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Q-analysis for modeling and decision making

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Abstract

This paper provides an overview of existing and potential applications of a system–theoretic approach called Q-analysis, using the examples of design and analysis of expert systems in medical image processing and analysis: namely the organization of a histopathologic knowledge base. Q-analysis is also applied to a multicriterion decision-making (MCDM) problem using a method called multicriterion Q-analysis (MCQA). A brief discussion of the advantages and limitations of Q-analysis is given, with suggestions for further applications.

Keywords: Computerized image analysis; Expert systems; Knowledge base construction; Multicriterion analysis; Q-analysis

1. Introduction

The purpose of this paper is to provide an overview of existing and potential applications of a system–theoretic approach, referred to as Q-analysis, using the design and analysis of expert systems in medical image processing and analysis to illustrate the methodology. Specifically, the approach deals with the organization of a histopathologic knowledge base; the specific knowledge base is a component of the histopathological and cytopathological diagnostic expert system developed at the University of Arizona, which has three subsystems or modules (Bartels et al., 1984). The first module guides the dynamic reconfiguration of processor elements in a multiprocessor computer system; the second module uses prior knowledge to guide the scene or image decomposition and the extraction of diagnostic information; the third module uses a rule-based procedure to obtain a classification of the available information for diagnostic assessment.

A specific example of Q-analysis of a knowledge base for the third module is given in Duckstein et al. (1988). In that example, expert opinion provided, for colonic sections, four diagnostic categories and 19 diagnostic clues; six of the clues concern overall tissue architecture, six concern characteristics of individual glands, and seven concern characteristics of nuclei (Paplanus

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et al., 1985). The technique of Q-analysis is applied to structuring the set of diagnostic clues with respect to the set of diagnostic categories. The first part of this paper describes Q-analysis in the context of a simplified example in histopathology. In the second part, Q-analysis is used in a multicriterion decision-making method and is applied to an example. A brief discussion of the advantages and limitations of the Q-analysis technique is also given, with suggestions for further applications for instance in developing expert systems for cytopathological and histopathological diagnosis.

2. Description of the Q-analysis technique

2.1. History

Q-analysis was originally developed by Atkin (1974, 1977), as an approach for studying the structural characteristics of social systems in which two sets of indicators, features, or characteristics are related to each other. Subsequently, Q-analysis has been applied in such diverse areas as chess (Atkin and Witten, 1975), flexible manufacturing systems (Robinson and Duckstein, 1986), sports events (Gould and Gatrell, 1980), and urban planning (Beaumont, 1984). Q-analysis is recognized as a useful tool in ecological studies, for example, in the evaluation of lake ecosystems (Casti et al., 1979) and in studying predator–prey relationships (Casti, 1979). Q-analysis has also been used in clinical psychology (Macgill and Springer, 1984), geology (Griffiths, 1983), transportation (Johnson, 1976), water distribution (Duckstein, 1983) and in a number of other contexts (Casti, 1979).

Q-analysis has proved especially useful in solving problems involving complex systems such as those generated by medical image processing. The technique requires a rigorous definition of data sets and their relations and encourages the investigation of the consequences of connectivity within the system. Q-analysis involves relatively simple calculations, once the approximate sets are defined and their relationships are assessed, no further information about the system is needed. The technique of Q-analysis provides an algebraic topological framework for data reduction that facilitates a macroscopic conceptualization of the systems. For this purpose, indices such as connectivity level, eccentricity and complexity can be defined and interpreted. The Q-analysis technique also provides ordering information: in our example, the members of the clue set can be ordered with respect to the members of the set of diagnostic categories. Q-analysis can be coupled with the analysis of dynamic patterns supported by the structural framework (called backcloth); this type of study (called traffic) is based on a discipline generally referred to as polyhedral dynamics (Casti et al., 1979; Johnson, 1981).

2.2. Data and incidence matrices

2.2.1. Matrix construction

In order to apply a Q-analysis approach to the study of the relationships between two finite sets, information concerning the interactions between elements of the sets is given in a data matrix. The application of Q-analysis will be demonstrated on an illustrative example using a data matrix $\mathbf{A}$, shown in Example 1, where the d_i's, $i = 1, \ldots, 4$, represent diagnostic categories such as normal, mild, severe and extreme while the c_j's, $j = 1, \ldots, 5$, represent diagnostic clues such as clinical observations. The matrix $\mathbf{A}$ represents values of the diagnostic clues for the diagnostic categories. The numbers in $\mathbf{A}$ correspond to either frequency data or subjective evaluations such as a grade between 0 and 20 of clue j for the diagnostic category i. The Q-analysis algorithm is given in Appendix A.

Example 1. A data matrix for the values of the five clues with respect to four diagnostic categories.

$$\mathbf{A} = \begin{array}{c} \\ d_1 \\ d_2 \\ d_3 \\ d_4 \end{array} \begin{array}{c} \begin{array}{ccccc} c_1 & c_2 & c_3 & c_4 & c_5 \end{array} \\ \begin{bmatrix} 12 & 13 & 1 & 12 & 20 \\ 7 & 4 & 2 & 15 & 11 \\ 1 & 0 & 9 & 0 & 0 \\ 7 & 4 & 11 & 13 & 16 \end{bmatrix} \end{array}$$

Formally, the sets of $\mathbf{A}$ are $\mathbf{D} = \{d_1, d_2, d_3, d_4\}$, the diagnostic categories and $\mathbf{C} = \{c_1, c_2, c_3, c_4, c_5\}$, the diagnostic clues.

The data matrix, **A**, can be mapped into an incidence matrix **B** through application of a mapping function defined by a so-called slicing or threshold parameter θ. In the present example, the mapping is defined as

$$b_{ij} = \begin{cases} 1 & \text{if } a_{ij} \geqslant \theta, \\ 0 & \text{otherwise,} \end{cases}$$

where b_{ij} is the element in the ith row and jth column of the incidence matrix (zero or one) and a_{ij} is its counterpart in the data matrix. Using $\theta = 7$, the resultant incidence matrix **B** is shown in Example 2.

Example 2. Incidence matrix corresponding to a slicing of the data matrix of Example 1, with $\theta = 7$.

$$\mathbf{B} = \begin{array}{c} \\ d_1 \\ d_2 \\ d_3 \\ d_4 \end{array} \begin{array}{c} \begin{array}{ccccc} c_1 & c_2 & c_3 & c_4 & c_5 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array}.$$

The elements of the incidence matrix represent a binary relationship, λ, between the sets **D** and **C**. Thus, the incidence matrix can also be defined in terms of b_{ij}, where

$$b_{ij} = \begin{cases} 1 & \text{if } d_i \text{ is } \lambda - \text{related to } c_j, \\ 0 & \text{otherwise.} \end{cases}$$

The element $b_{ij} = 1$ if and only if the ith element of the set **D** interacts with the jth element of the set **C**.

2.2.2. Geometrical representation

The multidimensional features of the system are defined by the simplicial complex $K_D(\mathbf{C}, \lambda)$, represents the elements of the set **D** as simplices $\sigma_p(d_i)$ and the elements of the set **C** as vertices. The simplices of the complex are geometrical figures representing the relations found in the incidence matrix. For example, a three-dimensional simplex is a tetrahedron, whereas a single-dimensional simplex consists of a line segment. The end points of the line segment in these figures are the vertices. The conjugate complex represents a transposition of the simplices and the vertices.

To illustrate the formation of the complex, a relation $\lambda : \mathbf{D} \times \mathbf{C}$ can be specified such that each diagnostic category is a simplex defined by the clues that are the vertices. Diagnostic categories, which share a value or attraction characteristic relative to the clues, are connected to form the simplicial complex $K_{\mathrm{D}}(\mathbf{C}, \lambda)$.

2.2.3. Dimensionality and q-connectivity

Each simplex $\sigma_p(d_i)$ of $K_D(\mathbf{C}, \lambda)$ is λ-related to a number of vertices. In Example 1, diagnostic category d_1 is described by four clues, so that the simplex representing category d_1, $\sigma_3(d_1)$ is λ-related to four vertices. By convention, p, the dimension of the simplex is indicated by a subscript and the simplex is named by the represented set element in parentheses (Johnson, 1981). The dimension of the simplex is equal to the number of related vertices minus one. Thus $\sigma_3(d_1)$ in Example 1 is a three-dimensional simplex for diagnostic category d_i.

Whereas dimensionality is indicative of the relationship between simplices and vertices, q-connectivity is a measure of the relationship between simplices with respect to shared vertices. In this sense, it measures the strength of the connection between the elements of **A** and those of **D**. The following definition of q-connectivity is taken from Atkin (1974).

The simplices $\sigma(p)$ and $\sigma(r)$ in the simplicial complex K, $\sigma(p)$ and $\sigma(r)$ are said to be q-connected in K, if and only if there exists a finite sequence of simplices $\{\sigma(\alpha_1), \ldots, \sigma(\alpha_n)\}$ such that:

1. $\sigma(p)$ is a face of $\sigma(\alpha_1)$,
2. $\sigma(\alpha_n)$ is a face of $\sigma(r)$,
3. $\sigma(\alpha_i)$ and $\sigma(\alpha_{i+1})$ have a common face (share a face) of dimension β_i, for $i = 1, \ldots, n$,
4. $q = \min\{p, \beta_1, \beta_2, \ldots, \beta_n, r\}$.

The interpretation of this definition is that the q-connectivity between a subset of the diagnostic category set is measured by the weakest relation (smallest number of clues shared) between any two consecutive d_i's in the chain $d_1, \ldots, d_n$. The simplicial relation described by q-connection is an equivalence relation, that is, a symmetric, reflective and transitive relation. This property is very useful

in eliminating redundant members of either the set of diagnostic clues or the set of diagnostic categories since one element can be exactly replaced by an equivalent one.

The concept of q-connection can be illustrated by continuing Example 1. Diagnostic categories d_1 and d_2 are described by three common clues. Therefore, simplices $\sigma_3(d_1)$ and $\sigma_2(d_2)$ are two-connected. If simplices $\sigma_3(d_1)$ and $\sigma_2(d_2)$ are q-connected, they are also connected at all lower levels, that is, they are $(q-r)$-connected, where $r = 1, \ldots, q$. Both dimensionality and q-connectivity, which are unique to Q-analysis, provide information and insight concerning the structure of a medical imagery knowledge base.

2.3. Structure vectors

For each dimension q of the complex K, we define the integer Q_q as the number of distinct equivalence classes, where each equivalence class is composed of q-connected simplices. This vector of Q_q's is the basis for an example illustrating the simplification that may be achieved through elimination of redundant clues in the same equivalence class simplices. The first structure vector, $\mathbf{Q}$, then describes the structure of simplicial connectivity,

$$\mathbf{Q} = (Q_{(\dim K)}, Q_{(\dim K-1)}, \ldots, Q_0).$$

The structure vector $\mathbf{Q}$, which is the immediate result of Q-analysis, can be used for gaining additional insight into the relationship between diagnostic clues and diagnostic categories. In Example 1, the dimension of the complex is 3 (see Example 2). However, the three-dimensional simplices, $\sigma_3(d_1)$ and $\sigma_3(d_4)$ share only vertices c_1, c_4 and c_5. Thus, $\sigma_3(d_1)$ and $\sigma_3(d_4)$ are only two-connected and the structure vector has two equivalence classes at dimension three, with the membership of each equivalence class represented by a single simplex.

Carrying out these calculations for each of the dimensions three through zero provides the first structure vector for Example 1,

$$\mathbf{Q} = \begin{matrix} 3 & 2 & 1 & 0 \\ (2 & 1 & 1 & 1), \end{matrix}$$

where the dimension is indicated above the dimension value.

A second structure vector, $\mathbf{P}$, can be defined as

$$\mathbf{P} = (P_{(\dim K)}, P_{(\dim K-1)}, \ldots, P_0),$$

where P_q is the number of simplices in the complex K with dimension greater than or equal to q (Johnson, 1978). The second structure for Example 1 is

$$\mathbf{P} = \begin{matrix} 3 & 2 & 1 & 0 \\ (2 & 3 & 3 & 4), \end{matrix}$$

where $\mathbf{P}$ indicates the frequency with which the simplices (diagnoses) are connected to the vertices (clues). The larger the values of $\mathbf{P}$ for higher dimension, the greater the connectivity. Conversely, vector $\mathbf{Q}$ indicates the extent of connectivity among the simplices relative to connectivity with the set of vertices.

Atkin (1974) gives an algorithm for performing Q-analysis to produce the structure and obstruction vectors. A concise description of this algorithm is provided in Duckstein (1983) and Featherlike and Duckstein (1986) and is provided in Appendix A.

2.4. Obstruction vector

Another vector generated by Q-analysis, the obstruction vector $\mathbf{Q}^*$, delineates restriction of information flow through the complex. $\mathbf{Q}^*$ can be defined by noting that the member simplices (diagnostic categories) within each equivalence class at dimension k may interact directly or indirectly at the kth level. The number of obstacles to interaction at dimension k is the number of "gaps" between equivalence classes. Therefore, $\mathbf{Q}^*$ is derived by subtracting a vector $\mathbf{U}$, consisting of all ones, from the structure vector. Thus, $\mathbf{Q}^* = \mathbf{Q} - \mathbf{U}$. The value of Q_k^* indicates the number of structural constraints to simplicial interaction at dimension k. Depending on the problem, high or low values of the elements of $\mathbf{Q}^*$ may be preferable. For example, it is preferable to have high

obstruction between diagnostic categories, so that they may be distinguished easily. The structure vectors, obstruction vector and equivalence classes at each q-level with slicing parameter $\theta = 7$ for Example 1 are shown in Table 1.

2.5. Eccentricity

Whereas the structure vectors and the obstruction vector describe global structural properties, eccentricity indicates the degree of integration of a specific simplex into the whole complex. The conventional measure of eccentricity for a simplex as defined in Casti et al. (1979) is denoted as ecc. Chin et al. (1991) suggest another measure of eccentricity called ecc′.

$$\mathrm{ecc}(\sigma) = \frac{\hat{q} - q^*}{q^* + 1} \tag{1}$$

and

$$\mathrm{ecc}'(\sigma) = \frac{2\sum_i q_i/\sigma_i}{q_{\max}(q_{\max}+1)}, \tag{2}$$

where $\hat{q}$ is the dimension of the simplex σ, q^* the highest dimension at which σ joins another simplex in an equivalence class, q_i each q-level where σ appears, σ_i the number of elements in σ_i's equivalence class at level q_i and $q_{\max}$ the maximum q-level of the complex. The difference $(\hat{q} - q^*)$ is a measure of the extent to which σ shares vertices with the simplex most highly connected with it. Therefore, ecc depends upon only a single simplex other than σ while ecc′ depends upon all the other simplices. Furthermore, ecc takes on values in $[0, \infty]$ and ecc′ in $[0, 1]$.

For the simplex $\sigma_3(d_1)$ of Example 1, the dimension is three and the highest dimension at which $\sigma_3(d_1)$ joins another simplex $\sigma_3(d_4)$ in an equivalence class is two. Thus, from Eqs. (1) and (2),

$$\mathrm{ecc}(\sigma_3(d_1)) = (3-2)/3 = 1/3;$$

$$\mathrm{ecc}'(\sigma_3(d_1)) = 2\left(3 + \frac{2}{3} + \frac{1}{3}\right)\Big/(3 \cdot 4) = 2/3$$

and

$$\mathrm{ecc}(\sigma_2(d_2)) = (2-2)/3 = 0;$$

$$\mathrm{ecc}'(\sigma_2(d_2)) = 2\left(\frac{2}{3} + \frac{1}{3}\right)\Big/(3 \cdot 4) = 1/6.$$

These eccentricities indicate that $\sigma_2(d_2)$ conforms to the overall structure of the complex better than $\sigma_3(d_1)$ does. If a simplex is isolated, $q^* = -1$, then the ecc is infinite. Here again, a high eccentricity value or a low eccentricity value may be preferable, depending upon the set considered. A highly eccentric diagnostic category is easy to identify; in contrast, a highly eccentric clue is useful for identifying a given diagnostic category but not for distinguishing between the other categories. The eccentricities of each diagnostic category with slicing parameter $\theta = 7$ for Example 1 are shown in Table 2.

2.6. Complexity

The results of Q-analysis can also be used to describe the complexity of the system structure. Numerous definitions of system complexity can be found in the literature; the appropriate definition depends on the type of problem considered. Complexity is discussed in this section in the context of comparing elements of the systems consisting of sets of diagnostic clues and diagnostic categories.

Table 1
The structure vectors, obstruction vector and equivalence classes at each q-level with slicing parameter $\theta = 7$ for Example 1

q	Q	P	Q^*	Equivalence classes
3	2	2	1	$\{d_1\}, \{d_4\}$
2	1	3	0	$\{d_1, d_2, d_4\}$
1	1	3	0	$\{d_1, d_2, d_4\}$
0	1	4	0	$\{d_1, d_2, d_3, d_4\}$

Table 2
Eccentricities of each diagnostic category with slicing parameter $\theta = 7$ for Example 1

σ	ecc	ecc′
d_1	1/3	2/3
d_2	0	1/6
d_3	0	0
d_4	1/3	2/3

Casti (1979) lists five axioms to be satisfied by any measure of complexity; the following three apply for polyhedral complexity.

1. A system consisting of one simplex has a complexity of one.
2. A subcomplex has a complexity no greater than that of the entire complex.
3. The combination of two complexes to form a new complex results in a level of complexity not greater than the sum of the complexities of the two components.

The complexity measure $\Psi(K)$ suggested by Casti (1979) is

$$\Psi(K) = 2\left[\sum_{k=0}^{\dim K}(k+1)Q_k/(\dim K+1)(\dim K+2)\right], \quad (3)$$

where Q_k is the kth component of the structure vector $\mathbf{Q}$. The measure $\Psi(K)$ satisfies the stated axioms. Explicit in $\Psi(K)$ is the recognition that both the dimensionality and the number of equivalence classes are related to the complexity of the structure. For Example 1,

$$\mathbf{Q} = \begin{matrix} 3 & 2 & 1 & 0 \\ (2 & 1 & 1 & 1) \end{matrix}$$

with dim $K = 3$. Thus, Eq. (3) yields,

$$\Psi(K) = 2[(1+2+3+8)/(4)(5)] = 1.4$$

As discussed in Duckstein et al. (1988), this complexity criterion does not appear to be of immediate use in the diagnostic clue/category problem. On the other hand, it may prove relevant for the purpose of comparing, for example, two microprocessor configurations. In fact, a tradeoff may be possible between some measure of complexity and the values of the **Q**- or **P**-structure vector components (Section 2.3), the obstruction vector (Section 2.4) or a stability index (Section 2.7).

2.7. Stability

Like complexity, stability has proved to be a difficult concept to apply (May, 1974). Svirezhev and Logofet (1983) discuss the controversy concerning the relationship of stability to complexity in the context of ecological systems. As pointed out in Casti et al. (1979), the Q-analysis technique appears to be appropriate for determining structural stability, i.e., the effect of perturbations on the structure vector **Q**.

Attention has been focused on two aspects of relative stability: the resistance of a system to displacement, also referred to as vulnerability, and the manner in which a displaced system returns to a reference state, that is, resilience (Patten and Witkamp, 1967; Holing, 1973; Duckstein et al., 1987). Q-analysis is particularly appropriate as a framework for developing techniques for determining the degree of resistance to displacement, which may be analyzed as follows. The resistance to changes of the structure vector **Q** depends on the resistance of the incidence matrix to changes in the data matrix. With a fixed slicing parameter, the incidence matrix is modified if a particular observation of a datum is increased over the threshold θ, for example, if it is found that the value (or frequency) of a given clue c_j for a selected diagnostic category d_i should be increased. Such a change in the incidence matrix may alter **Q**. The structural sensitivity to change can then be defined as the ease of displacement of a system structure due to perturbation of any of the relations which comprise that structure.

Alterations of the structure vector **Q** differ in their impact upon the system structure. In general, the lower the dimension of a modification of **Q**, the larger the number of simplices involved in the change and the more profound the effect upon system configuration. In the extreme, changes in **Q** at dimension zero may split or coalesce the system. Thus, a measure of sensitivity should depend upon the structural impact of changes in **Q**.

A method of calculating structural sensitivity with respect to **Q** is presented in Featherlike and Duckstein (1986), and a unitless measure of sensitivity is proposed as a basis for comparing structures of different sizes or scales. Stability is measured as sensitivity, rather than resistance, in order to include impact in the measure.

Given any data, the contribution to structural sensitivity can be attributed to any of three factors:

(1) weakness of connectivity between the simplices and vertices; (2) potential for the evolution of new associations; and (3) noise in the system, that is random errors resulting in imprecise measurements of the elements of the data matrix (Example 1) and, possibly, in modeling errors.

Weakness of connectivity in the only cause of sensitivity when an observation is greater or equal to θ. When an observation is smaller than θ but greater than zero, both noise and evolutionary potential contribute to sensitivity. The greater the ratio of the observation to θ, the greater the contribution of the noise. If the observation is zero, no noise occurs. An indication of the importance of noise in structural sensitivity is derived in Featherlike and Duckstein (1986).

2.8. Relationship between structural measures

The value of the (ij)th element of the data matrix indicates the strength of the relationship between, say, d_i and c_j. The mean interaction strength of a simplex is defined as the average value of the interactions of that simplex with the vertical set. A positive difference between an observation in the data matrix and the slicing parameter is a direct measure of the strength of the interaction between an observation in the data matrix; the slicing parameter is a direct measure of the strength of interaction between the respective simplex and vertex. The greater the strength of the interaction, the less sensitive the resultant vector is to perturbation in the data. For any simplex, various slicing parameters can be chosen to produce different interaction strengths without changing the dimensionality of the simplex.

A negative difference between an observation in the data matrix and the slicing parameter is interpreted as a representation of noise level, rather than as a measure of strength of interaction. Noise, as defined earlier, is an interference with the binary relationship upon which the structure of the complex is based. The slicing parameter acts as a filter, specifying the level at which an observation represents a simplex–vertex (or clue–diagnostic category) interaction. Data values smaller than this threshold are assumed to represent noise in the system. An increase in noise decreases the difference between the observed data and the threshold, consequently increasing the sensitivity of $\mathbf{Q}$.

Sensitivity is dependent upon dimensionality and q-connectivity, characteristics of $\mathbf{Q}$ that are the constraining parameters of the dynamic patterns sustained by the relational structure. Such patterns describe the dynamic behavior of the system, as discussed in some detail below.

2.9. Traffic and backcloth

The indicators discussed thus far reflect structural properties that generally change rarely or slowly over the lifespan of a system. Thus, these indicators would be useful primarily for a production-type expert system with an open loop property. On the other hand, various dynamic entities operate across the framework that these measurements describe. Atkin (1974, 1978), introduced the term "backcloth" and "traffic" to describe, respectively, the relatively static and relatively dynamic aspects of a system under study.

Atkin envisions the backcloth as setting a stage for the flow of traffic, the role the backcloth plays in the theater. The backcloth can exist without the traffic, but traffic requires the backcloth for support. The traffic, which introduces an adaptive feature (or closed loop mode of operation) into the system, consists of the behavioral characteristics of entities operating on the multidimensional structure of that system. For example, the "traffic" may be the evolution of a cancer under therapy, the backcloth being the data matrix representing a typical patient.

The traffic on the complex K is the behavior associated with K that is defined on the simplices of the complex, it can be described by a graded set function

$$\Pi = \Pi^0 \oplus \Pi^1 \oplus \cdots \oplus \Pi^N, \quad N = \dim K,$$

referred to as the pattern of the traffic. Each function Π^k maps the set of k-dimensional simplices into a specified number domain. The pattern Π^k is restricted to the k-dimensional simplices of K. The subgroups of Π can be weighted to account for peculiarities of the traffic. A change in the

pattern Π indicates a flow of traffic from one simplex to another, for example, from one diagnostic category to another. Traffic transmission across common faces of simplices at dimension k requires a $(k+1)$-connection between the simplices. The number of distinct $(k+1)$-connected components in K determines the extent of free changes in Π^k and Π.

The obstruction vector is an indicator of the gaps that occur in K, causing an obstruction to free change in the pattern Π. The greater the number of equivalence classes of simplices at dimension k, the larger the level k in $\mathbf{Q}^*$ and consequently, the more obstacles there are to the free flow of traffic. These obstacles, of course, are related to the basic geometry of the complex K. Other constraints to flow may result from specific characteristics of the pattern.

A change in pattern, $\Delta\Pi$, from dimension level k to $k+1$ results from a force in K. Thus the change, $\Delta\Pi$, may be attributable to a force directed toward (attractive k-force) or away from (repulsive k-force) a specific simplex.

The potential of Q-analysis to describe the way in which traffic and structure interact distinguishes this technique from similar techniques such as, for example, the single-link method of cluster analysis (Macgill, 1984). In particular, the description of dynamic patterns may be of considerable importance in histopathology, where system configuration determines the dynamic interplay between tissue, glands and cells.

3. Multicriterion decision making using Q-analysis

3.1. Q-analysis

Multicriterion Q-analysis (MCQA) has been used as a multicriterion decision-making (MCDM) method by Pfaff and Duckstein (1981), Hiessl et al. (1985) and Chin et al. (1991) to evaluate and select the "best" alternative or project. A data matrix $\mathbf{A}$ and a vector $\mathbf{w}$ are assumed to be given. Element a_{ij} in the matrix $\mathbf{A}$ rates alternative i for criterion j. In a medical environment, alternatives could be different types of treatment such as major surgery, drugs, radiation, diet change, nontreatment, etc. The vector $\mathbf{w}$ contains the criterion weights where a greater weight means a more important criterion. These weights could be the probabilities of recovery based on diagnostic clues.

MCQA I uses a project satisfaction index (PSI) and a project comparison index (PCI) while MCQA II also includes a project discordance index (PDI). PSI is a value or utility based index while PCI and PDI are so-called outranking types of indices. The PSI of an alternative is independent from the other alternatives while the PCI and PDI of an alternative are dependent on the criterion values for the other alternatives. Most MCDM problems have a weight on each criterion which are also used in the MCQA methods.

MCQA can accept nonnumerical scales for the criteria but it is better computationally to quantify all the ratings. Therefore, before starting MCQA, the values for each criterion should be quantified and normalized (usually linearly but not necessarily so) with the most desirable value for each criterion being one and the least desirable being zero. For example, the following numbers are assigned to the standard school grading system: (1), A-excellent; (0.75), B-good; (0.5), C-satisfactory; (0.25), D-poor; and (0), F-fail. For linearly normalizing a_{ij} into P_{ij} corresponding to alternative i and criterion j, when a higher value is more desirable (i.e. profit, power, speed), define:

$$p_{ij} = \frac{a_{ij} - \min_k a_{kj}}{\max_\ell a_{\ell j} - \min_k a_{kj}},$$

alternatively, when a lower value is more desirable (i.e. cost, distance, waiting time) then use

$$p_{ij} = \frac{\max_\ell a_{\ell j} - a_{ij}}{\max_\ell a_{\ell j} - \min_k a_{kj}}.$$

The preference matrix $\mathbf{P}$ rates each row of alternatives d_i with each column of criteria c_j with these quantified and normalized values. If $\max_\ell a_{\ell j} = \min_k a_{kj}$ for criterion c_j, then that criterion is deleted from the analysis since it has no role in ranking the alternatives.

Example 3. Ten portable microprocessors with the same major features (processor type, CD-ROM, color video screen, input/export ports, etc.) are evaluated using eight criteria. The criteria are price

(dollars), weight (grams), machine speed (hertz), video resolution (pixels), hard disk memory (bytes), dynamic memory (bytes), battery life (seconds) and battery recharging time (seconds). Let $\mathbf{w} = (4, 3, 4, 2, 2, 3, 2, 1)$ with w_j being the weight on criterion c_j. The so-called preference matrix is given in Table 3.

Now Q-analysis is applied to the preference matrix $\mathbf{P}$ where the alternatives are the simplices and criteria, the vertices. Several slicing parameters are used in this Q-analysis. In Example 3, ten slicing parameters are used in uniform intervals of one-tenth. The element of incidence matrix $\mathbf{B}$ at slicing level k which corresponds to slicing parameter $\frac{1}{10}k$ is formed as follows:

$$b_{ij}^k = \begin{cases} 1 & \text{if } p_{ij} \geqslant \theta^k, \\ 0 & \text{otherwise.} \end{cases}$$

One of the ten $\mathbf{B}^k$'s ($k = 5$) corresponding to Table 3 is shown in Table 4.

The PSI for alternative d_i shows how well d_i satisfies the criteria and is defined as

$$\mathrm{PSI}_i = \sum_{j,k} \theta^k \, w_j \, b_{ij}.$$

After computing the PSI for all the alternatives, PSI_i is then normalized,

$$\mathrm{PSIN}_i = \frac{\mathrm{PSI}_i}{\max_\ell \mathrm{PSI}_\ell}.$$

PSI's and PSIN's for Example 3 are listed in Table 9.

PCI ranks the alternatives by comparing q-connectivity. $\mathbf{S}^k$ in Eq. (4) is used to find the number of common satisfied criteria between the alternatives and then the equivalence classes and q-connectivity can easily be found among the alternatives by using the definitions found in Section 2.2.3. Also, the structure vectors and obstruction vector defined in Sections 2.3 and 2.4 can be found. The resultant $\mathbf{S}^k$, the structure vectors, obstruction vector

Table 3
The preference matrix $\mathbf{P}$ for the ten alternatives and the eight criteria in Example 3

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8
d_1	0.80	0	0.12	0	0.44	0	0.17	0
d_2	0.27	0	0.71	1	0.32	0	0.17	0.33
d_3	1	1	0.53	1	1	0	0	0
d_4	0.13	0	0.94	0	0.24	1	0.17	0.40
d_5	0.80	0	0.35	0	0	1	0	0
d_6	0.53	1	0	1	1	1	0.33	0
d_7	0.07	0	0.53	1	0.32	1	0.33	0.42
d_8	1	0	0.76	0	0.12	1	0	0.43
d_9	0.33	0	0	0	0.44	0	1	0.33
d_{10}	0	0	0.76	0	0.24	1	1	1

Table 4
Incidence matrix $\mathbf{B}^5$ for Example 3 when $\theta^5 = 0.50$

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8
d_1	1	0	0	0	0	0	0	0
d_2	0	0	1	1	0	0	0	0
d_3	1	1	1	1	1	0	0	0
d_4	0	0	1	0	0	1	0	0
d_5	1	0	0	0	0	1	0	0
d_6	1	1	0	1	1	1	0	0
d_7	0	0	1	1	0	1	0	0
d_8	1	0	1	0	0	1	0	0
d_9	0	0	0	0	0	0	1	0
d_{10}	0	0	1	0	0	1	1	1

Table 5
The S^5 matrix when $\theta^5 = 0.50$ for Example 3

	d_1	d_2	d_3	d_4	d_5	d_6	d_7	d_8	d_9	d_{10}
d_1	0	−1	0	−1	0	0	−1	0	−1	−1
d_2	−1	1	1	0	−1	0	1	0	−1	0
d_3	0	1	4	0	0	3	1	1	−1	0
d_4	−1	0	0	1	0	0	1	1	−1	1
d_5	0	−1	0	0	1	1	0	1	−1	0
d_6	0	0	3	0	1	4	1	1	−1	0
d_7	−1	1	1	1	0	1	2	1	−1	1
d_8	0	0	1	1	1	1	1	2	−1	1
d_9	−1	−1	−1	−1	−1	−1	−1	−1	0	0
d_{10}	−1	0	0	1	0	0	1	1	0	3

and equivalence classes for the simplices when using Table 4 are shown in Tables 5 and 6.

$$\mathbf{S}^k = \mathbf{B}^k\left(\mathbf{B}^k\right)^{\mathrm{T}} - \mathbf{e}^{\mathrm{T}}\mathbf{e},$$
$$\text{where } \mathbf{e} = (1,1,1,1,1,1,1,1,1,1). \tag{4}$$

The PCI for alternative d_i is defined as

$$\mathrm{PCI}_i = \sum_k \theta^k \left[\hat{q}_{ik} - q^*_{ik}\right],$$

where

$$\hat{q}_{ik} = s^k_{ii} \text{ and } q^*_{ik} = \max_{j|j\neq i} s^k_{ij},$$

$\hat{q}$ is the dimension of simplex d_i and q^* is the highest dimension at which simplex d_i joins another simplex in an equivalence class.

After computing the PCI for all the alternatives, PCI_i is then normalized,

$$\mathrm{PCIN}_i = \frac{\mathrm{PCI}_i}{\max_\ell \mathrm{PCI}_\ell}$$

Table 6
First and second structure vectors and the obstruction vector of the simplices from the equivalence classes at slicing level $\theta^5 = 0.5$ for Example 3

q	Q	P	Q^*	Equivalence classes
4	2	2	1	$\{d_3\}$, $\{d_6\}$
3	2	3	1	$\{d_3,d_6\},\{d_{10}\}$
2	1	5	0	$\{d_3,d_6\},\{d_7\},\{d_8\},\{d_{10}\}$
1	1	8	0	$\{d_2,d_3,d_4,d_5,d_6,d_7,d_8,d_{10}\}$
0	1	10	0	All

The PCI's and PCIN's for Example 3 are listed in Table 9.

The eccentricities after Eqs. (1) and (2) are given as

$$\mathrm{ecc}_{ik} = \frac{\Delta q_{ik}}{\max_{j|j\neq i}\, s^k_{ij} + 1}$$

and

$$\mathrm{ecc}'_{ik} = \frac{2\sum_\ell \left[q_{i\ell k}/\sigma_{i\ell k}\right]}{\left[\max_j\, q_{j\ell k}\right]\left[\max_j\, q_{j\ell k} + 1\right]},$$

where

$$q_{j\ell k} = \begin{cases} \ell & \text{if } j \text{ belongs to an equivalence class} \\ & \text{at } q-\text{level } \ell \text{ at slicing level } k, \\ 0 & \text{otherwise} \end{cases}$$

and $\sigma_{i\ell k}$ is the number of elements in the equivalence class containing d_i at q-level ℓ for slicing level k. Table 7 shows the eccentricities corresponding to Table 5.

Table 7
Eccentricities of the alternatives at slicing level $\theta^5 = 0.5$

	ecc	ecc′
d_1	0	0
d_2	0	0.01
d_3	0.25	0.66
d_4	0	0.01
d_5	0	0.01
d_6	0.25	0.66
d_7	0.50	0.21
d_8	0.50	0.21
d_9	0	0
d_{10}	1.00	0.51

Table 8
Matrix $\mathbf{S}^{5*}$ when $\theta^5 = 0.50$ for Example 3

	d_1	d_2	d_3	d_4	d_5	d_6	d_7	d_8	d_9	d_{10}
d_1	6	4	2	4	5	2	3	4	5	2
d_2	4	5	2	4	3	1	4	3	4	2
d_3	2	2	2	1	1	1	1	1	1	−1
d_4	4	4	1	5	4	1	4	4	4	3
d_5	5	3	1	4	5	2	3	4	4	2
d_6	2	1	1	1	2	2	1	1	1	−1
d_7	3	4	1	4	3	1	4	3	3	2
d_8	4	3	1	4	4	1	3	4	3	2
d_9	5	4	1	4	4	1	3	3	6	3
d_{10}	2	2	−1	3	2	−1	2	2	3	3

PDI ranks the alternatives by comparing discordance q-connectivity. $\mathbf{S}^{k*}$ in Eq. (5) is used to find the number of common dissatisfied criteria between alternatives and then the discordance equivalence classes and q-connectivity can be found by using the same method as applied to finding PCI. The resultant $\mathbf{S}^{k*}$ is shown in Table 5, with

$$\mathbf{S}^{k*} = \mathbf{B}^{k*}(\mathbf{B}^{k*})^{\mathrm{T}} - \mathbf{e}^{\mathrm{T}}\mathbf{e}, \tag{5}$$

where

$$b_{ij}^{k*} = \begin{cases} 1 \text{ if } b_{ij}^{k} = 0, \\ 0 \text{ otherwise} \end{cases}$$

and $\mathbf{e} = (1,1,1,1,1,1,1,1,1,1)$.

The matrix $\mathbf{S}^{5*}$ when $\theta^5 = 0.50$ for Example 3 is shown in Table 8.

The PDI for alternative d_i is defined similar to the PCI,

$$\mathrm{PDI}_i = \sum_k \theta^k \left[\hat{q}_{ik} - q_{ik}^*\right],$$

where

$$\hat{q}_{ik} = s_{ii}^{k*} \text{ and } q_{ik}^* = \max_{j|j\neq i}\left[s_{ij}^{k*}\right].$$

After computing the PDI for all the alternatives, PDI_i is then normalized,

$$\mathrm{PDIN}_i = \frac{\mathrm{PDI}_i}{\max_\ell \mathrm{PDI}_\ell}.$$

The PDI's and PDIN's for Example 3 are listed in Table 9.

3.2. Ranking the alternatives

Two methods are used to rank the alternatives, as shown below: MCQA I uses PRI1(project rating index 1) and MCQA II uses PRI2. These PRI's are defined as follows:

$$\mathrm{PRI1}_i^p = \left([1 - \mathrm{PSIN}_i]^p + [1 - \mathrm{PCIN}_i]^p\right)^{1/p}$$

and

$$\mathrm{PRI2}_i^p = \left([1 - \mathrm{PSIN}_i]^p + [1 - \mathrm{PCIN}_i]^p + [\mathrm{PDIN}_i]^p\right)^{1/p}.$$

A lower value indicates a better alternative under these methods. Table 10 lists the results from using the indices listed in Table 9.

The ranking order (best to worst) for PRT1 with ties indicated in brackets is:

Table 9
The project indices for Example 3 using 10 equal slicing levels

	PSI	PSIN	PCI	PCIN	PDI	PDIN
d_1	17.0	0.31	0	0	5.0	1
d_2	23.4	0.42	1.6	0.22	3.7	0.74
d_3	55.5	1	4.5	0.61	0.1	0.02
d_4	33.7	0.61	1.8	0.24	3.5	0.70
d_5	30.0	0.55	0	0	1.1	0.22
d_6	52.2	0.94	5.1	0.69	0.1	0.02
d_7	31.9	0.57	1.8	0.24	1.3	0.26
d_8	43.9	0.79	3.6	0.49	0	0
d_9	14.0	0.25	0.9	0.12	3.1	0.62
d_{10}	38.8	0.70	7.4	1	0.1	0.02

Table 10
The project rating for Example 3 using 10 equal slicing levels

	PRT1[1]	PRT1[2]	PRT2[1]	PRT2[2]
a_1	1.69	1.22	2.69	1.58
a_2	1.36	0.97	2.10	1.22
a_3	0.39	0.39	0.41	0.39
a_4	1.15	0.85	1.85	1.10
a_5	1.45	1.10	1.67	1.12
a_6	0.37	0.32	0.39	0.32
a_7	1.18	0.87	1.44	0.91
a_8	0.72	0.55	0.72	0.55
a_9	1.63	1.15	2.25	1.31
a_{10}	0.30	0.30	0.32	0.30

$d_{10}, d_6, d_3, d_8, d_4, d_7, d_2, d_5, d_9, d_1$.

while for PRT2, it is:

$d_{10}, d_6, d_3, d_8, d_7, [d_4, d_5], d_2, d_5, d_9, d_1$

The order of preference based on the four results is:

$d_{10}, d_6, d_3, d_8, [d_2, d_4, d_5, d_7], d_9, d_1$.

4. Summary and discussion

Q-analysis appears to be helpful in organizing a knowledge base such as the one of the expert system which was developed at the University of Arizona for analyzing histopathological images, as discussed in Duckstein et al. (1988). Example 1 is a simplified version of the example in that paper, where the two sets used for Q-analysis are the diagnostic clues, set **C** (the vertices) and the diagnostic categories, set **D** (the simplices). The matrix **A** represents an indication (on an ordinal scale) of the value (or extent of usefulness) of each diagnostic clue for each diagnostic category.

The direct analysis of the complex, which considers the data from the viewpoint of the knowledge base, helps in determining a feasible number of diagnostic categories. Thus Q-analysis helps to identify the diagnostic categories which are more easily recognized.

Q-analysis of the conjugate complex provides an indication of the usefulness of the diagnostic clues. Thus, if it were possible to use only two diagnostic clues, it would be more advantageous to choose two from different equivalence classes than two from the same equivalence class.

In order to study how the simplices (diagnostic clues) conform to the complex (represented in the histologic example by the matrix given in Example 1) and to determine whether there are any simplices (clues) that are totally disconnected, the following indices have been proposed: q-connectivity, structure and obstruction vectors, eccentricity, complexity, stability, traffic and backcloth.

q-Connectivity describes the global relationship among equivalence classes. The structure and obstruction vector indicate the potential for simplifying the representation of the relationships. In the MCQA example, eccentricity expresses the extent to which a criterion stands out from other criteria, with respect to equivalence class membership. Specifically, given that a criterion first appears alone in a class, eccentricity indicates the numbers of levels for which this is the case before another criterion also appears in the class. Thus, a non-zero eccentricity is found only for a criterion that first appears alone in an equivalence class. If there is a criterion, c_j, with infinite eccentricity, then the q-connectivity of the simplicial chain that includes c_j is zero. Further discussion of the use of these indices to evaluate criteria in MCDM is found in Chin et al. (1991) and Ozelkan and Duckstein (1996).

Complexity provides a measure of the tightness of the interweaving between the elements of two sets, **C** and **D**. Complexity could be an extremely useful indicator in the automatic reconfiguration of microprocessors, as the type and duration of tasks evolve. Stability is related to complexity, and also to the sensitivity of Q-analysis results to noise. This sensitivity may be expressed as structural change, either for a constant slicing level when a disturbance occurs or when the slicing level is changed, even in the absence of a disturbance.

The previous indices represent the backcloth of the system under consideration. This backcloth may be used as a support for traffic (pattern), that is, a dynamic change of the systems structure, such as a disappearance of cancer following therapy.

Points to be investigated further thus include:

- When is a high value of the obstruction vector and/or the eccentricity desirable?
- How are the various definitions of complexity matched with the problem type and how are the results interpreted?
- What is the relationship between complexity and stability, that is, how are the sensitivity to noise or disturbance and the choice of slicing level (threshold) measured?
- How can global result combine the Q-analyses corresponding to different thresholds? Here the indices used in Pfaff and Duckstein (1981) and Hiessl et al. (1985) may be useful.
- What is backcloth and what is traffic when studying, for example, the results of therapy?
- How can a trade-off be effected between various indices provided by Q-analysis? Here, a multicriterion decision-making method with non-numerical or qualitative criteria (Goicoechea et al. (1982) or Szidarovszky et al. (1986)) may be appropriate.

To end this discussion, the step by step procedure for performing Q-analysis may be described as follows, using the two examples of histopathological expert systems: the clue–diagnostic category module and the microprocessor-task module.

- Define the set of simplices:
  Case A: Diagnostic categories **D**.
  Case B: Tasks **T**.
- Define the set of vertices:
  Case A: Clues **C**.
  Case B: Microprocessors **M**.
- Establish threshold levels, select definition of indices such as complexity and traffic.
- Perform Q-analysis and calculate the indices.
- Interpret the results and if necessary, repeat the analysis changing either the threshold or index definitions (especially the definition of complexity).

The drawbacks of Q-analysis include the following.

1. Only qualitative measures of relationships are obtained, in contrast with statistical analysis; however, statistical analysis would require replications of homogeneous sets of data.
2. The mathematical theory behind the technique is not simple; however, a deep understanding of the theory is not required for application and correct interpretation of the results.
3. There are many indices that can be used, so conventions must be adopted for future comparisons of results.
4. The interpretation of results is not always straightforward.

The advantages of Q-analysis that have been pointed out in the body of this paper (simplicity, flexibility, use in many fields, etc.) are listed in Conclusions.

5. Conclusions

1. Referring to the analysis presented in Duckstein et al. (1988), the results of applying Q-analysis to aid in organizing a knowledge base are encouraging; equivalence classes of the diagnostic clues that cut across histological categories are defined, as well as clues whose usefulness might be reexamined. The effect of the slicing parameter appears to be quite substantial, so a careful prior and on-going analysis of this aspect of the problem is necessary.
2. Q-analysis provides a unique multidimensional view of the structural relationship between two sets, here the set of clues, **C**, and the set of diagnoses, **D**. In the dynamic reconfiguration problem, the sets are tasks, **T** and microprocessors, **M**.
3. The advantages of the Q-analysis technique are as follows.
   3.1. It is simple to use, requiring only "bookkeeping" types of calculations.
   3.2. It is flexible, there is no problem in changing slicing levels or criteria definitions.
   3.3. It provides ordering on both direct and conjugate complexes; for example, q-levels, eccentricity, equivalence classes, and elements of the obstruction vector provide an order on the c_j's and d_i's.
   3.4. It is applicable to several aspects of knowledge base analysis; for example, computerized histopathological image analysis.
   3.5. It can be used in multicriterion decision-making methods as demonstrated in

Example 3, as well as many other problems involving the dynamic analysis of the structure of a binary relation.

Appendix A. The *Q*-analysis algorithm

1. Let data matrix $\mathbf{A}_{(m\times n)}$ consist of m categories(alternative) and n clues(criteria).
2. $\mathbf{P}_{(m\times n)}$ is formed by quantifying $\mathbf{A}$ and normalizing the values for each criterion.
3. $\mathbf{B}_{(m\times n)}$ is the (0-1) incidence matrix where each element is equal to one if the corresponding element in $\mathbf{P}$ is equal to or greater than a threshold parameter, else it is equal to zero.
4. $\mathbf{S}_{(m\times m)} = \mathbf{B}\mathbf{B}^{\mathrm{T}} - \mathbf{E}_{(m\times m)}$ where $\mathbf{E}$ is a matrix whose elements are all one.
5. The structure vector $\mathbf{Q}_{(m)}$ is taken from the diagonal (top left corner to bottom right corner) of $\mathbf{S}$.

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