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David Lewin

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A FORMAL THEORY OF GENERALIZED TONAL FUNCTIONS

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Our point of departure is a common method of constructing a system of tonal functions and relationships given a tonic pitch-class T, a dominant interval d and a mediant interval m. The method is portrayed visually by Figure 1.

In the figure we see a C major system constructed given the tonic pitch class $T=C$, the dominant interval d=fifth (modulo the octave), and the mediant interval m=major third (modulo the octave). From the tonic note C we construct notes a dominant and a mediant interval away, that is $G=C+d$ and $e=C+m$. Together with the tonic note, these dominant and mediant notes constitute the tonic triad of the system. Next we construct a dominant triad, comprising the dominant note G together with the notes $D=G+d$ and $b=G+m$ which lie the intervals d and m respectively from that G. Similarly, we construct a subdominant triad comprising the tonic note C, the subdominant note F from which C lies the interval d away, and the submediant note a which lies the interval m from the subdominant note F. Since F is constructed to satisfy the relation $F+d=C$, we may symbolically write $F=C-d$; then $a=F+m=C-d+m$.

The set-theoretic union of the three triads so far generated can be called the "diatonic set" for the system, that is, the unordered set

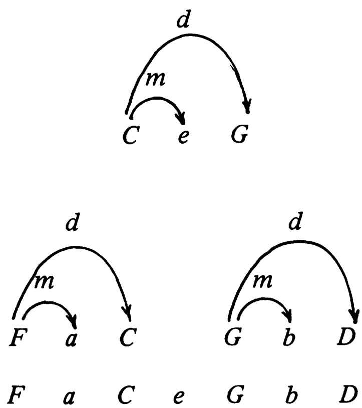


Figure 1

comprising the seven pitch classes under present consideration. The method of generation suggests a specific ordering for the diatonic set, condensing and collating the structure developed in the upper part of Figure 1. This ordering is displayed by the series *FaCeGbD* at the bottom of the Figure. The series will be called the "canonical listing" for the diatonic set of the system under construction. Upper-case letter names are used to denote pitch classes generated by the tonic note and various multiples of the dominant interval: C, $F=C-d$, $G=C+d$, $D=G+d=C+2d$. Lower-case letter names denote pitch classes whose generation involves the mediant interval: $e=C+m$, $b=G+m$, $a=F+m$.

Certain features of the system, it will be noted, are necessary consequences of the constructive method itself, whatever the sizes of d and m. For example, the dominant and subdominant triads must necessarily be transposed forms of the tonic triad. The interval from the mediant note $T+m$ to the tonic note T must be $-m$, and the interval from the mediant note $T+m$ to the submediant note $T-d+m$ must be $-d$; from these relations it follows that the "submediant triad," that is, that set comprising the notes $T-d+m$, T, and $T+m$, must be an inversion of the tonic triad. And so on.

Beyond such necessary internal features of the system, the constructive method also induces necessary formal relations between the system and certain other similarly constructed systems. For example, the system of Figure 1, whose canonical listing is *FaCeGbD*, bears certain necessary relations to other systems of similar construction, for example, those with canonical listings *FA♭Ce♭G♭♭D*, *D♭♭Ge♭Ca♭F*, *CeGbDf♯A*, *DfAcEgB*, and so forth.

In this paper we shall generalize the constructive method for Figure 1. We shall see to what extent traditional features of tonal theory depend formally only upon that method, rather than upon the specific d and m involved. We shall explore a number of non-tonal systems which can be constructed by the method, systems for which pertinent analogous features obtain. Such systems, while shedding light on the analogous formal features of tonal theory, also suggest interesting possibilities for composition and analysis in their own right.

There is a long and rich tradition in the history of tonal theory behind Figure 1. The formalistic approach of the discussion above to matters such as the interrelations of triads and systems can be particularly associated with the spirit and work of Hugo Riemann. For that reason, I shall call the basic abstract structure we will study a "Riemann System," (hereafter, "RS").[1]

Riemann systems. All notes and intervals in the following text will be understood modulo the octave, unless otherwise qualified. To adopt one convention consistently, we shall use additive notation for intervals, as in the discussion of Figure 1. Thus if X is some pitch class and i is some interval (each modulo the octave), then $X+i$ is the pitch class lying the interval i from X, and X lies the interval i from the pitch class $X-i$. As the context may suggest, i can be imagined to be measured in equally tempered semitones *modulo 12,* or in cents *modulo 1200,* or as the logarithm of a just ratio *modulo log*2, and so forth.

DEFINITION 1. *By a* ***Riemann System*** *(RS) we shall understand an ordered triple (T,d,m), where T is a pitch class and d and m are intervals, subject to the restrictions that* $d \neq 0$, $m \neq 0$, *and* $m \neq d$.

The restrictions are necessary and sufficient to guarantee that the pitch classes T, $T+m$ and $T+d$ will be distinct, so that we can talk of a "triad" without awkwardness. T will be called the "tonic pitch class" of the system; d and m are its "dominant and mediant intervals."

DEFINITION 2. *The* ***tonic triad*** *of the RS (T,d,m) is the unordered set* $(T, T+m, T+d)$. *The* ***dominant triad*** *of the system is* $(T+d, T+d+m, T+2d)$. *The* ***subdominant triad*** *of the system is* $(T-d, T-d+m, T)$. *These triads are the* ***primary triads*** *of the system.*

One sees that the primary triads are transposed forms, each of any other.

DEFINITION 3. *The* ***diatonic set*** *of (T,d,m) is the* ***unordered*** *set-theoretic union of the primary triads, comprising the various pitch classes* $T-d$, $T-d+m$, T, $T+m$, $T+d$, $T+d+m$, *and* $T+2d$.

Given the restrictions of Definition 1, there may be anywhere from 3 to 7 distinct notes (pitch classes) in the diatonic set. For instance, if d is exactly one-third of an octave, then $T-d$ and $T+2d$ will be the same note, here representing two distinct functions: subdominant and dominant-of-the-dominant.

DEFINITION 4. *The* ***canonical listing*** *for (the diatonic set of) the RS (T,d,m) is the* ***ordered series*** $(T-d, T-d+m, T, T+m, T+d, T+d+m, T+2d)$.

This ordered series will always have exactly seven entries, even if the same note be entered more than once. For example, if d is exactly half an octave and m is exactly a quarter of an octave, then the canonical listing for (C,d,m) can be written as *F♯aCe♭G♭b♭♭D♭♭*, with the convention that *F♯* and *G♭* are the same pitch class, and so forth. This series has seven entries, representing the seven functions of the system, but the unordered diatonic set itself has only four distinct notes.

The constructive methods so far discussed are visually manifest in Figure 2, which generalizes Figure 1.

The primary triads are displayed in the canonical listing at the bottom of Figure 2, as entries #1 through #3, #3 through #5, and #5 through #7 on that listing. The 3-note sets comprising entries #2 through #4 and entries #4 through #6 also play formal roles generalizing the tonal situation. Definition 5 provides the formalities.

DEFINITION 5. *The* ***mediant triad*** *of the RS (T,d,m) is the unordered set $(T+m, T+d, T+d+m)$. The* ***submediant triad*** *is the unordered set $(T-d+m, T, T+m)$. These are the* ***secondary triads*** *of the system.*

As we have already noted, the secondary triads are necessarily inverted forms of the primary triads.

The unordered set comprising entries #7, #1 and #2 on the canonical listing, that is $T+2d$, $T-d$ and $T-d+m$, is not necessarily of secondary-triad form. In fact, it need not even comprise three distinct pitch classes. For instance, if $m=3d$ (modulo the octave), then $-d+m=2d$ and $T-d+m=T+2d$. In order for $T+2d$, $T-d$, and $T-d+m$ to form a set of secondary-triad type, it is both necessary and sufficient for $T-d+m$ to lie the dominant interval away from $T+2d$, that is for $T-d+m$ to equal $(T+2d)+d$, that is for $T-d+m$ to equal $T+3d$. A little arithmetic shows that this will be the case if and only if m equals $4d$. The condition is satisfied by major tonality *with temperament,* using any temperament in which the "major third" is equivalent to four "fifths," modulo the octave. The condition is, however, not characteristic of Riemann Systems in general.[2]

Redundant and irredundant systems.

DEFINITION 6. *A Riemann System will be called* ***redundant*** *if its diatonic set has fewer than seven members,* ***irredundant*** *if its diatonic set has exactly seven members.*

Table 1 lists a few redundant Riemann Systems, using twelve-tone equal temperament and measuring intervals in equal semitones. The canonical listings in the Table use letter names for pitch classes, with the usual conventions regarding enharmonic spelling. The diatonic sets in the Table are written using integer notation for pitch classes (0 for *C*, 1 for *C*♯, . . . , 11 for *B*).

Table 1 shows that the diatonic set of a redundant Riemann System may have 3, 4, 5 or 6 members. The Table also shows how some interesting pitch-class sets of these cardinalities can be generated as diatonic sets for various Riemann Systems. This might prove interesting in the analysis or composition of music involving these sets. A particular

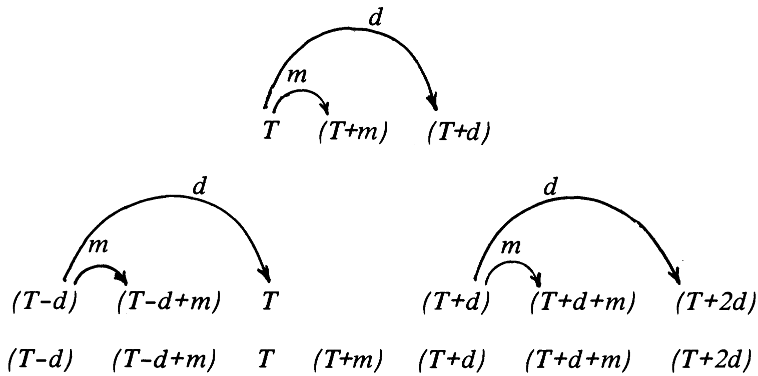


Figure 2

Table 1

RS	canonical listing	diatonic set
$(C, 8, 4)$	*F♭a♭CeG♯b♯D♯*	(0, 4, 8)
$(C, 6, 3)$	*F♯aCe♭G♭b♭♭D♭♭*	(0, 3, 6, 9)
$(C, 7, 2)$	*FgCdGaD*	(0, 2, 5, 7, 9)
$(C, 2, 4)$	*B♭dCeDf♯E*	(10, 0, 2, 4, 6)
$(C, 4, 7)$	*A♭e♭CgEbG♯*	(11, 0, 3, 4, 7, 8)
$(C, 7, 10)$	*Fe♭Cb♭GfD*	(10, 0, 2, 3, 5, 7)
$(C, 2, 6)$	*B♭eCf♯Dg♯E*	(0, 2, 4, 6, 8, 10)
$(C, 4, 2)$	*A♭b♭CdEf♯G♯*	(0, 2, 4, 6, 8, 10)

pitch-class set can be the diatonic set for more than one kind of Riemann System: for example, the whole-tone scale is generated by two essentially different systems on the Table.

A natural theoretical question presents itself: what are necessary and sufficient conditions on d and m, in order that the Riemann System (T,d,m) should be irredundant? Theorem 1 answers this question.

THEOREM 1. *For the RS (T,d,m) to be irredundant it is necessary and sufficient that conditions (1) and (2) following obtain.*

(1) For $N=1$, 2 or 3, Nd is not zero. (That is, one, two or three dominant intervals do not come out an exact number of octaves. That is, d is not 0, 6, 4, or 8 equally tempered semitones.)

(2) For $N=0$, ± 1, ± 2, or 3, m is not equal to Nd. (This condition does not exclude the possibility that m may equal $-3d$.)

The theorem can be verified by a systematic inspection of various intervallic relations among the various notes $T-d$, $T-d+m$, T, $T+m$, $T+d$, $T+d+m$, and $T+2d$. For example, $T-d$ and $T+d+m$ are the same note if and only if $-d=d+m$, which is the case if and only if $m=-2d$. Table 2 below essentially lists all irredundant Riemann Systems in twelve-tone equal temperament. The precise meaning of "essentially" here will become clear during subsequent discussion.

The first two columns of Table 2 contain all combinations of values for d and m that satisfy the conditions of Theorem 1, up through $d=5$. For instance, $d=1$ is allowed by condition (1) of the Theorem; for $d=1$, condition (2) disallows $m=1$, $m=-1(=11)$, $m=2$, $m=-2(=10)$, and $m=3$. The remaining possible values for m, that is $m=4,9,5,8,6$ and 7, are listed in the second column of the Table opposite $d=1$ in the first column. The order in which these values for m are listed will be discussed later. We shall see that it is sufficient to list on the Table values of d through $d=5$ only: irredundant systems with $d=7,9,10$ or 11 can be derived from systems with $d=5,3,2$, or 1 respectively.

The third column of Table 2 contains the names of the Riemann Systems with tonic note C and with dominant and mediant intervals corresponding to the entries in the first and second columns of the table. The fourth column contains the canonical listings generated by the Riemann Systems entered in column three. The usual spelling conventions for enharmonic equivalence are assumed. The last column contains the Forte-labels for the chord types of the diatonic sets at issue.[3] Where no label appears in column five of the Table, the last-written label is understood to carry on down. For example, the type of the diatonic set for $(C,1,8)$ is 7-5; the type of the diatonic set for $(C,3,10)$ is 7-31.

With two exceptions, each system on Table 2 is paired with another system, which we shall call its "conjugate system." The exceptions are

Table 2

d	m	system	canonical listing	Forte-label of diatonic set
1	4	$(C, 1, 4)$	*B e♭ C e C♯ f D*	7-1
1	9	$(C, 1, 9)$	*B g♯ C a C♯ b♭ D*	
1	5	$(C, 1, 5)$	*B e C f C♯ f♯ D*	7-5
1	8	$(C, 1, 8)$	*B g C g♯ C♯ a D*	
1	6	$(C, 1, 6)$	*B f C f♯ C♯ g D*	7-7
1	7	$(C, 1, 7)$	*B f♯ C g C♯ g♯ D*	
2	1	$(C, 2, 1)$	*B♭ b C c♯ D e♭ E*	7-1
2	3	$(C, 2, 3)$	*B♭ c♯ C e♭ D f E*	7-2
2	11	$(C, 2, 11)$	*B♭ a C b D c♯ E*	
2	5	$(C, 2, 5)$	*B♭ e♭ C f D g E*	7-23
2	9	$(C, 2, 9)$	*B♭ g C a D b E*	
2	7	$(C, 2, 7)$	*B♭ f C g D a E*	7-35
3	1	$(C, 3, 1)$	*A b♭ C d♭ E♭ e F♯*	7-31
3	2	$(C, 3, 2)$	*A b C d E♭ f F♯*	
3	4	$(C, 3, 4)$	*A c♯ C e E♭ g F♯*	
3	11	$(C, 3, 11)$	*A g♯ C b E♭ d F♯*	
3	5	$(C, 3, 5)$	*A d C f E♭ g♯ F♯*	
3	10	$(C, 3, 10)$	*A g C b♭ E♭ c♯ F♯*	
3	7	$(C, 3, 7)$	*A e C g E♭ b♭ F♯*	
3	8	$(C, 3, 8)$	*A f C a♭ E♭ b F♯*	
5	1	$(C, 5, 1)$	*G g♯ C c♯ F f♯ B♭*	7-14
5	4	$(C, 5, 4)$	*G b C e F a B♭*	
5	6	$(C, 5, 6)$	*G c♯ C f♯ F b B♭*	7-7
5	11	$(C, 5, 11)$	*G f♯ C b F e B♭*	
5	8	$(C, 5, 8)$	*G e♭ C a♭ F d♭ B♭*	7-35
5	9	$(C, 5, 9)$	*G e C a F d B♭*	

$(C,2,1)$ and $(C,2,7)$: each of these systems is its own conjugate. To see the salient mathematical aspect of the conjugate relationship, notice that $(C,1,4)$ and $(C,1,9)$ are paired, and that $4+9=1$. $(C,1,5)$ is paired with $(C,1,8)$, and $5+8=1$. $(C,2,3)$ is paired with $(C,2,11)$, and $3+11=2$. $(C,3,7)$ is paired with $(C,3,8)$, and $7+8=3$. $(C,2,1)$ is self-conjugate, and $1+1=2$. In general, (T,d,m) and (T,d,m') are conjugate systems if $m+m'=d$. Some formal definitions are in order.

DEFINITION 7. *The* ***conjugate system*** *of the RS (T,d,m) is the Riemann System $(T,d,d\text{-}m)$. The operation that transforms a given RS into its conjugate will be called "CONJ". We shall write, symbolically, $CONJ(T,d,m)=(T,d,d\text{-}m)$.*

The conjugate operation can be applied to any Riemann System, irredundant or redundant. Formally, it is necessary to verify that $(T,d,d\text{-}m)$ is indeed always a "Riemann System" whenever (T,d,m) is. That is, one must verify that $d\neq 0$, $d-m\neq 0$ and $d\neq d-m$, supposing that $d\neq 0$, $m\neq 0$ and $d\neq m$. This is easily done. In discussing subsequent transformations of Riemann Systems, we shall always understand such methodological niceties to have been carried out.

Given a Riemann System (T,d,m), let us set $m'=d-m$. The conjugate system articulates the dominant interval d, within each primary triad, as $d=m'+m$, rather than $d=m+m'$. See Figure 3.

The relation between original and conjugate subdivisions of the interval d thus generalizes such traditional notions as arithmetic-versus-harmonic division of d, and such traditional relations as that of *CeG* to *Ce♭G*.[4]

Extending the format of Figure 3 to cover the complete canonical listings of the systems under examination, we obtain a relationship portrayed visually by the upper two-thirds of Figure 4.

Let us now read the conjugate listing backwards. This produces a listing for a "retrograde conjugate" system. As the bottom part of Figure 4 indicates, the latter system has dominant interval $-d$ and mediant interval $-m$. In this sense, it is an inversion of the given system. In fact, the canonical listing for the *RET CONJ* system is a serial inversion of the canonical listing for the original system. Figure 4 makes this relation clear: the seven successive notes of the original listing span the six successive intervals m, m', m, m', m, m', while the seven successive notes of the retrograde conjugate listing (reading from right to left at the bottom of Figure 4) span the six successive intervals $-m, -m', -m, -m', -m, -m'$.

As the Figure indicates, the tonic note for the retrograde-conjugate system is the dominant note of the original system; likewise the dominant note of the retrograde-conjugate system is the tonic note of the

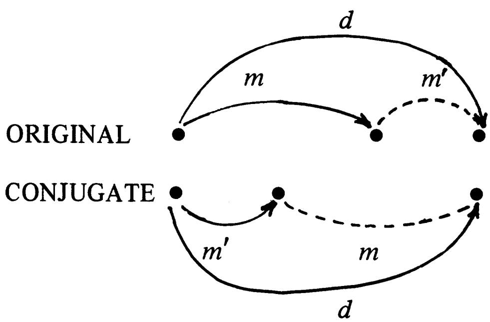


Figure 3

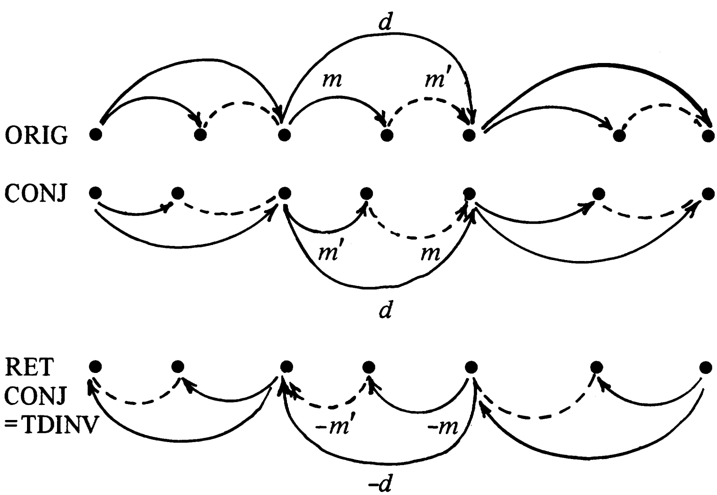


Figure 4

original system. Since the retrograde-conjugate relation interchanges tonic and dominant notes of the systems involved, while serially inverting the canonical listings involved, we shall call the transformation "TD-inversion."

DEFINITION 8. *The TD-inversion of the RS (T,d,m) is the Riemann System $(T+d, -d, -m)$. The transformation which takes any RS into its TD-inversion will be called "TDINV." We shall write, symbolically, $TDINV(T,d,m)=(T+d, -d, -m)$.*

Let us now return to Table 2. As we read the various canonical listings in the fourth column of that table backwards, we shall thereby be reading, en masse, the canonical listings for the *TD*-inversions of the various Systems appearing in column 3 of the Table. For instance, the *F* major Riemann System $(F,7,4)$ has dominant note *C*; hence it is the *TD*-inversion of the System $(C,-7,-4)=(C,5,8)$. As the *TD*-inversion of $(C,5,8)$, $(F,7,4)$ is also the retrograde-conjugate of $(C,5,8)$. So the listing for $(F,7,4)$ can be read, on Table 2, as the retrograde of the listing for the conjugate system of $(C,5,8)$, that is as the retrograde of the listing for $(C,5,9)$.

Generalizing this example, we can now see in what sense Table 2 lists "essentially" all types of irredundant Riemann Systems in twelve-tone equal temperament. If $d=1,2,3$ or 5 and the pair (d,m) satisfies the conditions of Theorem 1, then the System (C,d,m) appears directly on Table 2. If $d=7,9,10$ or 11 and the pair (d,m) satisfies the conditions of Theorem 1, then the pair $(-d,-m)$ will also satisfy the conditions of the Theorem, so that the System $(C,-d,-m)$ will appear on the Table. One can then read the canonical listing for a System with dominant interval *d* and mediant interval *m* by reading backwards the canonical listing for the conjugate system of $(C,-d,-m)$ off the Table.

Since the retrograde of the conjugate listing is a serial inversion of a given listing, it follows that the conjugate listing itself is a retrograde-inversion of the given listing. This property is easily seen and heard as manifested among the listings of Table 2; it explains why the listings paired together on the Table sound so closely related.

Given a Riemann System and its conjugate, since the canonical listings of their diatonic sets are retrograde-inversions, each of the other, it follows that the unordered diatonic sets themselves must be pitch-class inversions, each of the other. Hence, if the Systems are irredundant, they will have the same Forte-number entered in column 5 of Table 2.

One can also see why every irredundant Riemann System appearing on the Table with dominant interval $d=3$ must have 7-31 as the Forte-number of its diatonic set. In such a System, the notes $T-d$, T, $T+d$ and $T+2d$ will be $T-3$, T, $T+3$ and $T+6$; these four notes will constitute a "diminished seventh chord." And the three notes $T\text{-}d\text{+}m$, $T\text{+}m$ and $T\text{+}d\text{+}m$ will constitute a "diminished triad" which will be disjunct from the diminished seventh chord (since the System is irredundant.) Hence the seven notes of the diatonic set will be an octotonic scale missing one note, and any such collection must be of Forte-form 7-31.

Ignoring conjugate relationships, the Forte-number 7-7 appears essentially twice in the fifth column of Table 2. This is evidently related to the presence of tritones in the canonical listings involved. The Riemann Systems whose diatonic sets have this Forte-type are precisely those irredundant (!) Systems (T,d,m) such that either $m=6$ or $m'=d-m=6$.

It is interesting that the Forte-number 7-35 appears essentially twice on the Table, once for the "circle of fifths" System $(C,2,7)$ and once for the (inverted) "tonal" Systems $(C,5,8)$ and $(C,5,9)$. One sees here the importance of temperament as a means of compromising the Pythagorean and just intonations that the respective canonical listings more "naturally" suggest. The systems that share the Forte-number 7-1 in column 5 are the circle-of-fifths transforms of those that share the Forte-number 7-35.

The irredundant systems of Table 2, like the redundant systems of Table 1, might prove useful for analysing appropriate music; they also suggest interesting compositional possibilities.

The serial group. When we say that the *TD*-inversion of a Riemann System is the "retrograde of the conjugate" of that system, we are implicitly invoking the notion of transforming a *System*, not just a listing, by an operation called "retrogression." We should now be more formal about this notion. By the "retrograde system" of a given *RS* (T,d,m) we clearly mean that *RS* whose canonical listing is the retrograde of the canonical listing for (T,d,m). Figure 5 displays some salient features of the structure involved.

Inspecting the Figure, we see that the tonic note of the retrograde system will be the dominant note $T\text{+}d$ of the original system. The dominant interval of the retrograde system will be $-d$, and its mediant interval will be $-m'$, where $m'=d-m$. We can put all of this into a formal definition.

DEFINITION 9. *The* ***retrograde*** *of the RS (T,d,m) is the Riemann System $(T\text{+}d, -d, m-d)$. The* ***transformation*** *which takes any RS into its* ***retrograde*** *will be called "RET". We shall write, symbolically, $RET(T,d,m)=(T\text{+}d, -d, m-d)$.*

It will be helpful to collate various aspects of Figures 3, 4, and 5 into Figure 6, which displays some interrelationships among the primary triad structures of an original Riemann System, its retrograde, its conjugate and its *TD*-inversion.

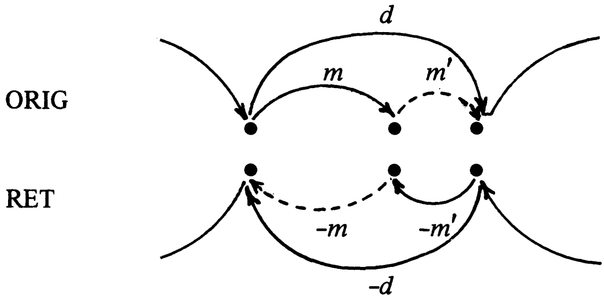


Figure 5

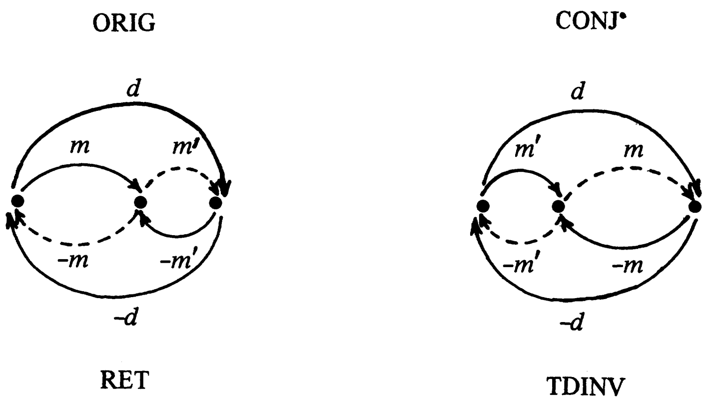


Figure 6

As Figure 6 suggests visually, the conjunction of a Riemann System with its conjugate, *TD*-inverted and retrograde systems gives rise to a closed and symmetrical collection of mutually interrelated Riemann Systems. One can hear this in the features that group together aurally any pair of conjugate listings and their retrogrades, on Table 2, as distinct from all other listings on that table. The closure and symmetry suggested by Figure 6 reflect the fact that the operations *CONJ, TDINV* and *RET*, along with the operation of "letting stand" a system, form what mathematicians call a "group of operations" on the family of all Riemann Systems. We shall pursue farther the implications of this fact. First, we need a formal label for the operation that "lets stand" any given Riemann System, producing the identical given System as a result of the operation.

DEFINITION 10. *By the* ***identity operation*** *on the family of Riemann Systems, we mean the operation IDENT which, when applied to the specimen RS* (T,d,m), *yields* (T,d,m) *itself as a result. We write symbolically* $IDENT(T,d,m)=(T,d,m)$.

We will now explore various relationships which hold true among combinations of the operations *IDENT, TDINV, RET* and *CONJ*. The sorts of relations that will concern us here are those which can be expressed by what we shall call operational equations. An example of such an equation is the symbolic assertion *RET CONJ=TDINV*. This equation states that the retrograde of the conjugate, of any given *RS*, is the same System as the *TD*-inversion of that given System. The assertion is true, as we recall from the pertinent discussion earlier (see pp. 31-33). We could also demonstrate the truth of the assertion by applying certain substitutions and algebraic manipulations to the definitions of the operations involved. I shall work out that demonstration in the next paragraph, for those readers who are interested; those who are not may omit or defer study of this exercise, which is paradigmatic for the rigorous demonstration of the truth of such equations.

We are to show that, given any Riemann System (T,d,m), the System $RET(CONJ(T,d,m))$ is the same as the System $TDINV(T,d,m)$. Let us denote by $(T\sharp,d\sharp,m\sharp)$ the System $CONJ(T,d,m)$; then we are to show that $RET(T\sharp,d\sharp,m\sharp)$ is the same system as $TDINV(T,d,m)$. Invoking Definition 9, we infer that we are to show the relation $(T\sharp+d\sharp,-d\sharp,m\sharp-d\sharp)=TDINV(T,d,m)$. Now $(T\sharp,d\sharp,m\sharp)$ is the conjugate of (T,d,m); by Definition 7 we know the latter System to be $(T,d,d-m)$; hence $T\sharp=T$, $d\sharp=d$ and $m\sharp=d-m$. Making these substitutions as appropriate, we see that the System $(T\sharp+d\sharp,-d\sharp,m\sharp-d\sharp)$ is the System $(T+d,-d,(d-m)-d)$ or $(T+d,-d,-m)$. So what we are supposed to show is that $(T+d,-d,-m)=TDINV(T,d,m)$. And that is indeed the case, as stated by Definition 8.

Another example of an operational equation is *RET RET=IDENT.* This equation asserts that the retrograde system of the retrograde system, of any given *RS,* is identically the given *RS* itself. The truth of the equation is easily intuited by considering the canonical listings involved. One can also prove the truth of the equation by the technique of substitution and algebra used for the proof of the preceding example. One sees in the format of the present equation the utility of having a formal operation called *IDENT,* to participate in such relationships.

The basic equations holding true among members of our family of four operations can be presented synoptically by Table 3, which mathematicians might call a "combination table" for the family.

The format of Table 3 is similar to that of an ordinary multiplication table: if the operation *Z* is entered on the table in the row headed by *X* on the left and in the column headed by *Y* at the top, then the operational equation *X Y=Z* is true. For instance, in the row of Table 3 with *TDINV* to the left, and in the column headed by *RET*, we find the entry *CONJ.* This means that the equation *TDINV RET=CONJ* is true. I.e.: the *TD*-inversion of the retrograde system, of any given *RS*, is the conjugate system of that given *RS*.

By inspecting Table 3 it is easy to verify that the family comprising *IDENT, TDINV, RET* and *CONJ* forms a group of operations in the mathematical sense. It satisfies the two requirements which a (non-empty) family of operations must satisfy to form a group. First, members of the family, combined among themselves, will always produce a net result which is operationally equal to some member of the family: they will not generate any "new" operations when applied one after another. Formally, if *X* and *Y* are members of the family (possibly the same member), then there is some *Z* in the family (possibly *X* or *Y* itself) which satisfies the operational equation *XY=Z*.

Second, for every member *X* of the family there is some member *Y* of the family (possibly *X* itself) which satisfies the equations *XY=YX=IDENT.* In this situation, *Y* is uniquely determined by *X*; it is called the "inverse (operation)" of *X*. Given *X* and its inverse *Y*; if *X* transforms operand 1 into operand 2, then *Y* transforms operand 2 into operand 1. For *Y*(operand 2)=*Y*(*X*(operand 1))=*IDENT*(operand 1)=operand 1. Thus, given a pair of inverse operations, each "undoes" the effect of the other. In the case of the family involved in Table 3, each member operation combines with itself to form the identity operation, and hence is its own inverse, "undoing" itself when applied a second time after its first application.

DEFINITION 11. *The family comprising the four operations IDENT, TDINV, RET and CONJ will be called the **serial group** of operations*

Table 3

	IDENT	TDINV	RET	CONJ
IDENT	IDENT	TDINV	RET	CONJ
TDINV	TDINV	IDENT	CONJ	RET
RET	RET	CONJ	IDENT	TDINV
CONJ	CONJ	RET	TDINV	IDENT

on (the set of all) Riemann Systems. The group will be denoted by GSER.

The name I have given this group arises from the effects its operations induce on the canonical listing of a given System. As we have already noted, the operations cited in Definition 11 respectively preserve, invert, retrograde and retrograde-invert the canonical listing of a given *RS* into the canonical listings of the respectively transformed Systems. This aspect of the serial group relates certain traditional tonal transformations, when generalized, to other well-known transformational structures usually considered only in the context of more recent Western musics. The relationship, in both its formal and cultural-historical aspects, is certainly more than fortuitous. It will, I think, repay deeper thought. For the present, let us only observe that the "serial" aspect of *GSER* is crucially dependent on the ordering for diatonic sets given by the concept of canonical listing; scalar ordering of the diatonic sets involved would obscure, rather than reveal, the structural functions of the serial transformations.

Given Riemann Systems *RS*1 and *RS*2, let us say that "*RS*2 is a form of *RS*1 (mod *GSER*)" when some member of *GSER* transforms *RS*1 into *RS*2, that is, when some member *X* of *GSER* satisfies $X(RS1) = RS2$. Because of the homogeneity induced by group structure, this relationship among Riemann Systems is both symmetric and transitive. That is, if $X(RS1)=RS2$, then $Y(RS2)=RS1$ where *Y* is the inverse of *X*; so if *RS*2 is a form of *RS*1 then *RS*1 is a form of *RS*2 (mod *GSER*). And supposing that $X(RS1)=RS2$ and $Y(RS2)=RS3$, *X* and *Y* being members of *GSER*, it follows that $Z=YX$ is a member of *GSER* such that $Z(RS1)=RS3$; thus if *RS*2 is a form of *RS*1 and *RS*3 is a form of *RS*2, then *RS*3 is a form of *RS*1 (mod *GSER*).

From these properties, an important fact follows. Given four (or two) Riemann Systems interrelated by the four operations of *GSER*, we can not attribute structural or formal priority to any one of those Systems solely on the basis of the network of *GSER*-relations obtaining among them; all can be equally considered forms of any one. For instance, given the Systems (*F*,2,3), (*F*,2,11), (*G*,10,1) and (*G*,10,9), we can indeed say that the second, third and fourth Systems are respectively the conjugate, retrograde and *TD*-inversion of the first; but we can equally well say that the first, second and fourth are respectively the retrograde, *TD*-inversion and conjugate of the third, and so forth. Unless we want to attribute special priority on some *other* basis to one of the pitch classes *F* or *G*, or to some of the intervals 2, 10, 3, 11, 1 and 9 at the expense of others, we have no way of asserting that any one of the four Systems should be considered "prime" or "basic" and the other three "derived" in some subordinating sense; as far as the group relations are concerned, any of the four Systems can be derived from any other one in absolutely egalitarian fashion.

It is useful to have made this observation before going on to investigate specifically the four Riemann Systems in twelve-tone equal temperament that can be derived from the "*C* major" System (*C*,7,4) by the operations of *GSER*: *C* major is equally a derived form of any of the other three Systems, which are equally derived forms of each other. The preceding discussion makes it clear that, if we want to attribute priority to *C* major in preference to the other three Systems, that is only because we wish to attribute priority to the "fifth" and the "major third" as structural intervals in this connection.[5]

The System (*C*,7,3) is the conjugate of *C* major. We shall call it "*C* minor." As already noted, the conjugate relationship puts into the terminology of the present paper such notions as the harmonic and arithmetic division of the fifth, or the inversion (*renversement*) of major and minor thirds within the fifth.

The System (*G*,5,8) is the *TD*-inversion of *C* major. We shall call it "dual *G* minor." Its tonic triad projects, from the tonic note *G*, the dominant and mediant notes $C=G+5$ and $e\flat=G+8$, that is, the notes respectively a fifth and a major third "down" from the tonic. The structure will be familiar to students of Riemann's theories; Riemann recommended this triad as the "natural" form for minor tonality, though he shrank rather illogically from calling *Ca♭F* the "dominant triad" of *Ge♭C* in this context.[6]

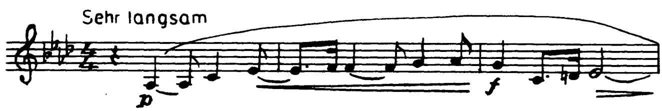
The System (*G*,5,9) is the retrograde of *C* major. We shall call it "dual *G* major." Its tonic triad projects, from the tonic note *G*, the dominant and mediant notes $C=G+5$ and $e=G+9$, that is, the notes respectively a fifth and a minor third "down" from the tonic. As far as I know, no aspect of this system has hitherto been investigated, much less recommended, by any theorist of tonality. The formalities of the present context make that observation quite striking. If one admits the musical cogency of both the relation "*C*min=*CONJ*(*C*maj)" and the relation "dual *G* min=*TDINV*(*C*maj)," and if in addition one considers minor tonality to be as basic as major, rather than strictly subordinate to major, then one ought to admit as equally cogent relations like "dual *G*maj=*CONJ*(dual *G*min)" or "dual *G*maj=*TDINV*(*C*min)." Dual *G* major would thus naturally come into consideration and, having admitted two species of minor which retrograde each into the other, it would be only logical to admit two species of major which retrograde each into the other, always assuming that one puts minor on a structural par with major.

Thus one can analyze the absence of dual *G* major from the literature of tonal theory in the following way. First, some theorists consider ordinary major tonality, with $d=7$ and $m=4$, to have strong priority because of certain aspects of the intervals 7 and 4 themselves, or the just intervals they approximate. For these theorists, any species of minor tonality is to be considered as derived from major in an asymmetrical relation of strict subordination. Hence one cannot necessarily do to *C* major the things one can do to *C* minor, or to dual *G* minor; nor can one necessarily do to either of the minor systems those things which may be done to *C* major. Second, among the theorists who do consider *either* *C* minor *or* dual *G* minor to be of equal structural priority with *C* major, none considers *both* the minor *and* the dual minor systems to be such. Thus a theorist who gives *C* minor equal status with *C* major, in a symmetric relation, can stress the conjugate transformation as all-important while denying implicitly the musical significance of *TD*-inversion and retrogression. *C* major and *C* minor are then symmetrically related, each as the conjugate of the other. Dual *G* minor and dual *G* major are also in this relationship, but that fact is of no more musical significance, for such a tonal theorist, than is the fact that (*C*,2,11) and (*C*,2,9) are conjugates: none of these Systems has anything to do with "tonality" as conceived by such a theorist. Mathematically, the group asserted as significant for tonal theory would comprise only the two members *IDENT* and *CONJ*. In similar spirit, a different theorist (such as Riemann) can accept *C* major and dual *G* minor as of equal priority, but rule out *C* minor and dual *G* major as "tonal" structures. In this case, *C* major and dual *G* minor are symmetrically related by the transformation *TDINV*. *C* minor and dual *G* major are also related by that transformation, but this fact is of no more significance, for such a theory, than is the *TDINV* relation obtaining between the equally "atonal" Systems (*C*,1,9) and (*C*♯,11,3). The group this theory asserts as significant comprises only the two members *IDENT* and *TDINV*.

It is worth exploring the extent to which dual major makes sense as a concept for the analysis of various tonal passages. Personally, I think it makes as much sense as dual minor. For that reason, and also because most contemporary American readers will doubtlessly share my own disinclination to "believe in" dual minor, let alone dual major, I shall discuss a number of musical examples in which I personally feel it makes sense to assert the functioning of either dual minor or dual major or both, at least as regards triadic structure and to a certain extent beyond that.

Example 1 is taken from the opening of Stravinsky's *Sacre du Printemps.* Within the sonority grouped by the sixteenth notes, the tone of reference is clearly *B*4, not *E*4. *B*4 participtes in the overall basic linear gesture, from *C*5 through *B*4 to *A*4. *E*4 hangs beneath *B*4, as does *G*4, both in consonant relation to the referential tone *B*4. Accordingly one would analyze the sixteenth-note sonority as a dual *B* minor triad: *E*

Example 1



Example 2

and *G* appear as fifth and major third below *B*. Certainly one does violence to the sense of the passage if one tries to hear an *E* root for the triad.

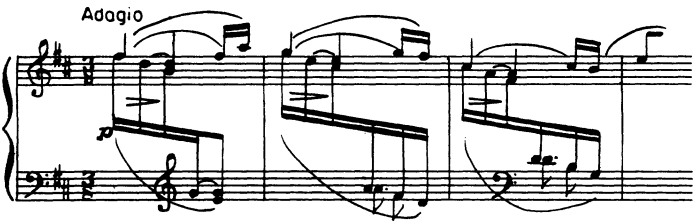
Example 2, taken from the opening of Wagner's *Parsifal,* shows a similar state of affairs as regards the structure outlined by the succession *G*4-*C*4-(*D*4)-*E*♭4. Our attention here is on *G*4, not on *C*4. *G*4 was just heard as leading tone to *A*♭4; now *A*♭4 becomes an upper neighbor to *G*4 and in fact resolves to *G*4, reversing the earlier tendency-relationship. *C*4 and *E*♭4, recently heard as 3rd and 5th of an *A*♭ major triad, now are heard as 5th and 3rd underneath *G*4, in a dual *G* minor triad. All of this is a textbook example of what Riemann called *Leittonwechsel.*

Example 3 is from the opening of Brahms's *Intermezzo* Op. 119, no. 1. It almost seems that the composer has constructed an academic exercise in the compositional projection not just of dual minor triads, but further, of large serial segments from the entire canonical listing of the dual *F*♯ minor Riemann System. One can read these segments from left to right on Figure 7, progressing from top to bottom.

The analysis makes good sense because one orients oneself harmonically, in this context, by the tops and beginnings of the 3rd-chains, more than by their (anticipated) bottoms and ends. The passage suggests modulation to a related major System, and the piece as a whole exploits such modulations. Given the analysis of Figure 7, the related System is surely not *D* major; rather it is dual *A* major: the major triads at issue are presented and developed in exactly the same way as are the minor triads. Figure 8 tries to analyze the whole passage as projecting segments of the canonical listing for dual *A* major. It is this structure, not ordinary *D* major, which is in contention with the structure of Figure 7.

The dual *A* major structure is not very powerful in the first four lines of Figure 8; it becomes strong however in the fifth line of Figure 8, stronger than the dual *F*♯ minor asserted by the fourth line of Figure 7 for the same music. The sixth line of Figure 8 and the fifth line of Figure 7 are of equal power in contending to assert their respective structures. The contention here is not only between tonic notes but also, perhaps more, between upper case and lower case for the names of all the notes at issue: *C*♯ or *c*♯? *a* or *A*?, and so forth. That is, is a given note hereabouts a root-or-fifth of something (upper case) or is it a third of something (lower case)? This question, which would have delighted Hauptmann, highlights the unique status of the note *E:* it is upper case on both Figures.

The *E* is thus uniquely released from the upper case/lower case tension, and in fact we find it used to initiate cadential material immediately following the passage at issue, now that complete canonical listings of



Example 3

F♯ d B g E

a

g E

C♯ a F♯ d

C♯ a F♯ d B g

E

Figure 7

f♯ D b G

E

A

G

E c♯ A f♯ D

c♯ A f♯ D b G

E

Figure 8

both dual *F*♯ minor and dual *A* major have been exposed. The reader who consults the score will find the uniquely pivotal character of the *E* confirmed by the elided half cadences, in dual *F*♯ minor at measure 4 and in dual *A* major in measure 6: both cadences start from the crucial *E* with similar music.

The relation in this piece between the canonical listings of dual *F*♯ minor and dual *A* major exemplifies a transformation we will later call "shifting." Each listing can be shifted into the other by moving it, so to speak, a stage farther along to its left or right, upper and lower cases being adjusted accordingly. See Figure 9.

Example 4 shows the subject of the *C* major Fugue from Book II of Bach's *Well-Tempered Clavier,* together with a version of that subject transformed by "tonal inversion at the fifth." In our terminology the relation is rather than of retrogression, obtaining between the structure of the *C*♯ major and dual *G*♯ major triads. See Figure 5 earlier. The situation is clearly typical: "tonal inversion at the fifth" will retrograde major or minor structures into dual major or dual minor structures, and vice-versa.

Example 5 is interesting in this connection because it exhibits such a relation (involving a similar motive, curiously,) between two different but related works by the same composer. The Example shows aspects of the themes from the two German Dance movements by Beethoven. The first, in *G* major, is from the *Piano Sonata* op. 79. The second, in "dual *D* major", is from the *String Quartet* op. 130.[7] The first segment ends with an appoggiatura into the subdominant harmony of *G* major. The second segment similarly (or "dually") ends with an appoggiatura into the "subdominant" harmony of dual *D* major. The reader, if confused by the last sentence, can review Definition 2 and those following to verify that *Af*♯*D* is indeed the "subdominant triad" of the Riemann System whose canonical listing is *Af*♯*DbGeC,* that is of dual *D* major.

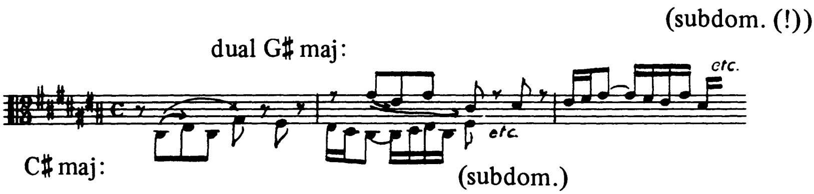
Example 6 shows the theme of Rachmaninoff's *Rhapsody on a Theme of Paganini* op. 43, together with some aspects of Variation XVIII. The situation is complicated here by the remoteness of the tonal relationship and also by the use of *f*♯ and *g*♯ in the original theme. Nevertheless, the melodic technique clearly exemplifies the notion that inverting minor will produce dual major, just as inverting major produces dual minor. The idea of dual *A*♭ major for the variation is supported harmonically: at just the point where the theme moves harmonically from dominant to tonic (see the Example), the variation also moves from "dominant" to tonic harmony (see the Example). One must of course recall that *D*♭*b*♭*G*♭ is the "dominant" harmony of the Riemann System whose canonical listing is *E*♭*cA*♭*fD*♭*b*♭*G*♭, that is of dual *A*♭ major.

The shift group. In connection with our analysis of the Brahms dual *F*♯ minor

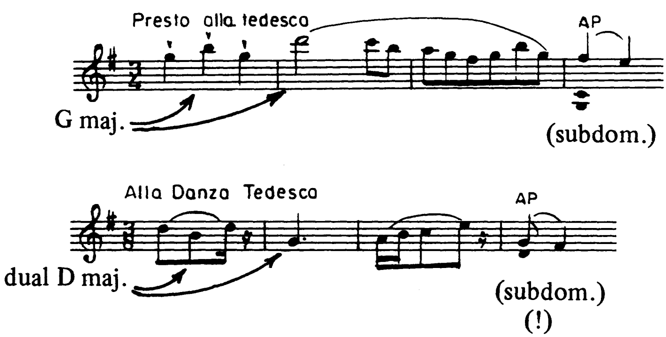
. . . *G*♯ *e* *C*♯ *a* *F*♯ *d* *B* *g* *E* *c* . . .
. . . *g*♯ *E* *c*♯ *A* *f*♯ *D* *b* *G* *e* *C* . . .

dual *A* major

Figure 9

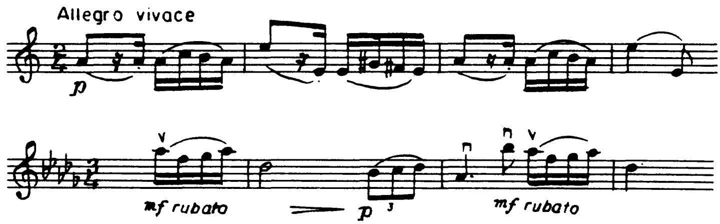


Example 4



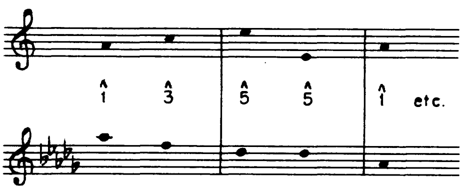
Example 5

A min: dom → tonic



dual A♭ maj.: "dom" → tonic

A minor



dual A♭ major

Example 6

Intermezzo, we introduced the idea of "shifting" the canonical listing for one Riemann System into the canonical listing for another. Figure 9 displayed the dual $F\sharp$ minor and dual A major systems in a shifted relationship. The shift there was by only one place, along an extended listing. We can generalize that situation to advantage.

Given an integer N, positive, zero or negative, we will define a formal operation $SHIFT(N)$ which operates on any given RS to produce a transformed RS whose canonical listing is "shifted N places" from that of the given System. Let (T,d,m) be a specimen RS, and let us write m' for d-m. The canonical listing for (T,d,m) spans the six successive intervals m,m',m,m',m,m'. Let us now imagine that canonical listing extended indefinitely both forward and backward so as to generate an indefinite alternation of successive intervals m and m'. The canonical listing for the N-shifted System is that series of seven notes which appears as a segment of the extended list N stages to the right of the original canonical listing. If N is negative, "N stages to the right" will be construed as "$-N$ stages to the left." If N is odd, it will be necessary to imagine the conceptual roles of upper and lower case notation reversed, in visualizing the canonical listing for the transformed system. (This reversal is understood not to affect the actual intonation.) Figure 10 illustrates these ideas, using the C major Riemann System as a point of reference.

From Figure 10 we can read off the relationships $B\flat$ maj$=SHIFT(-4)(C$maj$)$, $F\sharp$min$=SHIFT(4)(E$min$)$, Emin$=SHIFT(5)(B\flat$maj$)$, and so forth. For the minor systems at issue, we imagine the conceptual roles of upper and lower case reversed in the affected canonical listings, as they appear on Figure 10. In certain intonational systems, this may require further notational finesses; for instance; in just intonation the note written "*g*" on Figure 10 will not belong to exactly the same pitch class as the note written "*G*" on that Figure. However, we shall not concern ourselves with such notational problems here.

Inspecting Figure 10 and imagining its structure generalized to that of an extended listing for a generic (T,d,m), we can see how to calculate the effect of $SHIFT(N)$ on (T,d,m). First, if N is even, we see that the shifted System will have the same dominant and mediant intervals as the original, and that the new tonic note will be the old tonic note shifted N places along the list, and hence transposed by $(N/2)$ dominants. Thus $SHIFT(N)(T,d,m)=(T+i,d,m)$, where the interval of transposition is $i=(N/2)d$. Note that the interval of transposition depends not simply on N, but also on the dominant interval d of the particular System on which the shift is operating.

Second, in the case $N=1$, we see that the shifted System will have dominant interval d, mediant interval $m'=d-m$, and tonic note $T+m$, that is the mediant note of the original System. Thus $SHIFT(1)(T,d,m)$ $=(T+m,d,d-m)$.

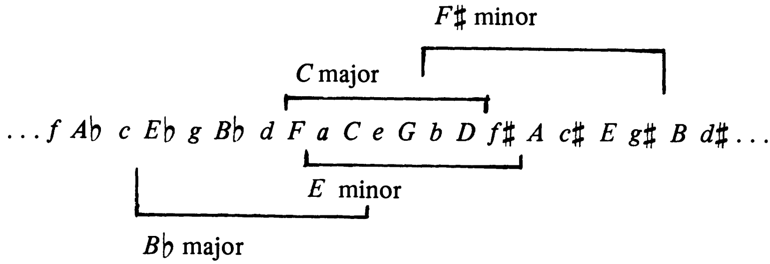


Figure 10

Finally, if N is any odd number whatsoever, we can write *SHIFT*(N) =*SHIFT*(1)*SHIFT*(N-1) in the operational sense. That is, *SHIFT* (N) (T,d,m)=*SHIFT*(1)(*SHIFT*(N-1)(T,d,m)). Since $N-1$ is even, we have seen that *SHIFT*(N-1)$(T,d,m)=(T+i,d,m)$, where $i=((N-1)/2)d$. The System we want is then *SHIFT*(1)$(T+i,d,m)$ which, as we observed in the preceding paragraph, is $(T+i+m,d,d-m)$. We can summarize all the foregoing calculations in a formal definition.

DEFINITION 12. *The operation of "shifting by N places" a Riemann System (T,d,m) is performed algebraically as follows. If N is even, SHIFT(N)(T,d,m)=(T+i,d,m), where i=(N/2)d. If N is odd, SHIFT (N)(T,d,m)=(T+i+m,d,d-m), where i=((N-1)/2)d.*

If we shift by M places the result of shifting a given *RS* by N places, the outcome will be to have shifted the given *RS* by a net displacement of (M+N) places. One easily intuits this by inspecting Figure 10 and generalizing the picture. (One must recall that a negative value for M or N means a displacement left by (-M) or (-N) places.) Our intuition, that is, argues the validity of the operational equation *SHIFT*(M)*SHIFT*(N) =*SHIFT*(M+N). The truth of the equation can be verified formally by using the standard techniques of substitution and algebra in conjunction with Definition 12. (The reader is, however, urged to forego that exercise, which involves taking up separately all the possible subcases contingent on M's being odd or even and N's being odd or even.)

SHIFT (0) is clearly the identity operation *IDENT* on Riemann Systems. It follows that *SHIFT*(N)*SHIFT*(-N)=*SHIFT*(0)=*IDENT*; thence it follows that *SHIFT*(-N) is the inverse operation of *SHIFT*(N). We have noted that the result of following one shift operation by another is (operationally equal to) a shift operation; we have also noted that every shift operation has an inverse which is itself a shift operation. Consequently, the shift operations form a mathematical group.

DEFINITION 13. *The group comprising all operations of form SHIFT (N) will be called the **shift group** of operations on Riemann Systems; it will be denoted by GSHIFT.*

There may be a finite or an infinite number of shifted forms of a given Riemann System, depending on the exact size of its dominant and mediant intervals. Intonation is important in this connection. For example it is clear that if we consider Figure 10 in just intonation there will be no duplication of exact pitch classes on the extended listing; hence there will be an infinite number of both major and minor Systems among the shifted forms of just *C* major. There will also be an (even greater) infinite number of just major and just minor Systems which are *not* shifted forms of just *C* major, namely all just major and minor Systems whose tonics can not be derived from the note *C* by adding and subtracting just fifths or just major thirds modulo the octave. While such Systems are indeed transpositions of just *C* major, or of just *A* minor, the transpositional relationships have no functional significance in the context of Riemann Systematics. Note in particular that the "just *C* minor" System we can read off Figure 10 as just *C* major shifted by -7 places is *not* the conjugate System of just *C* major, since the lower case *c* and the upper case *C* on Figure 10 differ, in just intonation, by a syntonic comma.

On the other hand, if we consider Figure 10 in equal temperament it is clear that there will be only 24 distinct Riemann Systems whose canonical listing are embedded in the extended listing; that listing will repeat itself indefinitely at every 24th entry. In this case any minor System involved will indeed be the conjugate of the major System involved that has the same tonic note, and vice-versa. However, even on this small family of 24 Riemann Systems, *SHIFT*(-7) and *CONJ* operate with different effect. That is, while *C* minor in this context is equally *SHIFT*(-7)(*C*major) and *CONJ*(*C*major), it is not true that *C* major is both *SHIFT*(-7)(*C*minor) and *CONJ*(*C*minor). Rather, *C* major is *SHIFT*(7) (*C*minor); *SHIFT*(-7)(*C*minor) is not *C* major but *C*♭ major= *B*major. So the operations *SHIFT*(-7) and *CONJ* do not have the same effect on all 24 Riemann Systems at issue.

For another example, let us study the System (*C*,3,7) in equal temperament and inspect its shifted forms. One sees that the canonical listing *AeCgE*♭*b*♭*G*♭ will extend on to the right as . . . *d*♭*AeCgE*♭. . . . The extended listing is thus a serial ordering of an octotonic scale-set which repeats at every eighth entry ad infinitum. Hence there will be only eight shifted forms of (*C*,3,7). (*C*,3, 7) itself will reappear as *SHIFT*(8)(*C*,3,7). However its conjugate System (*C*,3,8) will not appear among its shifted forms. Odd-numbered shifts of (*C*,3,7) will indeed produce Systems with dominant and mediant intervals 3 and 8 respectively, but the available tonic notes for those Systems will be only *E*,*G*,*B*♭ and *D*♭, never *C*. The extended canonical listing imposes a certain symmetrical ordering on the octotonic scale, a feature which might have interesting analytical or compositional implications.

Let us now return to the basic formula *SHIFT*(*M*)*SHIFT*(*N*)=*SHIFT*(*M*+*N*), and to the special case *SHIFT*(*N*)*SHIFT*(-*N*)=*SHIFT*(0)= *IDENT*. If *M* and *N* are both even, then *M*+*N* and -*N* will also be even. We can conclude that the even shifts, combining among themselves, form a group of operations. Mathematicians would call this a "subgroup" of *GSHIFT*.

DEFINITION 14. *The family of all operations SHIFT(N) such that N is even will be called the* ***even shift group*** *and denoted GEVSHIFT.*

It is clear from Definition 12 that the even-shifted forms of a *RS* (T,d,m) will be precisely all transpositions of that System by any number of its own dominant intervals d. One recognizes the significant role of *GEVSHIFT* in the context of tonal theory.

Among the even shifts, *SHIFT*(2) and its inverse *SHIFT*(-2) enjoy a special status. They transform a given System (T,d,m) respectively into its "dominant System" $(T+d,d,m)$ and its "subdominant System" $(T-d,d,m)$. These transformations interact so idiomatically with the constructive relationships of the primary triads with (T,d,m) that it makes sense to give the operations special names. We may as well also give special names to *SHIFT*(1) and *SHIFT*(-1).

DEFINITION 15. *As synonyms for SHIFT(1), SHIFT(2), SHIFT(-1) and SHIFT(-2) we shall write MED, DOM, SUBM and SUBD respectively. These are the "mediant," "dominant," "submediant" and "subdominant" operations which respectively transform a given Riemann System* (T, d, m) *into its "mediant System"* $(T+m,d,d-m)$, *its "dominant System"* $(T+d,d,m)$, *its "submediant System"* $(T-d+m, d,d-m)$, *and its "subdominant System"* $(T-d,d,m)$.

Any group of operations that contains *DOM* must also contain *DOM DOM, DOM DOM DOM,* and so forth; it must also contain *SUBD,* the inverse of *DOM,* and hence also *SUBD SUBD,* and so forth; thus it must contain every even shift operation. In this sense we say that *DOM* "generates" *GEVSHIFT.* In the same sense, *MED* generates *GSHIFT.*

Some other inversional transformations. While *MED* and *SUBM* generalize ways of relating the *C* major and *A* minor Systems to each other, neither transformation generalizes the symmetric relationship we think of as "taking the relative key." That is, *C* major and *A* minor are "relatives," each of the other, but neither *MED* nor *SUBM* adequately expresses the mutuality of this relation. *C*major=*MED*(*A* minor) but *A* minor≠*MED*(*C*major); *A* minor=*SUBM*(*C*major) but *C*major≠*SUBM* (*A* minor).

To obtain a symmetric operation which transforms "relative" tonal Systems indifferently into each other and which is formally plausible in the general context of the present study, we shall generalize Riemann's notion of the *Parallelklang* relationship. In connection with *C* major one uses dual *E* minor rather than *A* minor; one then inverts the *C* major and dual *E* minor triads, listings and Systems, each into the other. Figure 11 illustrates the idea.

As Figure 11 shows, the inversional relationship exchanges the tonic and mediant notes of the Systems involved: the tonic note of one becomes the mediant note of the other and vice-versa. So we can call the generalized operation "tonic-mediant inversion" or "*TM*-inversion."

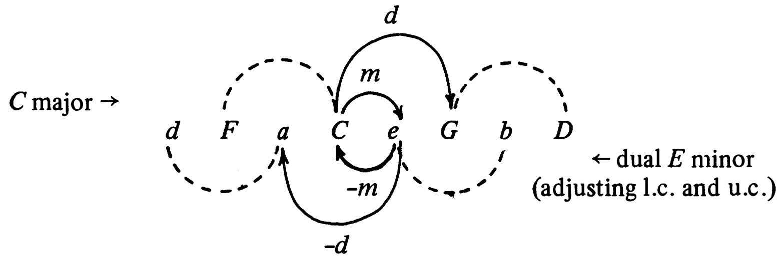


Figure 11

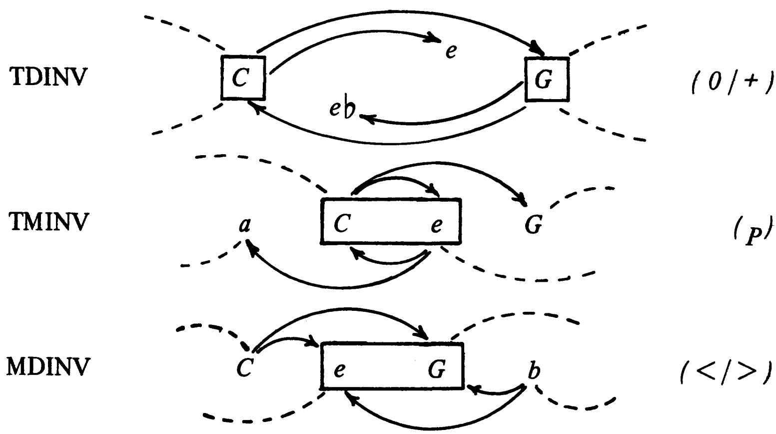


Figure 12

DEFINITION 16. *TM-inversion is that operation TMINV which transforms a given RS (T,d,m) into the Riemann System $TMINV(T,d,m) = (T+m,-d,-m)$.*

The correctness of the algebraic definition can be seen from inspecting Figure 11 and generalizing the picture. In similar fashion, we can define "mediant-dominant inversion."

DEFINITION 17. *MD-inversion is that operation MDINV which transforms a given RS (T,d,m) into the Riemann System $MDINV(T,d,m) = (T+d+m,-d,-m)$.*

The dominant note of the transformed System here is $(T+d+m)+(-d)=T+m$, that is the mediant note of the original System. The mediant note of the transformed System is $(T+d+m)+(-m)=T+d$, that is the dominant note of the original System. *MDINV* transforms *C* major into dual *B* minor and dual *B* minor into *C* major. That is, on major and dual minor Systems it corresponds formally with Riemann's *Leittonwechsel* operation. Of course, *MDINV* has no general implications involving voice-leading, as the *Leittonwechsel* does.

We have now studied three particular inversion operations on Riemann Systems, *TDINV, TMINV* and *MDINV*. Each generalizes, at least formally, a relationship singled out by Riemann as important in relating major with dual minor Systems. Beyond that, these three particular inversions enjoy a special and privileged formal status in the generalized theory. Each of the three holds invariant one of the three dyads embedded in the tonic triad of a given System, transforming it into a dyad of equivalent function in the tonic triad of the transformed System. *TDINV* holds invariant the tonic-and-dominant dyad, *TMINV* the tonic-and-mediant dyad, and *MDINV* the mediant-and-dominant dyad. Figure 12 illustratres these properties, using *C* major and its inverted transforms as examples. In each case, I have used a box to indicate the invariant dyad. The names of the transformations appear to the left on the Figure; at the right I have written the conventional Riemann symbols for the tonal relations involved, to indicate how our transformations generalize his.

Assorted further considerations. The transformations so far studied combine among themselves in a variety of ways, some simple and some complicated. For instance $DOM\ MDINV(T,d,m)=DOM(T+d+m,-d,-m)$ by Definition 17; the latter $=(T+d+m+(-d),-d,-m)=(T+m,-d,-m)$, which in turn is $TMINV(T,d,m)$ by Definition 16. Hence $DOM\ MDINV = TMINV$. In similar fashion, one can show that $MDINV\ SUBD=TMINV$, that $MDINV=SUBD\ TMINV=TMINV\ DOM$, that $TMINV\ MDINV= DOM$, and that $MDINV\ TMINV=SUBD$. The reader will recall the intuitive significance of such operational equations. The last equation above, for instance, states that the *MD*-inversion of the *TM*-inversion of any given Riemann System is the subdominant System of that given System.

From the equations just discussed, together with the fact that *TMINV* is its own inverse, it follows that the even shifts together with transformations of form *SHIFT*(2*N*) *TMINV* constitute a group of operations. The group contains *MDINV* as well as *TDINV*, since *MDINV*=*SHIFT* (-2)*TMINV*. When applied to *C* major, the members of this group generate *G* major, *F* major, *D* major, and so forth, as the evenly-shifted forms of *C* major, and also dual *E* minor, dual *B* minor, dual *A* minor, dual *F*♯ minor, and so forth, as the evenly-shifted forms of *TDINV*(*C* major). Any one of these Systems is a "form" of any other, modulo this group.

Another reasonably simple network of interrelations among our transformations arises from the fact that *DOM CONJ*=*CONJ DOM*. (The validity of the equation can be established by the usual techniques.) It follows that the even shifts, together with transformations of form *SHIFT*(2*N*) *CONJ*, constitute a group of operations. The forms of *C* major, mod this group, are *C* major, *C* minor, *G* major, *G* minor, *F* major, *F* minor, and so forth. Any one of these Systems is a form of any other, modulo the group.

Another interesting group comprises all the shifts together with all transformations of form *SHIFT*(*N*) *RET*. One can prove that *SHIFT* (*N*) *RET*=*RET SHIFT*(-*N*). The canonical listings for the various forms of a given System, mod this group, are the various seven-note segments of the extended canonical listing, read either forwards or backwards. Thus the forms of *C* major, for instance, are *C* major, dual *G* major, *E* minor, dual *B* minor, *A* minor, dual *E* minor, *G* major, dual *D* major, *F* major, dual *C* major, etc. The transformations *TMINV* and *MDINV* are members of this group, since *TMINV*=*SHIFT*(1) *RET* and *MDINV*= *SHIFT*(-1) *RET*. (The equations can be proved in the usual way. The visual format of Figure 11, when generalized, makes intuitively clear the validity of the former equation.)

Other ways of combining the transformations at hand lead to more remote transformations and to larger, more complex, groups of operations. For example, let us temporarily define an operation *X*: transpose a given Riemann System by its mediant interval. Thus *X*(*C*major)=*E* major, *X*(*C*minor)=*E*♭ minor, and so forth. In general, $X(T,d,m)=(T+m,d,m)$. Let us also define *Y*: transpose a given Riemann System by its dominant-minus-mediant interval. Thus *Y*(*C*major)=*E*♭ major, *Y*(*C* minor)=*E* minor, etc. In general, $Y(T,d,m)=(T+d-m,d,m)$. The following equations can be proved to hold true. *CONJ MED*=*X*, *MED CONJ*= *Y*, *TMINV TDINV*=*Y*, *MDINV TDINV*=the inverse of *X*, and so forth.

It follows: any group that contains both *CONJ* and *MED* must contain both X and Y; likewise, any group that contains *TMINV, MDINV* and *TDINV* must contain both X and Y, and so forth. And any group that contains both X and Y must contain the operation $Z = Y$-inverse X. Since X transposes the System (T,d,m) by m, and Y-inverse transposes the resulting System by the complement of m', that is by $m-d$, Z transposes the given System by $m-m'$, that is by $m-(d-m)$ or $2m-d$. Z, for instance, takes C major into $C\sharp$ major and C minor into $C\flat$ minor. Leaving further exploration of such generalized "post-Wagnerian" transformations to those who may be interested, let us now pass to another topic.

DEFINITION 18. *The **type** of the Riemann System (T,d,m) is the ordered pair of intervals (d,m). Two Riemann Systems "have the same type" or "are of the same type" if they have the same dominant interval and the same mediant interval.*

Thus C major and $F\sharp$ major are of the same type, supposing the same intonation. In twelve-tone equal temperament, $(C,2,5)$ and $(B\flat,2,5)$ are of the same type, namely the type $(2,5)$. Also of type $(2,5)$ is $(C+j,2,5)$, where j represents a just fifth. The concept of type enables us to relate the intervallic structure of $(C+j,2,5)$ to the intervallic structure of $(C,2,5)$, without having to assert any functional significance for the transpositional relationship between them.

The concept of type is also useful to make generalizations about the intervallic structures of Systems that *are* functionally related, without having to worry about their tonic notes. For example, given any *RS* (T,d,m), its conjugate System will be of type $(d,d-m)$, and so will its odd-shifted Systems. In that sense, we can say that the odd-shifted Systems are all "of conjugate type," even though the conjugate System itself may not appear as a shifted System.

In similar vein, we can say that the *TD*-inverted, *TM*-inverted and *MD*-inverted forms of (T,d,m) are all of one type, namely $(-d,-m)$, the "inverted type" of (d,m). And we can say that any minor System is "of retrograde type" to any dual minor System, in the same spirit. To make such discourse formally precise, we need only define the operations of conjugation, inversion and retrogression on *types.*

DEFINITION 19. *Given a type (d,m), the **conjugate type** is $conj(d,m)=(d,d-m)$. The **inverted type** is $inv(d,m)=(-d,-m)$. The **retrograde type** is $ret(d,m)=(-d,m-d)$. The group comprising the three operations conj, inv and ret, along with the identity operation ident, will be called the "serial type-group" of operations on System-types.*

It is easy to verify that the operations as defined do form a group. One verifies equations among them analogous to those that were collated in Table 3 for the analogous System-operations; the analog of Table 3 is valid for these type-operations.

The serial type-group in of very basic importance. One sees that all the operations we have so far examined, even our exotic "post-Wagnerian" operations, take Riemann Systems into Riemann Systems of either identical, or conjugate, or inverted, or retrograde type.

We pass again to another topic. It is interesting to note that Riemann's categories of *Ueberklang* and *Unterklang* can be generalized in our present terminology, at least formally.

DEFINITION 20. *The RS (T,d,m) will be called "directed above" when, given a pitch representing the pitch class T, the next-highest pitch representing T+m is lower than the next-highest pitch representing T+d. The RS (T,d,m) will be called "directed below" when, given a pitch representing the pitch class T, the next-lowest pitch representing T+m is higher than the next-lowest pitch representing T+d. The type (d,m) will be called "directed above/below" if all Riemann Systems of that type are directed above/below.*

One can verify that every *RS* is either directed above or directed below, but not both. The same is true of every type. The conjugate of any System or type is directed in the same sense as that System or type; inverted or retrograded Systems and types are directed in the opposite sense.

It should hardly be necessary to point out that there are important traditions in tonal theory which the work of this paper does not generalize. The most significant such tradition involves the study of voice-leading and counterpoint in relation to tonal functionality; our generalized theory, with its canonical listings, must perforce remain mute on such matters in generality, though one could of course work out protocols for voice-leading and counterpoint in connection with specific individual Riemann Systems other than the tonal ones.

The literature of another important tradition expounds certain systems in which one measures various harmonic intervals of interest (octaves, dominants, mediants) not as coming *from* a common *generator* but as going *to* a common *generatee.* To see the distinction, note that our formalism enabled us to analyze the minor triad as comprising a fifth "up" and a minor third "up" from a common generator (for instance, *C* to *G* and *C* to *e♭*). Our formalism also enabled us to analyze the minor triad as a "dual" structure, comprising a fifth "down" and a major third "down" from a common generator (for instance, *G* to *C* and *G* to *e♭*). But our formalism did not and can not enable us to analyze the minor triad as comprising a fifth "up" and a major third "up" *to* a common *generatee* (for example, *C* to *G* and *e*♭ to *G*).

Systems involving common-generatee relationships, whether in connection with minor triads or in other ways, can be called "phonic" as opposed to "sonic."[8] A formalism more general yet than that of the present paper could perhaps be developed to generalize phonic, as well as sonic, intervallic systems, and to interrelate all systems, phonic as well as sonic, among themselves and each other. However, the formalities of the present paper are, I imagine, amply general to satisfy most readers for the nonce.

NOTES

1. A bibliography and critique of writings by and pertinent to Riemann, convenient for American readers, is to be found in William C. Mickelsen, *Hugo Riemann's Theory of Harmony* (Lincoln and London: University of Nebraska Press, 1977). Among the variety of works providing foundations for the approach taken by Riemann and continued in the formalities of the present paper, one should particularly draw attention to the work of Jean-Philippe Rameau in his *Nouveau système de musique théorique et pratique* (Paris: Ballard, 1726). Here Rameau coins the word "subdominant." He analyzes the importance of the proportional relation: subdominant root is to tonic root as tonic root is to dominant root. This relation can be seen to underlie the triadic aspect of the constructive method portrayed in Figure 1; it will later be generalized by Figure 2. In the *Nouveau système,* Rameau analyzes the scale (better, our "diatonic set") as arising from the conjunction of the three primary triads, a feature also manifest in Figure 1 and to be generalized in Figure 2. Before this work, Rameau's contention that melody is theoretically subordinate to harmony had coexisted very uncomfortably with his adoption of the Ptolemaic, rather than just, major scale. See Jean-Philippe Rameau, *Traité de l'harmonie réduite à ses principes naturels* (Paris: Ballard, 1722), book 1, chapter 5.

   Another important work requiring citation here is that of Moritz Hauptmann, *Die Natur der Harmonik und der Metrik* (Leipzig: Breitkopf und Haertel, 1853). By adopting a philosophical, rather than numerical and acoustical, approach to constructions such as that of Figure 1, Hauptmann takes a considerable step towards formalism, incidentally avoiding some methodological problems that beset Rameau. In particular, his conception of the minor triad as a philosophical "negative unity" rather than an acoustical structure of "undertones" compares to great advantage with the conceptions of Riemann regarding that triad, conceptions which lead after all to the same formal analysis.

   The format of the canonical listing in Figure 1 is taken directly from Hauptmann, along with the upper-case and lower-case notation. As we shall see, the latter notation is convenient and suggestive in many contexts. The idea of using the canonical listing, rather than scalar order, as a fundamental ordering for the diatonic set has many important and profound formal implications, as we shall also see.
2. In fact, if we restrict our attention to twelve-tone equal temperament and enquire what Riemann Systems have the property under discussion and also have seven distinct notes in their diatonic sets, we shall find essentially only four such systems. One is the major tonal system, for example (C,7,4) with canonical listing *Fa*CeGb*D*. (The notes of the triad at issue are italicized.) Another is the "dual minor" tonal system, which we shall study more later, for example (G,5,8) with canonical listing *Dbb* Geb Cab *F*. The other two systems are the circle-of-fifths (or "multiplication") transforms of the above two, for example (C,1,4) and (C♯,11,8) with listings *Beb* CeC♯f*D* and *Dbb* C♯aCab *B*.
3. By the "Forte-label" corresponding to a set of pitch classes, I mean the identifying numerical label assigned to it in the work by Allen Forte, *The Structure of Atonal Music* (New Haven and London: Yale University Press, 1973).
4. Regarding the historical shadow of Figure 3, one would mention in particular

the discussion of harmonic and arithmetic division of the fifth in connection with major and minor triad structure given by Gioseffo Zarlino in the third edition of *Le Istitutioni Harmoniche* (Venice: Senese, 1573), book 3, chapter 31.

The relation of parallel major and minor triads in the manner of Figure 3 is discussed by Rameau in book 1, chapter 3, article 5 of the *Traité.* Rameau calls the interchange of m with m' a "new species of *renversement,*" meaning a redisposition of intervals in register purportedly analogous to the redisposition of pitches in register that produces chord-inversion. This attitude toward minor was, however, to be superseded in Rameau's later works by a large variety of other ideas.

5. Equal temperament is only a notational convenience here. The work that follows could be carried through using any other reasonable sizes of "fifths" and "major thirds" as dominant and mediant intervals for the major System.

   It would be a well-nigh endless task to catalog all the reasons that have been advanced in the history of tonal theory for attaching special priority to these intervals. New reasons are still being advanced actively in the literature. Peter Westergaard, for example, follows the grand tradition of constructing tonality ex nihilo in *An Introduction to Tonal Theory* (New York: W. W. Norton & Company, Inc., 1975), pp. 411–427. His discussion of the (just) fifth is a sensitive and attractive amalgam of many historical attitudes, numerical, philosophical and psycho-acoustical. His discussion of the major third, along with the syntonic comma and the need for temperament, also invokes some traditional ideas. It is however unique to my knowledge in the piquant additional argument that, if we did not have available a consonant interval of about this size, we would be unable to make Schenkerian analyses for the backgrounds of tonal pieces.

   A recent theory of Benjamin Boretz provides an interesting argument, rigorously anti-historicist and anti-acoustical, for attaching special priority to the fifth as a dominant interval in tonal music ("Musical Syntax (II)," *Perspectives of New Music* 10/1 [1971]:232–270). Starting (!) with twelve-tone equal temperament and certain formal desiderata for a certain type of musical system, Boretz shows that the interval 7 is the only available choice for dividing the octave, and being itself divided, in such a way as to satisfy those desiderata.
6. The reader will find an ample exposition of these ideas in Mickelsen, *Theory of Harmony.* As indicated in Note 1, Hauptmann had earlier favored this formal analysis of the minor triad. Rameau himself had flirted with it for a time, in *Génération harmonique* (Paris: Prault, 1737).
7. The curious relationship between the themes was pointed out by Ludwig Misch in his article "Alla danza tedesca," *Beethoven Studies,* trans. G. I. C. DeCourcy (Norman: University of Oklahoma Press, 1953), pp. 14–18.
8. The nomenclature generalizes the terminology of Arthur von Oettingen, *Harmoniesystem in dualer Entwicklung* (Leipzig: W. Glaser, 1866). Rameau himself presented a phonic theory of minor among his many attempts to reconcile minor with major tonality; it can be found in the *Démonstration du principe de l'harmonie* (Paris: Durand, 1750). In more recent times, phonic explanations for various tonal phenomena were advanced by Paul Hindemith in *The Craft of Musical Composition,* trans. Arthur Mendel (London: Associated Music Publishers, 1942).