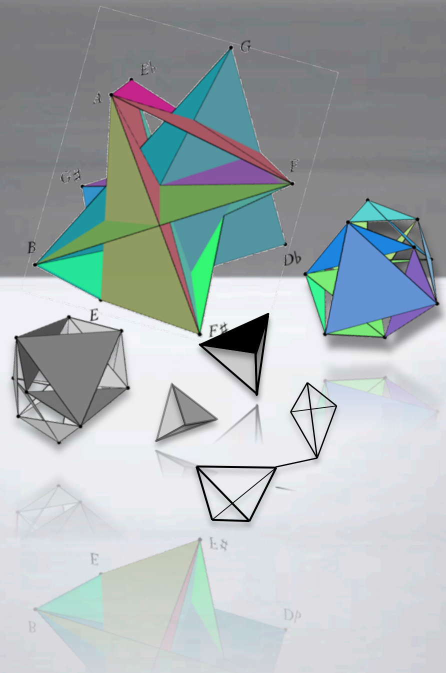
A topological Approach of Musical Relationships



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Outline

1. Beyond Graphs: Simplicial Complexes
    1. All-Interval Series
    2. Representation of Chords
        1. Path transformation in Tonnetz
        2. Space transformation
    3. Architectonic of a musical piece
2. Simplicial Complexes & Binary Relationships
    1. Q-analysis
    2. Formal Concept analysis
3. Conclusion

A topological manifesto

- The logical approach in computer science

  computation = deduction

  (the Curry-Howard isomorphism)

- Other paradigms can be fruitfull : topology

  computation = moving in some space

- Try to perceive space (and time) in programs (rather than logical operations)

  purposes: heuristic, technical, didactical

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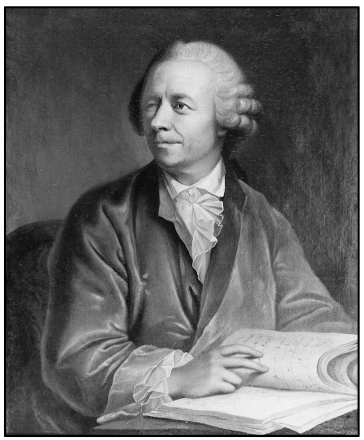
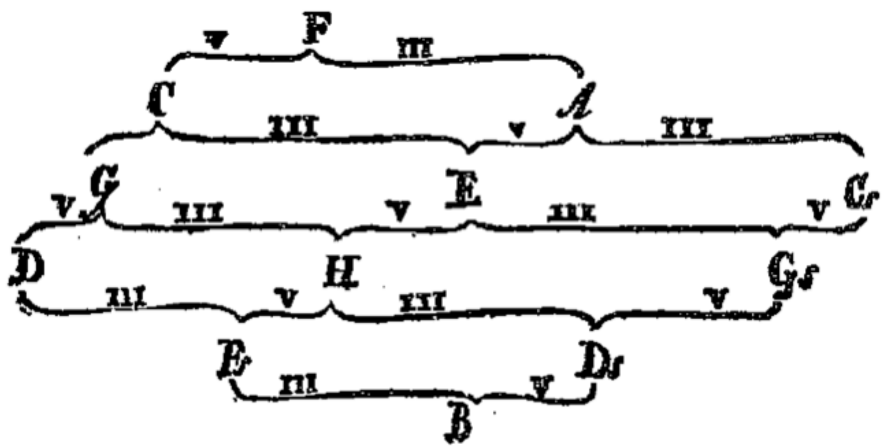
"Graphical" representation of intervals

The spatial metaphor:
the use of spatial relationships (connectedness, boundary, incidence, hole, dimension…) to grasp/catch/formalize relationships in another domain.





Putting close, notes that are consonant

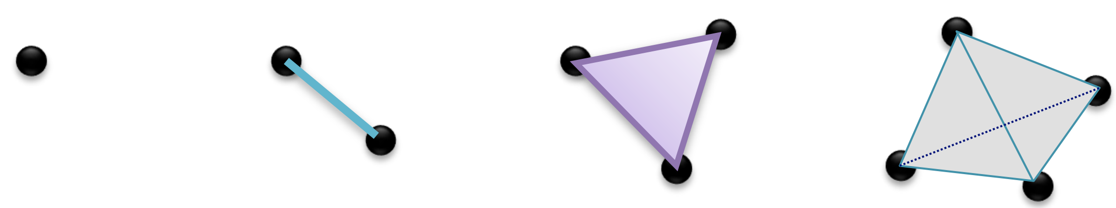
- notes are nodes
- intervals are (label of) edges
- close = two distinguished intervals

Beyond graphs

- Motivations
  - n-ary constraints between notes
    e.g., from intervals to *n*-chords (interval = 2-chords)
  - more sophisticated constraints
  - more sophisticated objects

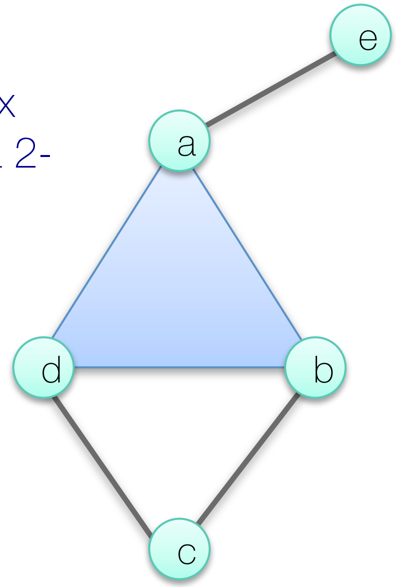
- Cellular complexes : natural generalization of graphs

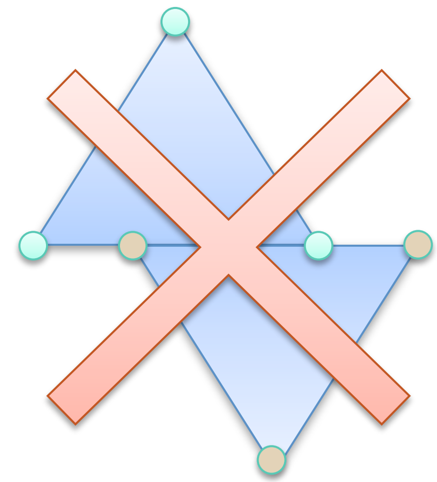
abstract simplex	{a}	{a,b}	{a,b,c}	{a,b,c,d}	
	point	edge	face	volume	...
euclidien simplex					



Cellular Complexes

an euclidean simplicial complex of dimension 2 embedded in a 2-dimensional euclidean space





An abstract simplicial complex

{

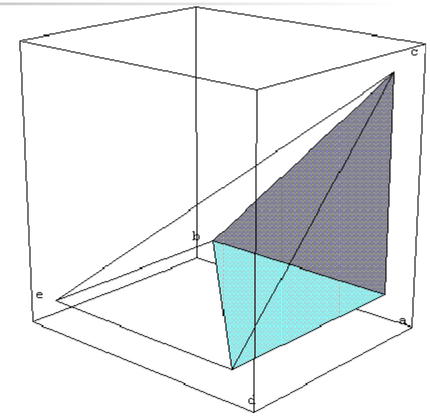
{a}, {b}, {c}, {d}, {e}, *dim* 0

{a,b}, {b,c}, {b,d}, {a,d}, {a,e}, {c,d}, *dim* 1

{a,b,d} *dim* 2

}

Topology ⟺ simplicial complex ⟺ lattice



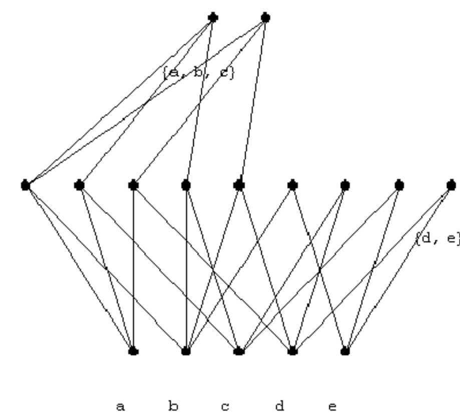
Topology (open and closed sets)

↓ simplices = closed sets ↑

Simplicial Complex (set of sets closed by inclusion/intersection)

↓ simplices = Lattice cones ↑

Lattice (order relation: ∧ , ∨)



- Reformulation of classical approaches (alternative technics)
- Extension

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Topological Representations in Music

- Examples
  - all interval series
  - chord spaces
    - chord spaces arising from set motivated my music theory
(a musical piece is a path in this space)
    - chord space arising from a piece itself
(the space is the "unfolding" of the musical space)
  - Julia Blondeau's compositional spaces
(the architecture of a musical piece)

- What can we catch ?
  - Q-analysis and Formal Concept Analysis

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Application I

ALL INTERVAL SERIES

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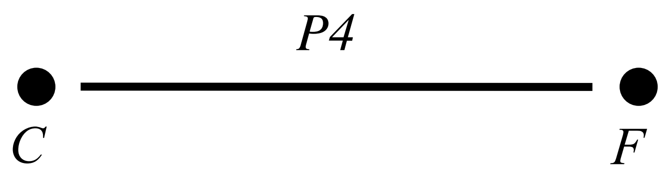
Spatial Interpretation of AIS

- AIS as *paths* in some *space*
- What kind of *space*?
  - Search space
    - Pitch classes: 0-cells
    - Intervals: 1-cells



- What kind of *path* ?

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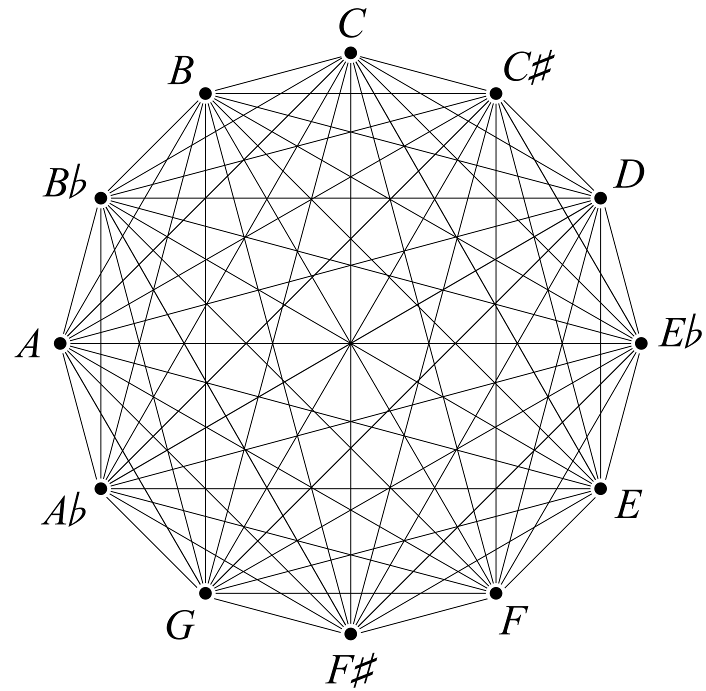

Complete graph K_{12}

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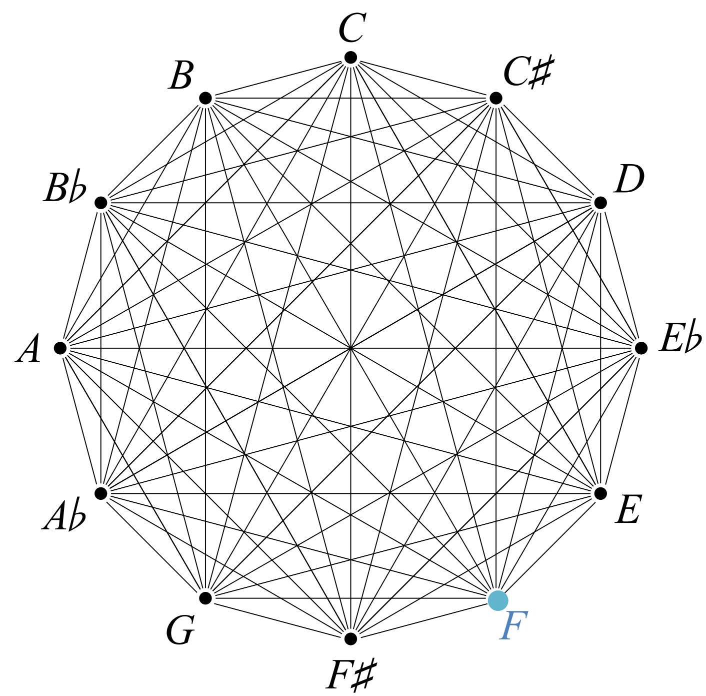

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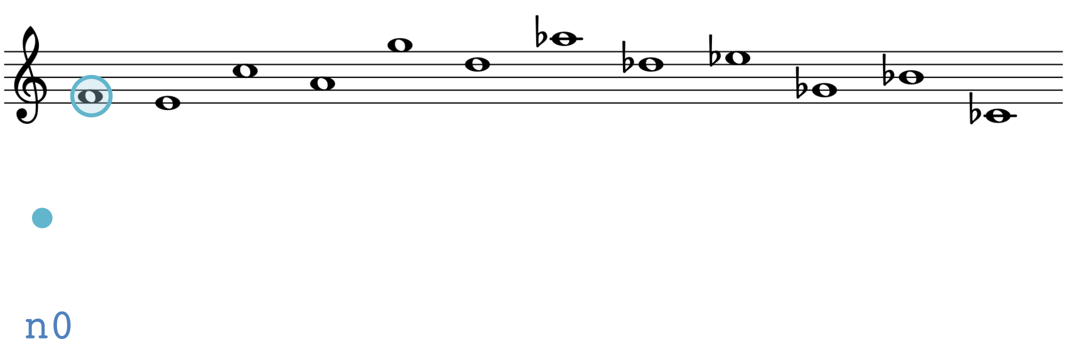

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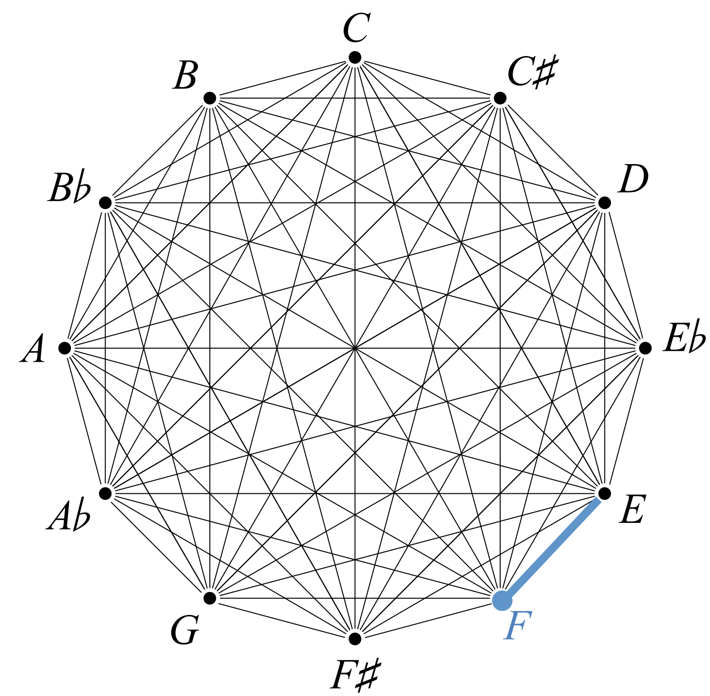

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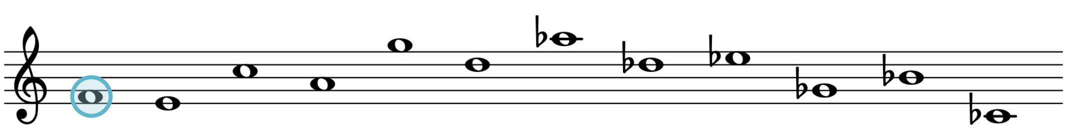

```
n0 < i1
```

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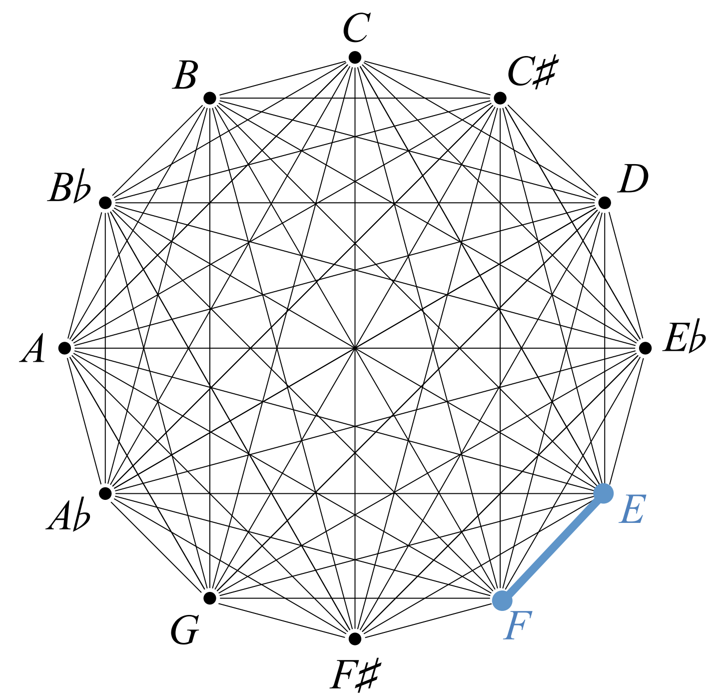

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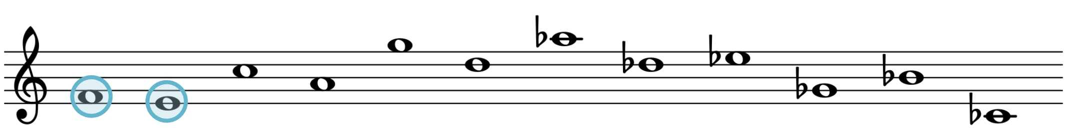

```
n0 < i1 > n1
```

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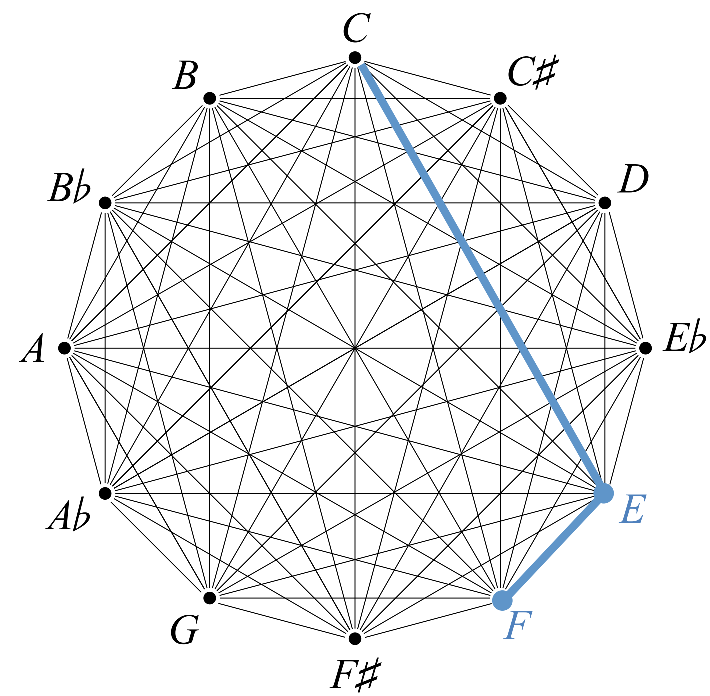

Complete graph K_{12}

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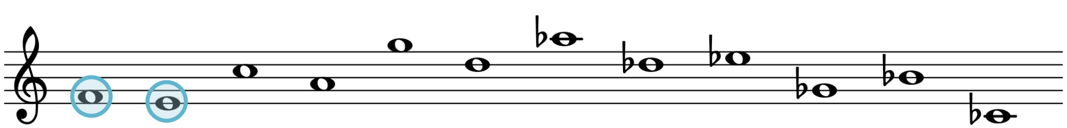

```
n0 < i1 > n1 < i2
```

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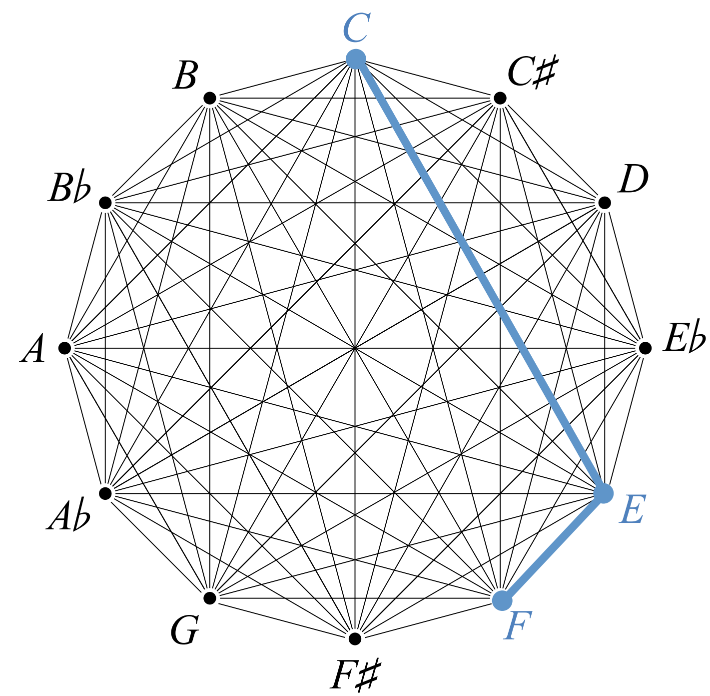

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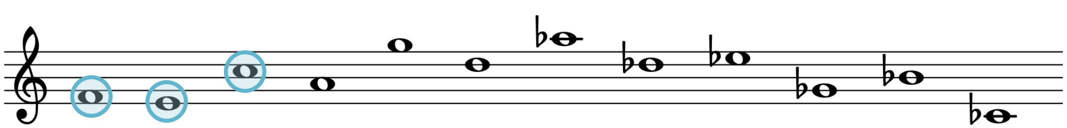

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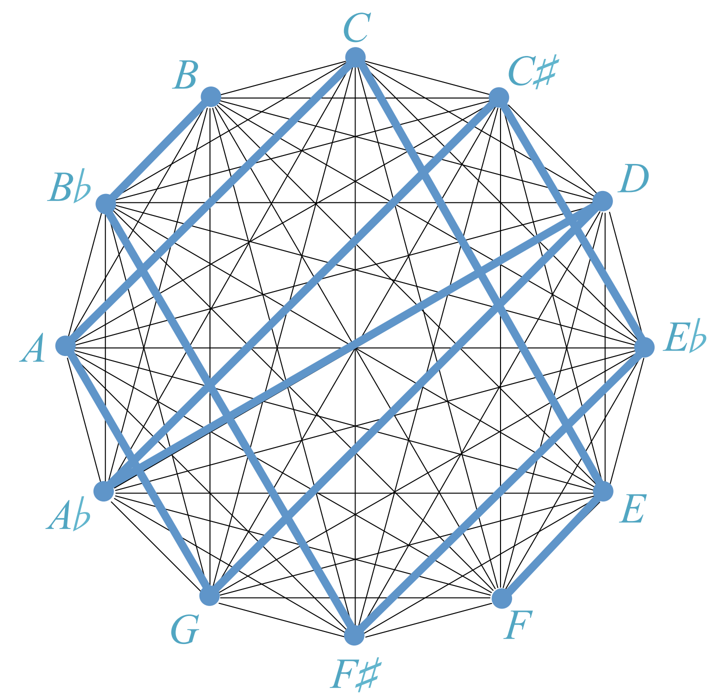

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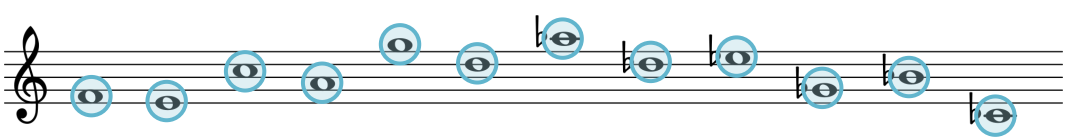

Hamiltonian path

`n0 < i1 > n1 < i2 > n2 < … > n10 < i11 > n11`

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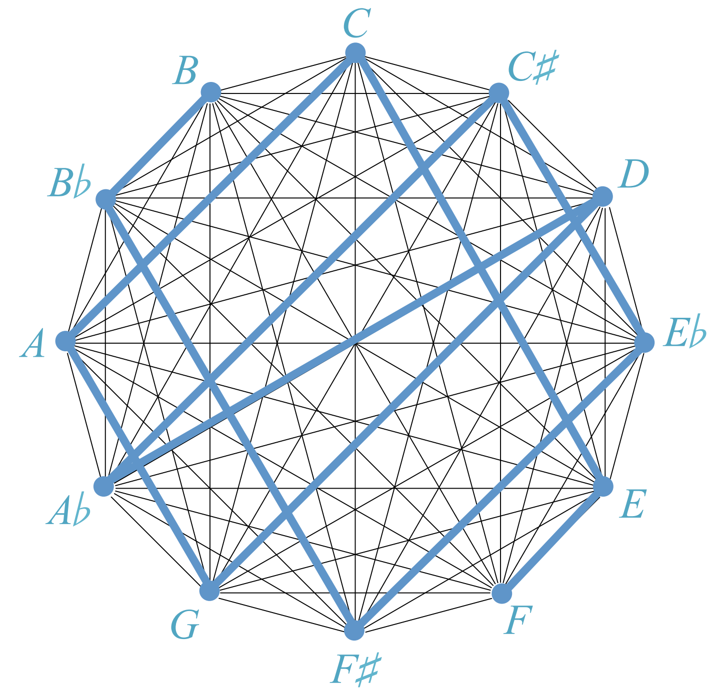

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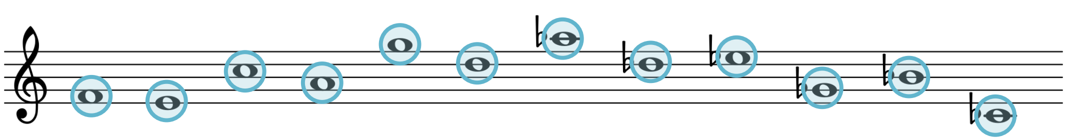


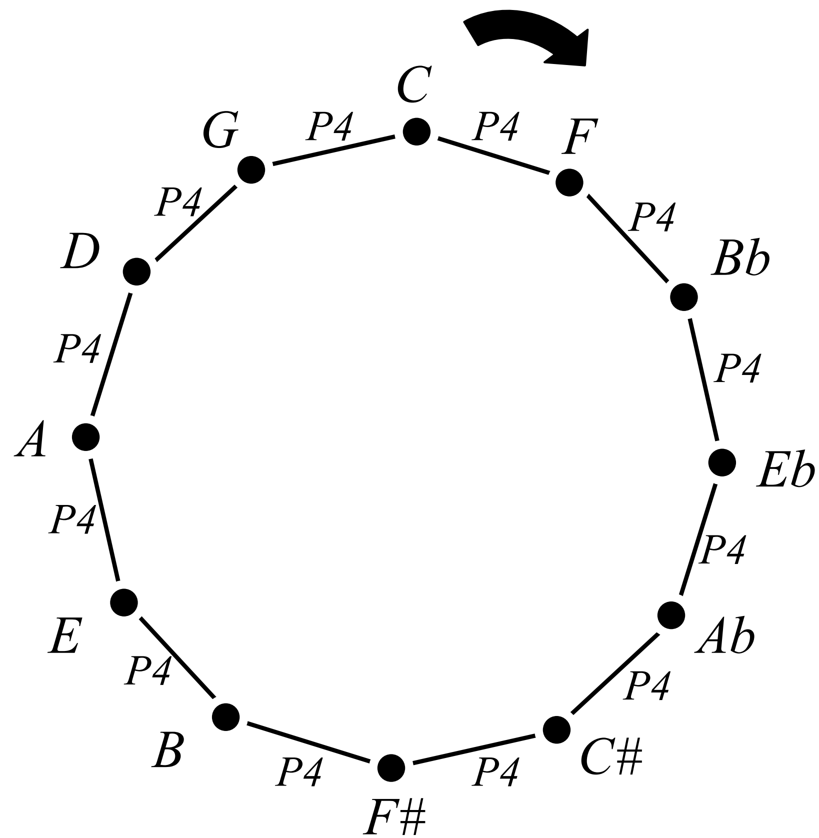
Hamiltonian path

such that edges have different labels

`n0 < i1 > n1 < i2 > n2 < … > n10 < i11 > n11` **`/ unique(i1,i2,…,i11)`**

Spatial Interpretation of AIS

- AIS as *paths* in some *space*
- What kind of *space*?
  - Search space
    - Pitch classes: 0-cells
    - Intervals: 1-cells (not unique)
    - Interval classes: 2-cells
- What kind of *path* ?



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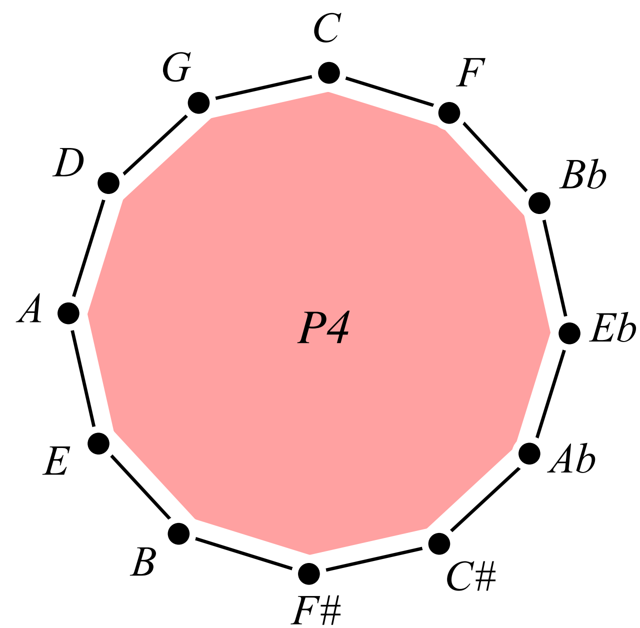

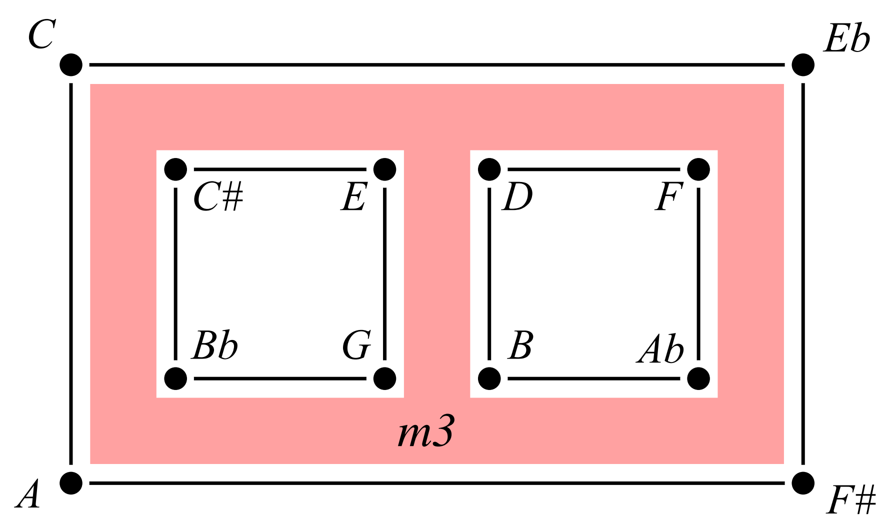

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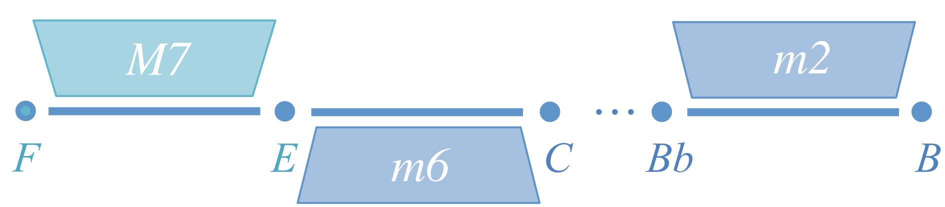
AIS Complex
(too difficult to draw)

<0, 1>-*Hamiltonian path*
<0, 2>-*Eulerian path*

- What kind of *path* ?



n0 < i1 × n1 >   … n10 < i11 < I11 > i11 > n11

Application II

CHORD COMPLEXES

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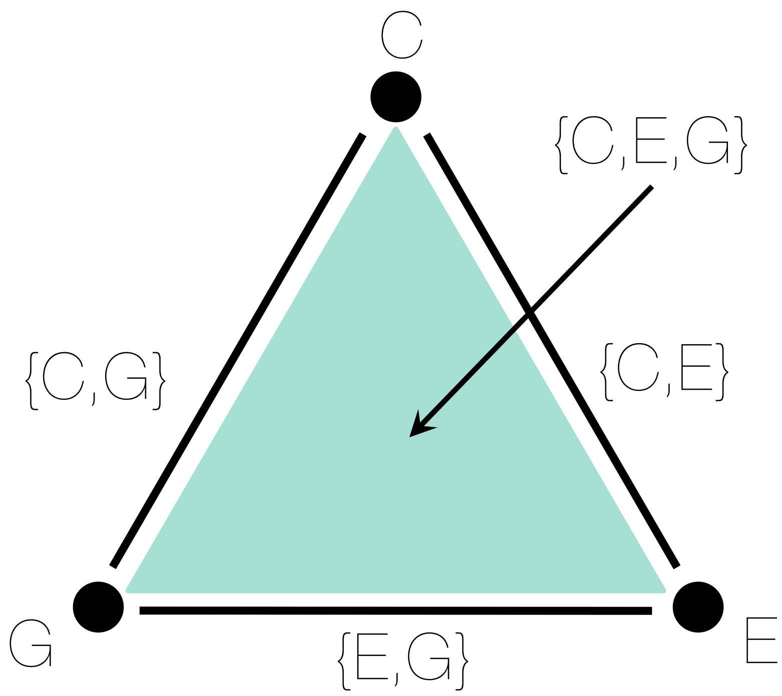
Simplex-based chord representation

- A n-pitch class set is represented by a (n-1)-simplex
  - Note: 0-simplex (vertex)
  - 2-note chord: 1-simplex (edge)
  - 3-note chord: 2-simplex (triangle)



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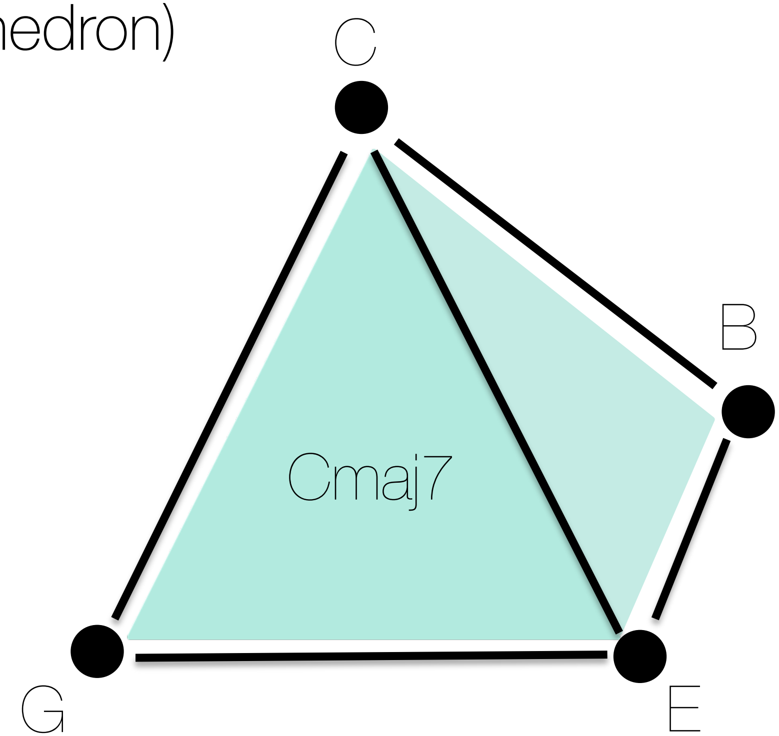
  - 4-note chord: 3-simplex (tetrahedron)



Building Chord Complexes

- *Chord population from chord transformation orbits*

  Transposition, inversion, permutation, etc.

- *Chord population from a piece segmentation*

  Analysis of Bach, Chopin, Schoenberg, Webern, Glass, etc.

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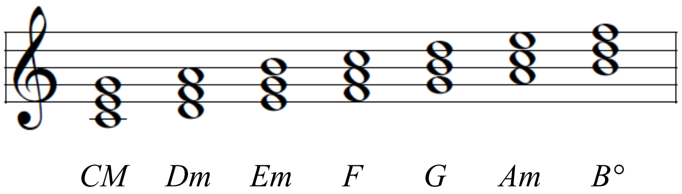
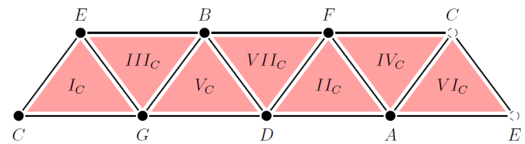
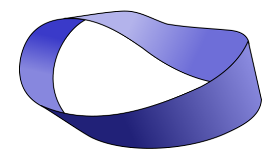
Application: Representation of a tonality

- Degrees of the diatonic scale

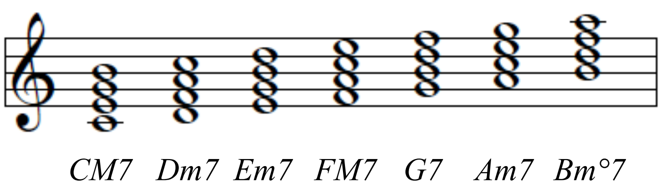




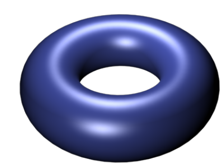


G. Mazzola

- Four note degrees

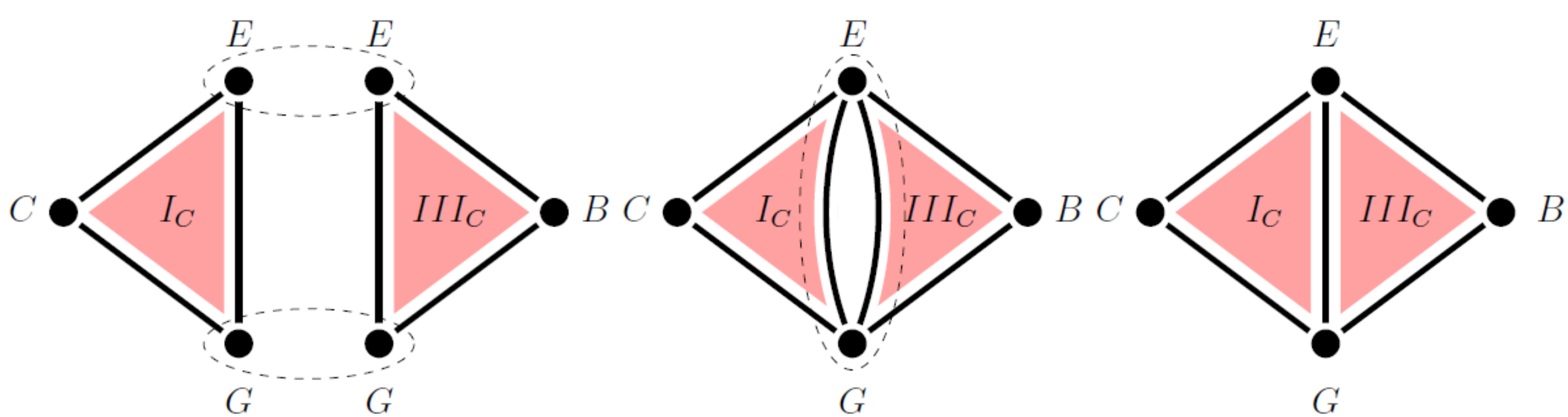


spatial programming



Building Chord Complexes

- by self-assembly (topological rewriting *à la* MGS)
  - basic elements: a population of chords (as simplices)
  - a rule that "*zip*" two simplices *a and* b
    - Assembly predicate: same pitch-class subset
    - the cofaces of the new simplex are the union of the cofaces of *a* and *b*



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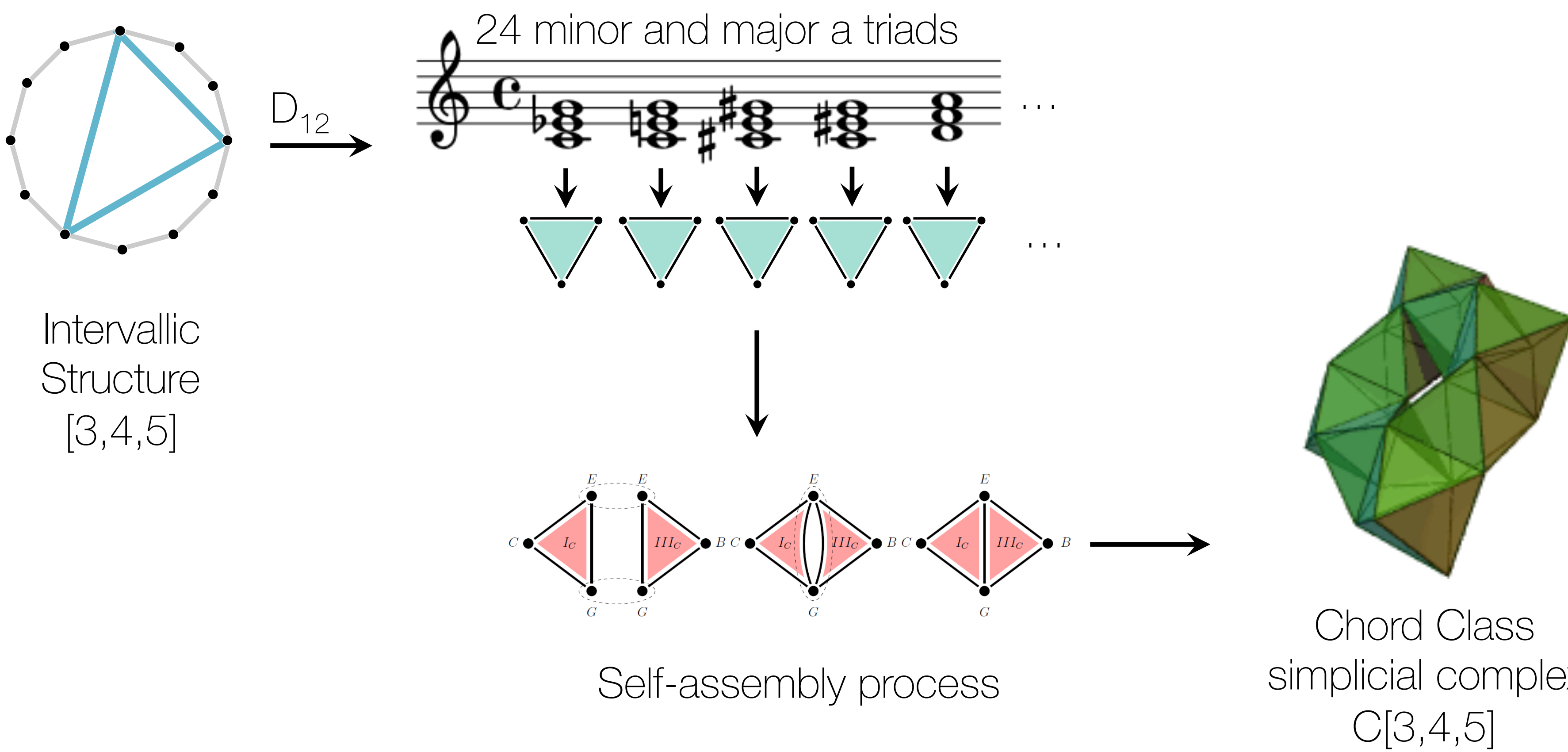
Application: Chord class representation

- population of chords = the orbit of a chord transformation
- example: representation of Forte class



Application: Chord class representation

- Complexes enumeration in heptatonic/chromatic systems (Louis Bigo)

Equivalence relation	Chromatic system (Z_{12})	Heptatonic system (Z_7)
Transposition	352 complexes	20 complexes
Transposition / inversion	224 complexes	18 complexes
Permutations	77 complexes	16 complexes
Affine application	157 complexes	16 complexes

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Application: Chord class representation

- Complexes enumeration in heptatonic/chromatic systems

$S_1(K_{TI}[3,4,5])$
[Cohn – 1997]

$S_1(K_{TI}[2,3,3,4])$
[Gollin - 1998]

$K_T[3,4]$, $K_T[2,2,3]$, $K_T[1,2,2,2]$
[Mazzola – 2002]

$K_{TI}[1,1,10]$ $K_{TI}[4,4,4]$
[Catanzaro - 2011]

$S_1(K_{TI}[1,2,4])$
[Hook – 2013]

d	complexe	taille	b_n	p-v	χ
-	$\mathcal{K}_\emptyset$	0	0		0
0	$\mathcal{K}_{TI}[0]$	0	[0]		0
1	$\mathcal{K}_{TI}[1,11]$	12	[1, 1]	x	0
	$\mathcal{K}_{TI}[2,10]$	12	[2, 2]		0
	$\mathcal{K}_{TI}[3,9]$	12	[3, 3]		0
	$\mathcal{K}_{TI}[4,8]$	12	[4, 4]		0
	$\mathcal{K}_{TI}[5,7]$	12	[1, 1]	x	0
	$\mathcal{K}_{TI}[6,6]$	6	[6, 0]		6
2	$\mathcal{K}_{TI}[1,1,10]$	12	[1, 1, 0]	x	0
	$\mathcal{K}_{TI}[1,2,9]$	24	[1, 2, 1]	x	0
	$\mathcal{K}_{TI}[1,3,8]$	24	[1, 2, 1]	x	0
	$\mathcal{K}_{TI}[1,4,7]$	24	[1, 2, 1]	x	0
	$\mathcal{K}_{TI}[1,5,6]$	24	[1, 1, 6]		6
	$\mathcal{K}_{TI}[2,2,8]$	12	[2, 2, 0]		0
	$\mathcal{K}_{TI}[2,3,7]$	24	[1, 2, 1]	x	0
	$\mathcal{K}_{TI}[2,4,6]$	24	[2, 2, 6]		6
	$\mathcal{K}_{TI}[2,5,5]$	12	[1, 1, 0]	x	0
	$\mathcal{K}_{TI}[3,3,6]$	12	[3, 0, 3]		6
	$\mathcal{K}_{TI}[3,4,5]$	24	[1, 2, 1]	x	0
	$\mathcal{K}_{TI}[4,4,4]$	4	[4, 0, 0]		4
	$\mathcal{K}_{TI}[1,1,1,9]$	12	[1, 1, 0, 0]	x	0
	$\mathcal{K}_{TI}[1,1,2,8]$	24	[1, 1, 12, 0]		12
	$\mathcal{K}_{TI}[1,1,3,7]$	24	[1, 2, 13, 0]		12
	$\mathcal{K}_{TI}[1,1,4,6]$	24	[1, 1, 18, 0]		18
	$\mathcal{K}_{TI}[1,1,5,5]$	12	[1, 1, 6, 0]		6

Louis Bigo

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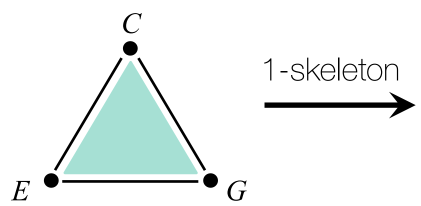
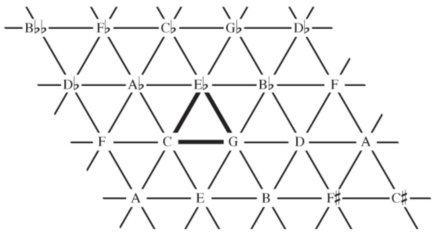
Application: Chord class representation

- Relation with *Generalized Tonnetze:* the 1-skeleton of a chord class simplicial complex is a graph in which all pitch are neighbor following a same set of intervals

C[3,4,5] (Forte class 3-11)

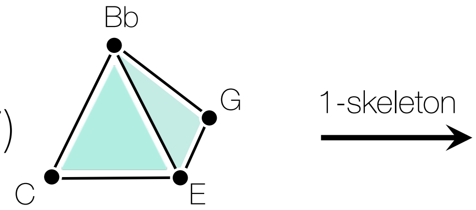


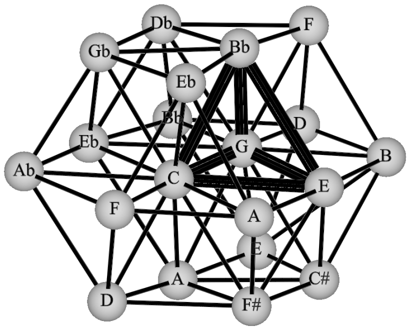


Original Tonnetz

minor and major chords

C[2,3,3,4] (Forte class 4-27)





3D Tonnetz

half-diminished seventh and seventh chords

Golin, E. (1998). Some Aspects of Three-Dimensional" Tonnetze"

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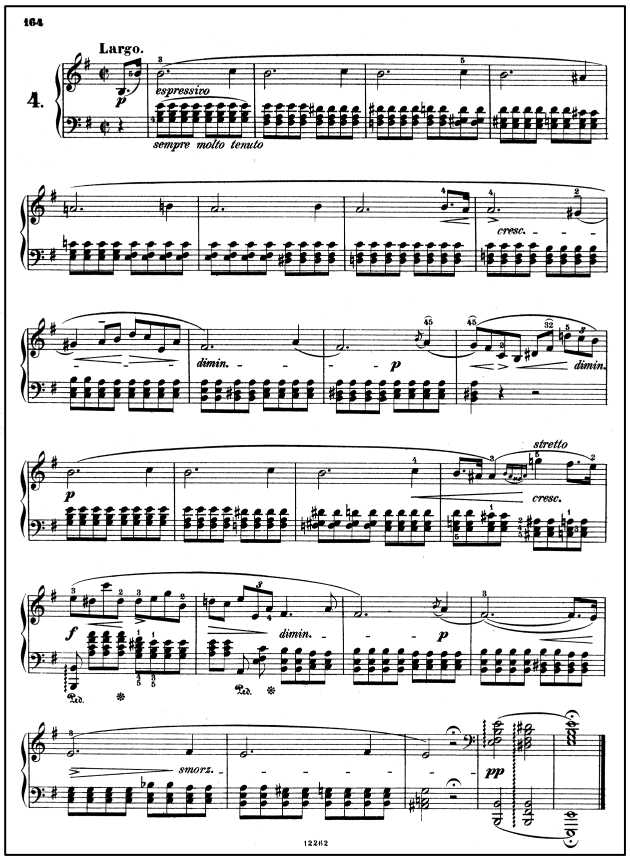
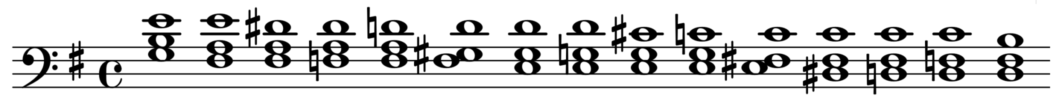
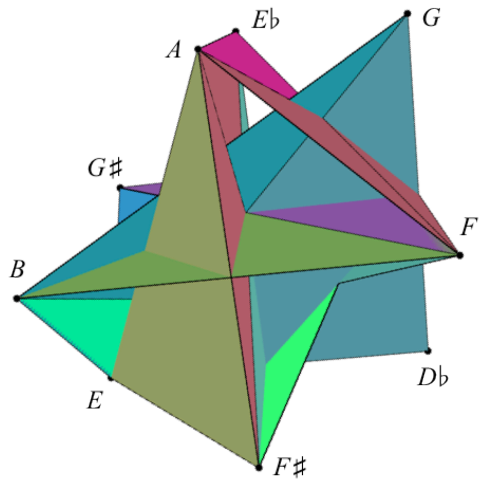
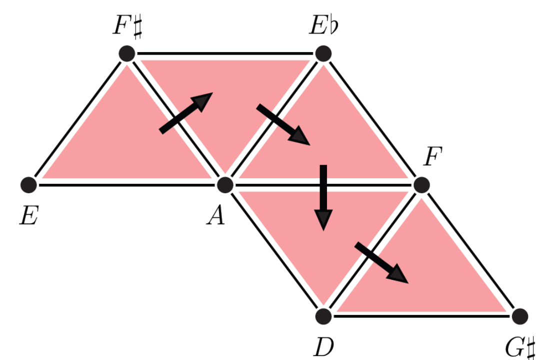
Application: Chord progression

- Extract of the Prelude No. 4 Op. 28 of F. Chopin

→







120 (2,1)-Hamiltonian paths

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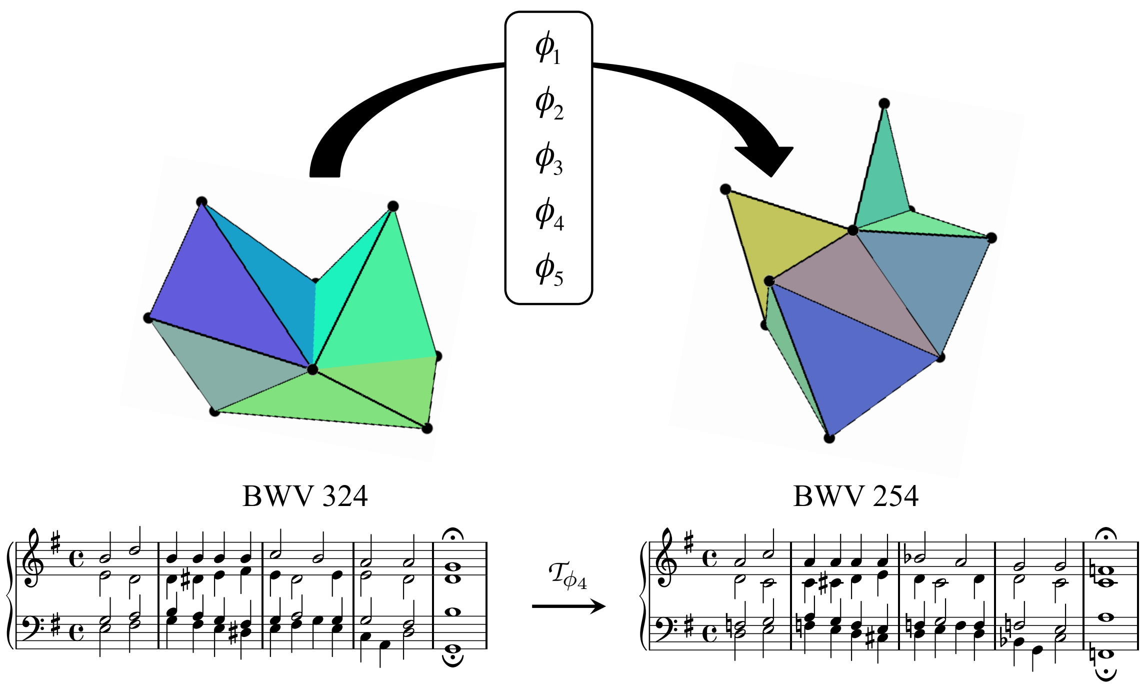
A musical piece as a SPACE

- a musical piece is a space
- this space can be characterized (Betti number, etc.)
- the performance of the piece is a path in this space
- the space can be relabeled



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A musical piece as a PATH in some space

the performance of the piece is

a *path* in some *space*

- the path can be transformed
- the underlying space can be changed

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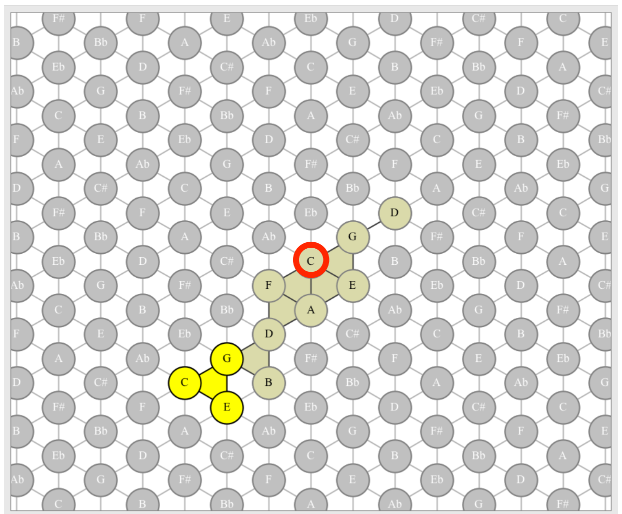
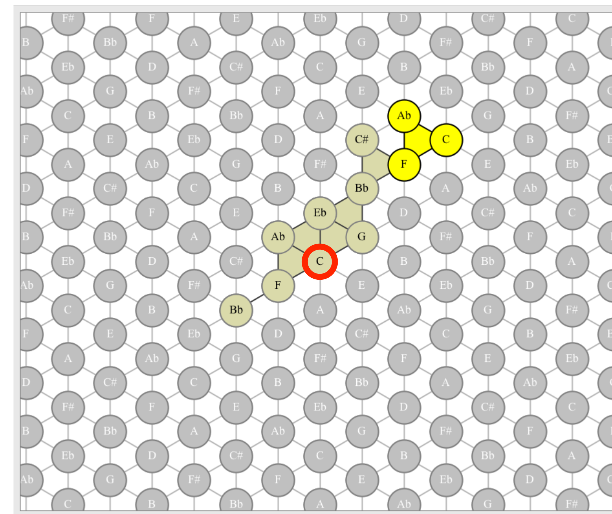
Applications: Chord Classes

- Path transformations

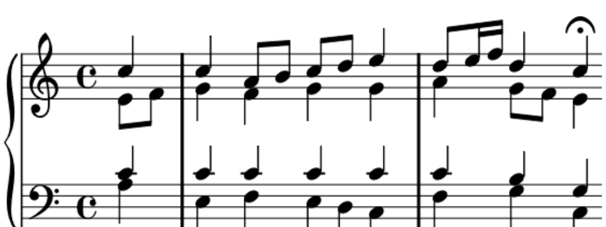
  Application of "geometrical" transformations (endomorphisms)

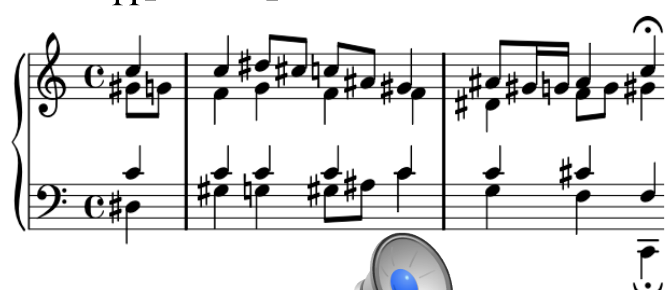


point reflection →



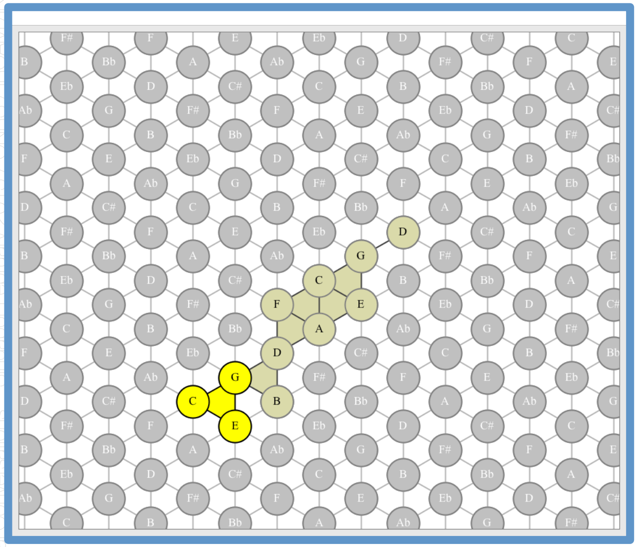
↑ $K^u_{TI}[3,4,5]$

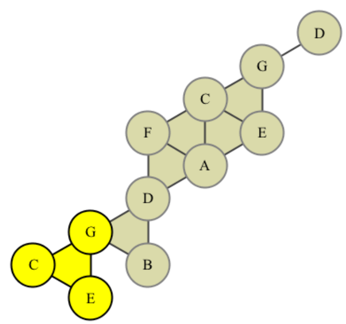
$K^u_{TI}[3,4,5]$ ↓

J.-S.Bach - Choral BWV 256

→ *pitch inversion*

Paths transformations

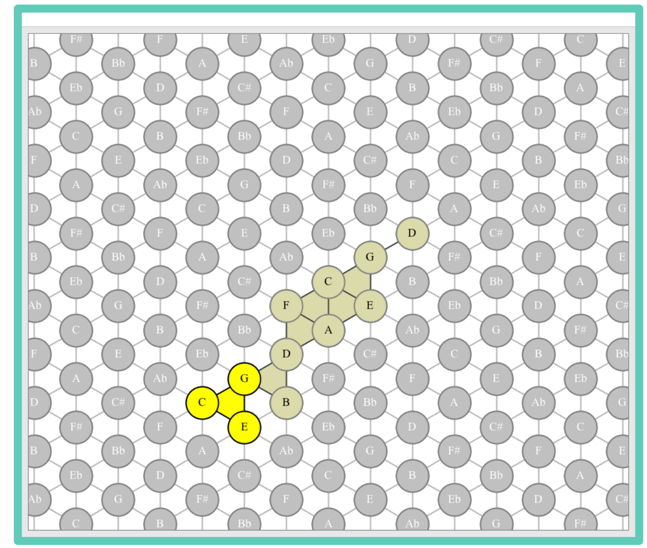
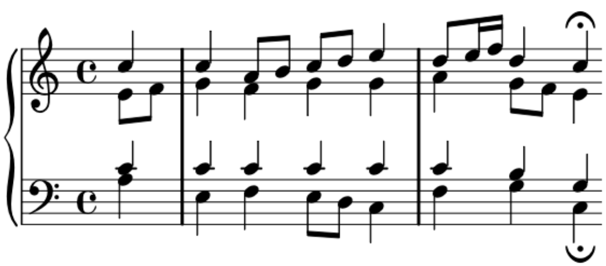
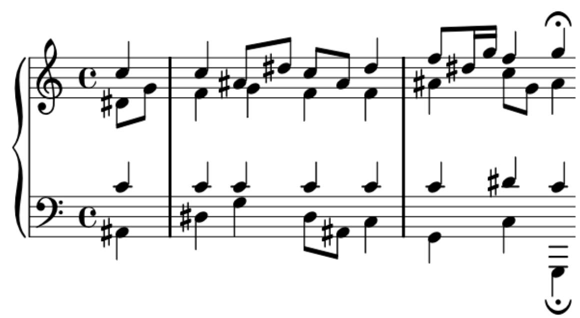
- transformation of the *initial space* of the path

C(3,4,5)





C(2,3,7)



Paths & space transformations

- Some audio results

original	
R(π) in C(3,4,5)	
T(1,-2) in C(1,2,4)	
R(π) in C(1,2,4)	

W.A. Mozart
Piano Sonata N° 16 - Allegro

original	
R(π) in C(3,4,5)	
R(2π/3) in C(3,4,5)	
C(2,3,7) ↓ C(1,2,4)	

C. Corea
Eternal Child

original	
R(π) in C(3,4,5)	
T(1,-2) in C(1,2,4)	
C(1,2,9) ↓ C(1,2,4)	

The Beatles
Hey Jude

Trajectory Transformations

- Musical interpretation of some spatial transformations

Trajectory transformation	Musical meaning
Translation in a chromatic space	Transposition
π – Rotation in a chromatic space	Inversion
Translation in a diatonic space	Modal transposition
π – Rotation in a diatonic space	Modal Inversion
ϕ – Rotation in a chromatic space (with $\phi \neq \pi$)	?
ϕ – Rotation in a diatonic space (with $\phi \neq \pi$)	?
Transformation of the underlying space	?

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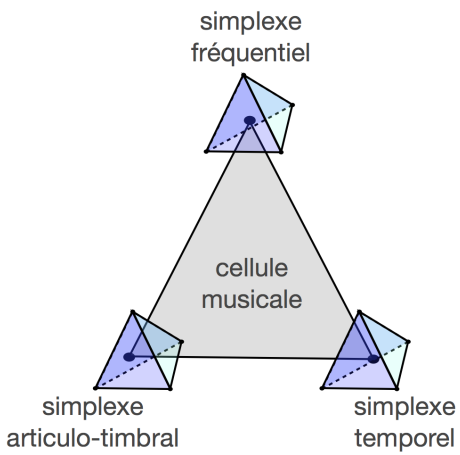
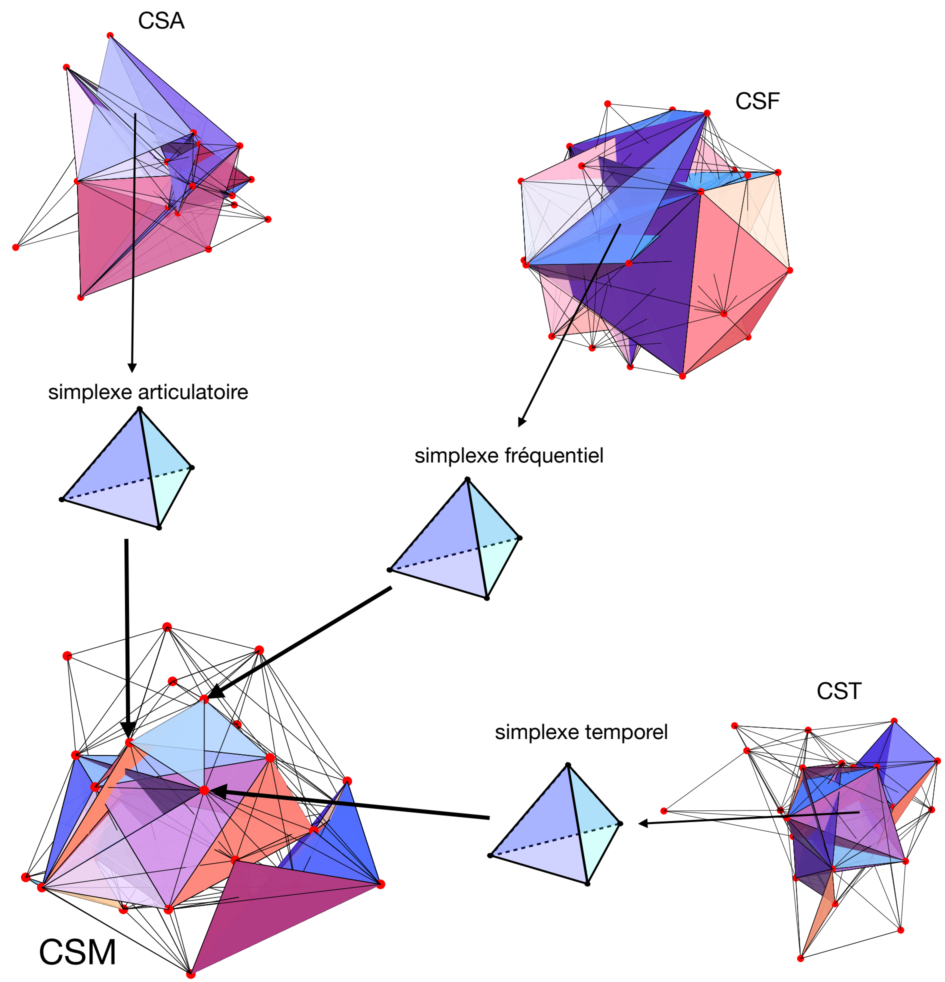
Application III

JULIA BLONDEAU'S COMPOSITIONAL SPACES

Julia Blondeau: Compositional Spaces





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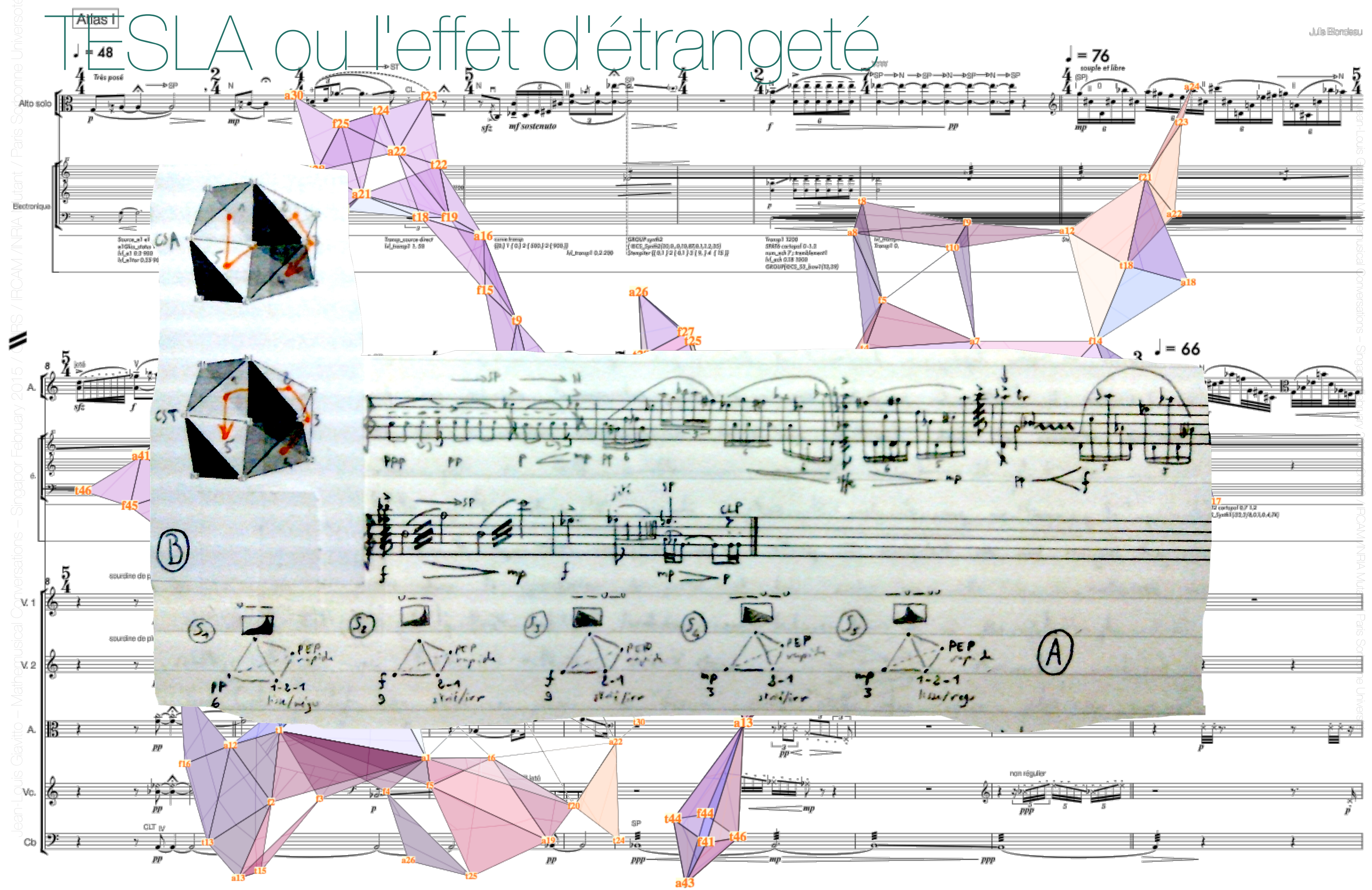
Julia Blondeau

Tesla
ou l'effet d'étrangeté



Many musical applications:

- classification using topological characteristics: can we catch the notion of style ?
- generative power of topological transformation ?
- representation of other musical relationships…

- … What kind of relationships ?

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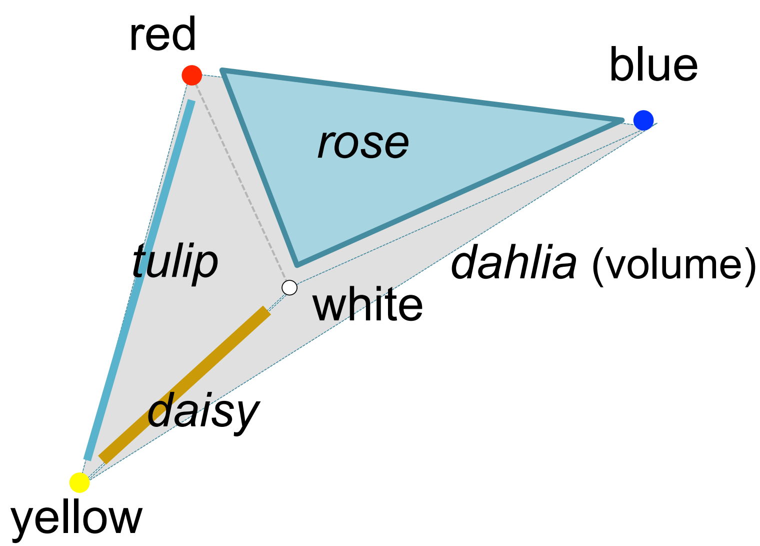
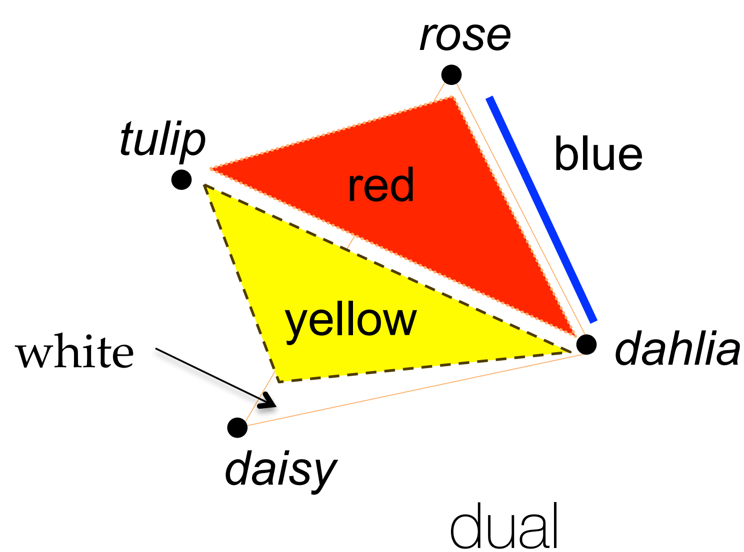
Q-analysis

a topological representation of a binary relationship

λ ⊂ {tulipe, rose, marguerite, dahlia} × {rouge, bleu, blanc, jaune}

λ	red	blue	white	yellow
tulip	1	0	0	1
rose	1	1	1	0
daisy	0	0	1	1
dahlia	1	1	1	1



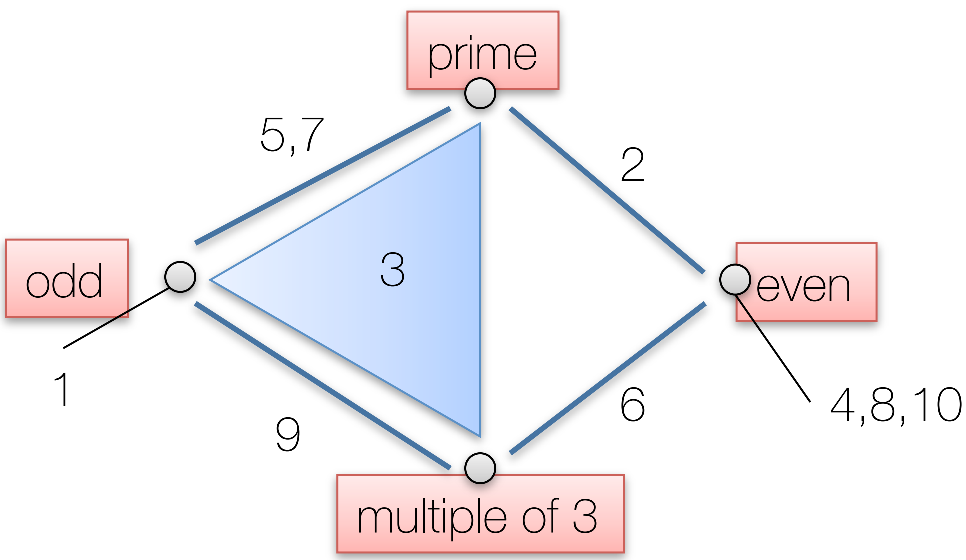


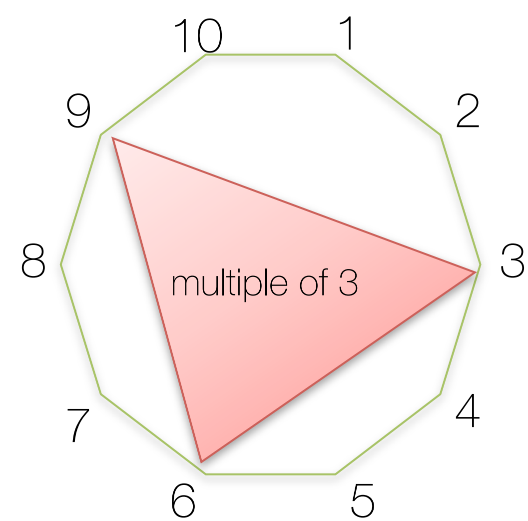
Representation of a set of predicates

$\lambda \subset \text{Objects} \times \text{Predicates} : (o,p) \in \lambda \Leftrightarrow p(o)$

Objects = {1, 2, 3, …, 10}

Predicates = {prime, even, odd, multiple-of-3}





Formal Concept Analysis

ircam Centre Pompidou

it can also be represented as a lattice

λ: interval g_i appear in a musical motive m_j

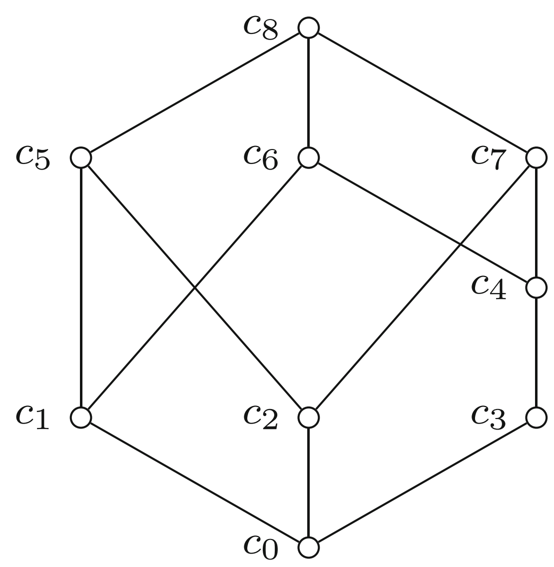
	m_1	m_2	m_3	m_4
g_1	×	×		×
g_2	×	×		
g_3	×		×	
g_4		×	×	
g_5			×	

let H ∈ G and N ∈ M

- H' = { m in M | (h, m) ∈ λ for all h ∈ H }
- N' = { g in G | (g, n) ∈ λ for all n ∈ N }

- (H, N) is a *formal concept* iff H' = N and H = N'

- formal concept are ordered: (H, N) < (I, P) iff H ⊂ I



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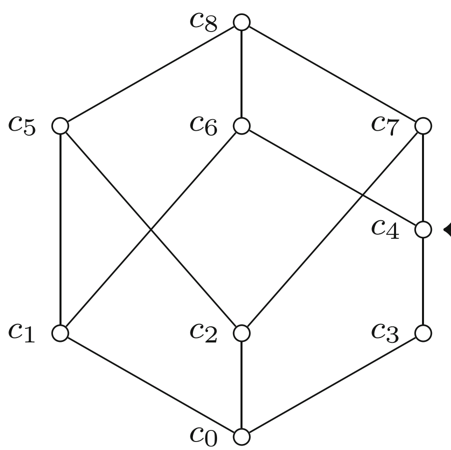
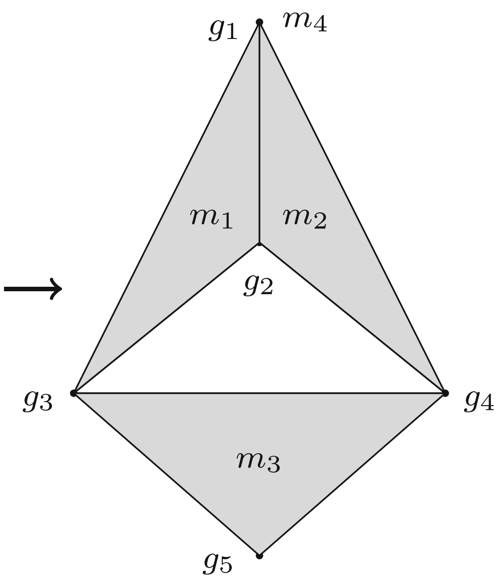
Relation between FCA and Q-analysis ?



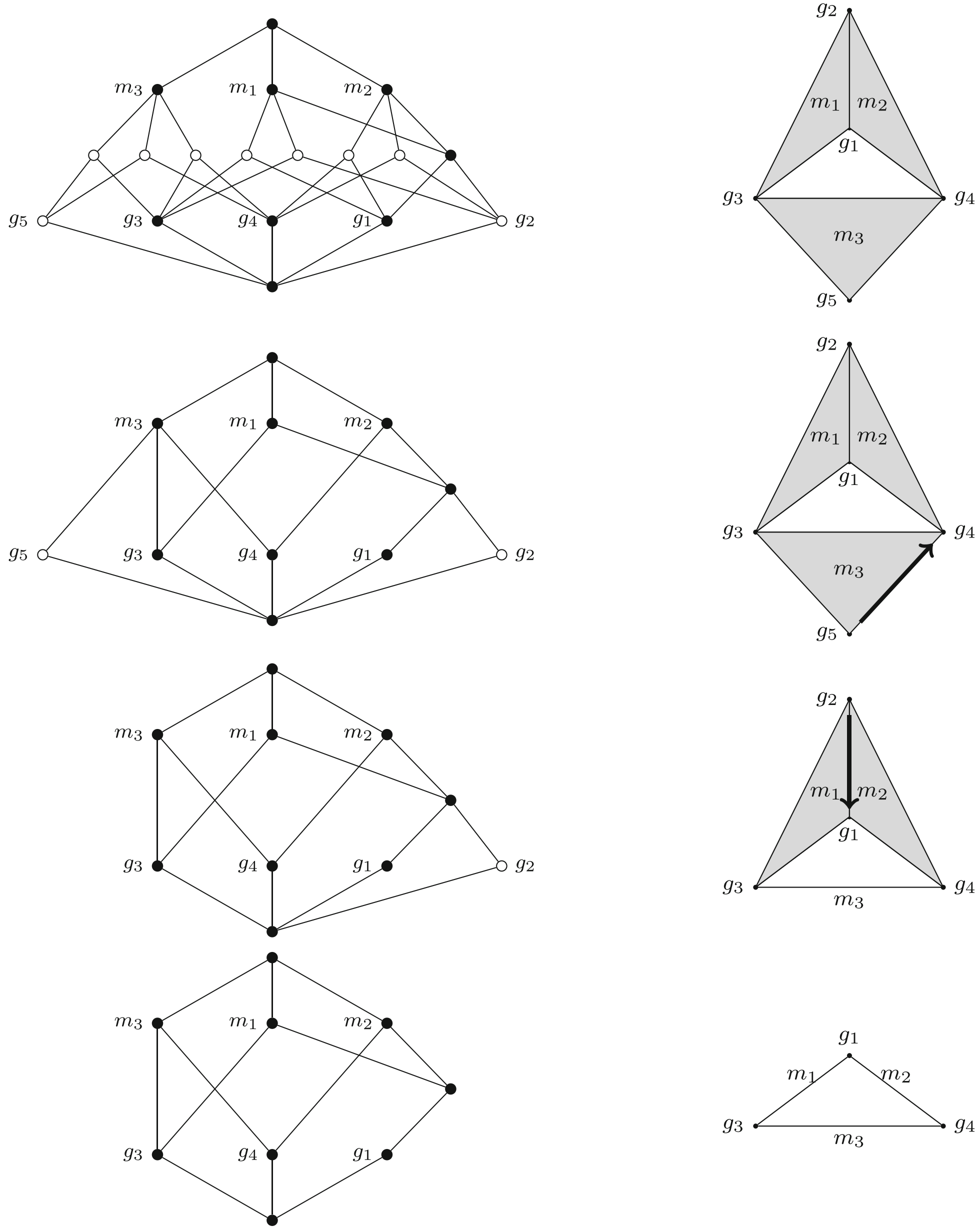
$\longleftarrow$

	m_1	m_2	m_3	m_4
g_1	×	×		×
g_2	×	×		
g_3	×		×	
g_4		×	×	
g_5			×	

$\longrightarrow$



?



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3 comments:

- Jean Nicod
- a geometry of time
- escaping the dispute

CONCLUSIONS

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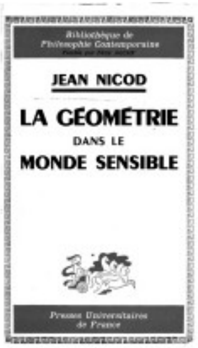
Jean Nicod

La géométrie dans le monde sensible

- La Géométrie des sensations de mouvement (1921)
  Imaginons un être pouvant se mouvoir librement dans un milieu homogène et immobile, par exemple un poisson au sein d'un océan tranquille. Faisons correspondre à chacun de ses mouvements, compliqué ou simple, prolongé ou bref, rapide ou lent, une sensation totale distinctive. Douons-le de mémoire et de raison, et demandons-nous s'il pourrait appliquer une géométrie à son expérience.

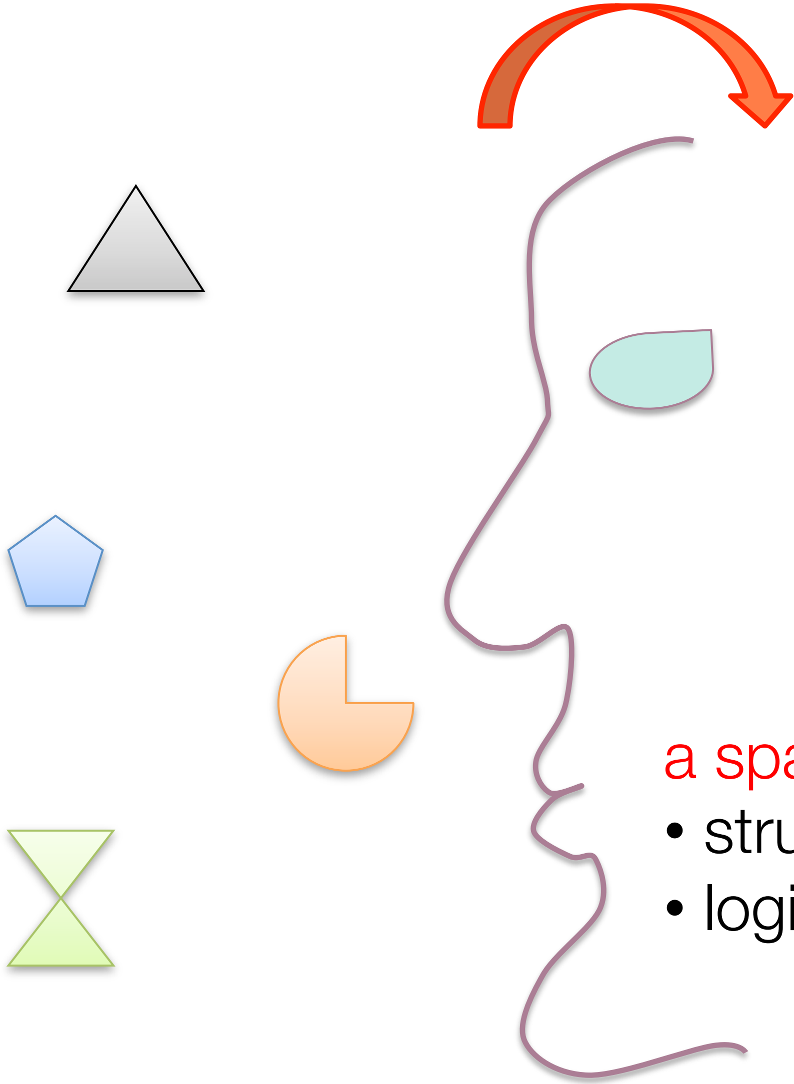
- *La Géométrie dans le monde sensible* (1923)
  Imaginons un auditeur transporté sans nulle autre sensation le long d'une ligne divisées en petite sections telles que à chaque passage sur une section quelconque (soit A), un sont d'une qualité particulière *a* se fasse entendre. […] Au sujet des sons produits, qui composent toute sa perception […], il ne conçoit, supposons nous, que deux questions : *le son* y *fut-il après le son* x ? *Le son* y *fut-il semblable au son* x ? Cherchons si le monde sensible ainsi défini a des lois ; s'il y a pour ce sujet, une physique.

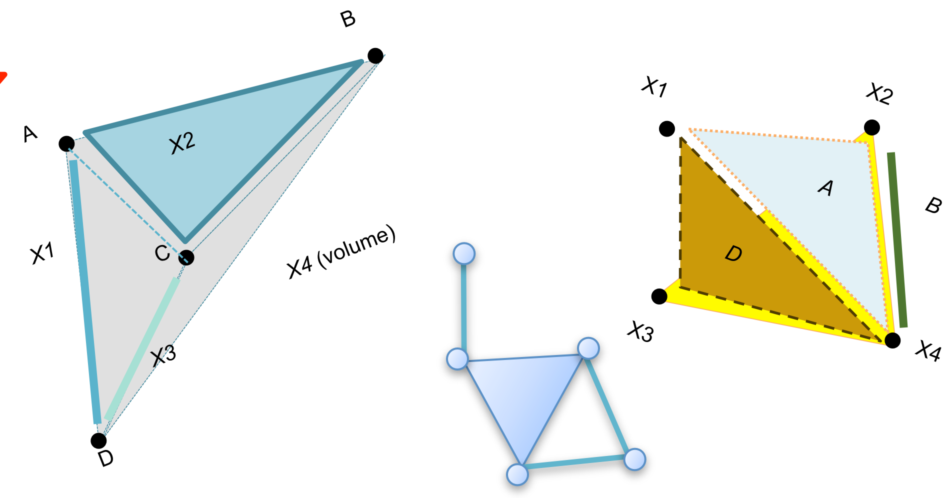


From percept to concept (~ Jean Nicod)

Percepts **Concepts**

pumping percepts into concepts





a spatial organization:

- structure of co-occurence of percepts
- logic of observations (= the open sets)

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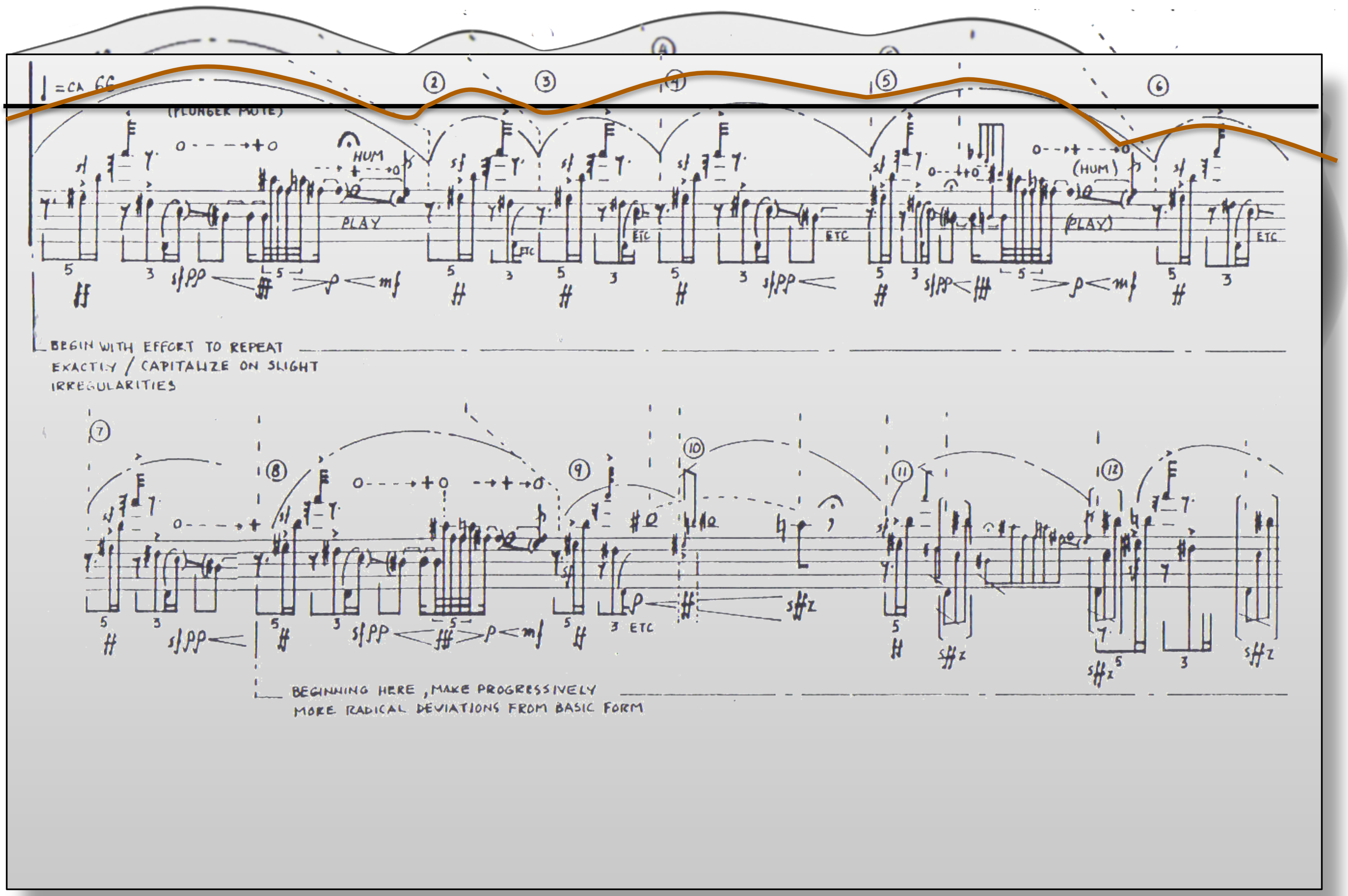
Interpretation & Playing together: the topology of time



What remains invariant ?

- event order
- relative duration
- the concurrent organization of musical objects (chords, phrases…) w.r.t. synchronization points:
  - note onset
  - start or end of a bar
  - maximal point of a dynamic
  - timber change
  - …
- qualitative first order property (shorter / longer) but also second order property (slower / faster)

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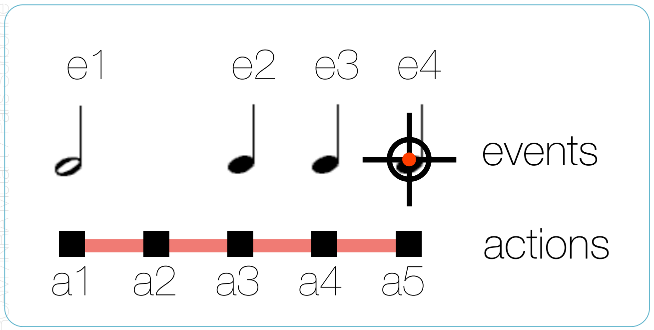
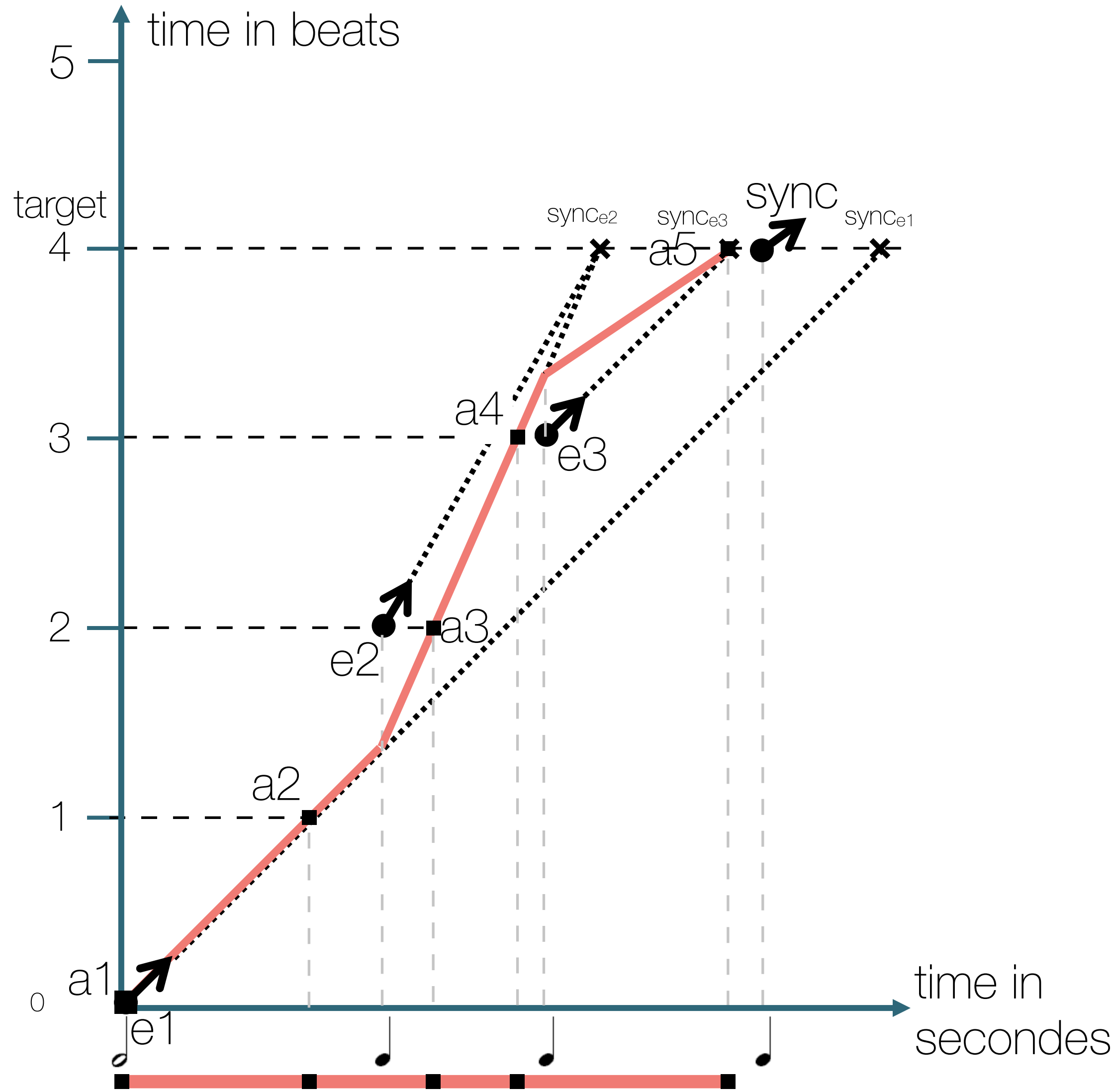
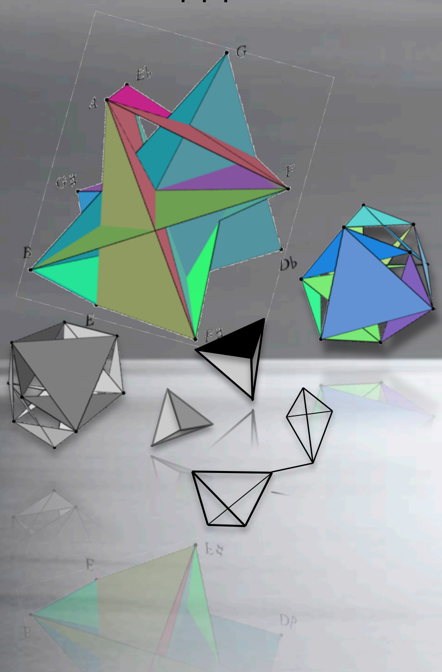
José Echeveste's time-time diagram



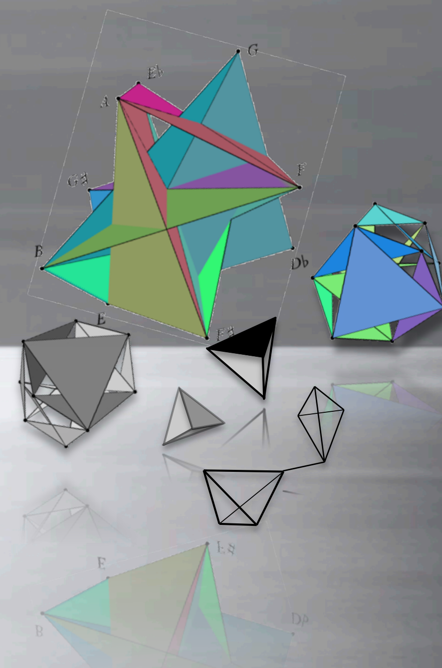

```
NOTE 60 2.0                ; e1
     group @target:={sync}
     {
          0.0 a0
                    1.0 a1
                    1.0 a2
                    1.0 a3
                    1.0 a4
     }
NOTE 60 1.0          ; e2
NOTE 60 1.0                ; e3
NOTE 60 1.0 sync           ; e4
```



transcendant
idealism
representationalism
...



immanent
materialism
emergentisme
...



Looking at the question, two ways

- Which geometry
  in pitch spaces, in rythmic spaces… ?
  (what is the "logic" of our sensations)

- What is the sensation
  in pitch spaces, in rythmic spaces… ?

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