

Hypernetworks for reconstructing the dynamics of multilevel systems

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Abstract

Networks are fundamental for reconstructing the dynamics of many systems, but have the drawback that they are restricted to binary relations. Hypergraphs extend relational structure to multi-vertex edges, but are essentially set-theoretic and unable to represent essential structural properties. *Hypernetworks* are a natural multidimensional generalisation of networks, representing n -ary relations by simplices with n vertices. The assembly of vertices to make simplices is key for moving between levels in multilevel systems, and integrating dynamics between levels. It is argued that hypernetworks are necessary, if not sufficient, for reconstructing the dynamics of multilevel complex systems.

1. Introduction

Many systems are complex because they are multilevel and the intra-level dynamics at lower levels constrain and are constrained by the intra-level dynamics at higher levels. Examples include biological systems in which cell dynamics are co-constrained by organism dynamics, road systems where the behaviour of drivers is co-constrained by the traffic dynamics, and multi-robot systems where the behaviour of individual robots is co-constrained by the dynamics of the team. A mathematical formalism is needed to integrate intra-level and inter-level dynamics.

This paper introduces what will be called *hypernetworks*. These are a natural extension of networks with two-vertex edges. In hypernetworks a hyper-edge, more commonly known as a *simplex*, can have many vertices. Hypernetworks are related to hypergraphs, and have a similar Galois lattice structure. However, there are subtle and important differences that make them more powerful for representing and reconstructing the dynamics of complex multilevel systems.

The simple idea of representing relationships between n things is our starting point. From this simple beginning all else follows: we can build a mathematical theory of multilevel systems with its own theory of discrete multidimensional time and multilevel multidimensional dynamics as system activity flows through the polyhedra within and between levels.

2. Networks, graphs and hypergraphs

Let V be a set of objects called *vertices*, and E a set of objects called *edges*. Each edge e_i is associated with a pair of vertices, (a, b) . We write $\rho: e_i \rightarrow (a, b)$. The pair (V, E) is called a *graph*. We allow $\rho(e_i) = \rho(e_j) = (a, b)$ for $i \neq j$, with more than one edge between pairs of vertices.

Generally dynamics *emerge* from interactions between parts of systems, sometimes with associated flows. The abstraction of edges, (a, b) , is ideally suited for representing *interactions* and *flows* between a and b . A general *network* is a graph in which the edges are *oriented* so that $(a, b) \neq (b, a)$. Usually a network has mappings from its vertices and edges to number sets representing flows and related entities, and these are constrained by the topological connectivity of the network.

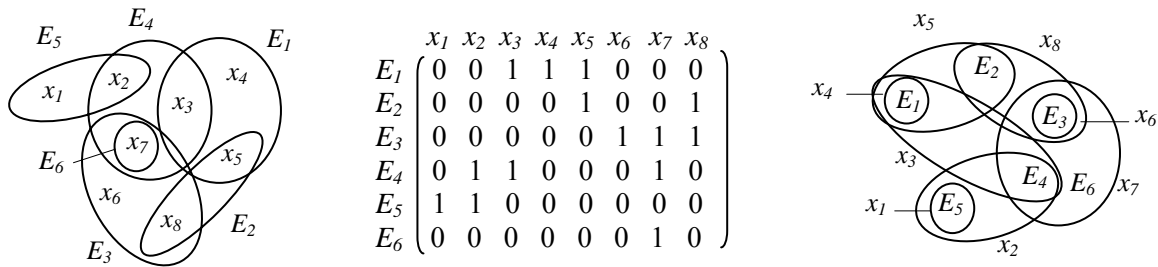
Despite the analytic power of graphs and networks they have the major limitation that they can only represent *binary relations* between pairs of things. For example, Figure 2(a) shows three binary telephone conversation relations between a mum, dad and daughter. This is an excellent structure for daughters to manipulate their parents because pairwise interaction is an inefficient way for three people to communicate. In contrast, the conference call structure of Figure 2(b) is that of a *3-ary relation* which enables everyone to hear what the others say, and introduces much less ambiguity and misunderstanding: $\langle \text{mum, daughter} \rangle + \langle \text{dad, daughter} \rangle + \langle \text{mum, dad} \rangle \neq \langle \text{mum, daughter, dad} \rangle$.



(a) three binary relations (2-ary relations) (b) a 3-ary relation

Figure 1. Some n -ary relations cannot be expressed as the composition of binary relations

The possibility of generalising graph edges to more than pairs of vertices has a long history: ‘For the past forty years Graph Theory has provided to be an extremely useful tool for solving combinatorial problems in areas as diverse as Geometry, Algebra, Number Theory, Operations Research and Optimization. It was thus natural to try and generalise the concept of a graph, in order to attack additional combinatorial problems. ... The idea of looking at families of sets from this standpoint took shape in around 1960. In regarding each set as a “generalised edge” and in calling the family itself a “hypergraph”, the initial idea was to try to extend certain classical results of results of Graph Theory ...’ [Berge, 1989].



(a) a hypergraph $H_E(X)$ (b) The incidence matrix of $H_E(X)$ (c) The dual hypergraph, $H_X(E)$

Figure 2. A hypergraph, its incidence matrix and its dual (Adapted from Berge 1989)

Berge defines a *hypergraph* on the finite set $X = \{x_1, x_2, \dots, x_n\}$ to be a family of subsets of X , $H = (E_1, E_2, \dots, E_m)$ such that (1) $E_i \neq \emptyset$ for all $i = 1, 2, \dots, m$, and (2) $X = \bigcup_i E_i$ for $i = 1, 2, \dots, m$. A *simple* hypergraph has the property that no hyper-edge is a proper subset of another. The sets, E_i are called *hypergraph edges* or simply *edges*. Later we relax constraint (1) and allow the empty set to belong to hypergraphs. Thus, any finite class of finite sets, $\{E_1, E_2, \dots, E_m\}$, will be called a *finite hypergraph* with vertex set $X = \bigcup_i E_i$, for $i = 1, 2, \dots, m$.

There is a binary relation, R , between the vertices and edges of a hypergraph with $E_i R x_j$ if and only if x_j belongs to E_i . This can be represented in the usual way by an *incidence matrix*, M , where $m_{ij} = 1$ if $x_j R E_i$ and $m_{ij} = 0$ otherwise. This is illustrated in Figure 2(b) where the rows define classes of sets in terms of their vertices. This example illustrates the duality often associated with relations, and the *dual hypergraph* of the transposed incidence matrix is illustrated in Figure 1(c).

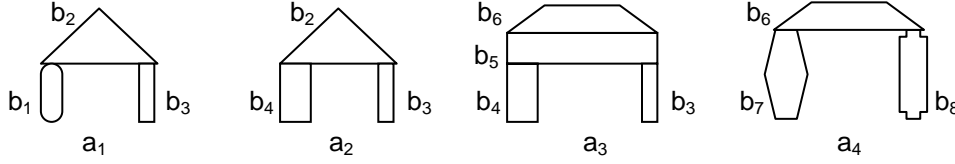


Figure 3. Four arches, a_1, a_2, a_3, a_4 , assembled from blocks $b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8$.

Consider a relation between a set of arch-shapes A and a set of blocks B (Fig 3). The incidence matrix for these relationships is given in Figure 4, and from this Figure 5 shows the bipartite graph of the binary relation between the parts and wholes.

	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8
a_1	1	1	1	0	0	0	0	0
a_2	0	1	1	1	0	0	0	0
a_3	0	0	1	1	1	1	0	0
a_4	0	0	0	0	0	1	1	1

Figure 4. Incidence matrix for the arch-block assembly relation in Figure 3.

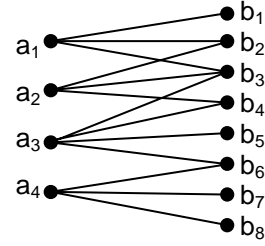
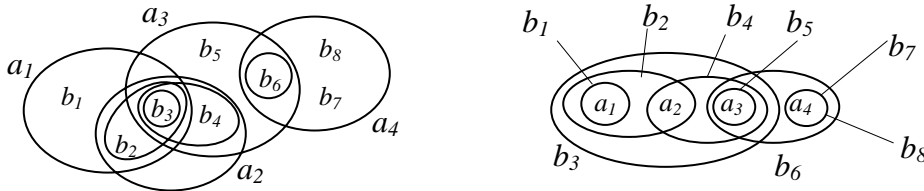


Figure 5. The bipartite graph of the assembly relation

Figure 5 defines a class of subsets of B given by $\{a_1 \rightarrow \{b_1, b_2, b_3\}, a_2 \rightarrow \{b_2, b_3, b_4\}, a_3 \rightarrow \{b_3, b_4, b_5, b_6\}, a_4 \rightarrow \{b_6, b_7, b_8\}\}$, which can be considered to be a hypergraph. A more interesting hypergraph also includes all the intersections of the sets, including the empty set, \emptyset (Fig 6(a)). Let this hypergraph be denoted $H_A(B, R)$. This hypergraph is defined by the elements of A being related to subsets of B . The *conjugate hypergraph*, $H_B(A, R)$, is defined by the elements of B being related to subsets of A , as shown in Figure 6(b).



(a) The hypergraph $H_A(B, R)$

(b) The hypergraph $H_B(A, R)$

Figure 6. The dual hypergraphs of the relation R between blocks and arches

3. Hypergraph Connectivity and the Galois Lattice

Two hypergraph edges are k -near if they share k vertices. This is an immediate generalisation of two network edges being connected by a single vertex. The edges E_i and E_j are k -connected if there is a chain of edges $E_{(1)}, E_{(2)}, \dots, E_{(n)}$, with $E_i = E_{(1)}, E_j = E_{(n)}$, and $E_{(h)}$ κ -near $E_{(h+1)}$, $\kappa \geq k$, for $h = 1, \dots, n-1$. This is analogous to the existence of a path in a network, and generalises the notion of connectivity. For example, E_1 is 2-connected to E_4 in Figure 7(a). Intuitively, two edges are ‘more highly connected’ for larger values of k . However, there is another issue illustrated in Figure 7(b). Here all four edges share the sub-edge $\{x_1, x_2, x_3\}$, and this is a ‘stronger’ connectivity than k -connectivity. We call this a *star-hub* structure where the class $\{E_1, E_2, E_3, E_4\}$ form a star-like configuration, with the set $E_5 = \{x_1, x_2, x_3\}$ as its hub.

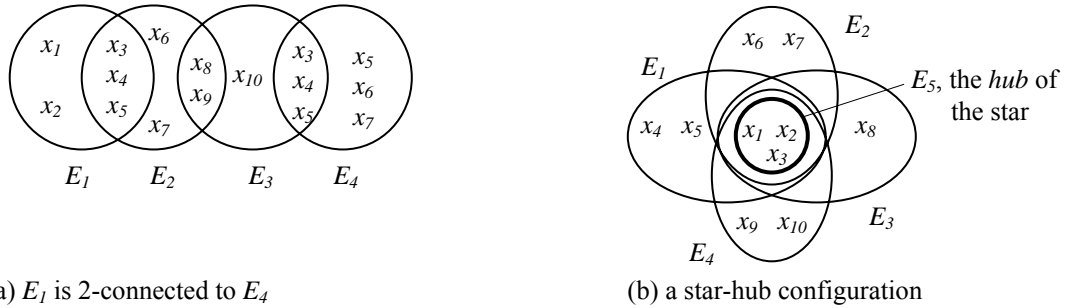


Figure 7. New kinds of higher level connectivity in hypergraphs

There is a 1-1 correspondence between the edges of the hypergraphs $H_A(B, R)$ and $H_B(A, R)$ in Figure 6, called a *Galois connection* [Barbut *et al*, 1970]. For example, $\{a_1, a_2\}$ is associated with $\{b_2, b_3\}$, since both of a_1 and a_2 related to both of b_1 and b_2 . These paired sets can be arranged in what is called a *Galois lattice*, illustrated in Figure 8. The expressions $(\{a_1, a_2, a_3, a_4\}, \emptyset)$ means that no member of B is related to every member of A , and $(\emptyset, \{b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8\})$ means that no member of A is related to every member of B , where \emptyset is the empty set .

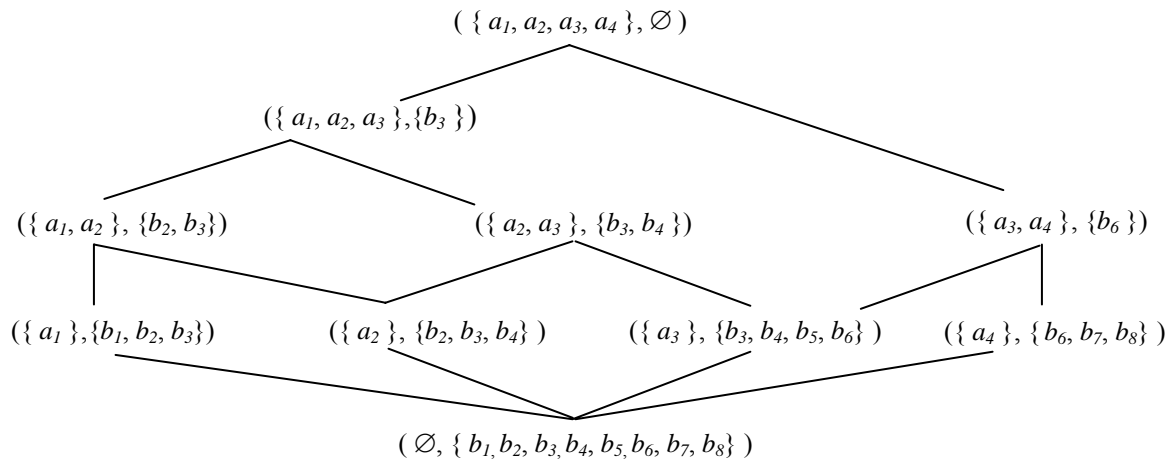


Figure 8 The Galois lattice of the relation R between A and B .

A *lattice* is defined to be a partially ordered set in which every two elements have a supremum and an infimum. Define the partial order \leq by $(A, B) \leq (A', B')$ if $A \subseteq A'$ and $B \supseteq B'$. For all (A_h, B_i) and (A_j, B_k) , $(A_h \cap A_j, B_i \cup B_k)$ is the supremum of (A_h, B_i) and (A_j, B_k) , since $(A_h, B_i) \leq (A_h \cap A_j, B_i \cup B_k)$ and $(A_j, B_k) \leq (A_h \cap A_j, B_i \cup B_k)$. Similarly $(A_h \cup A_j, B_i \cap B_k)$ is the infimum of (A_h, B_i) and (A_j, B_k) , since $(A_h \cap A_j, B_i \cup B_k) \leq (A_h, B_i)$ and $(A_h \cap A_j, B_i \cup B_k) \leq (A_j, B_k)$. Therefore the paired sets with this partial order form a lattice. In Figure 8 a line is drawn between each pair of sets and their supremum and infimum, to produce a picture of the resulting *Galois Lattice*.

4. From Networks to Hypernetworks

N -ary relations are a natural generalisation of binary relations, and they can be represented by a natural generalisation of the vertices and directed edges of network: a 3-ary relation can be represented by a triangle, a 4-ary relation can be represented by a tetrahedron, a 5-ary relation can be represented by a 5-hedron, and in general an n -ary relation can be represented by an n -hedron, as illustrated in Figure 9.

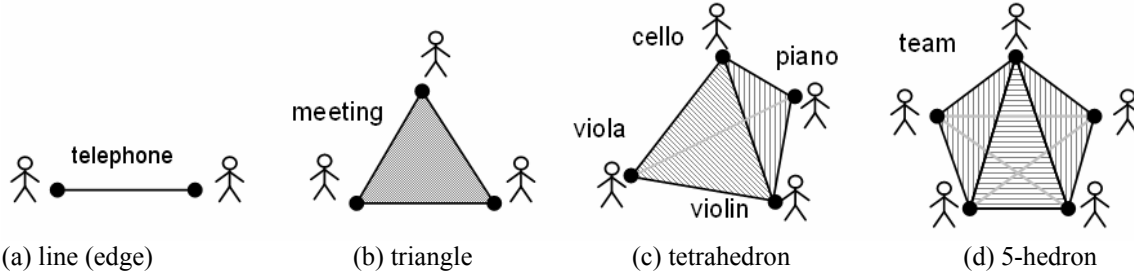


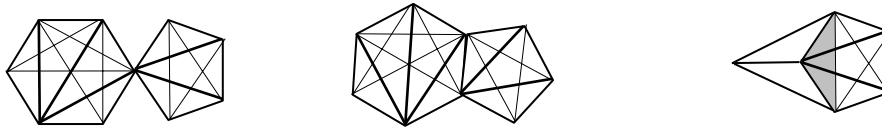
Figure 9. Representing relationships by multidimensional polyhedra

Polyhedra can be viewed as a multidimensional generalisation of vertices and edges in a network: vertices have dimension zero, lines have dimension one, triangles have dimension-two, tetrahedra have dimension three, ..., and $(p+1)$ -hedra have dimension p .

An abstract p -dimensional simplex, or p -simplex, σ_p , is defined by an ordered set of vertices, $\sigma_p = \langle v_0, v_1, v_2, \dots, v_p \rangle$. As we have seen, simplices can be represented graphically by polyhedra in multidimensional space. Although these pictures can carry the intuition of relational structure, their algebraic formulation provides the basis for rigorous theory.

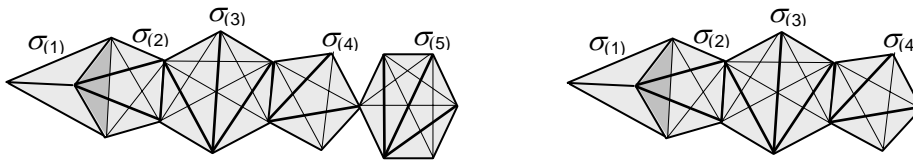
Let $\langle v_0, v_1, \dots, v_p \rangle$ be a simplex. Then $\{v_0, v_1, \dots, v_p\}$ is defined to be its *vertex set*. The simplex $\sigma_q = \langle v'_0, v'_1, \dots, v'_q \rangle$ is a **face** of $\sigma_p = \langle v_0, v_1, \dots, v_p \rangle$, written $\sigma_q \lesssim \sigma_p$, iff $\{v'_0, v'_1, \dots, v'_q\} \subseteq \{v_0, v_1, \dots, v_p\}$. A set of simplices is called a *simplicial family* [Johnson, 1982a]. A set of simplices with all its faces is called a *simplicial complex*.

Let σ_q be a face of both σ and σ' . Then σ and σ' are said to be q -near. σ_q is a q -dimensional *shared face*, or q -face of σ and σ' . If σ_q is the largest shared face between σ and σ' we write $\sigma_q = \sigma \cap \sigma'$. For example, the simplices in Figure 10(a) share a vertex and they are 0-near, those in Figure 10(b) share an edge and they are 1-near, while the simplices in Figure 4(c) share a triangle and are 2-near.



(a) 1 shared vertex (0-near) (b) 2 shared vertices (1-near) (c) 3 shared vertices (2-near)

Figure 10. Simplicies can be connected at different dimensions



(a) 0-connected

(b) 1-connected

Figure 11. Chains of q -connected simplicies

We say two simplices are q -connected if there is a chain of pairwise q -near simplices between them. More formally, two simplices σ and σ' are q -connected if there is a chain of simplices $\sigma_{(1)}, \sigma_{(2)}, \dots, \sigma_{(n)}$ with $\sigma = \sigma_{(1)}$, $\sigma_{(n)} = \sigma'$, and $\sigma_{(i)}$ q -near $\sigma_{(i+1)}$ for $i = 1, \dots, n-1$. For example, in Figure 11(a) $\sigma_{(1)}$, and $\sigma_{(5)}$ are 0-connected through $\sigma_{(2)}$, $\sigma_{(3)}$, and $\sigma_{(4)}$. In Figure 5(b) $\sigma_{(1)}$, and $\sigma_{(4)}$ are 1-connected through $\sigma_{(2)}$, and $\sigma_{(3)}$.

The q -connected components in simplicial complexes are the sets of mutually q -connected simplices. For example, Figure 12(b) shows the 1-connected components in Figure 12(a).

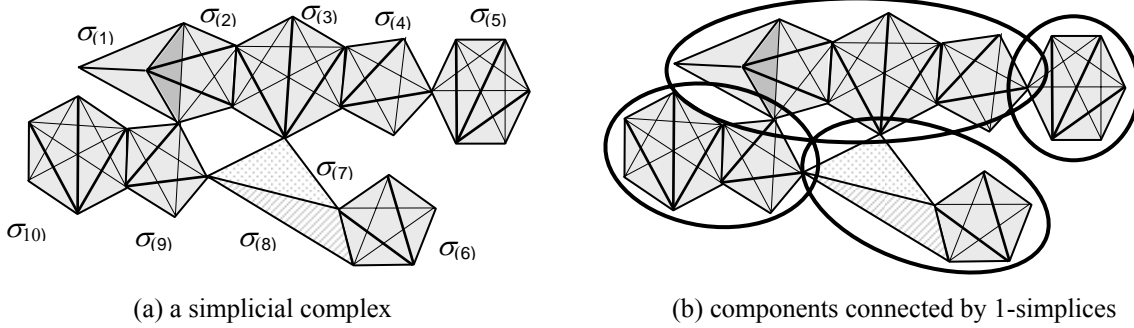


Figure 12. A simplicial complex of connected simplices

Table 1 shows what we call the *shared vertex matrix*, which shows the number of shared vertices between the simplices in Figure 12(a). From this we can list the q -connected components as the Q -analysis shown in Table 2. For example, there are four distinct components at $q = 1$, which become connected as a single component at $q = 0$ (Figure 12(b)).

	$\sigma_{(1)}$	$\sigma_{(2)}$	$\sigma_{(3)}$	$\sigma_{(4)}$	$\sigma_{(5)}$	$\sigma_{(6)}$	$\sigma_{(7)}$	$\sigma_{(8)}$	$\sigma_{(9)}$	$\sigma_{(10)}$
$\sigma_{(1)}$	4	3	0	0	0	0	0	0	1	0
$\sigma_{(2)}$	3	4	2	0	0	0	0	0	1	0
$\sigma_{(3)}$	0	2	5	1	0	0	1	0	0	0
$\sigma_{(4)}$	0	0	1	4	1	0	0	0	0	0
$\sigma_{(5)}$	0	0	0	1	6	0	0	0	0	0
$\sigma_{(6)}$	0	0	0	0	0	6	2	0	0	0
$\sigma_{(7)}$	0	0	1	0	0	2	3	2	1	0
$\sigma_{(8)}$	0	0	0	0	0	2	2	2	5	2
$\sigma_{(9)}$	1	1	0	0	0	0	0	0	5	2
$\sigma_{(10)}$	0	0	0	0	0	0	0	0	2	6

Table 1. Shared vertex matrix for Figure 12.

$q = 5$ (4 shared vertices)	$\{\sigma_5\}, \{\sigma_{10}\}$
$q = 4$ (5 shared vertices)	$\{\sigma_2\}, \{\sigma_3\}, \{\sigma_4\}, \{\sigma_5\}, \{\sigma_6\}, \{\sigma_9\}, \{\sigma_{10}\}$
$q = 3$ (4 shared vertices)	$\{\sigma_1\}, \{\sigma_2\}, \{\sigma_3\}, \{\sigma_4\}, \{\sigma_5\}, \{\sigma_6\}, \{\sigma_9\}, \{\sigma_{10}\}$
$q = 2$ (3 shared vertices)	$\{\sigma_1, \sigma_2\}, \{\sigma_3, \sigma_4\}, \{\sigma_5, \sigma_6\}, \{\sigma_7, \sigma_8\}, \{\sigma_9, \sigma_{10}\}$
$q = 1$ (2 shared vertices)	$\{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}, \{\sigma_5, \sigma_6, \sigma_7, \sigma_8\}, \{\sigma_9, \sigma_{10}\}$
$q = 0$ (1 shared vertex)	$\{\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7, \sigma_8, \sigma_9, \sigma_{10}\}$

Table 2. Q-analysis for Figure 12.

Q -analysis is based on the assumption that change can be transmitted between q -near simplices, and we define q -transmission fronts, illustrated in Figure 13, as hypernetwork *backcloth* structure that constrains the *traffic* of changes in the mapping values and flows [Johnson, 1982b].

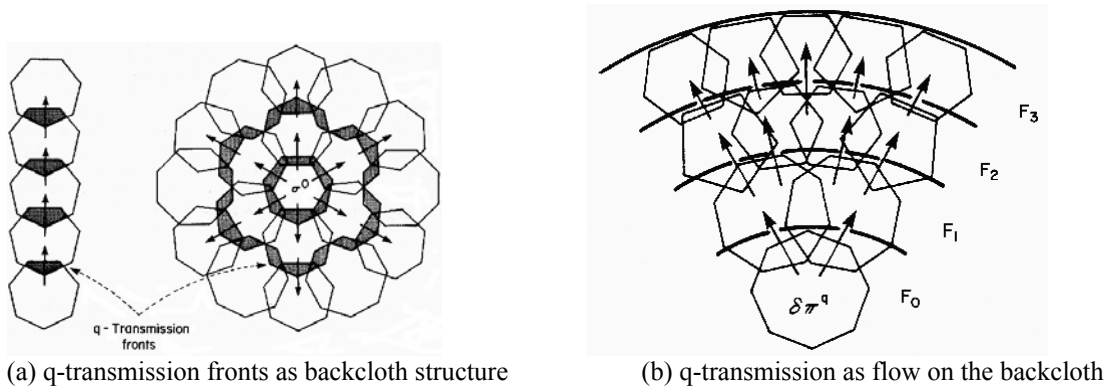
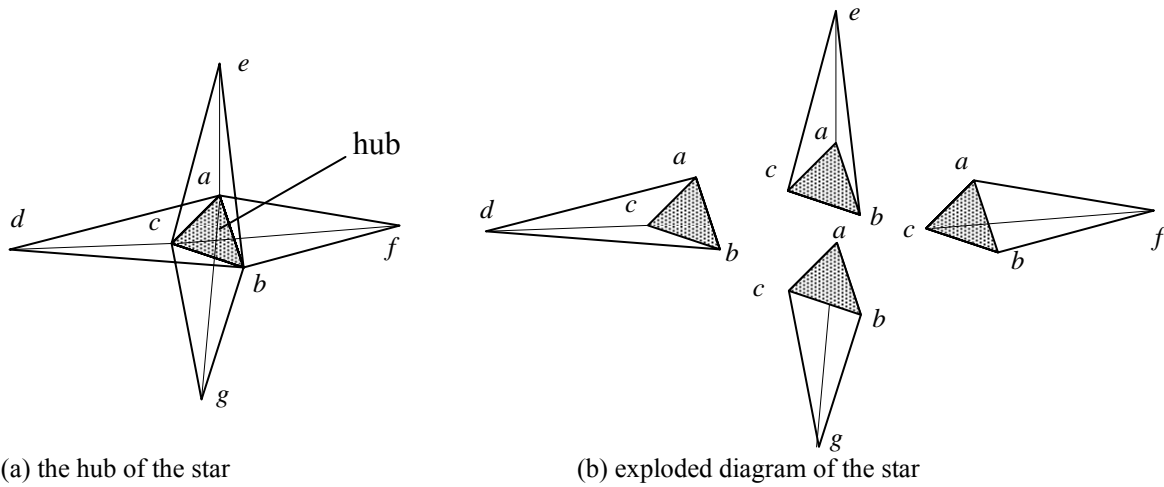


Figure 13. q-transmission fronts

5. Stars, Hubs, Maximal Rectangles, and Galois lattices



(a) the hub of the star

(b) exploded diagram of the star

Figure 14. A star-hub configuration

Q-analysis is based on the connectivity of pairs of simplices, but it is more general to consider the intersection of sets of simplices. This leads to the concept of stars and hubs, as illustrated in Figure 14 where the simplices, $\langle a, b, c, d \rangle$, $\langle a, b, c, e \rangle$, $\langle a, b, c, f \rangle$, and $\langle a, b, c, g \rangle$ share the face $\langle a, b, c \rangle$. The set of the four simplices is called a *star* and their intersection is called their *hub*.

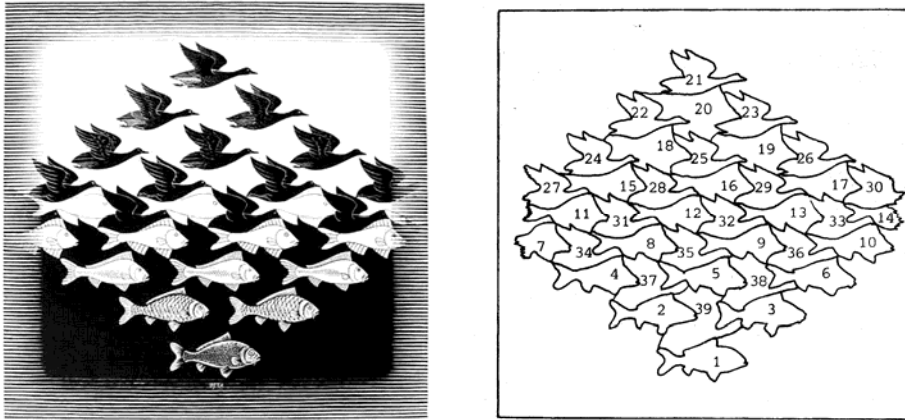


Figure 15. An analysis of Escher's Sky and Water (Source: Johnson, 1985)

To illustrate, Figure 15 shows thirty nine shapes abstracted from Escher's Sky and Water. The relation between these and the set of visual features $\{\text{scales, mouth, gills, fish-tail, fins, fish-shape, eye, duck-shape, two-wings, feathers, beak, legs}\}$ is given by the incidence matrix in Table 3. If the rows and columns of the incidence matrix are arranged appropriately, hubs correspond to rectangular regions in which the values are all 1, called *maximal rectangles* [Johnson, 1986].

The matrix shows that the simplices for the shapes numbered 1 to 6 are all the same, with a block of 1s corresponding to the features $\{\text{scales, mouth, gills, fish-tail, fins, fish-shape, eye}\}$, with all the shapes being good examples of fish. There is another group, shapes all related to the hub features $\{\text{fins, fish-shape, eye}\}$, which includes the less perfect fish shapes 8 to 13. The following major *star-hub pairs* can be abstracted as follows:

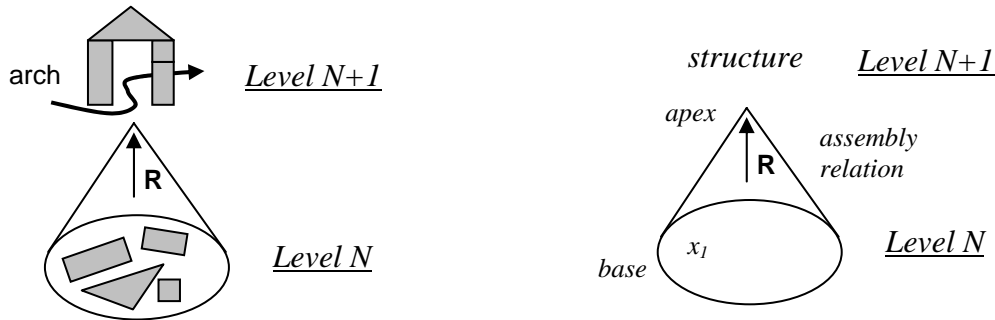
- $\langle 1, 2, 3, 4, 5, 6 \rangle \leftrightarrow \langle \text{scales, mouth, gills, fish-tail, fins, fish-shape, eye} \rangle$
- $\langle 1, 2, 3, 4, 5, 6, 8, 9, 10, 11, 12, 13 \rangle \leftrightarrow \langle \text{fish-tail, fins, fish-shape, eye} \rangle$
- $\langle 21, 22, 23, 24, 25, 26, 28, 29 \rangle \leftrightarrow \langle \text{eye, duck shape, two wings, feathers, beak, legs} \rangle$
- $\langle 21, 22, 23, 24, 25, 26, 28, 29, 31, 32, 33, 27 \rangle \leftrightarrow \langle \text{eye, duck shape, two wings} \rangle$

	1	2	3	4	5	6	8	9	10	11	12	13	7	21	22	23	24	25	26	28	29	31	32	33	27	30	34	35	36	37	38	14	15	16	17	18	19	20	39						
scales	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0		
mouth	1	1	1	1	1	1	1	1	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	
gills	1	1	1	1	1	1	1	1	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	
fish-tail	1	1	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	
fins	1	1	1	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	
fish-shape	1	1	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	
eye	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	
duck-shape	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	0	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	
two-wings	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
feathers	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
beak	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
legs	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Table 3. The incidence matrix of shape-feature relationship

The final weak structure is given by the shapes 15-19 which are related only to fish-shape, and shapes 34-38 which are only related to duck shape. Thus Escher has bird shapes highly connected at the top of the picture and fish shapes highly connected at the bottom, connected through the ambiguous shapes at the centre.

6. The Fundamental Diagram of Multilevel Systems



(a) parts aggregate into a whole (b) The Fundamental Diagram of Multilevel Systems

Figure 16. Hierarchical cones: the n -ary relation R maps the set of block to an arch at a higher level

At the heart of the multilevel mathematical theory being developed there is a very simple idea: wholes are assembled from parts. Let P be the set of parts of an object, W . Then these parts have to be *assembled* into W under a relation R . In other words, if R holds then the parts have been assembled into the whole. We write $R : P \rightarrow W$. If P has n elements, then R is an n -ary relation. For simplicity, this section will be developed using artificial physical systems as examples.

The concept of *emergence* is closely tied up with the idea of multilevel systems. If vertices exist at one level then structures assembled from vertices, exist at a higher level. Thus the mapping from the set to the simplex moves up the hierarchy of representation, from *Level N* to *Level N+1*, as shown in Figure 16. In Figure 16(a) an arch emerges as the blocks are assembled by the relation. Figure 16(b) presents what will be called the *Fundamental Diagram of Multilevel Systems*. In this the set is an Euler circle (ellipse) at the *base* of the cone, and the assembly relation maps it to a structure at a higher level in the representation.

The cone construction illustrates a number of interesting and important possibilities. Figure 17(a) shows that the same set can be assembled in different ways. Thus the set of vertices alone is not sufficient to represent a simplex. For full knowledge we need to know the relation, and therefore we use the notation $\langle v_0, v_1, \dots, v_n; R \rangle$, which provides information on both the vertices of the simplex and on the relation that assembles those vertices into the structure.

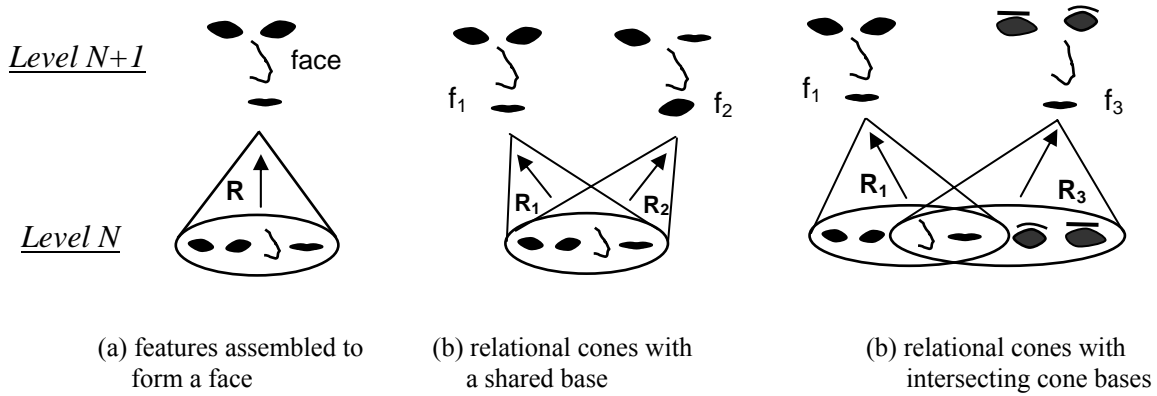


Figure 17. Hierarchical cones representing assembly of parts into holes.

Figure 17(a) shows how a set of visual features can be assembled to form a face. Figure 17(b) shows the possibility of two different relations, R_1 and R_2 aggregating the same set in different ways to form different higher level objects. Thus $R_1(P) \neq R_2(P)$, even though the n -ary relations R_1 and R_2 operate on the same domain, P . Figure 17(c) shows that the bases of hierarchical cones may intersect, and this leads to the Galois lattice discussed previously. They are sites of interaction and support the system dynamics. They are the generalisation of shared vertices that connect links in graphs and networks.

Figure 18(a) illustrates very simply the debate concerning the evolution of bird wings, and the gap between a form that gives significant evolution advantage resulting in a wing and the precursor to the wing which may have little or no advantage. [Gould *et al*, 1982] suggested *exaptation* as an explanation, where a form that gives an advantage for one purpose (a feather-covered moveable flap used as a thermoregulator) can be subject to evolutionary pressures giving different advantages (a wing for flying). In this case evolution can operate on the whole organism and its parts at all levels. Figure 18(b) illustrates exaptation as using components for new purposes in new contexts, which can be very advantageous and is very common in human systems. Here a Victorian kitchen sink used to contain flowers in a garden.

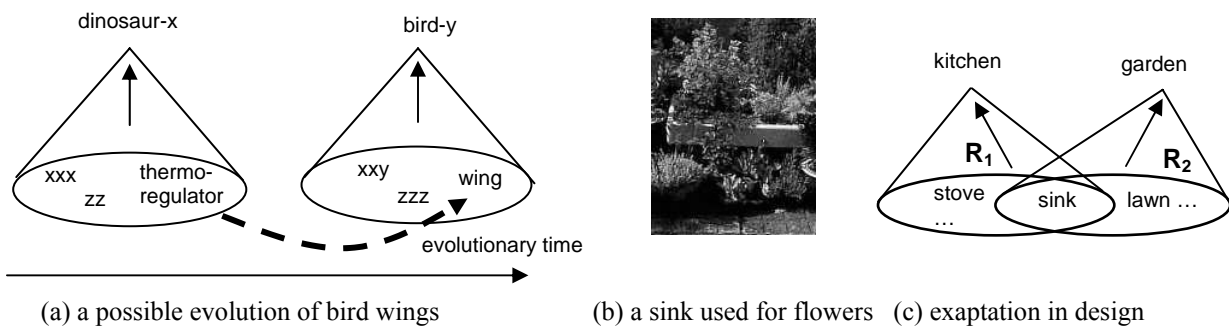


Figure 18. Exaptation as the evolution of parts able to aggregate into new wholes.

When discussing the evolution of complex artificial systems, Simon (1969, page 201, 1984 edition) tells a story of two watchmakers, each making fine watches with about a thousand parts. Tempus made his watches in a way that meant they fell to pieces if he had to put the assembly down, as he did when the phone rang with a new order. Hora designed his watches putting together subassemblies of about ten elements each. Ten of these subassemblies could be put together into a larger subassembly; and a system of ten of the latter subassemblies made the whole watch. When Hora put down a partly assembled watch to answer the phone, he lost only a small part of his work, and he assembled his watches in only a fraction of the time it took Tempus. This is a compelling argument for designing multilevel systems, such as the car shown in Figure 19.

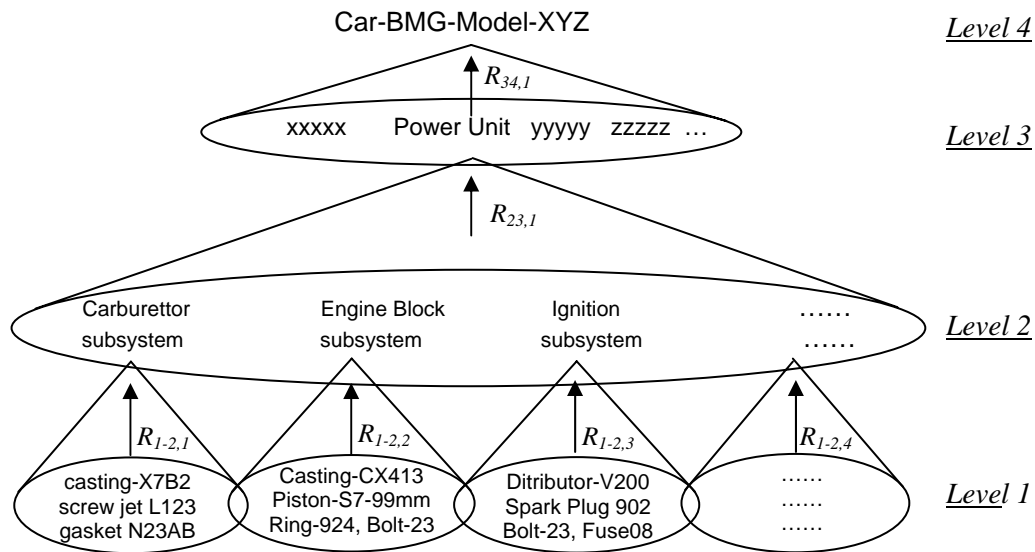


Figure 19. A car represented as a multilevel assembly system – a systems of systems

As Figure 19 shows, the generality the *bases* of the cones (the sets of parts) may intersect, inducing lattice structures and connectivity; that systems of systems may have many assembly relations; and that multilevel systems have *compositions* of assembly relations with one aggregation, R_2 , applied after another, R_1 , written as $R_2 \circ R_1$.

Compared to natural language such as English, by virtue of being mathematical, composition immediately resolves many problems that cause great confusion in the analysis of multilevel systems. For example, below are *meronymic* (part-whole) relation ‘problems’ taken from (Winston *et al*, 1987), and proposed mathematical resolutions to them:

Simpson’s finger is part of Simpson	Simpson’s finger R_1 aggregates into Simpson
Simpson is part of the Philosophy Department	Simpson R_2 into the Philosophy Department
Simpson’s finger is part of the Philosophy Department	Simpson’s finger $R_2 \circ R_1$ aggregates into the Philosophy Department
A windshield is part of a car	A windshield R_1 aggregated into a car
This shard was part of a windshield	This shard R_2 aggregated into a windshield
This shard was part of a car	This shard $R_2 \circ R_1$ aggregated into a car
Water is part of the cooling system	Water R_1 aggregates into the cooling system
Water is partly hydrogen	Hydrogen R_2 aggregates into Water
Hydrogen in part of the cooling systems	Hydrogen $R_2 \circ R_1$ aggregates into the cooling system

It is perfectly meaningful, if uninteresting, to say that Simpson’s finger is $(R_2 \circ R_1)$ -related to the Philosophy Department. It is one of many true but uninformative things. It does not imply that Simpson’s finger is R_1 -related to the Philosophy Department, which is where the apparent problem arises. This example illustrates how mathematics can resolve problems which are very puzzling when expressed in words. Altogether some twelve troublesome meronymic relations are given involving ‘part of’ relationships. In every case the ‘problem’ is unnecessary, being due to the same symbol, ‘part of’, ambiguously representing different things.

7. Building Multilevel Representations

The complex systems literature often disparages what is seen as a reductionist approach when parts of systems are identified. Ross Ashby [1955, Page 1/7] writes:

‘Science stands today on something of a divide. For two centuries it has been exploring systems that are either intrinsically simple or that are capable of being analysed into simple components. The fact that such a dogma as “vary the factors one at a time” could be accepted for a century, shows that scientists were largely concerned in investigating such systems as *allowed* this method; for this method is often fundamentally impossible in the complex systems.’

Such sentiments are sometimes taken to mean that one should not look the parts of systems or try to identify them as relatively autonomous subsystems, and the term ‘reductionist’ is widely used in a pejorative way. However, we *do* look at the parts of systems, and often it is essential to do so in order to gain any kind of understanding.

Disaggregation of a system is a very different process to aggregation. Aggregations are usually one-to-one in terms of classes and behaviours. It is usually useful to manufacture things the same way, so that individual things have known properties characteristic of the class. There are usually many ways to take things to pieces, and there are usually many choices of what those pieces might be.

This has been called the *Intermediate Word Problem*, as illustrated in Figure 20. Here the analyst is faced with a large complex system such as the city of Stockholm. Such as system can only be observed by looking at its parts. At the highest level there is ‘the system’. At lower levels, even before formally collecting data, the analyst is aware of some parts of the system and their dynamics. However, these are all jumbled together into what is called the *hierarchical soup*, because it is a heterogeneous mixture of sets and elements at many different levels (Gould *et al*, 1984). Forming the right vocabulary to model Stockholm in any meaningful scientific way is very difficult. The vocabulary alone contains many thousands of word, each expressing some part or behaviour of the multilevel systems of systems. By hypothesis, in multilevel systems all the lower minutiae *do* have an impact on the higher level macro dynamics, and these details cannot be filtered out as ‘insignificant’. Constructing multilevel vocabulary by solving the intermediate word problem is an essential part of trying to understand the dynamics of complex systems.

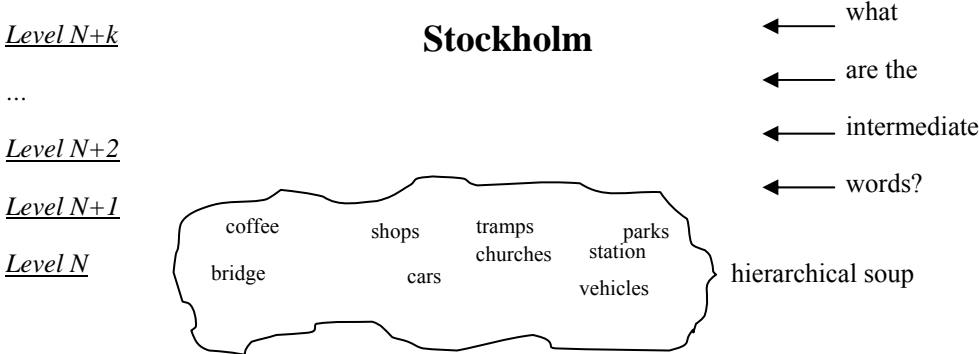


Figure 20. The intermediate word problem

9. Alpha- and Beta-aggregations in lattices hierarchies

The use of n -ary relations to build objects out of their parts establishes hierarchical levels. However, there is a subtlety in hierarchical aggregation involving another kind of aggregation. This is illustrated in Figure 21 where three arches are assembled from their components. These assemblies require *all* the parts for their n -ary relation to hold. We call this an α -aggregation, or an *AND-aggregation*. At the next level the arches are gathered up to form a set. In this case A-1 *or* A-2 *or* A-3 is sufficient for an arch. We call this a β -aggregation, or an *OR-aggregation*. Thus the set of arches is defined by a disjunction of conjunctions, $\vee_j \langle \vee_{j1}, \dots, \vee_{jn} ; R_j \rangle$.

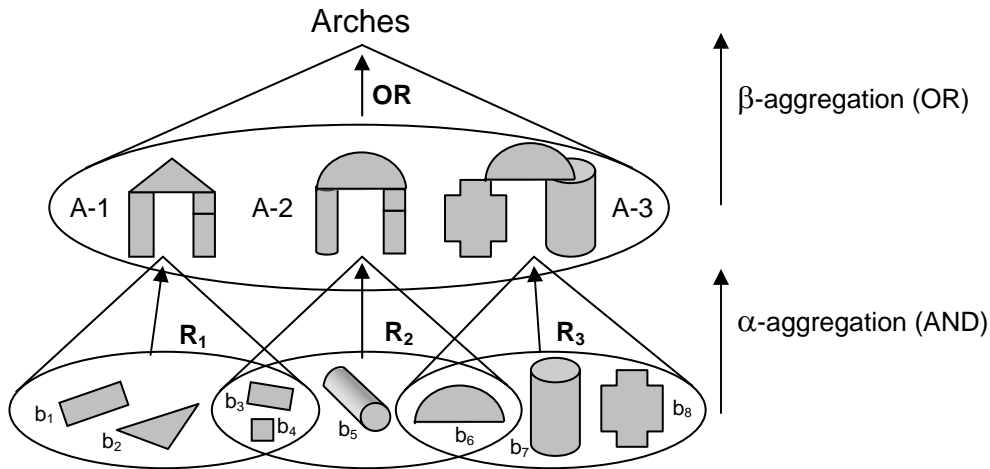


Figure 21. Two different types of multilevel aggregation

10. Backcloth, Traffic, and Type-1 Dynamics

In network theory, connectivity generally underlies *flows* represented by numbers associated with the vertices and the edges. For example, in electrical networks the flows are of electrical current on the links between potential differences across the vertices. In road systems the flows on the links are numbers of vehicles. In human systems the flows include information. We call the network the *backcloth* of the systems, and the flows the *traffic* on it.

Network theory is powerful because the connectivity structure of the network constrains the flows. This can be seen clearly by the way electrical engineers design the connectivity in order to make the components interact in the right way to achieve desirable electrical traffic.

Multidimensional networks also have traffic defined on them, and their q -connectivity also constrains the flows. For example, Figure 22 shows multidimensional traffic at three levels in an organisation. In the case of wages traffic, on the left, the aggregation is linear. In the case of making profits, on the right, the relationship between costs is non-linear through time.

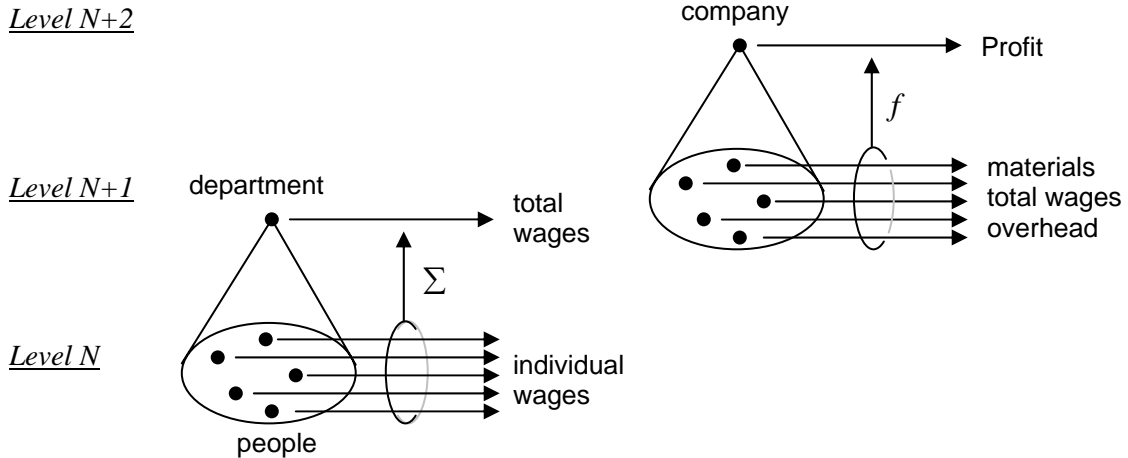


Figure 22. Multidimensional traffic on the multidimensional multilevel backcloth

When considering the dynamics of systems, we make a distinction between changes in the traffic (functions) and changes in backcloth (relations). We use the terms *Type 1 Dynamics* for changes in the traffic on an unchanging backcloth, and *Type-2 Dynamics* when the relational backcloth changes. Generally, Type-1 dynamics are ‘fast’, with the values of functions changing in micro-time. In contrast, changes in the backcloth often involve the assembly of *infrastructure* which can be ‘slow’ to achieve in clock time. Type-2 changes can be relatively expensive.

Figure 23 illustrates that in the kind of hierarchical structure considered here the sets at any level may intersect, giving a *lattice hierarchy* rather than a *tree* hierarchy. Furthermore, the intersections of the sets induce connections between the simplices supporting various flows and transmission dynamics of the system.

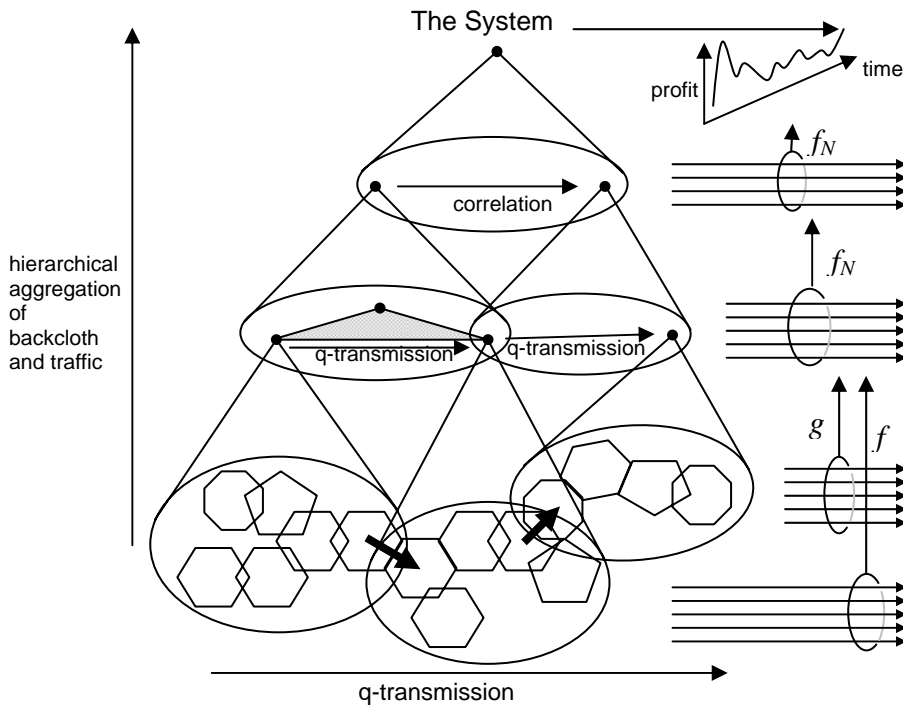


Figure 23. Coherent aggregation of Type-1 dynamics (traffic) over the hierarchical backcloth

Type-1 dynamics are constrained by the backcloth and its connectivity, in the same way that the connectivity structure of electrical circuits constrains their behaviour. Sometimes changes in the traffic can induce Type-2 dynamics, for example when telephone traffic jams a switchboard, or when an Internet server crashes through excessive demands. Much less is known about Type-2 dynamics than is known about Type-1 dynamics

11. Multidimensional Events and Type-2 Dynamics

The formation of a relational polyhedron can be considered to be an *event* in a system. The event gives a way of marking system time, as shown in Figure 24. *Before* there is no polyhedron, *after* there is a polyhedron. Here we have a 2-dimensional ‘arch-is-built’ event. The event of a pendulum bob can be used to measure time in physical systems, and structural events can be used to measure time in other kinds of system, especially socio-technical systems. Moreover, system events measure time in systems in a natural way, *e.g.* the event of parts arriving will determine when things can be made from them, not a schedule in clock time.

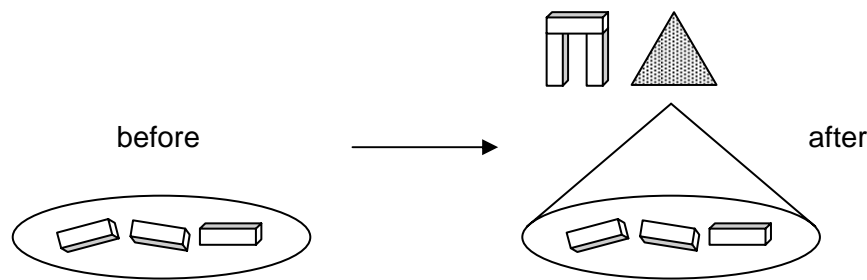


Figure 24. The formation of relational polyhedra marks an event in system time

Figure 25 illustrates the example of assembling four people with job-1, job-2, job-3, and job-4, into a ‘team’ simplex $\langle \text{job-1}, \text{job-2}, \text{job-3}, \text{job-4} \rangle$. Before they come together they are just a set of vertices, $\langle \text{job-1} \rangle, \langle \text{job-2} \rangle, \langle \text{job-3} \rangle, \langle \text{job-4} \rangle$. After the relational structure of the team has formed, the people meld together as a simplex $\langle \text{job-1}, \text{job-2}, \text{job-3}, \text{job-4} \rangle$. So, the situation before and after can be discriminated, and we say that the formation of the simplex is a *polyhedral event*. Polyhedral events mark the passage of *system time*. Events occur at different levels on multilevel systems, and they have to be coordinated.

It is supposed that before the team can be formed, one of the potential members needs to be trained to acquire a new skill for their job. For this person the training is an event, marked by the change of acquiring the new job skill and integrating it with those already possessed. It is also assumed that a new person has to be appointed for one of the jobs. Then the team-building involves getting all four people to work together.

Thus forming the team defined by $\langle \text{job-1}, \text{job-2}, \text{job-3}, \text{job-4} \rangle$ involves a skill training event (*Level N-1 to Level N*), a job training event (*Level N to Level N+1*), an appointment event (*Level N to Level N+1*), and a team-building event (*Level N+1 to Level N+2*). Each of these events creates higher level structure in the hierarchy.

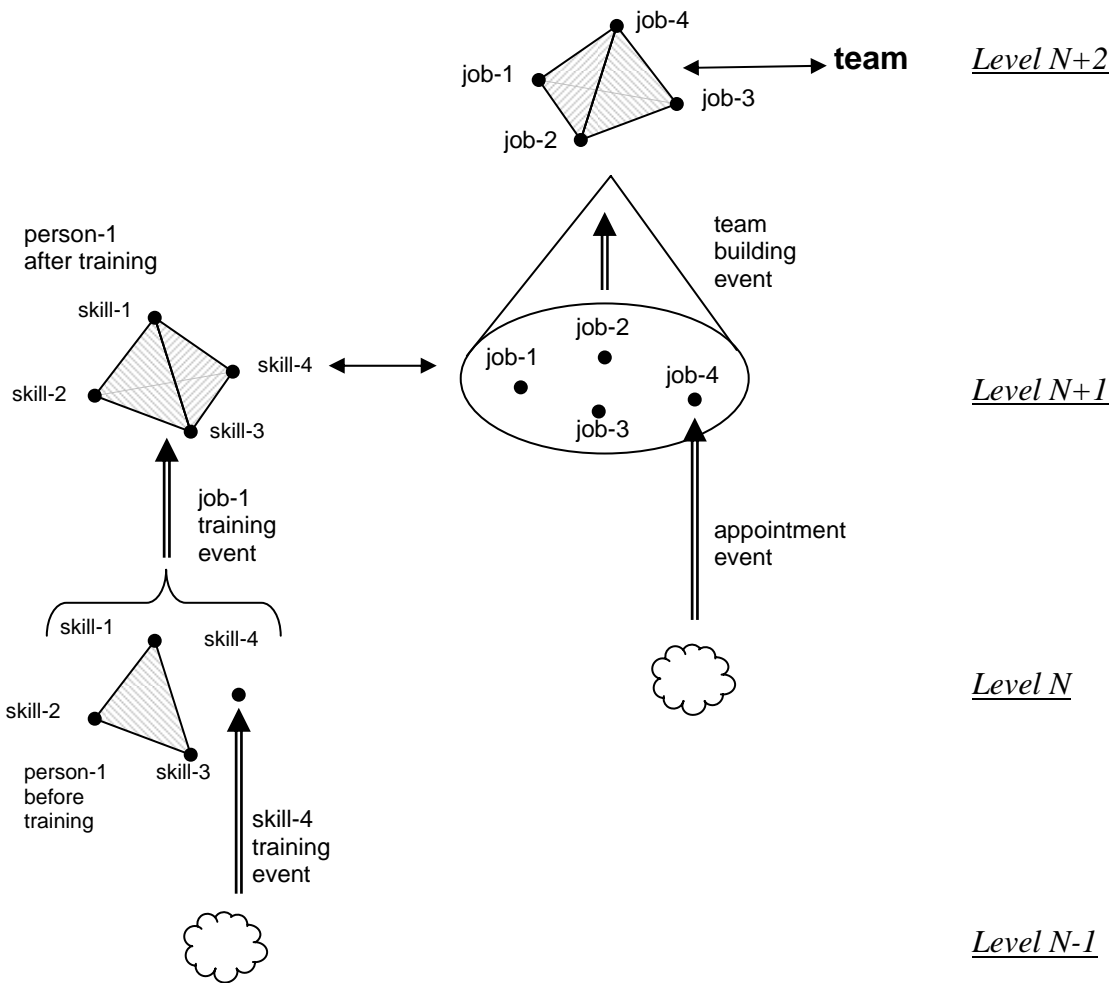


Figure 25. The formation of a team as series of multilevel multidimensional events

Decision makers and managers have to take clock time into account, even when they are responsible for complex systems with their own structural dynamics. People expect to be paid in clock time, rent has to be paid in clock time, and people expect delivery in clock time. Thus managers have the problem of establishing relationships between multidimensional system time and clock time, as illustrated in Figure 26. This picture is simplistic of course. In reality the polyhedral trajectory would be one of many trajectories, with many dependencies between them.

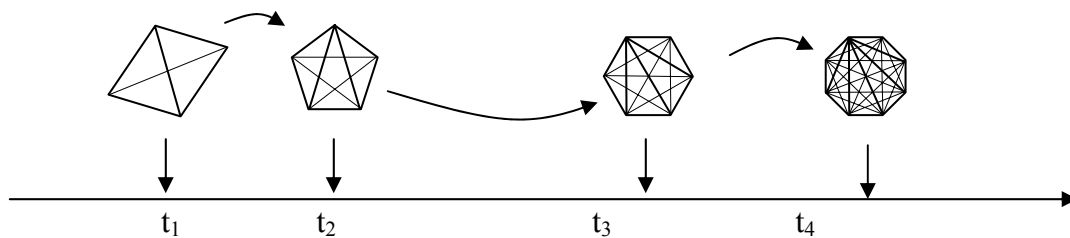


Figure 26. Polyhedral dynamics form trajectories in a non-linear way in clock time

In his book *Multidimensional Man*, Atkin develops this theory of polyhedral events and makes a convincing argument that structural events are related to clock time in a nonlinear way related to their dimensions. He gives a convincing explanation why higher dimensional events take a lot longer to occur in clock time than simple events [Atkin, 1981, pages 191-196].

Example

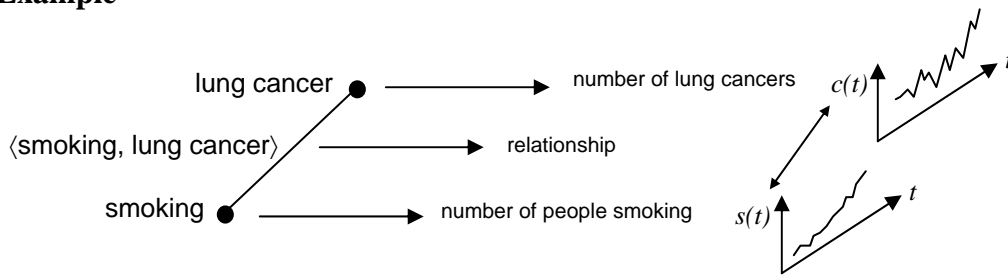


Figure 27. Establishing statistical relationship at relatively high levels of representation

One of the reasons for systems appearing to behave in unpredictable ways may be that attention is focused at an inappropriate level in the system, with the real interactions generating emergence at different levels. Very often scientists focus on the behaviour of functions at very high levels (*e.g.* time series of aggregate quantities) rather than investigate the dynamics at lower levels that generate the values of those functions.

To illustrate this, consider the relationship between smoking tobacco and lung cancer discovered in the nineteenth century, and vigorously denied by some until recently. Before the 1950s a number of studies were conducted into the statistical relationship between smoking and lung cancer, but the results were inconclusive. In 1951 Doll and Hill found that of 1,357 men with lung cancer, 99.5% were smokers, and in 1964 they published ten years observations on British doctors which gave convincing evidence of a smoking-lung cancer link (Ash, 2005). The backcloth and traffic of such studies is illustrated in Figure 27.

There are often *mechanisms* underlying statistical relationships at lower levels in the representation. In 1953 Wynder reported that painting cigarette tar on the backs of mice creates tumours (Ash, 2005), effectively shifting the focus of the research down the hierarchy of representation to establish a relationship between cigarette tar and tumours.

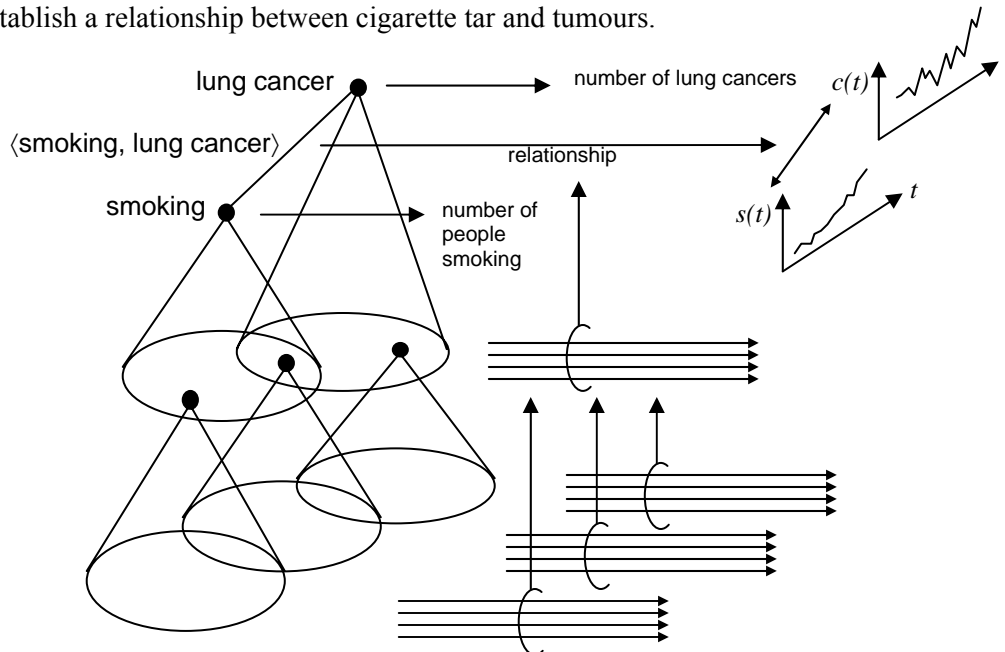


Figure 28. Mechanisms underlying risk may exist at lower level in the hierarchy

The pressure group, Action on Smoking and Health publishes 'key dates in the history of anti-tobacco campaigning' on its website (Ash, 2005). The remarkable thing about this document is that almost all the entries concern the relationship between smoking and health at a relatively high level in the representation: either scientists publishing statistical interpretations of observations, members of the tobacco industry and lobbying groups disagreeing or agreeing with the scientists' conclusions, and many lawyers giving advice and appearing in court cases.

It is interesting to note that in 1856 there was a debate about the health effects of smoking in the *Lancet* medical journal, and that Langley and Dickinson worked on the effect of nicotine on nerve cells in 1889, at the lowest levels in the hierarchy of representation. In 1908 the sale of tobacco to children under 16 was banned. By 1912 Adler strongly suggested that lung cancer is related to smoking. More recently it has become known that carbon monoxide from tobacco smoke combines with haemoglobin in the blood, putting stress on the circulation system.

Today it is widely accepted that smoking tobacco carries high risks. The remarkable thing about this story in which the risks of tobacco have been debated for one and a half centuries, is that conducting the debate at a high representational level has prolonged it considerably. The relationship between smoking, lung cancer and heart disease lies in multi-agent biochemical systems much lower down in the hierarchy of representations. This is certainly more complex than the one-simplex (smoking, cancer) backcloth, but ultimately where the answers lie.

12. Conclusions

In this paper it has been argued that hypernetworks are able to represent the multilevel dynamics of complex systems in a way that combines dynamics at all levels. Many research problems remain, but hypernetworks appear to be necessary if not sufficient for reconstructing the multilevel dynamics of complex systems.

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