Available online at www.sciencedirect.com
SCIENCE DIRECT
Discrete Mathematics 298 (2005) 39–61
DISCRETE MATHEMATICS
www.elsevier.com/locate/disc

Perspectives on *A*-homotopy theory and its applications

Hélène Barcelo[a,1], Reinhard Laubenbacher[b]

[a] *Department of Mathematics and Statistics, Arizona State University, Tempe, AZ 85287-1804, USA*
[b] *Virginia Bioinformatics Institute at Virginia Tech, Washington St. (0477), Blacksburg, VA 24061, USA*

Received 10 March 2003; received in revised form 20 February 2004; accepted 10 March 2004
Available online 26 July 2005

Abstract

This paper contains a survey of the *A*-theory of simplicial complexes and graphs, a combinatorial homotopy theory developed recently. The initial motivation arises from the use of simplicial complexes as models for a variety of complex systems and their dynamics. This theory diverges from classical homotopy theory in several crucial aspects. It is related to prior work in matroid theory, graph theory, and work on subspace arrangements.

MSC: 37F20

Keywords: Combinatorial topology; Homotopy; Simplicial complex; Graph

0. Introduction

In his book [3] Atkin says: "In order to capture the geometric essence of any natural system *N*, we must choose an appropriate formal geometric structure into which the observables of *N* can be encoded. It turns out to be useful to employ what is termed a *simplicial complex* as our formal mathematical framework. . . . A simplicial complex . . . is a natural generalization of the intuitive idea of a Euclidean space, and is formed by interconnecting a number of pieces of varying dimension. The mathematical apparatus, which has its roots

E-mail addresses: barcelo@asu.edu (H. Barcelo), reinhard@vbi.vt.edu (R. Laubenbacher).
[1] The first author was partially funded by NSA, Grant 04G-110.

0012-365X/$ - see front matter © 2005 Elsevier B.V. All rights reserved.
doi:10.1016/j.disc.2004.03.016

in algebraic topology, gives us a systematic procedure for keeping track of how the pieces fit together to generate the entire object, and how they each contribute to the geometrical representation of *N*."

Atkin proceeded to model a variety of social and technological networks using simplicial complexes. Examples range from soccer and its strategic subtleties to the committee structure at the University of Essex. (In the latter case simplices correspond to committees, with the members represented by the vertices. Combinatorial "holes" in the complex correspond to "missing" committees, that is, committees with a membership suitable for certain issues to be addressed.) In order to analyze and compare social structures he developed a measure on complexes which he termed *Q*-analysis [1,2]. It is reminiscent of measuring the connected components of a topological space, except that Atkin was interested in measuring the *combinatorial* connectivity of the complex.

The central object of *Q*-analysis is an integer vector associated with a simplicial complex Δ as follows. Suppose the dimension of Δ is d. Let $0 \leqslant q \leqslant d$, and let $\sigma, \tau \in \Delta$ be two simplices. Call σ and τ *q-near* if they share a simplex of dimension q, that is, if their intersection contains at least $q+1$ elements. The two simplices are *q-connected* if there is a sequence

$$\sigma, \sigma_1, \ldots, \sigma_n, \tau,$$

such that consecutive simplices are q-near. This notion of connectivity generates an equivalence relation on the simplices of Δ for each choice of q. Define

$$Q(\Delta) = (q_0, q_1, \ldots, q_d),$$

where q_i is the number of equivalence classes obtained by choosing $q = i$. Observe that for $q = 0$ one obtains exactly the number of connected components of Δ viewed as a topological space, and for $q = d$ one simply obtains the number of simplices of maximal dimension. Atkin and others used *Q*-analysis to study phenomena such as traffic flow and television viewing habits (see e.g. [20]).

Laubenbacher became interested in *Q*-analysis as a potential tool to analyze the dynamic network of interactions in socio-technical complex systems. One goal was to associate qualitative measures with different dynamic modes of the system. As an example, consider a collection of stock traders, say at the New York Stock Exchange. The buying and selling decisions of each individual trader depend in part on information obtained from a variety of sources, on software that analyzes market trends, and on the actions of other select traders. How is the system affected when, for instance, one or more traders are equipped with faster data links than others? As another example, consider the drug traffic interception efforts of government authorities in the Southwestern US. Through a variety of means, including blimps stationed in strategic positions along the US–Mexican border, data are collected on air and ground traffic bringing illegal drugs into Arizona, California, New Mexico, and Texas. One smuggling method is to fly drugs to clandestine air strips on the US side of the border and then use other planes and ground transport for further distribution. Is it possible to use observed air traffic patterns of a partially known network of clandestine airstrips to reconstruct the unknown part? Finally, these kinds of questions have counterparts in other systems of interactions, such as the gene regulatory network of an organism or the interaction of species in an ecosystem.

While Q-analysis is sometimes useful for questions of this sort, it is a very crude invariant of a complex, just like the set of connected components of a topological space does not contain a great deal of information about the space. Atkin had realized this and proposed a definition for a group associated with a simplicial complex, similar to the fundamental group of a pointed topological space [2]. But it too should be an invariant of certain aspect of the combinatorial rather than the topological structure of the complex. A rigorous definition of such a group was given in [22], together with an algorithm for its computation. At that point it had become clear that this group had to be part of a general theory, with Atkin's Q-analysis representing dimension zero. The theory should be similar to the classical homotopy theory of a pointed topological space. However, it should depend on the combinatorial structure of the complex, rather than on its properties when viewed as a topological space. In applications, the individual simplices have interpretations that should not be lost in the computation of invariants. For instance, topologically, any two triangulations of a 2-sphere are equivalent, whereas combinatorially they will in general be very different.

Such a new combinatorial homotopy theory,

$$A_n^q(\Delta, \sigma_0), \quad n \geqslant 1, \ \ 0 \leqslant q \leqslant \dim(\Delta), \ \ \sigma_0 \in \Delta,$$

was presented in [6], termed A-theory, in honor of Atkin. It is similar to classical homotopy theory in some respects and different in others. Similarities include such properties as a Seifert–van Kampen Theorem for the combinatorial fundamental group, a long exact sequence associated with the relative theory, and the fact that the higher dimensional groups are abelian. Differences include, for instance, the fact that complexes that are contractible as topological spaces can have nontrivial A-groups, and lack of invariance under triangulation. Using a completely different definition, a combinatorial homotopy theory $A_n^G(\Gamma)$ for graphs Γ was also defined and related to the A-theory of simplicial complexes.

A fascinating aspect of A-theory is that once it was well defined and applied to different simplicial complexes, it was discovered that, in fact, it is related to constructions arising in quite diverse contexts. We briefly describe three examples that will be revisited in greater depth in the last section of this paper. In the early 1970s, Maurer, in his study of matroid basis graphs, was led to develop a homotopy theory for matroid complexes (see [27, Section 4]). As Maurer mentions, the classical notion of path homotopy applies to graphs, but his notion is not the same, nor is it the same as Tutte's [30]. It turns out that the A_1-group of a matroid corresponds exactly to Maurer's graph homotopy group. Later on, in 1977, Lovász [24] introduced new topological methods for proving some connectivity results in graph theory. One of his techniques consists of attaching 2-cells to all 3- and 4-cycles of a graph, before computing its (classical) fundamental group. It so happens that this computation is equivalent to calculating the A_1-group of the original graph. More recently, Babson et al. [5] discovered that the A_n-groups of the order complexes associated with the intersection lattice of some arrangements of linear subspaces coincide with the homotopy groups of the (real) complements of those arrangements, a fact also (independently) proved by Björner [9], for the case $n = 1$. All these constructions, and several more, can be formulated within the framework of A-theory, proving it to be an interesting theory.

In the next section we give the definition of A-theory, both for simplicial complexes and for graphs. The definitions are illustrated with examples. It is worthwhile to note that both definitions are important to understand all aspects of the theory, so both should be kept in mind. In Section 3, an algorithm for computing the abelianization of the A_1-groups is described, while in Section 4, we recall some classical behavior exhibited by the A-groups, and explain how the two definitions are related. The last two sections are devoted to several applications of A-theory.

1. Definitions and theorems

As mentioned in the introduction, there are two frameworks for A-theory, one using simplicial complexes and the other using graphs. The two approaches are closely related and we will recall them here. All details and proofs can be found in [6].

1.1. A_1 of simplicial complexes

We begin with a simplicial complex Δ of dimension d, a fixed integer q, with $0 \leqslant q \leqslant d$, and a given maximal simplex σ_0 (with respect to inclusion) of dimension greater than or equal to q. For further details regarding the following definitions see Section 2 of [6].

Definition 1.1. (1) Two simplices σ and τ of Δ are *q-connected*, if there is a sequence of simplices (in Δ)

$$\sigma, \sigma_1, \sigma_2, \ldots, \sigma_n, \tau,$$

such that any two consecutive ones share a q-face, that is, they have at least $q + 1$ vertices in common. Such a chain will be called a *q-chain*.

(2) The complex Δ is *q-connected*, if any two simplices in Δ of dimension greater than or equal to q are q-connected.

(3) A *q-loop in Δ based at* σ_0 is a q-chain beginning and ending at σ_0. Denote a q-loop $\sigma_0, \sigma_1, \ldots, \sigma_n, \sigma_0$ by $(\sigma_0, \sigma_1, \ldots, \sigma_n, \sigma_0) = (\sigma)$. Its *length* is n. (Note that the σ_i need not be distinct.)

Two such combinatorial q-loops of simplices are A-homotopic if they can be deformed into each other without breaking any q-dimensional connections. More precisely, we have the following definition.

Definition 1.2. Let $\simeq_A$ be the equivalence relation on the collection of q-loops in Δ, based at σ_0, generated by the following three conditions.

(1) The q-loop

$$(\sigma) = (\sigma_0, \ldots, \sigma_i, \sigma_{i+1}, \ldots, \sigma_n, \sigma_0)$$

is equivalent to the q-loop

$$(\sigma') = (\sigma_0, \ldots, \sigma_i, \sigma_i, \sigma_{i+1}, \ldots, \sigma_n, \sigma_0).$$

That is, loops can be “stretched” by repeating a simplex without changing its equivalence class.

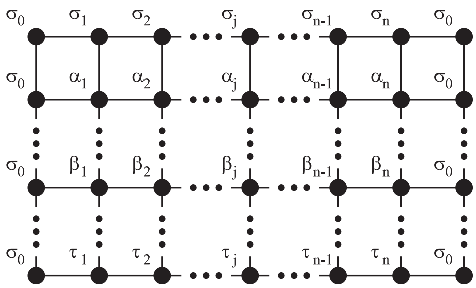

Fig. 1. σ and τ, two equivalent q-loops.

(2) Suppose that (σ) and (τ) have the same length. They are equivalent if there is a diagram as in Fig. 1. The diagram is to be interpreted as follows. A horizontal or vertical edge between two simplices indicates that they share a q-face. Each row in the diagram is a q-loop based at σ_0, while each column represents a q-chain starting at σ_i and ending at τ_i. Thus, (σ) is equivalent to (τ) $((\sigma) \simeq_A (\tau))$ if there is a sequence of q-loops based at σ_0 connecting them. Such a diagram is said to be an A-homotopy between (σ) and (τ).

(3) A q-loop is called *A-contractible* if it is A-homotopic to the constant q-loop at the base simplex σ_0.

This equivalence relation is called *A-homotopy*, and the equivalence class of a loop (σ) is denoted by $[\sigma]$, while the set of all equivalence classes is denoted by $A_1^q(\Delta, \sigma_0)$.

The next natural step is to concatenate q-loops based at σ_0 in order to obtain a *product* operation on $A_1^q(\Delta, \sigma_0)$. Having done so, it is easily shown that $A_1^q(\Delta, \sigma_0)$ is a group with unit element the equivalence class of the constant (or trivial) loop (σ_0). In this group, the inverse of an element $[\sigma]$ is given by the equivalence class of the same loop traversed in the opposite direction. So, we have obtained a family $\{A_1^q(\Delta, \sigma_0)\}$ of groups, one for each $0 \leqslant q \leqslant d = \dim(\Delta)$.

The subscript suggests that these definitions and groups might be extended to higher dimensions. Indeed, this is the case, and in fact, $\{A_1^q(\Delta, \sigma_0)\}$ is the A-counterpart of the fundamental group of a simplicial complex, $\pi_1(\Delta, \sigma_0)$. The generalization of these concepts to $\{A_n^q(\Delta, \sigma_0)\}$ groups can be found in [6] and will not be reproduced here. As it turns out, they are also the A-counterpart of the higher homotopy groups of a simplicial complex, $\pi_n(\Delta, \sigma_0)$. We now describe A-theory of graphs.

1.2. A_1 *of graphs*

The definition of A-theory for graphs parallels closely that of the homotopy groups of a topological space. We start by recalling some elementary constructions from graph theory. For more details see Section 5 of [6].

Definition 1.3. Let $\Gamma_1 = (V_1, E_1)$, $\Gamma_2 = (V_2, E_2)$ be simple graphs, that is, graphs without loops and multiple edges.

(1) The *Cartesian product* $\Gamma_1 \times \Gamma_2$ is the graph with vertex set $V_1 \times V_2$. There is an edge between (u_1, u_2) and (v_1, v_2) if either $u_1 = v_1$ and $u_2v_2 \in E_2$ or $u_2 = v_2$ and $u_1v_1 \in E_1$.
(2) A *graph map* $f : \Gamma_1 \longrightarrow \Gamma_2$ is a set map $V_1 \longrightarrow V_2$ such that, if $uv \in E_1$, then either $f(u) = f(v)$ or $f(u)f(v) \in E_2$.
(3) Let $\mathbf{I}_m$ be the graph with $m+1$ vertices labeled $0, 1, \ldots, m$, and edges $(i-1)i$ for $i = 1, \ldots, m$.
(4) Let $v_1 \in \Gamma_1$, $v_2 \in \Gamma_2$ be distinguished base vertices. A *based* graph map $f : (\Gamma_1, v_1) \longrightarrow (\Gamma_2, v_2)$ is a graph map such that $f(v_1) = v_2$.

Next, we recall G-homotopy of graph maps and G-homotopy equivalence of graphs.

Definition 1.4. (1) Let $f, g : (\Gamma_1, v_1) \longrightarrow (\Gamma_2, v_2)$ be based graph maps. Then f and g are called *G-homotopic*, denoted by $f \simeq_G g$, if there is an integer $m \geqslant 1$ and a graph map

$$\phi : \Gamma_1 \times \mathbf{I}_m \longrightarrow \Gamma_2,$$

such that $\phi(-, 0) = f$, and $\phi(-, m) = g$, and such that $\phi(v_1, i) = v_2$ for all i.

(2) We call (Γ_1, v_1) and (Γ_2, v_2) *G-homotopy equivalent* if there exist based graph maps $f : \Gamma_1 \longrightarrow \Gamma_2$ and $g : \Gamma_2 \longrightarrow \Gamma_1$ such that $gf \simeq_G \mathrm{id}_{\Gamma_1}$ and $fg \simeq_G \mathrm{id}_{\Gamma_2}$. The maps f and g are called *G-homotopy inverses* of each other.

(3) A graph map $f : (\Gamma_1, v_1) \longrightarrow (\Gamma_2, v_2)$ is *G-contractible* if it is G-homotopic to the graph map that sends all vertices (thus edges) to the base vertex v_2.

The base point for the graph $\mathbf{I}_m$ will be the vertex labeled 0, and the boundary $\partial(\mathbf{I}_m)$ of $\mathbf{I}_m$ consists of the vertices labeled 0 and m. Given this, $A_1^G(\Gamma, v_0)$ is the set of G-homotopy classes of graph maps

$$f : (\mathbf{I}_m, 0) \longrightarrow (\Gamma, v_0),$$

for all $m \geqslant 1$, such that $f(\partial \mathbf{I}_m) = v_0$. Note that we allow m to vary, that is, we allow arbitrarily fine subdivisions of the discrete "unit" interval, for it can be shown that two maps from the discrete unit interval of different heights can be viewed as being defined on the highest one, without change of homotopy type.

The equivalence class of a map f in $A_1^G(\Gamma, v_0)$ is denoted by $[f]$. For $A_1^G(\Gamma, v_0)$ to become a group, one needs an operation on its equivalence classes. Intuitively, if one represents a map $f : (\mathbf{I}_m, 0) \longrightarrow (\Gamma, v_0)$, by the chain $\mathbf{I}_m$ whose vertex i is labeled by $f(i)$, for all $0 \leqslant i \leqslant m$, and where $f(0) = f(m) = v_0$, then the group operation $[f] * [g]$ simply corresponds to "stacking" up the two labeled chains corresponding to f and g, in this order. It is a routine exercise to show that the stacking operation is well defined, and that $A_1^G(\Gamma, v_0)$ is a group.

2. Examples

2.1. Simplicial A-theory

(1) Consider the 2-dimensional simplicial complex, with four maximal faces of dimension 2, shown in Fig. 2, and let $q = 1$.

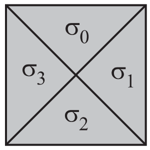

Fig. 2. A 2-dimensional complex with $A_1^1 = 1$.

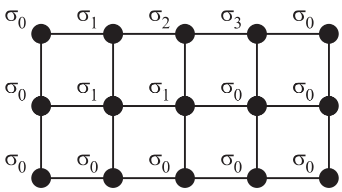

Fig. 3. Contraction of the 4-loop.

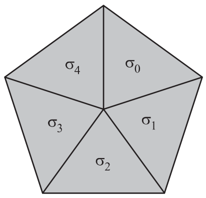

Fig. 4. A 2-dimensional complex Δ_2, with $A_1^1(\Delta_2) \simeq \mathbf{Z}$.

It is not difficult to "contract" the loop $(\sigma) = (\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_0)$ to the trivial constant loop (σ_0). Such a contraction is illustrated in Fig. 3. Moreover, one also easily sees that, for this complex, all the loops are A-contractible, thus making the A_1^1 group trivial. Note that the (classical) fundamental group of this complex is also trivial.

(2) On the other hand, if we look at the 2-dimensional simplicial complex Δ_2, shown in Fig. 4, which has five faces of dimension 2, one realizes (after some calculations) that the loop

$$(\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_0)$$

is not A-contractible. A combinatorial explanation in terms of a "gangster problem" is given in example (5). In fact, it can be shown that the A_1^1-group for this simplicial complex is isomorphic to $\mathbf{Z}$. In comparison, the (classical) fundamental group for this complex is clearly trivial, since the complex is contractible as a topological space.

But there is a way to modify this complex so that the non-contractible loop becomes A-contractible. Simply "fill in the combinatorial hole" of the complex by adding a new 2-dimensional simplex as is done in Fig. 5. A contraction of the loop $(\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_0)$ is shown in Fig. 6.

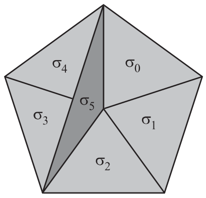

Fig. 5. Filling the combinatorial hole in Δ_2. $A_1^1(\Delta_2') \simeq *$.

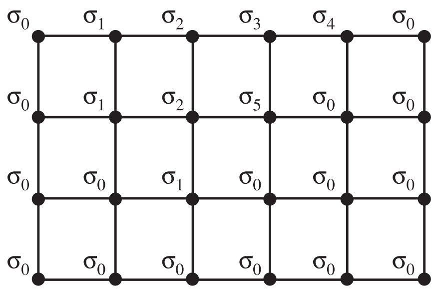

Fig. 6. Contraction of the loop $(\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_0)$.

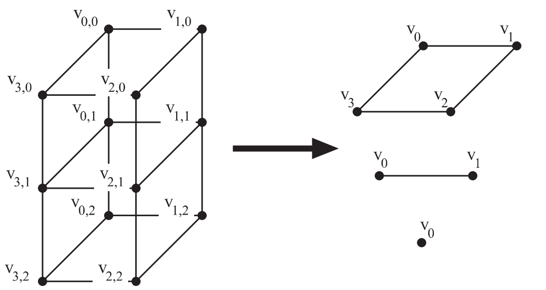

Fig. 7. Contraction of the square.

In this case the A_1^1-group becomes trivial as is the (classical) fundamental group of this modified complex Δ_2'.

2.2. A-theory of graphs

(3) The *A*-theory for graphs and for simplicial complexes are very similar. Consider the graph Γ consisting of a single cycle on four vertices $v_0 - v_1 - v_2 - v_3$. This cycle is *G*-contractible, as can be seen in Fig. 7. Indeed, a *G*-homotopy is given by the map

$$\phi : \Gamma \times \mathbf{I}_2 \longrightarrow \Gamma,$$

where on $\Gamma\times\{0\}$ the map is the identity. On $\Gamma\times\{1\}$, ϕ is defined by $\phi(v_0,1)=\phi(v_3,1)=v_0$, and $\phi(v_1,1)=\phi(v_2,1)=v_1$. Finally, on $\Gamma\times\{2\}$ all vertices are sent to v_0. Note that for esthetic purposes, in Fig. 7 the ordered pairs (v_i, j) are labeled $v_{i,j}$. Thus the A_1^G-group of the 4-cycle graph is trivial. One also notes that there are no *graph* maps

$$\phi : \Gamma \times \mathbf{I}_1 \longrightarrow \Gamma$$

that would "retract" the 4-cycle. Indeed, assuming that the interval $\mathbf{I}_m$ has length $m=1$ and that ϕ' is such a map, we must have $\phi'(v_i,0)=v_i$ and $\phi'(v_i,1)=v_0$ for all $0\leqslant i\leqslant 3$. But then, ϕ' is not a graph map, for while $(v_2,0)$ and $(v_2,1)$ are adjacent in the graph $\Gamma\times\mathbf{I}_1$ their images, v_2 and v_0 (respectively), are not adjacent in Γ. One realizes that the situation in the simplicial case is analogous. That is, the 4-loop in example (1) could not be contracted to the trivial loop without going through the intermediate loop, $(\sigma_0,\sigma_1,\sigma_1,\sigma_0,\sigma_0)$. One additional remark is worth mentioning. While interpreting the 4-cycle graph Γ as a loop in $\mathbf{R}^2$, one realizes that its (classical) fundamental group is nontrivial, and that in fact it is isomorphic to $\mathbf{Z}$. Note, however, that attaching a 2-cell to this cycle would yield a space with (classical) fundamental group trivial, and thus equal to $A_1^G(\Gamma)$.

(4) In a similar manner, one can easily verify that the 3-cycle is G-homotopy contractible as well. In this case, the obvious map $\phi' : \Gamma\times\mathbf{I}_1 \longrightarrow \Gamma$ (similar to the one defined in example (3)) is indeed the correct graph map, which contracts the 3-cycle to a point. Again, the situation is analogous to that of the A-contraction of loops with three simplices for the simplicial approach.

(5) On the other hand, the n-cycle graph, for $n\geqslant 5$, does not G-contract to a point. One way of seeing it for $n=5$ is via the creative interpretation given by Malle [25], and known as the *gangster problem*:

Suppose the vertices of a graph are towns and the edges, roads connecting the towns. In each town there is a member of a gangster syndicate. The gangsters decide to meet in one of the towns. For safety reasons they decide that each day they will move from one town to an adjacent one or rest in the same town and if two of the gangsters are in adjacent towns originally, then at all steps of the journey these gangsters must be in adjacent towns, or in the same town. The problem is: For which graphs is it possible for the gangsters to meet in one of the towns?

It is not difficult to see that, indeed, the restrictions on the gangsters' movements do correspond to our notion of G-homotopy of graphs. The days represent the interval $\mathbf{I}_m$ (if m days are needed) and the adjacency (or resting in the same town) restriction on the movements represent the notion of graph map. Thus, drawing a 5-cycle, as in Fig. 8, with vertices labeled 1, 2, 3, 4, 5 and with the additional edge $\{1,4\}$, one sees that the gangsters can all meet, for example, on the third day, in town 1. Indeed, on the second day, the gangsters from towns 4 and 5 moved to town 1, while the gangster from town 3 moved to town 2. On the other hand, it is clearly impossible for all the gangsters to meet at any time, in any town, if the additional edge (road) $\{1,4\}$ is not present. Again, one sees that the situation with the simplicial approach was similar. We had a non-contractible 5-loop of 2-dimensional simplices (sharing a 1-face) which could be A-contracted by filling a combinatorial hole with an additional 2-simplex.

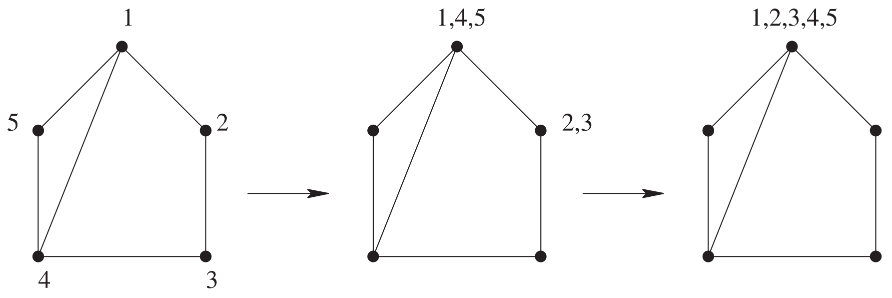

Fig. 8. A gangster meeting.

3. Calculation of $A_1^1(\Delta, \sigma_0)$

Computing the (abelianization) of A_1^q-groups turns out to be easier than one may first think. Moreover, it is via this computation that one is led to a deeper understanding of the link between the A_1^G- and A_1^q-groups. We quickly review this calculation here. For more details see [6]. In case the reader is wondering about the base simplex (or base vertex), it should be mentioned that if σ_0 and τ_0 (or v_0, t_0) are maximal simplices in Δ (vertices in Γ) that are q-connected (connected), then $A_1^q(\Delta, \sigma_0) \cong A_1^q(\Delta, \tau_0)$, (or, $A_1^G(\Gamma, v_0) \cong A_1^G(\Gamma, t_0)$).

Let $\Gamma = \Gamma^q(\Delta)$ be the graph with vertices corresponding to all simplices of Δ of dimension greater than or equal to q. Two vertices v and w are connected by an edge if and only if the corresponding simplices σ and τ share a q-face. Let v_0 be the distinguished vertex of Γ, corresponding to σ_0. This graph is said to be the q-connectivity graph of Δ. One realizes that there is a one-to-one correspondence between q-loops in Δ based at σ_0 and cycles in Γ that contain v_0. Recall that the topological fundamental group $\pi_1(\Gamma, v_0)$ is a free group with free generators. Moreover, to each cycle of Γ one can associate a specific element of $\pi_1(\Gamma, v_0)$. Let N be the normal subgroup of $\pi_1(\Gamma, v_0)$ generated by the elements corresponding to the 3- and 4-cycles of Γ.

Theorem 3.1 (*Barcelo et al.* [6, Theorem 2.7]). $A_1^q(\Delta, \sigma_0) \cong \pi_1(\Gamma, v_0)/N$.

It is also worth mentioning that one can replace Γ^q by the generally much smaller graph, $\Gamma^q_{\max}$, whose vertices correspond to all maximal simplices (with respect to inclusion) of Δ of dimension greater than or equal to q. From this theorem it is not too difficult to understand why the following one holds true.

Theorem 3.2 (*Barcelo et al.* [6, Theorem 5.16]). *Let Δ be a simplicial complex, with distinguished maximal simplex σ_0, $0 \leqslant q \leqslant \dim(\Delta)$. Let $\Gamma^q(\Delta)$ be the connectivity graph of Δ in dimension q, and $\Gamma^q_{\max}(\Delta) \subseteq \Gamma^q(\Delta)$ be the subgraph as defined above. Then*

$$A_1^q(\Delta, \sigma_0) \cong A_1^G(\Gamma^q(\Delta), v_0) \cong A_1^G(\Gamma^q_{\max}(\Delta), v_0).$$

Now that the relation between the A_1^G and A_1^q groups is well established, we can give a unified definition for the higher *A*-homotopy groups. Let

$$\mathbf{I}_m^n = \mathbf{I}_m \times \cdots \times \mathbf{I}_m$$

denote the *n*-fold Cartesian product of $\mathbf{I}_m$ for some *m*. $\mathbf{I}_m^n$ is called an *n-cube* of *height m*. Its *distinguished base point* is $\mathbf{O} = (0, \ldots, 0)$, and its *boundary*, $\partial \mathbf{I}_m^n$, is the subgraph of $\mathbf{I}_m^n$ containing all vertices with at least one coordinate equal to 0 or *m*. This being said, one can show that the subscript *m* in the above notation can be "omitted".

Definition 3.3. Let $A_n^G(\Gamma, v_0)$, $n \geqslant 1$, be the set of homotopy classes of graph maps

$$f : (\mathbf{I}^n, \mathbf{O}) \longrightarrow (\Gamma, v_0),$$

such that $f(\partial \mathbf{I}^n) = v_0$. For $n = 0$, $A_0^G(\Gamma, v_0)$ is the pointed set of connected components of Γ, with the component containing v_0 as distinguished element. The equivalence class of a map f in $A_n^G(\Gamma, v_0)$ is denoted by $[f]$.

Since all the boundary points of an *n*-cube are given the value v_0, one easily sees that the operation of "stacking" cubes makes sense (for $n \geqslant 1$). As one expects, it can be shown that the sets

$$A_n^G(\Gamma, v_0)$$

($n \geqslant 1$) are groups, and that Theorem 3.2 holds true for all $n \geqslant 1$.

So, if one computes the fundamental group of a graph, $\pi_1(\Gamma, v_0)$, and quotients out the normal subgroup generated by all 3- and 4-cycles, one obtains the A_1-group of this graph. But, inspired by Lovász' technique introduced in [24], one sees that the fundamental A_1^G-group of a graph Γ is isomorphic to the classical fundamental group of the topological space X_Γ obtained from Γ by attaching 2-cells along the boundary of each 3- and 4-cycle of Γ:

$$A_1^G(\Gamma, v_0) \cong \pi_1(X_\Gamma, v_0).$$

This was an important milestone enabling one to connect the *A*-theory to a wide variety of situations. Of course, the next natural question is whether there is an analogous topological space that can be constructed for all $n \geqslant 2$. The answer is yes, but requires more detail and (hard) work than one would initially envision. The details can be found in [4]. Intuitively, the space X_Γ is a cell complex obtained by successively attaching (for $m = 1, 2, \ldots$) *m*-dimensional cells to the *m*-cubes of Γ (possibly degenerate *m*-cubes; this is analogous to the fact that the 3-cycle (triangle) can be viewed as a degenerate 4-cycle (square)), yielding a *cubical* complex, that bears some resemblance to Kan complexes. By this we (roughly speaking) mean that if all the faces of an *m*-cube belong to the space X_Γ then the *m*-cube itself belongs to the space. This is another important step, for it connects *A*-theory to the realm of (real) arrangements of linear subspaces.

4. Classical behavior

As we saw through the examples of Section 2, A-theory and classical homotopy theory behave quite differently at times. It is now time to explore their similarities. As we saw in the last section, the $A_n^q(\Delta, \sigma_0)$ and $A_n^G(\Gamma, v_0)$ groups are intimately related. Thus, we shall refer to both of them at once, using A_n-groups for notation.

One of the first similarities comes from the fact that the A_1-fundamental groups, like their π_1 counterparts, are *not abelian* in general. Moreover, for all $n \geqslant 2$, the A_n-groups are abelian, likewise for the π_n groups of topological spaces.

Another common characteristic is a Seifert–van Kampen theorem. That is, one may ask whether one can compute A_1 of a simplicial complex (graph) as the free product of A_1 of appropriate subcomplexes (subgraphs), modulo A_1 of their intersection. This is indeed true, even though one must add an extra condition on the intersection of the subcomplexes (subgraphs), in addition to requiring q-connectivity of the complex (graph), the subcomplexes (subgraphs), and the intersection. This extra condition amounts to requiring that if there are 3- or 4-cycles with some of their simplices (vertices) belonging to the intersection then these 3- and 4-cycles must entirely lie in one of the subcomplexes (subgraphs) that are being intersected.

The last similarity we shall mention here is in connection to a relative A-theory. Relative A-groups with respect to a subcomplex (subgraph) have been defined. Briefly, given a subgraph (subcomplex) $\Gamma' \subset \Gamma$, and a distinguished $(n-1)$-face F of I^n, $(n \geqslant 2)$, the relative $A_n(\Gamma, \Gamma', v_0)$ group is the set of all A-homotopy classes of graph maps

$$f : (\mathbf{I}^n, \mathbf{O}) \longrightarrow (\Gamma, v_0),$$

such that F is mapped into Γ', together with the natural multiplication. One familiar with the classical homotopy theory will recognize this definition as the A-analog of the relative homotopy theory for simplicial complexes. So we do obtain a long exact sequence, and, as expected, the A_n-relative groups are abelian.

5. Applications

In this section we review recent applications of A-theory and connections to work related to A-theory.

5.1. Maurer's approach

In [27] Maurer studied matroid basis graphs. The basis graph of a matroid has a vertex for each basis and an edge for each pair of bases that differ by the exchange of a single pair of elements. It was well known that, for any connected graph Γ, the spanning trees, viewed as sets of edges, form the bases of a matroid. The basis graphs, called tree graphs (for the vertices correspond to the spanning trees of Γ and are denoted here by Γ_{T}), of such matroids had been already extensively studied [15,17,19,28]. Maurer's contribution was to completely characterize basis graphs for all matroids. For this, he studied the common neighbor (CN) subgraphs of the basis graph of a matroid. In a graph G, if the distance between two vertices v, v' is 2, then the set of vertices consisting of v, v' and all vertices adjacent to both is called a common neighbor subgraph. Maurer's main theorem (in its first form) is the following:

Theorem 5.1 (*Maurer [27, Theorem 2.1]*). *G is a basis graph of a matroid if and only if*:

- *it is connected*;
- *each common neighbor subgraph is a square, a pyramid, or an octahedron*;
- *in every leveling each common neighbor subgraph satisfies the positioning condition*; *and*
- *for some v_0 the neighborhood subgraph $N(v_0)$ is the line graph of a bipartite graph.*

We shall not go into details of conditions (3) and (4). Suffice it to say that while the first two conditions are relatively easy to verify, these other two are generally not reasonably dealt with. In fact, this is the reason that motivated Maurer to look for a condition that might replace them. For this, he developed the following notion of homotopy.

(1) If the distance between two vertices v_{k-1} and v_{k+1} of G is equal to 2, then the paths $P_1 = v_1 \cdots v_{k-1} v_k v_{k+1} \cdots v_n$ and $P_2 = v_1 \cdots v_{k-1} v'_k v_{k+1} \cdots v_n$ are said to differ by a 2-switch. In our language this simply means that the 4-cycle $(v_{k-1} v_k v_{k+1} v'_k)$ allows us to A_1-deform P_1 into P_2 as seen in Fig. 3.
(2) If the distance between v_{k-1} and v_{k+1} is equal to 1 and $P_3 = v_1 \cdots v_{k-1} v_{k+1} \cdots v_n$, then P_1 and P_3 are said to differ by a shortcut. For us, this means that the 3-cycle $v_{k-1} v_k v_{k+1}$ allows us to directly A_1-deform P_1 into P_2.
(3) If $v_{k-1} = v_{k+1}$ and $P_4 = v_1 \cdots v_{k-1} v_{k+2} \cdots v_n$, then P_1 and P_4 are said to differ by a deletion. Again, for us this simply means that the path P_1 can be A_1-deformed into the pathP_4 by simply repeating the vertex v_{k-1}.
(4) Finally, Maurer declares two paths *homotopic* if one can be transformed into the other by a finite sequence of these elementary deformations.

It is easy to see that Maurer's notion of homotopy is equivalent to our notion of A_1-homotopy. Then Maurer goes on to prove that if Γ is a basis graph, then any two paths with the same end-points are homotopic. This is simply stating that $A_1^G(\Gamma)$ is trivial. A word of caution is in order here. While it is tempting to deduce that "surely" the A_1^G-group of a graph whose CN subgraphs are either a square, a pyramid or an octahedron must be trivial (since these subgraphs are clearly A-contractible, as they consist of 3- and 4-cycles), this is not necessarily the case. One look at Maurer's proof reveals that conditions (3) and (4) are needed to prove this theorem.

But the interesting fact is that Maurer hoped that his notion of homotopy would replace conditions (3) and (4) of his main theorem. Indeed, Maurer finishes his paper with the conjecture (still open) that Γ is a basis graph if and only if

(1) Γ is connected;
(2) each CN is a square, pyramid, or octahedron; and
(3) $A_1^G(\Gamma)$ is trivial.

One last important remark is that Maurer also claims (without proof) that *his* notion of homotopy (thus our notion of A_1^G) might be the right one for graphs since

$$A_1^G(\Gamma_1 \times \Gamma_2) = A_1^G(\Gamma_1) \times A_1^G(\Gamma_2),$$

where $(\Gamma_1 \times \Gamma_2)$ denotes the Cartesian product of graphs, and $A_1^G(\Gamma_1) \times A_1^G(\Gamma_2)$ represents the direct product of groups. This is not difficult to show using our definition of A_1^G.

5.2. Lovász's approach

In 1975, at the Fifth British Combinatorial Conference in Aberdeen, Frank [18] and Maurer [26] presented the following problem.

Theorem 5.2. *Let Γ be a k-connected graph, $\{v_1, \ldots, v_k\} \subseteq V(\Gamma)$, and $n_1, \ldots, n_k$ positive integers with $n_1 + \cdots + n_k = n = |V(\Gamma)|$. Then there exists a partition $\{V_1, \ldots, V_k\}$ of $V(\Gamma)$ such that*

(1) $v_i \in V_i$,
(2) $|V_i| = n_i$,
(3) V_i *spans a connected subgraph of* Γ $(i = 1, \ldots, k)$.

The case $k = 2$ is rather easy, and Frank, Milliken, Györi and Lovász independently provided solutions for the case $k = 3$. But the most interesting one was provided by Lovász, for it was the only solution that could be generalized to all $3 \leqslant k \leqslant n$. The idea of the proof was based on the following innovative concept.

Given a graph Γ, add a new point a and connect it to $v_1, v_2, \ldots, v_k$. Denote this new graph by Γ'. Next, construct a graph $\tilde{\Gamma}$ whose vertices are the spanning trees T_i of Γ', and whose edges (T_i, T_j) are pairs of spanning trees whose intersection $T_i \cap T_j$ contains a tree on $n-1$ vertices including a. So T_j can be obtained from T_i by replacing an endline by another endline. Then Lovász proved that if Γ is k-connected, then $\tilde{\Gamma}$ is connected. Finally, the last step in the proof consisted in constructing a cellular complex C_Γ (called the arborescence complex of Γ, relative to a) which is

- simply connected and
- for which the homology groups (relative to a) $H^0(C_\Gamma) = \cdots = H^{k-2}(C_\Gamma)$ are all trivial,

whenever Γ is k-connected.

One notices how similar the construction of $\tilde{\Gamma}$ is to the construction of the tree graph Γ_{T} associated to Γ (as described in the subsection on Maurer's work). In fact, this connection is further developed and studied by Björner et al. in [10]. As Björner mentions [8], in the language of greedoids (which matroids are), Lovász's arborescence complex is the basis complex (in Maurer's sense) of the branching greedoid determined by the rooted graph Γ'.

Moreover, also notice the strong similarity with our concept of q-connectivity. We build graphs from simplicial complexes by setting the vertices to be q-simplices and edges between two of them if they are *sufficiently* (q-) connected.

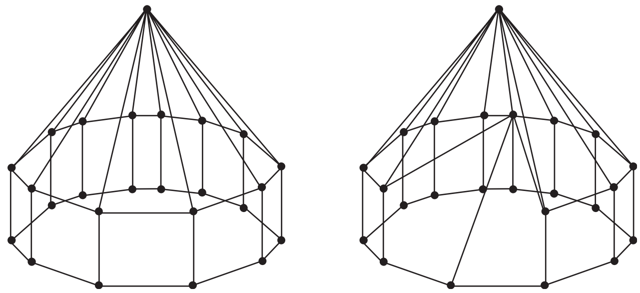

Fig. 9. Planar (left) and pseudoplanar (right) nets.

But the similarity does not stop here. While we do not want to recall the construction of the full arborescence complex C_Γ, it is instructive to recall it in the case $k = 3$. Consider the triangles (3-cycles) and quadrilaterals (4-cycles) in $\tilde{\Gamma}$ and span a 2-cell on each of them, to get the topological space C_Γ. On sees immediately that this is also the space $X_{\tilde{\Gamma}}$ described in Section 3. Lovász goes on proving that if Γ is a 3-connected (k-connected as well) graph then the (classical) fundamental group of the cell complex C_Γ is trivial, yielding (for the case $k = 3$) in the A-language, that the A_1-group of $X_{\tilde{\Gamma}}$ is trivial.

The fact that this group is trivial is the last ingredient enabling Lovász to prove Theorem 5.2. Even though we have not yet worked out all the details, we do believe that Lovász's arborescence complex would correspond to a subcomplex of our general infinite dimensional cell complex X_Γ, and that the property that the fundamental group of the arborescence complex is trivial would also translate to the A-group of the appropriate subcomplex of X_Γ being trivial, thus enlarging the pool of applications of A-theory.

5.3. *Malle's approach*

In 1983, Malle [25] developed a homotopy group for graphs, and defined what he calls the string group $S(\Gamma)$ of a graph. It turns out that $S(\Gamma) = A_1^G(\Gamma)$. While Malle realized that the (classical) fundamental group of a graph is isomorphic to his string group, $S(\Gamma)$, whenever a graph has girth greater than or equal to 5, he does not make the step of showing that $S(\Gamma)$ is indeed isomorphic to the quotient $\pi_1(\Gamma)/N$ where N is the normal subgroup of $\pi_1(\Gamma)$ generated by 3- and 4-cycles. He also does not generalize his notion of string groups to higher dimensions. On the other hand, Malle gives a complete description of graphs that have trivial A_1^G-group.

Theorem 5.3 (*Malle [25, Theorem 6]*). *A graph Γ has trivial A_1^G-group if and only if it is connected and each cycle of Γ has a pseudoplanar net in Γ.*

While we will not give the details of the definitions of planar and pseudoplanar nets, a look at Fig. 9 reveals how it works. From the planar net (left) one sees that the lower cycle is easily A-deformed into the upper one via a series of 4-cycles, then the upper cycle is A-contracted to a point via 3-cycles. In the pseudoplanar net (right), the situation is a bit different: the lower cycle is A-deformed (via 4-cycles) into two 6-cycles which in turn are both A-contracted to a point.

5.4. Link to subspace arrangements

After having developed A-theory it is natural to try it on diverse simplicial complexes, in particular those interesting to combinatorialists. The first author admits to being entirely biased toward the order complexes. Given a poset P, its *order complex* $\Delta(P)$ is the simplicial complex on the vertex set P whose k-faces are the k-chains $x_0 < x_1 < \cdots < x_k$ in P.

The first poset for which we computed (the abelianization) of A_1^q-groups was the Boolean lattice B_n, that is, the poset of all subsets of the set $\{1, 2, \ldots, n\}$, ordered by inclusion. The first computations gave the following results. (They were carried out with software written by Luis Garcia, available from the authors.)

- $A_1^0(\Delta(B_3))^{ab}$ is a free abelian group on 1 generator,
- $A_1^1(\Delta(B_4))^{ab}$ is a free abelian group on 7 generators,
- $A_1^2(\Delta(B_5))^{ab}$ is a free abelian group on 31 generators,
- $A_1^3(\Delta(B_6))^{ab}$ is a free abelian group on 111 generators,
- $A_1^4(\Delta(B_7))^{ab}$ is a free abelian group on ? generators?

For a long time the orders of the next groups, $A_1^q(\Delta(B_n))^{ab}$ (for $n \geqslant 7$ and $q = n - 3$), remained unknown, for the computational complexity associated with the construction of the order complex of a lattice grows very rapidly. No closed formula was known nor even conjectured despite attempts (prior to 2001) at finding such a sequence in the On-Line Encyclopedia of Integer Sequences by Sloane [29]. What was then known is the following: To compute $A_1^{n-3}(\Delta(B_n))$ one draws the graph $\Gamma_{\max}^{n-3}(\Delta(B_n))$ whose vertices correspond to the maximal chains of B_n, and whose edges correspond to pairs of maximal chains that differ in exactly one place. But one realizes that this coincides with the 1-skeleton of the permutahedron. Indeed, the permutahedron Π_{n-1} is defined as the convex hull of all vectors that are obtained by permuting the coordinates of the vector $(1, 2, \ldots, n)$. Its vertices can be identified with the permutations of S_n (the symmetric group over n elements) in such a way that two vertices are connected by an edge if and only if the corresponding permutations differ by an adjacent transposition. The permutahedron is a classical object; see [31] for a nice account of its combinatorial properties. One notes that there are no 3-cycles in the 1-skeleton of the permutahedron, for this would mean that one could write the identity permutation as a product of three transpositions; the same reasoning yields that in fact there are only even length cycles. A picture of the 3-dimensional permutahedron Π_3, that is, the convex hull of all the vectors obtained by permuting the coordinates of the vector $(1, 2, 3, 4)$ can be seen in Fig. 10.

Thus, computing (the abelianization of) $A_1^{n-3}(\Delta(B_n))$ corresponds to computing the fundamental group of the 1-skeleton of the permutahedron to which one attaches 2-cells to each 4-cycle. For example, in the case of Π_3 after filling the 6 squares, one is left with 8 hexagons, of which 7 are generators for $A_1^1(\Delta(B_4))$. Even though one had a nice description of the $X_{\Gamma(\Delta(B_n))}$-space no one knew how to compute its homotopy group. That

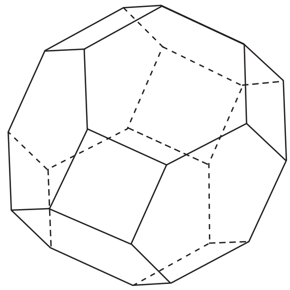

Fig. 10. The 3-dimensional permutahedron, Π_3.

is, until January 2001, when Eric Babson realized that this construction (adding 2-cells to each 4-cycle of the 1-skeleton of the permutahedron) yields a space which is homotopy equivalent to the complement $M_{n,3}$ of the (real) 3-equal arrangement!

The space $M_{n,3}$ is defined as follows: For $2 \leqslant k \leqslant n$, let $V_{n,k}$ be the set of points $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbf{R}^n$ such that $x_{i_1} = x_{i_2} = \cdots = x_{i_k}$ for some k-set of indices $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n$. $M_{n,k} = \mathbf{R}^n - V_{n,k}$. The k-equal arrangements have been extensively studied, and, for example, Björner and Welker [13] had a formula for the dimensions of the corresponding homology groups. Thus Babson was able to conclude that $A_1^{n-3}(\Delta(B_n))^{ab}$ is a free abelian group on

$$2^{n-3}(n^2 - 5n + 8) - 1$$

generators. In the meantime the "Bjorner–Welker" sequence of integers had also been entered into the Sloane collection of integer sequences, since it occurs in other contexts as well.

In the language of A-theory, Babson's result is as follows:

Theorem 5.4 (*Babson [5]*).

$$A_1^{n-3}(\Delta(B_n)) \cong \pi_1(M_{n,3}).$$

Proof. We shall give a very informal proof here, since the ideas are simple while the necessary notation (and formal concepts) would only obscure the argument.

First, recall that the braid arrangement consists of all the (real) hyperplanes $H_{i,j} = \{(x_1, \ldots, x_n) | x_i = x_j\}$, for $1 \leqslant i < j \leqslant n$. The 3-equal ($k$-equal) arrangement embeds in the braid arrangement as each subspace of $V_{n,3}$ ($V_{n,k}$) is an intersection of some of its hyperplanes. Given a (finite) hyperplane arrangement in $\mathbf{R}^n$, its intersection with the (real) $(n-1)$-dimensional sphere, S^{n-1} yields a cell complex. The dual of this cell complex is a zonotope (i.e., the Minkowski sum of the line segments which are normal to the hyperplanes in the arrangement). It turns out that for the braid arrangement the dual zonotope is the permutahedron.

Next, it is also known that the complement in $\mathbf{R}^n$ of the subspace arrangement is homotopic to the space obtained by removing the faces of the zonotope corresponding to the subspaces belonging to the arrangement. For a proof of this fact see Proposition 3.1

in [14]. Let us see what those faces are for the 3-equal arrangement when $n = 4$. First, one must remove the interior of the permutahedron (the unique 3-dimensional face), since it corresponds to the subspace of all vectors of the form (x, x, x, x), for $x \in \mathbf{R}$. Second, one must remove all 2-dimensional hexagons since they correspond to subspaces similar to $\{(x, x, x, y) | x, y \in \mathbf{R}\}$, that is, a set of vectors with three of their coordinates equal. One does not remove the 2-dimensional squares, for those correspond to subspaces similar to $\{(x, x, y, y) | x, y \in \mathbf{R}\}$ which do *not* belong to the 3-equal arrangement. One then realizes that this space is certainly homotopic to the one obtained by attaching 2-cells to the squares of the 1-skeleton of the permutahedron Π_3. □

With a bit more effort one can see how this argument generalizes to the following theorem, also independently proved by Björner [9].

Theorem 5.5 (*Babson [5]*).

$$A_m^q(\Delta(B_n)) \cong \pi_m(M_{n,n-q}).$$

We should remark that the interest in obtaining information on these spaces $M_{n,k}$ arose in connection with a problem from computer science (see [11,12]). It was shown that the Betti numbers of $M_{n,k}$ are the essential ingredient in finding a lower bound for the complexity of deciding membership in $V_{n,k}$, using linear decision trees. More precisely the link with the k-equal arrangement comes from the k-equal problem: given n real numbers $x_1, x_2, \ldots, x_n$, and an integer $k \geqslant 2$, how many comparisons $x_i \geqslant x_j$ are needed to decide if some k of them are equal? Note that the comparisons $x_i - x_j \geqslant 0$ are special cases of the so-called linear tests: $l(x) \geqslant 0$. Thus the geometric reformulation reads:

Given the subspace arrangement $V_{n,k}$, how many linear tests are needed (by the best algorithm in the worst case) to decide if $\mathbf{x} \in V_{n,k}$ for points $\mathbf{x} \in \mathbf{R}^n$?

Thus it was natural to expect that the topological complexity of the arrangement had some bearing on the complexity of the algorithm. This is where the Betti numbers of $M_{n,k}$ entered the scene. For more details on this topic see [7,11,12].

5.5. *Link to pseudomanifolds*

There is one last connection to yet another area of mathematics that deserves to be mentioned. Recently, Joswig [21] introduced a (finite) group of projectivities, $\Pi(\Delta)$, for each simplicial complex Δ that is strongly connected, finite dimensional, and pure. The motivation to introduce such groups was to solve a coloring problem for simplicial polytopes which arose in the area of toric algebraic varieties. Joswig's idea consists in associating a finite group to each facet (maximal face with respect to inclusion) of Δ. For this, he first constructs what he calls the *dual graph*, $\Gamma(\Delta)$, of Δ. The vertices of this graph are the facets of Δ and there is an edge between two facets if the two facets share a codimension 1 face! This is exactly our $\Gamma_{\max}^{d-1}(\Delta)$ if the dimension of Δ is equal to d. Even though Joswig works with pure simplicial complexes (all facets have the same dimension) his definition of the dual graph is valid for non-pure complexes as well. Next, for each codimension 1 face contained in two facets σ, τ there is a unique element $v(\sigma, \tau)$ which is contained in σ but not in τ. Joswig's *perspectivity* $[\sigma, \tau] : \sigma \to \tau$ is defined by setting

$$w \mapsto \begin{cases} v(\tau, \sigma) & \text{if } w = v(\sigma, \tau), \\ w & \text{otherwise.} \end{cases} \tag{5.1}$$

For a path $g = (\sigma_0, \sigma_1, \ldots, \sigma_n)$ in $\Gamma^{d-1}_{\max}(\Delta)$, the *projectivity* $[g]$ (from σ_0 to σ_n) along g is the concatenation

$$[g] = [\sigma_0, \sigma_1, \ldots, \sigma_n] = [\sigma_0, \sigma_1][\sigma_1, \sigma_2][\sigma_2, \sigma_3] \cdots [\sigma_{n-1}, \sigma_n]$$

of perspectivities. Thus the map $[g]$ is a bijection from σ_0 to σ_n. As usual, a loop based at σ_0 is simply such a path that starts and ends at σ_0. Joswig then realizes that the projectivity of the concatenation of two paths is simply the concatenation of the corresponding projectivities, that is, $[g * h] = [g][h]$. Then Joswig's *group of projectivities* of Δ at σ_0, $\Pi(\Delta, \sigma_0)$, is the set of projectivities along loops based at σ_0. It is a (permutation) subgroup of the symmetric group on the set of vertices of σ_0. Despite the strong similarity with the $A_1^{d-1}(\Delta, \sigma_0)$ the groups are not isomorphic in general. Nevertheless, both groups have similar properties. Moreover, one of the connections between these groups was discovered by De Longueville and Reiner [23]. They showed that if Δ is a d-simplicial pseudomanifold which does not contain a triangle as a minor (i.e., a link in some vertex-induced subcomplex) then there is a well-defined surjective homomorphism from $A_1^{d-1}(\Delta, \sigma_0)$, dealing with “galleries of facets” of Δ, to Joswig's group of projectivities $\Pi(\Delta, \sigma_0)$.

There are still several other applications of A-theory to other branches of mathematics such as the *wonderful models* of subspace arrangements, as introduced by De Concini and Procesi in [16], and buildings, to name a few. We close this paper with an application to the type of problems that provided the initial motivation for the development of A-theory.

6. Combinatorial time series analysis

In this example we use A-theory to analyze multivariate time series of data in cases where additional information about the local correlation between variables is available. This is the case, for instance, when the time series arises from agent-based computer simulations and we have knowledge about the interaction of agents at each time step. The techniques developed are applicable to a wide range of real and simulated systems, including such diverse examples as search-and-rescue operations and router networks for Internet packet traffic. Both examples can be represented as autonomous agent systems with a need for coordination through information exchange. Individual agents make decisions by interacting with other agents, and coordination, or control, of the system relies in an essential way on an understanding of the global structure of interaction flow.

We associate with a time series of system data a partially ordered set from which in turn we derive a simplicial complex that provides global models for the dynamic structure generated by local variable interdependence. The feature of special interest to us are the structural properties of the flow of interactions in such interaction networks, ways to measure and characterize it, and, ultimately, the ramifications of these measures for a control theory of interaction networks.

As an example we discuss here a computer simulation that the second author has co-developed, together with Michael Coombs at New Mexico State University's Physical Science Laboratory, and Abdul Jarrah, presently at VBI. First, we briefly describe the simulation, which we call AGENT, in its simplest form. AGENT consists of autonomous agents $x_1, \ldots, x_n$, each of which performs an unspecified task that takes *m* time steps. After completing the task, each agent follows a procedure whereby it must file a report in an external database before being allowed to continue. The database, however, can only process a limited number of agents at a time (has a limited channel capacity set by the parameter *S*). Thus, when the number of agents ready to report exceeds the size of the channel, a queue of waiting agents forms. The size of the queue at any one time will depend on both the size of the channel and the degree of agent synchronization (i.e., how many agents are in need of the database at any one time). An agent's goal is to maximize the percentage of time that it is working at its task (i.e., it is assumed to be delay adverse). Its "fitness" is, therefore, defined as the percentage of elapsed time that it is *not* delayed. Agents, therefore, have an interest in desynchronizing from those with whom they are frequently in contention for access to the database. Such desynchronization is regulated by a system-wide protocol whereby, having experienced a certain degree of contention, an agent can impose desynchronizing delays on the agent, or agents, who have blocked its database access. The research described here concerns the analysis of patterns of desynchronization interactions arising from agents following this protocol.

The state of agent x_i is a vector

$$w_i = (p_1, \ldots, p_{i-1}, p_{i+1}, \ldots, p_n),$$

where p_j is the number of times that agent x_i was delayed by agent x_j. Each agent x_i has assigned to it a threshold T_i which represents its tolerance to delay. When $\Sigma_j p_j > T_i$, then agent x_i selects a fixed number P_i of agents and delays them one time step (i.e., the response multiplier—"fan-out"—parameter). The agents to be delayed are selected by x_i using a decision function f_i attached to x_i. This function can be selected to be either deterministic or stochastic. After delaying the other agents, x_i changes its state w_i to the zero vector. The system is initialized by assigning vectors w_i to all agents. For the simulation results described here, the parameters were set to be the same for all agents, that is, $P_i = P$ and $T_i = T$ for all *i*. The channel capacity parameter *S* is set as constant for a simulation run.

For each simulation of *m* time steps we construct a poset as follows. It has elements x_{ij}, $i = 1, \ldots, n$, $j = 1, \ldots, m$, corresponding to agents x_i at time *j*. It is best to think of these as arranged in horizontal rows, with *j* indicating the row number. The order relation is generated by the following (covering) rule:

$$x_{ij} < x_{k,j+1},$$

if agent x_i delays agent x_k during the transition from time *j* to time $j + 1$. Associated with this poset, or segments of it, we can now consider two simplicial complexes, its order complex and its *covering complex*, which is the simplicial complex generated by the lower-order ideals. For purposes of computation the order complex is of limited value, since it is generally far too large. The findings below made use of the covering complex only.

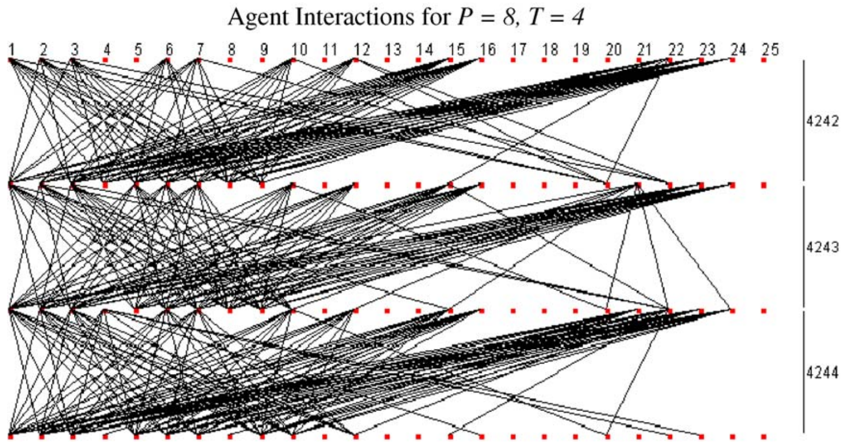

Fig. 11. Interaction poset over three time steps.

It was observed that the AGENT synchronization protocol yields many different mechanisms both across parameterizations, and within a single pair of parameter settings. The invariants of the simplicial complexes provide such a rich range of options for defining structure in collective desynchronization events within a single time step that the cataloging of mechanisms has only just begun. However, as an example of what there is to discover, we describe one mechanism. This involves a very interesting phenomenon we have termed "dynamic clustering". Let Δ be the covering complex for the interaction poset of AGENT for a specified time series, and fix q. A *dynamic q-cluster* of AGENT is a complete subgraph of the Γ^q-graph of Δ. Fig. 11 contains an example of a 3-time-step output from AGENT. Fig. 12 illustrates the dynamic q-clusters for each time step.

Dynamic clusters regularly appear in simulation runs. We have found that the participants in these clusters typically have lower than average fitness values. In other words, they form "frozen cores" of agents within the agent set, around which the other agents are more or less free to move. Since this typically happens for relatively high P values, delay events have a relatively high impact, and so agents will tend to freeze into permanent states of delay, and the clusters expand by attracting additional agents.

However, we have observed that, with very high frequency, these clusters are prevented from becoming permanently frozen by the evolution of nontrivial elements of the A_1-groups of the associated covering complexes; these act to free up agents. Fig. 12 contains such an example. Agent 18 has high fitness at time step 4242, and A_1 of the covering complex of the poset representing the interaction at time step 4242 is trivial. Then, in time step 4243, Agent 18 has become part of a large cluster, which reduced its fitness. At the same time A_1 of the order complex at step 4243 has a nontrivial element, with a representative indicated with bold edges, involving Agents 2, 9, 10, 4, and 1. In time step 4244, Agent 18 has left the dynamic cluster again, has regained its high fitness, and the A_1-group in time step 4244 is back to being trivial. Fig. 13 shows the parameter ranges that produce this phenomenon.

At present we do not have an explanation for this phenomenon and are unable to state and prove precise results about it. We view this as an important open research question.

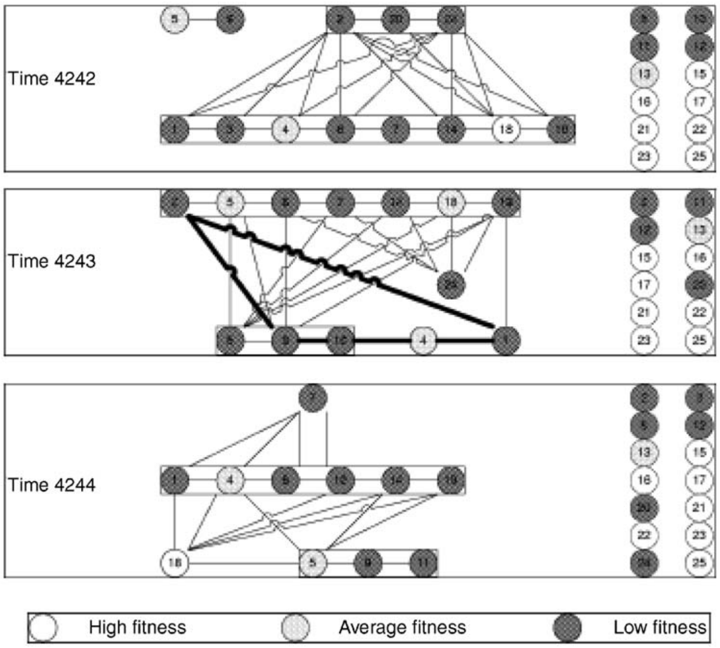

Fig. 12. Dynamic q-clusters in the interaction poset.

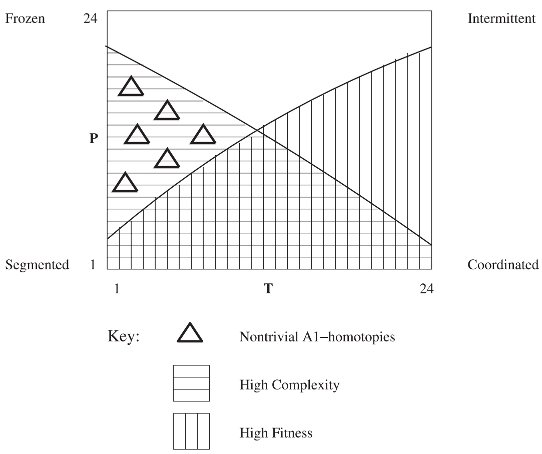

Fig. 13. Parameter space.

Acknowledgements

The authors would like to thank the referees for their helpful comments and suggestions and for the patience of one of the referees with the first author's written English.

References

[1] R. Atkin, An algebra for patterns on a complex, I, Internat. J. Man–Machine Studies 6 (1974) 285–307.
[2] R. Atkin, An algebra for patterns on a complex, II, Internat. J. Man–Machine Studies 8 (1976) 448–483.
[3] R. Atkin, Multidimensional Man, Penguin, London, 1981.
[4] E. Babson, H. Barcelo, M. de Longueville, R. Laubenbacher, A Homotopy Theory for Graphs, preprint, 2003.
[5] E. Babson, H. Barcelo, R. Laubenbacher, *A*-homotopy theory and arrangements of linear subspaces, in preparation.
[6] H. Barcelo, X. Kramer, R. Laubenbacher, C. Weaver, Foundations of a connectivity theory for simplicial complexes, Adv. in Appl. Math. 26 (2001) 97–128.
[7] A. Björner, Subspace arrangements, Proceedings of the First European Congress of Mathematics, vol. I, Paris, 1992, Progr. Math., ed., vol. 119, Birkhäuser, Basel, 1994, pp. 321–370.
[8] A. Björner, Topological methods, in: Handbook Of Combinatorics, vol. II, MIT Press, North-Holland, Cambridge, MA, Amsterdam, 1995, pp. 1819–1872.
[9] A. Björner, personal communication, 2001.
[10] A. Björner, B. Korte, L. Lovász, Homotopy properties of greedoids, Adv. in Appl. Math. 6 (1985) 447–494.
[11] A. Björner, L. Lovász, Linear decision trees, subspaces arrangements, and Möbius functions, J. Amer. Math. Soc. 7 (1994) 677–706.
[12] A. Björner, L. Lovász, A. Yao, Linear decision trees: volume estimates and topological bounds, Proceedings, 24th ACM Symposium on Theory of Computing, New York, ACM, New York, 1992, pp. 170–177.
[13] A. Björner, V. Welker, The homology of "*k*-equal" manifolds and related partitions lattices, Adv. in Math. 110 (1995) 277–313.
[14] A. Björner, G. Ziegler, Combinatorial stratification of complex arrangements, J. Amer. Math. Soc. 5 (1) (1992) 105–149.
[15] J.A. Bondy, Pancyclic graphs. II, in: R.C. Mullin, (Ed.), Proceedings of the Second Louisiana Conference on Combinatorics, Graph Theory and Computing, Baton Rouge, Louisiana, 1971 (summary).
[16] C. De Concini, C. Procesi, Wonderful models of subspaces arrangements, Selecta Math. I (1995) 459–494.
[17] R.L. Cummins, Hamiltonian circuits in tree graphs, IEEE Trans. Circuit Theory 13 (1966) 82–90.
[18] A. Frank, Some polynomial algorithms for certain graphs and hypergraphs, Proceedings of the Fifth British Combinatorial Conference, Aberdeen, 1975, Congr. Numerantium XV, 1976, pp. 221–226.
[19] C.A. Holzmann, F. Harary, On the tree graph of a matroid, SIAM J. Appl. Math. 22 (1972) 187–193.
[20] J. Johnson, The mathematical revolution inspired by computing, The Mathematics of Complex Systems, Oxford University Press, Oxford, 1991.
[21] M. Joswig, Projectivities in simplicial complexes and colorings of simple polytopes, Math. Z. 240 (2) (2002) 243–259.
[22] X. Kramer, R. Laubenbacher, Combinatorial homotopy of simplicial complexes and complex information networks, in: D. Cox, B. Sturmfels (Eds.), Applications of Computational Algebraic Geometry, vol. 53, Proceedings of the Symposium in Applied Mathematics, American Mathematical Society, Providence, RI, 1998.
[23] M. De Longueville, V. Reiner, personal communication, 2001.
[24] L. Lovász, A homology theory for spanning trees of a graph, Acta Math. Acad. Sci. Hungar. 30 (3–4) (1977) 241–251.
[25] G. Malle, A homotopy theory for graphs, Glasnik Mathematicki 18 (38) (1983) 3–25.
[26] S.B. Maurer, Problem presented, at the Fifth British Combinatorial Conference, Aberdeen, 1975.
[27] S.B. Maurer, Matroid basis graphs, I, J. Combin. Theory Ser. B 14 (1973) 240–261.
[28] H. Shank, A note on Hamiltonian circuits in tree graphs, IEEE Trans. Circuit Theory 15 (1968) 86.
[29] N.J.A. Sloane, On-line encyclopedia of integer sequences, http://www.research.att.com/njas/sequences/
[30] W.T. Tutte, A homotopy theorem for matroids, I–II, Trans. Amer. Math. Soc. 88 (1958) 144–174.
[31] G.M. Ziegler, Lectures on polytopes, Graduate Texts in Mathematics, Springer, Berlin, 1994.