Evaluation of the Performance of a Distribution System by Q-Analysis

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ABSTRACT

The performance of an existing goods distribution system is evaluated by means of a modern system-theoretic tool called Q-analysis. It is assumed that the system has been developed without an initial master plan, so that potential difficulties or breakdowns may occur. The case of a water distribution network which has been expanded stagewise in a southwestern U.S. city is used to illustrate the approach. The system consists of interconnected sources (reservoirs or "vertices") supplying a set of sinks (users or "simplices") through a capacitated network. Q-analysis, a technique which stems from algebraic topology, may be visualized as dealing with multidimensional graph theory, and yields indices that are used to measure performance. Specifically, the following indices are used: the q-vector, indicating the connectivity level between sources; the eccentricity, measuring whether or not elements are well integrated in the system (represented as a complex); the pattern, introducing dynamics into the analysis; the obstruction vector, identifying potential bottlenecks; and the complexity, measuring the length of various distribution paths and, indirectly, the stability. The analysis, first done from the viewpoint of source adequacy, is repeated by considering the conjugate complex, in which sources and sinks are interchanged: this introduces the users' viewpoint into the performance analysis. A numerical example shows how the above concepts provide a simple way to compare designs, identify problem areas, and improve operation characteristics of a distribution system.

1. INTRODUCTION

The purpose of this paper is to demonstrate how certain concepts of modern system theory can be used to study the performance of a distribution system, using the distribution of water as an example. The main tool in-

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troduced here is Q-analysis, which is used to characterize system structure, including the level of linkage between system elements. The Q-analysis technique is based on a branch of topology (polyhedral dynamics) dealing with multidimensional graphs, in contrast with usual models of distribution systems, based on two-dimensional network flow representations.

As an example of use of Q-analysis, consider a water distribution network which has resulted from growth without a water master plan. Such is the case for the network in the city of Tucson (Arizona, U.S.A.), which has resulted from a very small original system, to which sources and users have been added as the city grew and new areas were annexed or small water companies were bought out by the municipal water company. Such is also the case for New York City [4] and many other towns. At some point in time, the problem arises of determining the structural characteristics of the distribution system, namely, the weak points, the locations where redundancy or looping exists, and those where it does not. Another example where the structure of a water distribution network needs to be analyzed is the case where preventive measures against drought are sought. In this case it is essential to know which reservoirs or origins, links, or arcs are "controlling" the system. In the present investigation, it is shown how Q-analysis can easily answer such questions.

2. STUDY OF DISTRIBUTION-SYSTEM STRUCTURE BY Q-ANALYSIS

2.1. *Incidence Matrix and Structure of Complex*

The concept of Q-analysis, related to polyhedral dynamics [1, 2, 6], is based on a binary relationship between two sets A and B, called respectively the simplex set and the vertex set. For example, A may be a set of origins—here, water sources $(A(1), A(2), \ldots, A(n))$—and B, a set of destinations—sinks or users $(B(1), \ldots, B(n))$. Let λ be the set of linkages between the elements of A and the set of B. An incidence matrix $\Lambda = [x(j, k)]$ can thus be defined as follows:

$$x(j,k) = \begin{cases} 1 & \text{if} \quad (A(j), B(k)) \in \lambda, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

This relationship defines a complex K denoted $K_A(B, \lambda)$. Interchanging the vertex and simplex sets yields the conjugate complex $K_B(A, \lambda^{-1})$.

In a distribution system, the matrix describing the relationship between origins A and destinations B is initially composed of nonnegative numbers $X(j, k)$ such as flows and capacities. A binary matrix is then obtained by using a threshold X^* to define element $X(i, j)$ of the matrix as:

$$x(j,k)=\begin{cases}0 & \text{if} \quad X(j,k)<X^*,\\ 1 & \text{if} \quad X(j,k)\geqslant X^*.\end{cases} \tag{2}$$

The correspondence between Q-analysis and the geometry of polyhedra is illustrated by forming a polyhedron with vertices B and simplices A. Vertices are extreme points of a convex set, while simplices may be points, edges, or n-dimensional faces.

The complex K only shows the global relationship between the sets A and B. For a more detailed investigation of the relationship between simplices forming the complex, the notion of q-connectivity is introduced [1].

DEFINITION. Two simplices $\sigma(a)$ and $\sigma(b)$ are said to be q-connected in the complex K if and only if there exists a finite sequence of simplices $\{\sigma(\alpha(i));\ i=1,2,\dots,p\}$ in K such that

(1) $\sigma(a)$ is a face of $\sigma(\alpha(1))$,

(2) $\sigma(\alpha(p))$ is a face of $\sigma(b)$,

(3) $\sigma(\alpha(i))$ and $\sigma(\alpha(i+1))$ share a face of dimension $\beta(i)$, where the dimension of a face is its number of vertices minus one,

(4) $q=\min\{a, \beta(1), \beta(2),\dots,\beta(p-1), b\}$.

For example, if two simplices have m vertices in common, the two simplices are said to form a single component at the $(m-1)$-dimensional level.

2.2. *Algorithm for Q-analysis*

If $\sigma(a)$ and $\sigma(b)$ are q-connected, then they are also $(q-s)$-connected, $s=1,\dots,q$. Atkin [1] demonstrates that q-connectivity is an equivalence relationship on K and gives an algorithm to perform Q-analysis:

(1) Form $\Lambda\Lambda^T$ (an $m\times m$ matrix), where $\Lambda=$ the incidence matrix.

(2) Evaluate $\Lambda\Lambda^T-\Omega$, where Ω is an $m\times m$ matrix with all entries equal to 1.

(3) Retain only the upper triangular part (including the main diagonal) of the symmetric matrix $\Lambda\Lambda^T-\Omega$: this is the "shared-face" matrix, so called because it indicates the dimension of the faces shared by the different simplices.

(4) By reading from the main diagonal (up and to the left or to the right and down), the structure vector $Q(i)$ is obtained.

2.3 Eccentricity

The relationship between a given simplex σ and its complex K is now examined from two viewpoints [2]:

(a) to how many distinct elements of the complex is σ related?
(b) how well is σ integrated into K?

These two viewpoints are needed because, even though a particular simplex may have a high dimensionality (related to many vertices), it may not be related to the other simplices in K. An index synthesizing these two viewpoints is the so-called eccentricity of σ, denoted ecc(σ) and defined as follows [1]:

$$\mathrm{ecc}(\sigma) = \frac{\hat{q} - q^*}{q^* + 1}, \tag{3}$$

where

$\hat{q}$ = the dimension of the simplex
q^* = the largest q-value for which σ appears in a component with another distinct simplex.

Upon inspection it can be seen that $\hat{q} - q^*$ tells how many vertices the simplex σ does not share with any other simplex in the complex. But, for example, $\hat{q} - q^* \to 7 - 5$ and $\hat{q} - q^* \to 6 - 4$ yield the same result, so that it is appropriate to introduce a normalization factor $q^* + 1 \geq 0$, yielding $(7-5)/(5+1) = \frac{1}{3}$ and $(6-4)/(4+1) = \frac{2}{5}$. The former simplex is thus better integrated into K than the latter. Note that $\mathrm{ecc}(\sigma) = \infty$ means that σ is totally disconnected from the complex.

2.4. Example

Consider a water distribution network between 12 origins (4 wells, 5 reservoirs, 3 types of treated water) and 14 destinations (3 agricultural users, 3 industrial ones, 1 municipal one, 2 domestic ones, and the same 5 reservoirs). The incidence matrix corresponding to this network is shown in Table 1. In that table, let the origins or sources be the simplices, and the destination, the vertices. The table was obtained by slicing the origin-destination matrix indicating capacities of links in part of a Northern California water distribution system.

Performing up to step (3) of the Q-analysis algorithm described in Section 2.2, the shared-face matrix of Table 2 is obtained. Step (4) of the same algorithm yields the summary results shown in Table 3, in which it may be observed that the highest q-connectivity between different sources is 6:

TABLE 1
ORIGINAL INCIDENCE MATRIX[a]

	AG1	AG2	AG3	IND1	IND2	IND3	MUNI	DOM1	DOM2	RES1	RES2	RES3	RES4	RES5
WEL1	0	1	0	1	0	0	1	1	0	0	0	0	0	0
WEL2	1	0	1	0	1	0	1	0	1	1	0	0	0	1
WEL3	0	1	1	1	0	1	0	1	1	0	1	1	0	0
WEL4	0	1	1	0	1	0	1	1	0	0	1	1	0	0
RES1	1	0	0	1	1	0	0	1	0	1	1	0	1	1
RES2	1	1	0	0	1	0	1	0	1	0	1	1	0	0
RES3	1	1	0	1	1	1	0	1	0	0	1	1	0	0
RES4	1	0	1	0	1	0	1	0	1	1	0	0	1	1
RES5	0	0	1	1	0	1	0	1	1	0	1	0	0	1
II	1	1	0	0	1	0	0	0	0	0	0	1	0	0
III	1	1	1	0	1	1	1	0	0	0	0	1	0	0
IIICP	0	0	0	0	0	1	1	1	1	1	1	0	0	1

[a]WEL: well; RES: reservoir; II: secondary-treated water; III: tertiary-treated water; IIICP: tertiary-treated and chlorinated water.

TABLE 2

SHARED-FACE MATRIX FOR THE DISTRIBUTION SYSTEM (SOURCES)

	WEL1	WEL2	WEL3	WEL4	RES1	RES2	RES3	RES4	RES5	II	III	IIICP
WEL1	3	0	2	2	1	1	2	0	1	0	1	1
WEL2		6	1	2	3	3	1	6	2	1	3	3
WEL3			7	4	2	3	5	1	5	1	3	3
WEL4				6	2	4	4	2	2	2	4	2
RES1					7	2	4	4	3	1	1	3
RES2						6	4	3	1	3	4	2
RES3							7	1	3	3	4	2
RES4								7	2	1	3	3
RES5									6	−1	1	4
II										3	3	−1
III											6	1
IIICP												6

TABLE 3

SUMMARY OF RESULTS FOR THE DISTRIBUTION SYSTEM (SOURCES)

q	Q	
7	4	(RES1) (RES3) (WEL3) (RES4)
6	9	(IIICP) (III) (RES2) (RES1) (WEL4) (RES5) (RES3) (WEL3) (RES4 ,WEL2)
5	7	(IIICP) (III) (RES2) (RES1) (WEL4) (RES5 , RES3 , WEL3) (RES4 ,WEL2)
4	1	(IIICP , III , RES2 , RES1 , WEL4 , RES5 , RES3 , WEL3 , RES4 ,WEL2)
3	2	(IIICP , III , RES2 , RES1 , WEL4 , RES5 , RES3 , WEL3 , RES4 ,WEL2 ,II) (WEL1)
2	1	(IIICP , III , RES2 , RES1 , WEL4 , RES5 , RES3 , WEL3 , RES4 ,WEL2 ,II , WEL1)
1	1	(IIICP , III , RES2 , RES1 , WEL4 , RES5 , RES3 , WEL3 , RES4 ,WEL2 ,II , WEL1)
0	1	(IIICP , III , RES2 , RES1 , WEL4 , RES5 , RES3 , WEL3 , RES4 ,WEL2 ,II , WEL1)

(RES1), (RES3), (WEL3), and (RES4) are all 7-connected to themselves. Another example of interpretation of Table 3 is:

(RES4) is 6-connected to (WEL2) via

$$\langle \text{AG1, AG3, IND2, MUNI, DOM2, RES1, RES5} \rangle.$$

Physically, it means that reservoir 4 and well 2 supply, at least in part, 7 users out of 14. The structure vector $\mathbf{Q} = (Q7, Q6, \dots, Q1)$ indicates the number of q-connected components at each level; thus, $Q7 = 4$ means that there are 4 components connected at the 7th level. At levels 4, 2, 1, and 0, all the elements are in the same complex; at level $q = 2$, all sources are in a common complex meaning that 3 out of 14 users receive water from all 12 sources.

To find out how the individual simplices—or subsystems—are integrated into the complex, Equation (3) defining eccentricity is used, yielding the vector

$$(0.33, 0.0, 0.33, 0.40, 0.60, 0.40, 0.33, 0.143, 0.167, 0.0, 0.40, 0.40).$$

Thus the complex does not appear to be very homogeneous. In particular, ecc(RES1) = 0.60 means that reservoir 1 is not well integrated with the other sources. In this example, the Q-analysis reveals that some elements of the distribution system should be linked more tightly to the other elements to increase the resilience of the network.

Regarding the choice of a threshold function, great care must be taken that relevant data are not discarded when computing the $(0,1)$ incidence matrix. An illustration of this point is provided in [5].

Up to this point, one could have obtained all the results presented by inspection of the incidence matrix—albeit such an inspection certainly will become cumbersome and error-prone for a larger matrix. Furthermore, Q-analysis should be repeated at various slicing levels and with different configurations; hence the practicality of automating the analysis. Finally, the next concept can hardly be studied by inspection for any sizable distribution system.

2.5. *Patterns*

The operation of a water-resources system is dynamic, whereas the elements of Q-analysis presented so far deal only with static features. The concept of pattern has been introduced in Atkin and Casti [2] to model certain dynamic aspects of system structure.

Let $\Pi(i)$ be an integer-valued mapping defined on the subset of i-dimensional simplices, and the direct sum of subgroups be defined as follows.

Given a group Π containing the subgroups $\Pi(0), \Pi(1), \ldots, \Pi(K)$, let the only element common to all subgroups be zero, and every element of Π be the sum of one element of each subgroup $\Pi(0), \Pi(1), \ldots, \Pi(K)$. Then Π is called the direct *sum* of the subgroups and is written

$$\Pi = \Pi(0) \oplus \Pi(1) \oplus \cdots \oplus \Pi(K).$$

This quantity Π defines a pattern.

In the previous example, the sets $\Pi(i)$ could be the mappings from the i-dimensional sources to the flow from those sources. Thus:

$$\Pi(3)\begin{cases}\text{mapping WEL1} \to \text{flow from WEL1}\\ \text{mapping II} \to \text{flow from II}\end{cases}$$

The dynamics is introduced by means of a pattern change $\delta\Pi$ which, in our case, simply changes the amounts flowing from various sources. Thus, the original pattern Π is composed of mappings defined on the following sources:

$\Pi(0) = \Pi(1) = \Pi(2) = \varnothing$,

$\Pi(3)$: WEL1, II,

$\Pi 4 = \Pi(5) = \varnothing$,

$\Pi(6)$: WEL2, WEL4, RES2, RES5, III, IIICP,

$\Pi(7)$: WEL3, RES1, RES3, RES4.

The new pattern $\Pi + \delta\Pi$ may be

$\Pi(0) = \Pi(1) = \Pi(2) = \varnothing$,

$\Pi(3)$: WEL1, II,

$\Pi(4) = \varnothing$,

$\Pi(5) =$ RES2, RES4,

$\Pi(6) =$ WEL2, WEL4, RES5, IIICP,

$\Pi(7) =$ WEL3, RES1, RES3, III,

which can be interpreted physically by noticing that RES2 and RES4 have become 5-connected instead of 6, and III,7-connected instead of 6.

2.6 Obstruction Vector and Complexity

The obstruction vector is a measure of resistance to change of pattern (or obstruction to change) at level q. Let W be a vector of whose components are all 1's; the obstruction vector is defined as $\hat{Q}=Q-W$, in which Q is the vector of components $Q(i)$ found from the Q-analysis. The obstruction vector may be used to identify the flexibility of a water distribution system in case of emergency (floods, droughts, or other natural hazards).

The concept of complexity describes the density of interconnections between simplices and is defined in [3] as having three properties:

(1) A system consisting of a single simplex has a complexity equal to 1.

(2) A subsystem (subcomplex) has complexity no greater than that of the entire complex.

(3) The combination of two complexes results in a level of complexity no greater than the sum of the complexities of the components.

If N is the dimension of the complex $K, Q(i)$, then a measure of complexity satisfying the three properties just listed is

$$\Psi(k)=2\frac{\sum_{i=0}^{N}(i+1)Q(i)}{(N+1)(N+2)}. \qquad (4)$$

The structure vector of sources in this distribution problem is

$$Q(S,U)=(\overset{7}{3}\ 8\ 6\ 0\ 1\ 0\ 0\ \overset{0}{0}).$$

Thus, using Equation (4), one finds $\Psi(S,u)=3.33$, which is a high value, susceptible of causing system instability [3].

The conjugate complex of the example is now analyzed to shed some light on the interrelationship between users. For example, in case of drought, how could one user deliver water to another one, and how complicated would the transshipment scheme have to be?

The shared-face matrix of the conjugate complex is given in Table 4, and the summary of results in Table 5. The eccentricity is

$$0.0, 0.17, 0.50, 0.0, 0.14, 0.25, 0.40,$$

$$0.17, 0.50, 0.0, 0.17, 0.0, 0.0, 0.25,$$

which would tend to show that the destination complex is better integrated than the source one.

Finally, the obstruction vector is

TABLE 4

CONJUGATE SHARED-FACE MATRIX

	AG1	AG2	AG3	IND1	IND2	IND3	MUNI	DOM1	DOM2	RES1	RES2	RES3	RES4	RES5
AG1	6	3	2	1	6	1	3	1	2	2	2	3	1	2
AG2		6	2	2	4	2	3	3	1	−1	3	5	−1	−1
AG3			5	1	3	2	3	2	3	1	2	2	0	2
IND1				4	1	2	0	4	1	0	3	1	0	1
IND2					7	1	4	2	2	2	3	4	1	2
IND3						4	1	3	2	0	3	2	−1	1
MUNI							6	2	3	2	2	2	0	2
DOM1								6	2	1	5	2	0	2
DOM2									5	2	3	1	0	3
RES1										3	1	−1	1	3
RES2											6	3	0	2
RES3												5	−1	−1
RES4													1	1
RES5														4

TABLE 5

SUMMARY OF RESULTS FOR THE CONJUGATE COMPLEX (USERS)

q	*Q*														
7	1											(IND2)			
6	5				(RES2)	(DOM1)			(MUNI)	(AG2)		(IND2	, AG1)		
5	6		(DOM2)		(RES2	, DOM1)		(AG3)	(MUNI)	(AG2	, RES3)	(IND2	, AG1)		
4	6	(RES5)	(DOM2)	(IND3)	(RES2	, DOM1	,IND1)	(AG3)	(MUNI	, AG2	, RES3	, IND2	, AG1)		
3	1	(RES5	, DOM2	, IND3	, RES2	, DOM1	,IND1	, AG3	, MUNI	, AG2	, RES3	, IND2	, AG1	, RES1)	
2	1	(RES5	, DOM2	, IND3	, RES2	, DOM1	,IND1	, AG3	, MUNI	, AG2	, RES3	, IND2	, AG1	, RES1)	
1	1	(RES5	, DOM2	, IND3	, RES2	, DOM1	,IND1	, AG3	, MUNI	, AG2	, RES3	, IND2	, AG1	, RES1	, RES4)
0	1	(RES5	, DOM2	, IND3	, RES2	, DOM1	,IND1	, AG3	, MUNI	, AG2	, RES3	, IND2	, AG1	, RES1	, RES4)

$$\overset{7}{(0} \; 4 \; 5 \; 5 \; 0 \; 0 \; 0 \; \overset{1}{0}),$$

leading to a complexity of $\Psi = 2.30$, which may be interpreted physically as follows: the user-source system is less complex than the source-user system; hence it is less prone to instability.

In summary, an example of a distribution system with 12 origins and 14 destinations has been presented, and the structure of this system has been scrutinized by means of Q-analysis. A set of indices has been defined from the Q-analysis, namely:

(1) the structure vector $Q(i)$, indicating the connectivity level between sources,

(2) the eccentricity, measuring the level of integration of a given element into the distribution system or complex,

(3) the pattern, accounting for possible changes of links in the network,

(4) the obstruction vector, identifying critical links or bottlenecks,

(5) the complexity, yielding, indirectly at least, a measure of the system stability.

By means of these indices, the Q-analysis technique enables one to examine both the supply and the user viewpoint, to identify sources or destinations that are not well integrated into the network, and hence to pinpoint elements of the system that may require attention.

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