Leibniz vs. Newton
Leibniz vs. Newton

\[ x \Rightarrow . \]
Leibniz vs. Newton

\[ x \Rightarrow . \]
A graph

- is a pair $(V, E)$ where $V$ est is a set of vertices and $E$ is a set of edges
- An edge $E$ is a pair of vertices
- a vertex $V$ is a pair $(I, O)$ where $I$ is the set of the ingoing edges and $O$ is a set of the outgoing edges
Un graphe en soi : exemple

\[
\begin{align*}
\text{a} & \quad \text{c} \\
\text{d} & \quad \text{b} \\
\text{e} &
\end{align*}
\]
Un graphe en soi : exemple bis
Simplicial Complex

Brep
Gmap

...
Relational spaces

• A space is a closed world

• Each point in space is an observer of the other points

• Each point has its identity from the relationships it has with the other points

• This is not far from the concept of monad in Leibniz
Towards a formalization

• Which mathematical object may specify the internal structure of the points?

\[
\begin{align*}
a &= \{ b, d, c \} \\
b &= \{ d, c, a \} \\
c &= \{ a, b, d \} \\
d &= \{ c, a, b \}
\end{align*}
\]

• We need **multisets** because the equations are symmetric for all variable permutations and so \( a = b = c = d \)

• In fact we need more: a surface (not a graph, even if a surface can be “coded” by graph, cf. V-V system)

• But it is enough for a first approach
A 4-point space
• Dans un « GBF » (graphe de Cayley) tous les points sont indistingables (il y a un automorphisme qui transforme un point en n’importe quel autre)

• Suivant Leibniz : tous les états indistingables sont identifiés

• Barbour et Smolin se sont intéressés aux graphes de variété maximale dans le contexte de la physique (un tel graphe = un état intrinsèque de l’univers). On peut même définir des plus maximaux que d’autres avec le diamètre (l’horizon) nécessaire à distinguer les sommets.
A metaphysical conclusion
Les ensembles non-standards via Paul Finsler

• Ses travaux sur les ensembles circulaires l’invite à identifier un élément à une classe et, en l’occurrence, chaque homme à l’humanité tout entière.

• Ses travaux sur les espaces de Riemann lui montrent que le fini n’est pas nécessairement limité.

• De sorte qu’il imagine que l’autre vie n’est que la vie des autres.