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LATTICE COMPUTING:
LATTICE THEORY BASED COMPUTATIONAL INTELLIGENCE

MANUEL GRAÑA

Computational Intelligence Group
Universidad Pais Vasco Spain
web: www.ehu.es/ccwintco

Abstract. Defining Lattice Computing as the class of algorithms that either construct the computations using the lattice operators inf and sup, or use lattice theory to produce generalizations or fusions of previous approaches, we find that a host of algorithms for data processing, classification and signal filtering have been produced over the last decades. We give a fast and brief review, that by no means could be exhaustive, with the aim to show, first, that this area has been growing during the past decades, and, second, what we think are broad avenues for future research.

1. Introduction Many computational algorithms in the diverse fields of Computer Science, including Computational Intelligence, are defined assuming as the computational framework the algebraic structure given by the ring of the real numbers, the addition and the multiplication $(\mathbb{R}, +, \times)$. However there is a parallel line of works based on other algebraic structures, like $(\mathbb{R}, \vee, +)$ or its dual $(\mathbb{R}, \wedge, +)$, where the role of the addition is taken by the lattice operation inf or sup, and the multiplication role is taken by the addition. These works have evolved into the application of Lattice Theory as a framework to define new approaches and algorithms that either generalize previous ones [20] or fuse existing computational paradigms [22]. We call this broad class of algorithms and knowledge representation methods Lattice Computing. Of course, the works reviewed didn't refer themselves as belonging to this category.

We can, at a very abstract level, distinguish the kind of processes realized in Computational Intelligence applications and methods into three basic groups:

1. Filtering: maps of objects (i.e. signals) in a high dimensional space into objects of the same space, i.e. $F : \mathbb{R}^{\mathrm{N}} \to \mathbb{R}^{\mathrm{N}}$.

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2. Dimension reduction: maps of objects in a high dimensional space into ones in a lower dimension space, i.e. $F : \mathbb{R}^N \to \mathbb{R}^d$ with $d << N$.

3. Classification: Mapping objects in a (high dimension) space into categories, where the construction can be done in a supervised or unsupervised (clustering) way, i.e. $F : \mathbb{R}^N \to \Omega$ with $\Omega = \{\omega_1, \cdots, \omega_c\}$.

There have been instances of Lattice Computing algorithms on all these categories. Lattice Computing filtering approaches roughly correspond to Mathematical Morphology applied to image and signal processing [51, 24, 25], which has been quite a successful alternative to linear signal processing. The literature in Mathematical Morphology is enormous and, of course, we do not attempt to review it here. In fact this success has been the source of inspiration for researches trying to extend this approach to other Computational Intelligence problems and applications closer to Artificial Intelligence classical topics. There have been some approaches to the construction of classification systems based on Lattice Computing ideas which fall in a no man's land between Artificial Neural Networks and Fuzzy Systems, i.e.: [71, 52, 28]. The results reported by these classifiers sometimes improve conventional classifiers and sometimes fall behind. There are few instances of dimension reduction based on Lattice Computing ideas, may be [68, 69, 7, 9] are the only works falling in this category.

A lattice is a partially ordered set (poset) any two of whose elements have a supremum and an infimum. The inf and sup operations are binary relations that give, respectively, the infimum and the supremum of any pair objects in the set. The classical reference on Lattice Theory is the book [2]. Perhaps the first extensive work that proposed computational algorithms involving the shift from the conventional ring $(\mathbb{R}, +, \times)$ into the algebraic structures involving lattice operations, $(\mathbb{R}, \vee, +)$ or its dual $(\mathbb{R}, \wedge, +)$, is the definition of the Minimax Algebra [5] in the context of planning problems. The next historical landmark is the introduction of Mathematical Morphology [51]. Image Algebra [48, 50] is another early attempt to define lattice computing methods devoted to image processing. The Fuzzy-ART architecture [4, 3] maybe the earliest Lattice Computing learning approach. Recent works [18, 22, 21] identify the Lattice Theory and the algebraic structures based on lattice operators as a central concept for a whole family of methods and applications. In short, the Lattice Computing approach has served to bridge the gap between computational paradigms as diverse as Fuzzy Systems, morphological signal processing and Artificial Neural Networks [10, 14, 12, 20, 58, 60, 66, 67]. In section 2 we comment on the problems and the approaches followed to implement learning procedures in Lattice Computing algorithms. In section 3 we comment on some of the approaches that we find more promising and in section 4 we finish with some conclusions.

2. Learning and parameter estimation Construction by learning is the induction of the system parameters from the available data. The key to the success of the Artificial Neural Networks paradigm is the development of easy to implement, robust (up to some degree) methods to estimate the parameters of the system: backpropagation, competitive learning algorithms, among others. These algorithms were designed as gradient descent procedures for a given energy or cost function. Some Lattice Computing approaches had tried to follow this same pattern [28, 52, 53, 71]. For instance, [71] tries to minimize the MSE of the output, [28] tries to maximize an equality index between the output and the desired output of the system. The major difficulty in these approaches lies in the discontinuity of the min and max functions. It has been avoided proposing continuous approximations to the min and max functions [28, 71, 72] that allow the derivation of closed expressions for approximations of the gradient. Some approaches had added complications, such as the need to express the clause structure in a way that allows differentiation in [71]. Besides the construction of classification systems, there have been works [70, 29] that tried to define gradient descent algorithms for the estimation of the morphological filter structural element. In fact, [30, 31] propose a class of mixed morphological/linear systems which are trained with an algorithm analogous to the error backpropagation. Some authors have tried to apply non-differential learning approaches, based on heuristic reasoning. For instance, there have been attempts to mimic the perceptron rule for lattice based morphological networks [37, 54]. Among the heuristic approaches, one of the earliest and most successful is Fuzzy-ART [4, 3] which applies an ad-hoc heuristic category growing learning procedure. The Fuzzy-ART learning procedure produces hyperrectangles covering the data, corresponding to the categories. These hyperrectagles grow monotonically as the learning proceeds, only stopped by the application of the vigilance parameter. Both the resolution of the result and the number of categories found depend on this vigilance parameter. The learning process is highly dependent on the order of presentation of the data [6], and still research is going on the setting of the algorithm parameters. Another approach to build up systems is that of the Morphological Associative Memories [43, 38, 62, 46, 63, 59]. It is very light computationally and does not involve gradient descent algorithms. It consists on the morphological (lattice) analogy to the construction of linear associative memories and Hopfield networks. Most of the theoretical work on their behaviour have been devoted to explain their behaviour and to understand the shape and properties of their fixed point subspaces [49, 63, 55, 56]. The related dendritic morphological neural network [40, 44, 47] has a learning algorithm based on the incremental refining of the covering of the sets in the classes to be discriminated.

3. Review of some approaches The goal of this review is to give a flavour of some of the avenues of work and research in this broad area which we find most involving and promising of future results. It can not be exhaustive, as some fields like Mathematical Morphology have a large number of works, but we hope that it can be enough to motivate readers to pursue works in this area.

3.1. *Mathematical Morphology* In image and signal processing a big class of tasks to be solved are related to noise removal and/or to segmentation. The basic morphological operators are the erosion and dilation operators, while the basic morphological filters are opening and closing. Sophisticated morphological procedures like texture segmentation, granulometries, watershed segmentation, etc. are compositions of these filters. One of the venues of research is the definition of appropriate lattice ordering of high dimension spaces, such are colour spaces and other vector spaces found in images, so that filters can be generalized to them. The other venue of interest from our point of view is that of defining new morphological operators and filters [22, 10, 27, 26] and the approaches to estimate appropriate structural elements [29, 30, 31, 23]. Lattice Theory was already used to formalize morphological operators [10]. The use of Lattice Theory in [22, 26] to generalize the erosion and dilation operators and the rigorous construction of adjoin operator pairs to systematically construct generalized opening and closing operators is a breakthrough that paves the way for new research areas. The embedding of fuzzy intersection and union norms into the morphological framework gives new fuzzy morphological openings and closings with enhancing noise removal and edge detection properties. In the meantime, there have been proposals for adaptive morphological operators sometimes mixed with linear operators, using gradient descent algorithms for structural element estimation [29, 30, 31], and they were applied to image restoration and character recognition. Another example of the fusion of ideas are the Morphological Shared-weight networks [70] that were proposed for target recognition in images. There a morphological hit-or-miss transform is implemented through a shared-weight network and its structural elements are adaptively estimated to obtain an optimal target recognition.

3.2. *Morphological Associative Memories* The works on Image Algebra [48, 45, 50] were the prelude to the proposal of morphological neural networks, in the form of morphological perceptron [54, 37, 40] and of associative memories [43, 38, 42]. They were proposed for the storage of binary and gray patterns, with the aim of recovering the original clean image from noisy copies, which is an image restoration process. Also grayscale morphological associative memories are used in [61] to pre-process the data prior to classification with a simple Nearest Neighbour approach. The good properties of Morphological Autoassociative Memories (later renamed Lattice Autoassociative Memories) were hindered by their sensitivity to specific kinds of noise (erosive or dilative noise), so that a big deal of effort was addressed to obtain robust versions [41, 58, 46, 62, 63, 59, 57, 61, 36, 39, 64, 65]. These efforts produced a new kind of memories in the frontier between associative morphological memories and fuzzy systems. Our own works on spectral unmixing of hyperspectral images [8, 7] have led to the use of the convex coordinates produced by unmixing process as features for classification purposes [7, 68, 69]. The sample data points are expressed relative to the so-called endmembers in hyperspectral image processing. The idea of endmember is that they represent instances of pure elements, so that all the other elements in the image correspond to mixtures of these pure elements. Endmembers are vertices of a polygonal convex set covering data cloud. This is a new kind of feature extraction that we think deserves further research. It happens that Autoassociative Morphological Memories have specific noise sensitivities that allow the detection of endmembers in the data from hyperspectral images. Working ways to enhance the robustness of Associative Morphological Memories against all kinds of noise [36, 64, 65] led to the definition of "morphological independence" and latter to "lattice independence" which turns out to be related to affine independence [49], a condition to be fulfilled by sets of endmembers. This justifies an algorithm exploiting the Autoassociative Morphological Memories sensitivities to extract the endmembers from the data sample previously proposed in [7]. It is possible also to obtain these endmembers trough the construction of the Autoassociative Morphological Memories [49]. The dendritic neuron [44, 47] is a further development of the Morphological Neural Netwoks were the biological model is a dendrite and the learning procedure is an incremental procedure that is shown to converge, with some similarities to the Fuzzy-ART learning algorithm.

3.3. *Fuzzy Lattice Neurocomputing* The Fuzzy Lattice Neurocomputing paradigm [15, 18, 32, 33] arises from the generalization of Fuzzy-ART [4, 3]. The generalization comes from the use of a general measure of inclusion and definition of the vigilance parameter in terms of the diagonal of the generalized object. The general data structure that allows the modelling of several classical structures is called Fuzzy Interval Number [11, 34]. It allows the manipulation of rather different data objects in a common lattice framework. These ideas have matured to propose lattice computing as a framework for new inference systems [13] and for new version of well known algorithms, such as the the grSOM (granular SOM) [20], the grARMA (granular ARMA) [12], the unification of SOM and ART algorithms [19]. The authors also propose FLNMAP as the generalization of the supervised Fuzzy ARTMAP.

The works on Fuzzy Lattice Neurocomputing had a lot of applications in classification and prediction: bone drilling for epidural anaesthesia [16], text classification [35], sugar production prediction[17], air quality monitoring [1]. The FIN structure is a promising way to produce new generalized algorithms able to deal with heterogeneous data structures, different from the conventional Euclidean spaces.

4. Conclusions The aim of this paper was to give a glimpse of the computational field composed of algorithms that employ in any way sup and inf operators and can be, therefore, put into the framework of lattice theory. We have taken the liberty to call it Lattice Computing. The focus on Lattice Theory and lattice operators reveals the existence of a profound parallelism between areas as divergent as Fuzzy Systems, Mathematical Morphology, Min-max Algebra and Artificial Neural Networks, which still may be the source of new contributions and enhacements to existing methods.

References

[1] I. N. Athanasiadis and V. G. Kaburlasos. Air quality assessment using fuzzy lattice reasoning (flr). In *Fuzzy Systems, 2006 IEEE International Conference on*, pages 29–34, 2006.

[2] G. Birkhoff. *Lattice Theory.* American Mathematical Society, 1967.

[3] G.A. Carpenter, S. Grossberg, N. Markuzon, J.H. Reynolds, and D.B. Rosen. Fuzzy ARTMAP: A neural network architecture for incremental supervised learning of analog multidimensional maps. *IEEE Trans Neural Networks*, 3(5):698–713, 1992.

[4] G.A. Carpenter, S. Grossberg, and D.B. Rosen. Fuzzy ART: Fast stable learning and categorization of analog patterns by an adaptive resonance system. *Neural Networks*, 4(6):759–771, 1991.

[5] Raymond A. Cuninghame-Green. *Minimax Algebra.* LNCS. Springer Verlag, 1979.

[6] I. Dagher, M. Georgiopoulos, G.L. Heileman, and G. Bebis. An ordering algorithm for pattern presentation in fuzzy artmap that tends to improve generalization performance. *IEEE Trans Neural Networks*, 10(4):768–778, 1999.

[7] M. Grana, P. Sussner, and G. Ritter. Associative morphological memories for endmember determination in spectral unmixing. In *Fuzzy Systems, 2003. FUZZ '03. The 12th IEEE International Conference on*, volume 2, pages 1285–1290, 2003.

[8] Manuel Graña, Bogdan Raducanu, Peter Sussner, and Gerhard Ritter. On endmember detection in hyperspectral images with morphological associative memories. In F.J. Garijo, J.C. Riquelme, and M. Toro, editors, *Advances in Artificial Intelligence - IBERAMIA 2002: 8th Ibero-American Conference on AI, Seville, Spain, November 12-15, 2002. Proceedings*, volume 2527 of *LNCS*, pages 526–535. Springer Verlag, 2002.

[9] Manuel Graña, Ivan Villaverde, Ramon Moreno, and Francisco X. Albizuri. Convex coordinates from lattice independent sets for visual pattern recognition. In Vassilis G. Kaburlasos and Gerhard X. Ritter, editors, *Computational Intelligence Based on Lattice Theory*, pages 99–126. Springer Verlag, 2007.

[10] Henk J. A. M. Heijmans and Petros Maragos. Lattice calculus of the morphological slope transform. *Signal Processing*, 59(1):17–42, 1997.

[11] V. G. Kaburlasos. FINs: lattice theoretic tools for improving prediction of sugar production from populations of measurements. *Systems, Man and Cybernetics, Part B, IEEE Transactions on*, 34(2):1017–1030, 2004.

[12] V. G. Kaburlasos and A. Christoforidis. Granular auto-regressive moving average (grARMA) model for predicting a distribution from other distributions. real-world applications. In *Fuzzy Systems, 2006 IEEE International Conference on*, pages 195–200, 2006.

[13] V. G. Kaburlasos and A. Kehagias. Novel analysis and design of fuzzy inference systems based on lattice theory. In *Fuzzy Systems, 2004. Proceedings. 2004 IEEE International Conference on*, volume 1, pages 281–286, 2004.

[14] V. G. Kaburlasos and S. E. Papadakis. grSOM: a granular extension of the self-organizing map for structure identification applications. In *Fuzzy Systems, 2004. Proceedings. 2004 IEEE International Conference on*, volume 2, pages 789–794, 2004.

[15] V. G. Kaburlasos and V. Petridis. Fuzzy lattice neurocomputing (FLN) models. *Neural Networks*, 13(10):1145–1170, 2000.

[16] V. G. Kaburlasos, V. Petridis, P. N. Brett, and D. A. Baker. Estimation of the stapes-bone thickness in the stapedotomy surgical procedure using a machine-learning technique. *Information Technology in Biomedicine, IEEE Transactions on*, 3(4):268–277, 1999.

[17] V. G. Kaburlasos, V. Spais, V. Petridis, L. Petrou, S. Kazarlis, N. Maslaris, and A. Kallinakis. Intelligent clustering techniques for prediction of sugar production. *Mathematics and Computers in Simulation*, 60(3-5):159–168, 2002.

[18] Vassilis G. Kaburlasos. *Towards a Unified Modeling and Knowledge-Representation based on Lattice Theory*. Springer Verlag, 2006.

[19] Vassilis G. Kaburlasos. Unified analysis and design of ART/SOM neural networks and fuzzy inference systems based on lattice theory. In F. Sandoval, A. Prieto, J. Cabestany, and Manuel Graña, editors, *Computational and Ambient Intelligence*, volume 4507 of *LNCS*, pages 80–93. Springer Verlag, 2007.

[20] Vassilis G. Kaburlasos and S. E. Papadakis. Granular self-organizing map (grSOM) for structure identification. *Neural Networks*, 19(5):623–643, 2006.

[21] Vassilis G. Kaburlasos and Gerhard X. Ritter. *Computational Intelligence Based on Lattice Theory*. Springer Verlag, 2007.

[22] P. Maragos. Lattice image processing: A unification of morphological and fuzzy algebraic systems. *Journal of Mathematical Imaging and Vision*, 22(2-3):333–353, 2005.

[23] P. Maragos, M. Akmal Butt, and L. F. C. Pessoa. Two frontiers in morphological image analysis: differential evolution models and hybrid morphological/linear neural networks. In *Computer Graphics, Image Processing, and Vision, 1998. Proceedings. SIBGRAPI '98. International Symposium on*, pages 10–17, 1998.

[24] P. Maragos and R. Schafer. Morphological filters–part i: Their set-theoretic analysis and relations to linear shift-invariant filters. *Acoustics, Speech, and Signal Processing, IEEE Transactions on*, 35(8):1153–1169, 1987.

[25] P. Maragos and R. Schafer. Morphological filters–part ii: Their relations to median, order-statistic, and stack filters. *Acoustics, Speech, and Signal Processing, IEEE Transactions on*, 35(8):1170–1184, 1987.

[26] P. Maragos, V. Tzouvaras, and G. Stamou. Synthesis and applications of lattice image operators based on fuzzy norms. In *Image Processing, 2001. Proceedings. 2001 International Conference on*, volume 1, pages 521–524, 2001.

[27] Petros Maragos. Morphological systems: Slope transforms and max-min difference and differential equations. *Signal Processing*, 38(1):57–77, 1994.

[28] W. Pedrycz. Neurocomputations in relational systems. *IEEE Tras. Patt. Anal. Mach. Int.*, 13(3):289–297, 1991.

[29] L. F. C. Pessoa and P. Maragos. Morphological/rank neural networks and their adaptive optimal design for image processing. In *Acoustics, Speech, and Signal Processing, 1996. ICASSP-96. Conference Proceedings., 1996 IEEE International Conference on*, volume 6, pages 3398–3401, 1996.

[30] L. F. C. Pessoa and P. Maragos. MRL-filters: a general class of nonlinear systems and their optimal design for image processing. *Image Processing, IEEE Transactions on*, 7(7):966–978, 1998.

[31] Lucio F. C. Pessoa and Petros Maragos. Neural networks with hybrid morphological/rank/linear nodes: a unifying framework with applications to handwritten character recognition. *Pattern Recognition*, 33(6):945–960, 2000.

[32] V. Petridis and V. G. Kaburlasos. Fuzzy lattice neural network (FLNN): a hybrid model for learning. *Neural Networks, IEEE Transactions on*, 9(5):877–890, 1998.

[33] V. Petridis and V. G. Kaburlasos. Learning in the framework of fuzzy lattices. *Fuzzy Systems, IEEE Transactions on*, 7(4):422–440, 1999.

[34] V. Petridis and V. G. Kaburlasos. FINkNN: A fuzzy interval number k-nearest neighbor classifier for prediction of sugar production from populations of samples. *Journal of Machine Learning Research*, 4:17–37, 2003.

[35] V. Petridis, V. G. Kaburlasos, P. Fragkou, and A. Kehagias. Text classification using the σ-FLNMAP neural network. In *Neural Networks, 2001. Proceedings. IJCNN '01. International Joint Conference on*, volume 2, pages 1362–1367, 2001.

[36] Bogdan Raducanu, Manuel Graña, and Francisco X. Albizuri. Morphological scale spaces and associative morphological memories: Results on robustness and practical applications. *Journal of Mathematical Imaging and Vision*, 19(2):113–131, 2003.

[37] G. X. Ritter and T. W. Beaver. Morphological perceptrons. In *Neural Networks, 1999. IJCNN '99. International Joint Conference on*, volume 1, pages 605–610, 1999.

[38] G. X. Ritter, J. L. Diaz-de Leon, and P. Sussner. Morphological bidirectional associative memories. *Neural Networks*, 12(6):851–867, 1999.

[39] G. X. Ritter and L. Iancu. A morphological auto-associative memory based on dendritic computing. In *Neural Networks, 2004. Proceedings. 2004 IEEE International Joint Conference on*, volume 2, pages 915–920, 2004.

[40] G. X. Ritter, L. Iancu, and G. Urcid. Morphological perceptrons with dendritic structure. In *Fuzzy Systems, 2003. FUZZ '03. The 12th IEEE International Conference on*, volume 2, pages 1296–1301 vol.2, 2003.

[41] G. X. Ritter and M. S. Schmalz. Learning in lattice neural networks that employ dendritic computing. In *Fuzzy Systems, 2006 IEEE International Conference on*, pages 7–13, 2006.

[42] G. X. Ritter and P. Sussner. Associative memories based on lattice algebra. In *Systems, Man, and Cybernetics, 1997. 'Computational Cybernetics and Simulation'., 1997 IEEE International Conference on*, volume 4, pages 3570–3575, 1997.

[43] G. X. Ritter, P. Sussner, and J. L. Diaz-de Leon. Morphological associative memories. *Neural Networks, IEEE Transactions on*, 9(2):281–293, 1998.

[44] G. X. Ritter and G. Urcid. Lattice algebra approach to single-neuron computation. *Neural Networks, IEEE Transactions on*, 14(2):282–295, 2003.

[45] G. X. Ritter, J. N. Wilson, and J. L. Davidson. Image algebra: An overview. *Computer Vision, Graphics, and Image Processing*, 49(1):125–, 1990.

[46] Gerhard Ritter, L Iancu, and M.S. Schmalz. A new auto-associative memory based on lattice algebra. In *Progress in Pattern Recognition, Image Analysis and Applications*, volume 3287 of *LNCS*, pages 148–155. Springer Verlag, 2004.

[47] Gerhard X. Ritter and Gonzalo Urcid. Learning in lattice neural networks that employ dendritic computing. In V. Kaburlasos and G.X. Ritter, editors, *Computational Intelligence Based on Lattice Theory*, pages 25–44. Springer Verlag, 2007.

[48] G.X. Ritter and P. D. Gader. Image algebra techniques for parallel image processing. *J. Paral. Distr. Comput.*, 4:7–44, 1987.

[49] G.X. Ritter and P. D. Gader. Fixed points of lattice transforms and lattice associative memories. In *Advances in Imaging and Electron Physics*, volume 144, pages 165–242. Academic Press, 2006.

[50] G.X. Ritter and J.N. Wilson. *Handbook of Computer Vision Algorithms in Image Algebra, Second Edition*. CRC Press, Boca Raton: Fl, 2001.

[51] J. Serra. *Image Analysis and Mathematical Morphology*. Academic Press, 1982.

[52] P.K. Simpson. Fuzzy min-max neural networks - part1: Classification. *IEEE Trans Neural Networks*, 3(5):776–786, 1992.

[53] P.K. Simpson. Fuzzy min-max neural networks - part2: Clustering. *IEEE Trans Fuzzy Systems*, 1(1):32–45, 1993.

[54] P. Sussner. Morphological perceptron learning. In *Intelligent Control (ISIC), 1998. Held jointly with IEEE International Symposium on Computational Intelligence in Robotics and Automation (CIRA), Intelligent Systems and Semiotics (ISAS), Proceedings of the 1998 IEEE International Symposium on*, pages 477–482, 1998.

[55] P. Sussner. Fixed points of autoassociative morphological memories. In *Neural Networks, 2000. IJCNN 2000, Proceedings of the IEEE-INNS-ENNS International Joint Conference on*, volume 5, pages 611–616, 2000.

[56] P. Sussner. A relationship between binary morphological autoassociative memories and fuzzy set theory. In *Neural Networks, 2001. Proceedings. IJCNN '01. International Joint Conference on*, volume 4, pages 2512–2517, 2001.

[57] P. Sussner. Binary autoassociative morphological memories derived from the kernel method and the dual kernel method. In *Neural Networks, 2003. Proceedings of the International Joint Conference on*, volume 1, pages 236–241, 2003.

[58] P. Sussner. A fuzzy autoassociative morphological memory. In *Neural Networks, 2003. Proceedings of the International Joint Conference on*, volume 1, pages 326–331, 2003.

[59] P. Sussner. New results on binary auto- and heteroassociative morphological memories. In *Neural Networks, 2005. IJCNN '05. Proceedings. 2005 IEEE International Joint Conference on*, volume 2, pages 1199–1204, 2005.

[60] P. Sussner and M. E. Valle. A brief account of the relations between gray-scale mathematical morphologies. In *Computer Graphics and Image Processing, 2005. SIBGRAPI 2005. 18th Brazilian Symposium on*, pages 79–86, 2005.

[61] P. Sussner and M. E. Valle. Gray-scale morphological associative memories. *Neural Networks, IEEE Transactions on*, 17(3):559–570, 2006.

[62] Peter Sussner. Observations on morphological associative memories and the kernel method. *Neurocomputing*, 31(1-4):167–183, 2000.

[63] Peter Sussner. Associative morphological memories based on variations of the kernel and dual kernel methods. *Neural Networks*, 16(5-6):625–632, 2003.

[64] G. Urcid and G. X. Ritter. Noise masking for pattern recall using a single lattice matrix auto-associative memory. In *Fuzzy Systems, 2006 IEEE International Conference on*, pages 187–194, 2006.

[65] Gonzalo Urcid and Gerhard X. Ritter. Noise masking for pattern recall using a single lattice matrix associative memory. In Vassilis G. Kaburlasos and Gerhard X. Ritter, editors, *Computational Intelligence Based on Lattice Theory*, pages 81–100. Springer Verlag, 2007.

[66] M. E. Valle, P. Sussner, and F. Gomide. Introduction to implicative fuzzy associative memories. In *Neural Networks, 2004. Proceedings. 2004 IEEE International Joint Conference on*, volume 2, pages 925–930, 2004.

[67] Marcos Eduardo Valle and Peter Sussner. Fuzzy associative memories from the perspective of mathematical morphology. In *Fuzzy Systems Conference, 2007. FUZZ-IEEE 2007. IEEE International*, pages 1–6, 2007.

[68] I Villaverde, Manuel Graña, and Alicia d'Anjou. Morphological independence for landmark detection in vision based slam. In F. Sandoval, A. Prieto, J. Cabestany, and Manuel Graña, editors, *Computational and Ambient Intelligence*, volume 4507 of *LNCS*, pages 837–844. Springer Verlag, 2007.

[69] I Villaverde, Manuel Graña, and J.L. Jimenez. Lattice independence and vision based mobile robot navigation. In *Knowledge-Based Intelligent Information and Engineering Systems*, volume 4693 of *LNAI*, pages 1179–1186. Springer Verlag, 2007.

[70] Yonggwan Won, P. D. Gader, and P. C. Coffield. Morphological shared-weight networks with applications to automatic target recognition. *Neural Networks, IEEE Transactions on*, 8(5):1195–1203, 1997.

[71] Ping-Fai Yang and Petros Maragos. Min-max classifiers: Learnability, design and application. *Pattern Recognition*, 28(6):879–899, 1995.

[72] Xinghu Zhang, Chang-Chieh Hang, Shaohua Tan, and Pei-Zhuang Wang. The min-max function differentiation and training of fuzzy neural networks. *Neural Networks, IEEE Transactions on*, 7(5):1139–1150, 1996.