Klumpenhouwer Networks and Some Isographies that Involve Them

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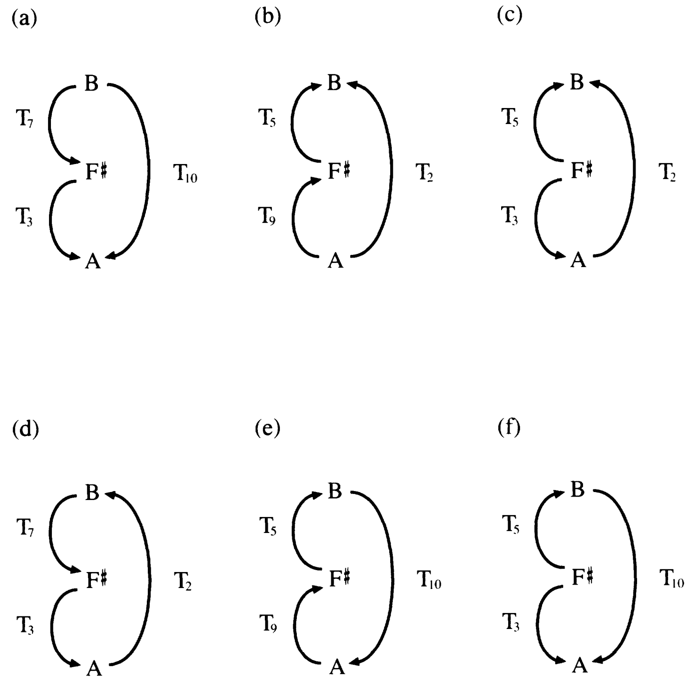
I. KLUMPENHOUWER NETWORKS DEFINED; SOME BASIC PROPERTIES

A convenient point of departure is the network of Figure 1a, which interprets the first chord of Schoenberg's Op. 19 No. 6. The vertical order of pitch classes B, F♯, and A on the figure corresponds to the registral order of pitches in the music. In the visual display all arrows point downward. This reflects a characteristic feature of the particular interpretation which the network imposes on the chord: implicit priority is attached to those intervals which are directed "downwards" in the registral ordering of pitch classes within the chord.[1]

For other purposes one might interpret the chord by other networks. For example, one might want to read intervals from

Figure 1.



Information on Klumpenhouwer and his work is provided in Part VII (pp. 114–116). Also provided there is a fairly detailed account of the circumstances under which we made our respective contributions to the study at hand.

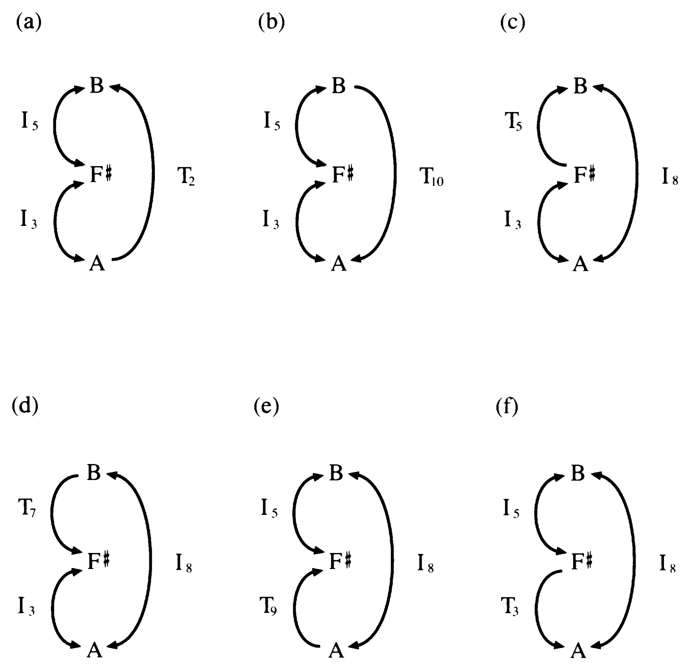
[1]In *Generalized Musical Intervals and Transformations* (New Haven and London: Yale University Press, 1987), 159–60, I discuss a related network involving pitches and pitch intervals, rather than pitch classes and pc intervals. There I explore some particular analytic implications of this priority.

Throughout the article I will use the vocabulary of *GMIT* with a fair amount of freedom.

the pitch class in the bass "upwards" in the registral ordering; this would generate the transpositional network of Figure 1b, a different network giving a different interpretation for the chord. Yet another interpretation is provided by Figure 1c, which measures intervals from F♯ to the other two pitch classes. Other transpositional networks interpreting the chord are illustrated in Figures 1d through 1f.

Klumpenhouwer's idea, both simple and profound in its implications, is to allow inversional, as well as transpositional, relations into networks like those of Figure 1. Some such networks, interpreting the chord of Figure 1, are illustrated in Figures 2a through 2f.

Figure 2.



The numerical labels for the inversion operations on the figure are affixed using the Babbitt/Forte convention: the pitch class C is provisionally imagined as labeled with the number 0, C♯ with 1, . . . , B with 11. If n is a generic number between 0 and 11 inclusive, the operation I_n exchanges those pitch classes whose numerical labels sum to n. For example, on Figure 2a, the operation I_5 exchanges the pitch classes B and F♯, whose numerical labels sum to 5.[2]

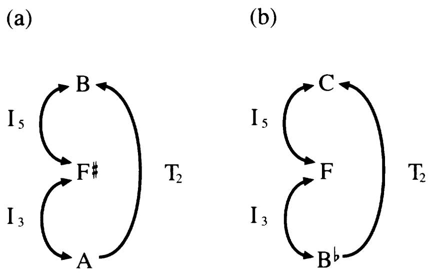
The networks of Figures 1 and 2 all interpret Schoenberg's chord; still other interpretive networks are possible, using more or fewer arrow-relationships. Any network that uses T and/or I operations to interpret interrelations among pcs will be called a *Klumpenhouwer Network*.

Using I operations, Klumpenhouwer was able to interpret chords of different set classes with isographic networks. Figure 3 illustrates this powerful aspect of the construction. Figure 3a displays the network of Figure 2a, interpreting the opening chord of Op. 19 No. 6. Figure 3b displays another network, interpreting a chord that appears later in the piece (m. 5). The two networks are *isographic*: that is, they have isomorphic graphs. Indeed, they have a stronger property yet; they are what Klumpenhouwer calls *strongly isographic*: not only are the two graphs isomorphic, the two graphs are strictly identical. That is, the configurations of nodes and arrows are the same, and so are the transformations associated with corresponding arrows.

The vertical arrangement of pitch classes in the display of Figure 3b does not correspond to the registral ordering of those

[2]Later figures explore what happens to certain numerical interrelations among such networks when the conventions we use to label pitch classes are varied. That is why I use letter names, rather than fixed numerical labels, for the pitch classes.

Figure 3.



Example 1.

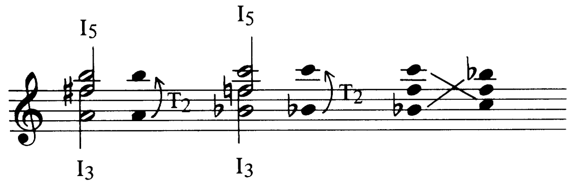
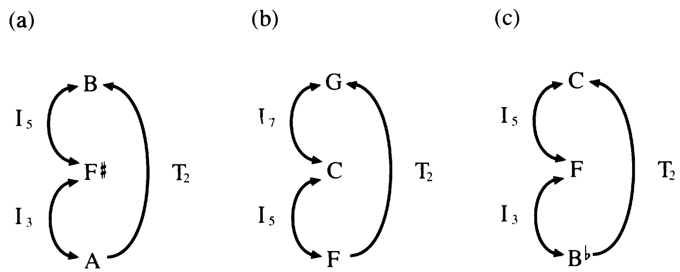


Figure 4.



pitch classes in the corresponding music. To the extent that one wants to assert the networks of Figure 3 analytically, one is thereby asserting a permutation that produces the registral ordering of the chord 3b in the music from the registral ordering of chord 3a in the music. Example 1 explores the phenomenology of this a bit further.

On the left of the example, the first three vertical dyads extract relationships from the network of Figure 3a. The I_3-dyad comprising A and F♯ is written with open noteheads and a stem down. The I_5-dyad comprising F♯ and B is written with open noteheads and a stem up. The T_2-dyad is written with solid noteheads, no stem, and an arrow pointing from A to B. The next three vertical dyads of Example 1 extract analogous relationships from the network of Figure 3b. The I_3-dyad comprising B♭ and F is written with open noteheads and a stem down, and so forth.

Playing the two I_3-dyads of the example in isolation, one can hear that the inversional relation of I_3 persists in the progression from A–F♯ to B♭–F. Playing the two I_5-dyads of the example in isolation, one can likewise hear that the inversional relation of I_5 persists in the progression from F♯–B to F–C. Finally, playing the two T_2-dyads of the example in isolation, one can hear that the transpositional relation of T_2 persists in the progression from A–B to B♭–C. The ear-training exercise helps reveal the strong isography between the networks of Figure 3. (Incidentally, the two chords are connected by a slur on the score.) At the right of Example 1, the pitch classes B♭, F, and C—those participating in the second network—are reordered in the symbolic registral space of the figure. The new ordering corresponds to the actual registral order of the corresponding pitches in the music of m. 5.[3]

[3]A rigorous group-theoretical study of such reorderings lies at the heart of Klumpenhouwer's dissertation. The present article will not concern itself further with the topic.

Figure 4a copies over Figure 3a; 4c copies over 3b. In between 4a and 4c a new Network 4b appears, which interprets the second chord of Op. 19 No. 6. The two chords in the music corresponding to Figures 4b and 4c are registrally ordered transpositions of each other. The vertical orderings in the visual displays of Networks 4b and 4c are analogous permutations of those registral orderings; the two networks are clearly "transpositions" of each other, in some sense.

This is the sense: to get from 4b to 4c,

1. Transpose each pc of 4b by T_5, as it lies.
2. Taking in turn each operation X that labels an arrow of 4b, replace it by the operation $(T_5)X(T_7)$, to label the corresponding arrow of 4c. In this way each T_n-arrow of 4b remains a T_n-arrow of 4c, since $(T_5)(T_n)(T_7) = T_n$. And each I_n-arrow of 4b becomes an I_{n+10}-arrow of 4c, since $(T_5)(I_n)(T_7) = (T_5)(T_5)(I_n) = (T_{10})(I_n) = I_{n+10}$.

The second step above instances a situation discussed elsewhere "in great generality: When a system modulates by an operation A, the transformation f′ = A f A-inverse plays the structural role in the modulated system that f played in the original system."[4]

Supposing any network of pitch classes, then, and any pc operation A, there is a special relation between the given network and the new network obtained as follows:

1. Replace each pc s of the given network by pc A(s) in the new network.
2. Taking in turn each operation X that labels an arrow of the old network, replace X by the operation AXA′, to label the corresponding arrow of the new network. (Here A′ means the inverse operation of A.)

Klumpenhouwer calls this way of deriving one network from another a *network isomorphism*. The specific relation between Network 4b and Network 4c instances a general rule. Given two pcsets of the same set class, and given any specific Klumpenhouwer Network that interprets one of the pcsets, then there exists a Klumpenhouwer Network with properties (a) and (b) following:

(a) it interprets the other of the two given pcsets; and
(b) it is network-isomorphic to the specific Klumpenhouwer Network originally given.

Let the family of transpositions and inversions on pitch classes be called "the T/I group." If A is a fixed member of the T/I group, let X vary through the group and define the function F(X) = AXA′. F is what is called an *automorphism* of the group: it is a one-to-one mapping of the group onto itself satisfying the rule F(XY) = F(X)F(Y). (That is easily seen, since F(XY) = A(XY)A′ = (AX) (YA′) = (AX) (A′A) (YA′) = (AXA′) (AYA′) = F(X)F(Y).) In mathematical parlance, an "automorphism" of a group is an "isomorphism" of a group with itself.[5]

[4] *GMIT,* 149. Here, suppose that s and t are sample pitch classes involved in network 4b. Then $T_5(s)$ and $T_5(t)$ will be the corresponding pitch classes of network 4c. Suppose that an arrow labeled by operation X goes from s to t on network 4b. This means that X(s) = t. Let Y be the operation $(T_5)X(T_7)$. Then Y will transform the pitch class $T_5(s)$ into the pitch class $T_5(t)$, in network 4c. That is, it will be true that $Y(T_5(s)) = T_5(t)$. (Computing: $Y(T_5(s)) = ((T_5)X(T_7)(T_5))(s) = ((T_5)X)(s) = T_5(X(s)) = T_5(t)$.) Hence the arrow on network 4c from $T_5(s)$ to $T_5(t)$ is labeled by the operation Y.

[5] Robert Morris discusses automorphisms of exactly this type (F(X) = AXA′) in *Composition with Pitch Classes* (New Haven and London: Yale University Press, 1987). On p. 167 he defines "an automorphism" to be precisely such a mapping; he works out the implications of these mappings in a rich variety of settings, manifest on his index under "automorphism."

Morris's "automorphisms" are more properly called "*inner* automorphisms." Every inner automorphism of a group is an automorphism—that is, it satisfies the law F(XY) = F(X)F(Y). However, not every automorphism

Network isomorphism is thus a particular case of *isography* between networks. To be isographic, two networks must have these features:

1. They must have the same configuration of nodes and arrows.
2. There must be some isomorphism F that maps the transformation-system used to label the arrows of one network, into the transformation-system used to label the arrows of the other.
3. If the transformation X labels an arrow of the one network, then the transformation F(X) labels the corresponding arrow of the other.

To recapitulate, Klumpenhouwer has singled out two particular kinds of isographies that can obtain between his networks. In the case of strong isography, illustrated by the relation of Network 4a to 4c, the two networks, sharing the same configuration of nodes and arrows, have in common as well exactly the same operations labeling corresponding arrows. As in this illustration, strongly isographic networks can interpret chords that are not of the same set class. Network isomorphism, illustrated by the relation of Network 4b to 4c, arises between networks using analogous configurations of nodes and arrows to interpret pcsets that are of the same set class.

The relation between Networks 4a and 4b is neither a strong isography nor a network isomorphism, but it is still an isography. The two networks still satisfy the three criteria for isography listed above:

1. They have the same configuration of nodes and arrows.

(one-to-one mapping of a group onto itself satisfying that law) is an inner automorphism.

2. There is an isomorphism F that maps the transformation-system used to label the arrows of one network, into the transformation-system used to label the arrows of the other. Here, the isomorphism F maps the T/I operation X into the T/I operation $F(X) = (T_5)X(T_7)$.
3. If the transformation X labels an arrow of Network 4a, then the transformation F(X) labels the corresponding arrow of Network 4b.

Inspecting Networks 4a and 4b, Klumpenhouwer noted that each T-number of 4a remained the same in 4b, while each I-number of 4a was exactly 2 less (10 more) than the corresponding I-number of 4b. This observation led him to formulate a conjecture.

KLUMPENHOUWER'S CONJECTURE: Klumpenhouwer Networks (a) and (b), sharing the same configuration of nodes and arrows, will always be isographic if each T-number of Network (b) is the same as the corresponding T-number of Network (a), while each I-number of Network (b) is exactly j more than the corresponding I-number of Network (a), where j is some constant number modulo 12.

The conjecture is true. In fact, a lot more is true, which will be explored at some length.

If the j of the conjecture is an even number, $j = 2k \bmod 12$, then the automorphism $F(X) = (T_k)X(T_{-k})$ establishes the desired isography. For $F(T_n) = (T_k)T_n(T_{-k}) = T_n$, and $F(I_n) = (T_k)I_n(T_{-k}) = (T_k)(T_k)I_n = T_jI_n = I_{n+j}$. Hence under the conditions of the conjecture, the isomorphism F between the T/I group for Network (a) and the T/I group for Network (b) satisfies the third condition for isography: each X-arrow of Network (a) is replaced by an F(X)-arrow at the corresponding location in Network (b).

The above argument, however, does not help prove the conjecture if j is an odd number. At this point, it is clear that we need to know exactly what all the automorphisms of the T/I group *are*.

II. MATHEMATICAL BACKGROUND (1); THE ISOGRAPHIES OF KLUMPENHOUWER NETWORKS

The Automorphisms of the T/I Group: Let u be 1, 5, 7, or 11 mod 12; let j be any number mod 12. Then an automorphism of the T/I group is given by the mapping F⟨u,j⟩, defined as follows:

$F\langle u,j\rangle(T_n) = T_{un}$
$F\langle u,j\rangle(I_n) = I_{un+j}$.

Furthermore: every automorphism of the T/I group is of the form F⟨u,j⟩, for some suitable values of u and j. That is, the various mappings F⟨u,j⟩, as u and j vary, exhaust *all* the automorphisms.

The assertions above are no mathematical news, and a proof of them in the main text here would be out of place for most readers. Readers who are interested will find a proof in Appendix A to this paper. The proof assumes a certain amount of familiarity with the basics of group theory.

The mathematical work provides a means to answer the conjecture and to add further information as well.

Five Rules for Isography of Klumpenhouwer Networks:

1. Klumpenhouwer Networks (a) and (b), sharing the same configuration of nodes and arrows, will be isographic under the circumstance that

each T-number of Network (b) is the same as the corresponding T-number of Network (a), and

each I-number of Network (b) is exactly j more than the corresponding I-number of Network (a).

The pertinent automorphism of the T/I group is F⟨1,j⟩:

$$F\langle 1,j\rangle(T_n) = T_n;\quad F\langle 1,j\rangle(I_n) = I_{n+j}.$$

2. Klumpenhouwer Networks (a) and (b), sharing the same configuration of nodes and arrows, will be isographic under the circumstance that

each T-number of Network (b) is the complement of the corresponding T-number in Network (a), and

each I-number of Network (b) is exactly j more than the complement of the corresponding I-number in Network (a).

The pertinent automorphism of the T/I group is F⟨11,j⟩:

$$F\langle 11,j\rangle(T_n) = T_{-n};\quad F\langle 11,j\rangle(I_n) = I_{-n+j}.$$

3. Klumpenhouwer Networks (a) and (b), sharing the same configuration of nodes and arrows, will be isographic under the circumstances that

each T-number of Network (b) is 5 times the corresponding T-number in Network (a), and

each I-number of Network (b) is exactly j more than the 5 times the corresponding I-number in Network (a).

The pertinent automorphism of the T/I group is F⟨5,j⟩:

$$F\langle 5,j\rangle(T_n) = T_{5n};\quad F\langle 5,j\rangle(I_n) = I_{5n+j}.$$

4. Klumpenhouwer Networks (a) and (b), sharing the same configuration of nodes and arrows, will be isographic under the circumstance that

each T-number of Network (b) is 7 times the corresponding T-number in Network (a), and

each I-number of Network (b) is exactly j more than the 7 times the corresponding I-number in Network (a).

The pertinent automorphism of the T/I group is F⟨7,j⟩:

$$F\langle 7,j\rangle(T_n) = T_{7n};\quad F\langle 7,j\rangle(I_n) = I_{7n+j}.$$

5. Klumpenhouwer Networks (a) and (b), even if sharing the same configuration of nodes and arrows, will not be isographic under any other circumstances.

Networks (a) and (b) of Figure 4 provided an example of isography under Rule 1 above:

each T-number of Network 4b is the same as the corresponding T-number of Network 4a, and

each I-number of Network 4b is exactly 2 more than the corresponding I-number of Network 4a.

The pertinent automorphism of the T/I group is F⟨1,2⟩:

$$F\langle 1,2\rangle(T_n) = T_n;\quad F\langle 1,2\rangle(I_n) = I_{n+2}.$$

Networks (a) and (b) of Figure 5 provide an example of isography under Isography-Rule 2 above:

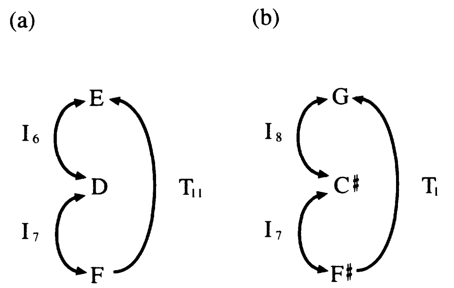
each T-number of Network 5b is the complement of the corresponding T-number in Network 5a, and

each I-number of Network 5b is exactly 2 more than the complement of the corresponding I-number in Network 5a.

The pertinent automorphism of the T/I group is F⟨11,2⟩:

$$F\langle 11,2\rangle(T_n) = T_{-n};\quad F\langle 11,2\rangle(I_n) = I_{-n+2}.$$

Figure 5.



The interested reader may work out further examples of isography under Rules 3 and 4 above. These cases will not be treated further in the present paper.

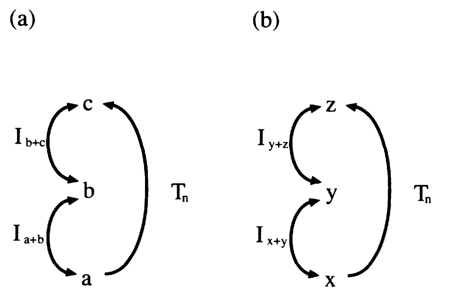
Let us call an isography under Rule 1 a *positive isography,* and an isography under Rule 2 a *negative isography,* of Klumpenhouwer Networks.

Klumpenhouwer observed that any two trichords which share a common dyad-class can be interpreted by positively isographic networks. To see this, suppose that the numerical labels for the pcs of the first trichord are a, b, and c; suppose that the numerical labels for the pcs of the second trichord are x, y, and z. Suppose further that these labels are chosen so that the interval c-a equals the interval z-x; call the number of this interval "n." Then Networks 6a and 6b of Figure 6 can be set up to interpret the two trichords.

Under the Babbitt/Forte system, the inversion operation that exchanges pitch class b with pitch class c will have the number (b + c) as its label; that is why "I_{b+c}" appears as a label for the arrow between pcs b and c on Network 6a. The labels for the other inversion operations of Figure 6 are computed in the same way.

Set j = (y + z) − (b + c). Then (y + z), the top I-number for Network 6b, is j more than (b + c), the top I-number for Network 6a. The following claim is true: (x + y), the bottom I-number for Network 6b, is also j more than (a + b), the bottom I-number for Network 6a. That claim, when verified, will set up the desired positive isography: T-numbers are the same in the two networks, and I-numbers will consistently be j more in the second network than in the first.

Figure 6.

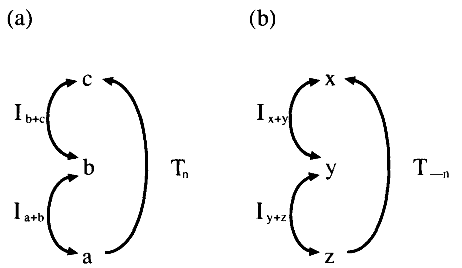


The claim is that $(a+b) + j = (x+y)$. Now j was defined as $(y+z) - (b+c)$; hence

$$
\begin{aligned}
(a+b) + j \\
&= (a+b) + (y+z) - (b+c) \\
&= (a-c) + (y+z) \\
&= (y+z) - n && [(c-a) = n] \\
&= (y+z) - (z-x) && [(z-x) \text{ also } = n] \\
&= (x+y) && \text{as claimed.}
\end{aligned}
$$

Using similar methods, one demonstrates that the two trichords can also be interpreted by negatively isographic Klumpenhouwer Networks, as in Figure 7, where it still holds that $n = c-a = z-x$.

Figure 7.



Incidentally, the following observation is true as well: If two trichords can be interpreted by positively or negatively isographic Klumpenhouwer Networks that have at least one T-arrow, then the trichords must share at least one common dyad, up to transposition. To verify this, suppose that one network involves a T_n-arrow. The other network, being positively or negatively isographic, must have either a T_n-arrow or a T_{-n}-arrow at the corresponding location. Hence each trichord contains some dyad spanning interval n.

Klumpenhouwer observed that any two tetrachords which share a common trichord, up to transposition, can be interpreted by positively isographic networks. By invoking negative as well as positive isographies, one can reach a stronger conclusion, since tetrachords that share a common trichord up to inversion can be interpreted by negatively isographic networks. Beyond that, if tetrachord (a) can be broken down into two dyads of interval classes m and n, and if tetrachord (b) can also be broken down into two dyads of the same two interval classes, then the two tetrachords can be interpreted by positively isographic networks, and also (differently) by negatively isographic networks.

And so forth; many such theorems can be proved and are of considerable interest in connection with larger sets.

III. ANALYTIC EXERCISES (1)

Because of the many ways in which a given pcset can be interpreted by Klumpenhouwer Networks, and because of the many ways in which those networks can enter into isographic relationships among themselves, very flexible and powerful resources become available for the analysis of atonal music.

Such resources are already evident in connection with the first chord of Schoenberg's Op. 19 No. 6: the network of Figure 1a interprets this chord with one network, and Klumpenhouwer's network of Figure 4a interprets the same chord with another network. The network of Figure 1a is isographic to a network that describes a certain progression of motive-forms during the piece as a whole. Klumpenhouwer's network is isographic to Networks 4b and 4c, networks that interpret prominent harmonies from elsewhere in the piece.

To be noted here is a shift in emphasis: these networks, instead of relating *pcsets* (chords), now relate *interpretations of pcsets*. Each pcset has many interpretations, and can thus behave as a sort of nexus (in the colloquial sense of the word): the pcset binds together each of its interpretations, along with all the pertinent network-isographies that may radiate from any particular interpretation.

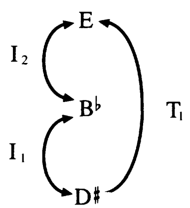
Another powerful feature of the system under study is the ease with which one can set up extensive systems of isographic networks, by making suitable Klumpenhouwer-interpretations for various pcsets over a span of music. As Klumpenhouwer observed in connection with his work, this enables networks to interrelate systematically, even where the pcsets being interpreted are not necessarily of the same set class. Figure 8 provides a more extensive example than any so far.

Figure 8.

Fl:	B♭4	A5	A5	E6	D♯6	F♯4
Cl:	E4	E4	B♭4	F5	B♭5	C♯3
Vn:	D♯5	D♯5	D5	D5	A6	G3
Und die	sanfte	Magd des	Himmels,	von den	Zweigen zart umschmeichelt,	
	E	D♯	A	E	A	G
	B♭	A	D	D	E♭	C♯
	D♯	E	B♭	F	B♭	F♯
	I_2	I_0	I_{11}	I_6	I_0	I_8
	T_1	T_{11}	T_{11}	T_{11}	T_{11}	T_1
	I_1	I_1	I_0	I_7	I_1	I_7
	g1	g2	g3	g4	g5	g6

The top part of the figure transcribes certain aspects of Schoenberg's *Pierrot Lunaire* No. 4 ("Eine blasse Wäscherin"), mm. 13–14. Below the text are produced network-interpretations for the chords of the music. Each network is symbolically represented here by a vertical display of three pitch classes, under which there is a slightly skewed vertical display of three T/I operations. Thus the network interpreting the first chord is represented by the vertical display, under the text, of E, B♭, and D♯, reading down, plus the skew vertical display below that of the symbols I_2, T_1, and I_1. This specific network is to be understood as follows: an I_2-arrow goes up and down between E and B♭; an I_1-arrow goes up and down between B♭ and D♯; a T_1-arrow goes up from D♯ to E. In short, the formatting of the symbols here is a version of Figure 9.

Figure 9.



The other vertical displays under the text of Figure 8 symbolize other such networks, interpreting the other chords in the music. The symbols g1 through g6 at the bottom of Figure 8 stand for "graph 1" through "graph 6."

All of the chords in Figure 8 contain a dyad of interval class 1. The second through fifth networks have been laid out so that their corresponding T-arrows are all T_{11}. As observed earlier (p. 89, Fig. 6), this interpretation for the second through fifth chords means that the second through fifth networks will all be mutually positively isographic.

The first chord is an inversion of the second as a pcset; the sixth chord is also a pc inversion of the fifth. The first and sixth networks have been laid out as negatively isographic to the others, reflecting that relationship. To exhibit the negative isography, it suffices that the T-arrow in g1 and g6 be T_1, where the T-arrow of g2 through g5 is T_{11}. This situation was also studied earlier (p. 90, Fig. 7); Figure 5 earlier illustrated the specific negative isography here between g4 and g6. The first, second, fifth, and sixth chords are forms of set 3–5. The third and fourth chord are each of different set classes, yet their interpretations fit in among the networks of Figure 8 very smoothly.

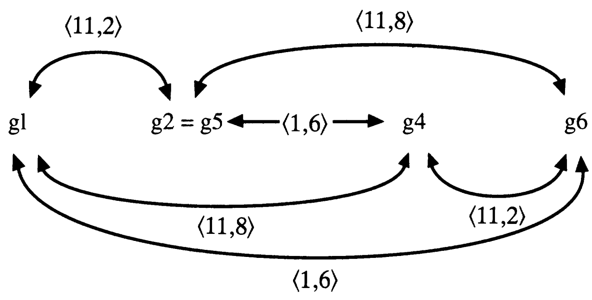
Another very powerful feature of the system under consideration now becomes apparent. Figure 8, graph g3 is isomorphic to graph g2, for each T_n-arrow of g2 remains a T_n-arrow of g3; each I_n-arrow of g2 becomes an I_{n+11}-arrow of g3. Here the automorphism F⟨1,11⟩ transforms the arrow-labels of g2 into the arrow-labels of g3: F⟨1,11⟩(T_n) = T_n; F⟨1,11⟩(I_n) = I_{n+11}. Thus F⟨1,11⟩ can be regarded as an operation on (T/I)-graphs, and one can symbolically write F⟨1,11⟩(g2) = g3.

In similar fashion, one can write F⟨11,2⟩(g1) = g2. The equation can be interpreted as follows: to get from g1 to g2, keep the same configuration of nodes and arrows; then apply F⟨11,2⟩ to each T or I of g1 to get the corresponding T or I of g2. F⟨11,2⟩ applied to T_n yields T_{-n}; F⟨11,2⟩ applied to I_n yields I_{-n+2}. So in particular, F⟨11,2⟩ applied to the T_1-arrow of g1 yields the T_{-1}-arrow of g2; F⟨11,2⟩ applied to the I_2-arrow of g1 yields the I_{-2+2}-arrow of g2; F⟨11,2⟩ applied to the I_1-arrow of g1 yields the I_{-1+2}-arrow of g2.

The networks for the second and fifth chords exemplify Klumpenhouwer's strong isography: though the *pcsets* are different, the fifth *graph* of Figure 8 is exactly the same as the second *graph*. That is, "g2 = g5."

Since the graphs of Figure 8 are all isomorphic by way of ap-

Figure 10.



propriate F⟨u,j⟩ operations, the graphs g1 through g6 *themselves* can be arranged into a variety of transformation-networks, using the group of F⟨u,j⟩ to label the arrows between different g-elements of the network. The next few figures will show several such structures.

On Figure 10 the symbol "F" has been left off all the automorphisms, to conserve space. Thus one reads from the figure that g2 = F⟨11,2⟩(g1), g4 = F⟨1,6⟩(g2), and so forth. (The figure reminds the reader that g2 and g5 are the same graph.) These relations can be verified by inspecting the graphs at the bottom of Figure 8. For instance, to verify that g4 = F⟨1,6⟩(g2), inspect g2 and g4 at the bottom of Figure 8. The T-numbers of g4 are the same as the corresponding T-numbers of g2; each I-number of g4 is 6 more than the corresponding I-number of g2; hence g4 = F⟨1,6⟩(g2).

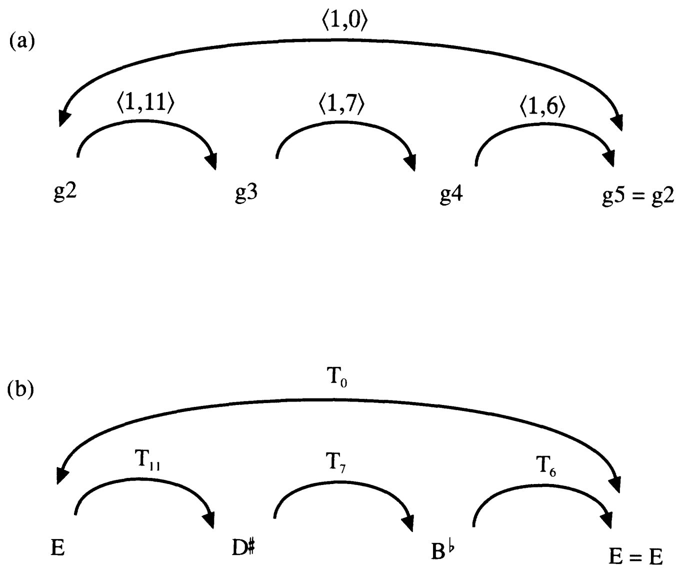
Figure 10 presents an interesting proportioning. Here g1 and g6 can be regarded as "extremes" of the progression; they are in a ⟨1,6⟩ relation with each other. The graph g2 = g5 can be regarded as one "mean" between the extremes: g2 = g5 articulates the overall ⟨1,6⟩ progression (g1 to g6) into subprogressions of ⟨11,2⟩ and ⟨11,8⟩ (g1 to g2 = g5, and then g2 = g5 to g6). Further, the graph g4 can be regarded as the complementary "mean" between the extremes: g4 articulates the overall ⟨1,6⟩ progression (g1 to g6) into complementary subprogressions of ⟨11,8⟩ and ⟨11,2⟩ (g1 to g4, and then g4 to g6). Curiously, unlike the harmonic and arithmetic means of traditional harmonic theory, the two means here enjoy the same relationship, ⟨1,6⟩, as do the two extremes.

Figure 11a isolates graphs g2 through g5 for study. The isomorphisms among these graphs are all positive, while graphs g1 and g6 are negative isomorphs. That articulates g2-through-g5 as a span. So does the fact that g5 = g2; under the current network-interpretations this section of the music projects a "circular progression" of *graphs* (even though the fifth pcset is different from the second pcset). The circular progression is indicated on Figure 11a by the transformation-symbol "⟨1,0⟩": F⟨1,0⟩ is the identity automorphism, leaving each T_n and each I_n unchanged. The ⟨1,0⟩ relation between two graphs indicates Klumpenhouwer's strong isography between the two Klumpenhouwer Networks employing those graphs.

The arrows and F-labels of Figure 11a show that the ⟨1,0⟩ relation is articulated into a ⟨1,11⟩ relation, followed by a ⟨1,7⟩ relation, followed by a ⟨1,6⟩ relation. More explicitly: the move from g2 to g3 adds 11 to the I-numbers; the move from g3 to g4 adds 7 to the I-numbers; the move from g4 to g5 = g2 adds 6 to the I-numbers; at this point, the I-numbers are all as they were originally, since 11 + 7 + 6 = 0 mod 12.

In this context, 11, 7, 6, and 0 seem to be acting suspiciously like transposition numbers. Furthermore, these particular transposition numbers are especially suggestive here, because they can be used to set up something like a transpositional Klumpenhouwer Network that interprets certain pcsets of form 3–5, the most prominent harmony of the musical passage under study. Figure 11b is such a network, using the pitch classes of the first chord from the passage.

Figure 11.



Of course Figure 11b interprets the pcset (B♭, E, D♯) quite differently from the interpretation of g1 in Figure 8. This illustrates the point made earlier (p. 91), that "each pcset has many interpretations, and can thus behave as a sort of nexus (in the colloquial sense of the word): the pcset binds together each of its interpretations, along with all the pertinent network-isographies that may radiate from any particular interpretation."

But *is* the relation of Figure 11b to Figure 11a an isography in the sense defined above? Yes, it is. To establish that claim rigorously one must demonstrate a mathematical isomorphism between the group of F⟨1,j⟩'s, which label the arrows of Figure 11a, and the group of T_j's, which label the arrows of Figure 11b.

This demonstration will appear in due course; meanwhile, let us study another interesting way of arranging the graphs g1 through g6.

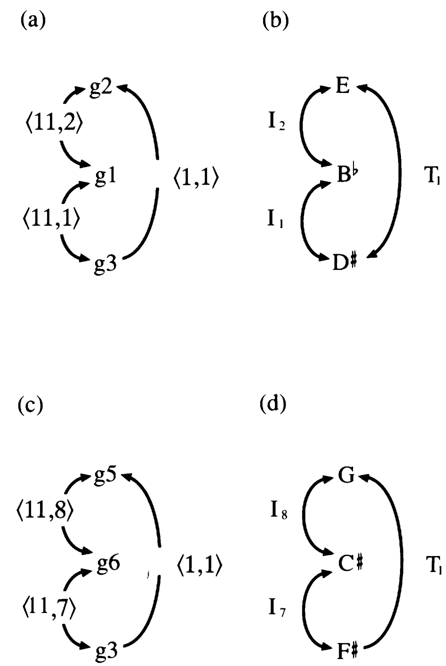
In Figure 12a, the symbols g1 through g3 have been arranged in a new, vertical format. The ⟨u,j⟩-arrows mean what they always have meant so far: g1 = F⟨11,1⟩(g3), and so forth. The point of this format is to emphasize visually a resemblance to Figure 12b. The latter figure is the Klumpenhouwer Network that appeared in Figure 8 (and Figure 9) to interpret the first chord of the "Wäscherin" passage. As with Figure 11a and Figure 11b, Networks 12a and 12b appear to be isographic. To establish the isography, one must demonstrate a pertinent isomorphism between the group of F⟨1,j⟩'s and F⟨11,j⟩'s, on the one hand, and the group of T_j's and I_j's, on the other hand. The isomorphism does exist and will be demonstrated later.

Once those formalities are worked out, a powerful statement can be made about Figures 12a and b. Figure 12a, a network of interrelations-among-interpretations, is isographic to Figure 12b, the particular Klumpenhouwer Network that interprets the opening chord. More colloquially, a certain structure of interrelations among interpretations of the first three chords, Figure 12a, "prolongs" at a higher hierarchical level the interpretation of the first chord, Figure 12b. The notion of "prolongation" is powerful here. So is the recursive aspect of the situation: graph g1, which appears as a "thing" filling one node of Figure 12a, also appears as the entire graph for Figure 12b, a graph whose nodes there are filled by pitch classes. And, because 12a is isographic with 12b, graph g1 is *also* (isomorphic to) the graph of Figure 12a itself.

Figures 12c and d display an analogous situation. 12c shows F-interrelations among g3, g5, and g6, while Figure 12d reproduces the Klumpenhouwer Network of Figure 8 for the sixth chord of the "Wäscherin" passage. Figures 12c and 12d are isographic, exactly as were 12a and 12b.

In the isography between 12b and 12a, the opening chord generates from its own internal structure a progression of chords that includes and follows the incipit chord. This is very like the situation in Rameauean tonal theory, where an opening tonic triad generates a rising-fifth progression in the fundamental bass through an aspect of the triad's own internal harmonic structure. In the isography between 12c and 12d, the final chord of the "Wäscherin" phrase "pulls into" its own internal structure a progression of chords that precedes and includes the cadence chord itself. This, in turn, is very like the Rameauean authentic cadence, where the final tonic triad pulls into its internal harmonic structure the fifth that preceded it in

Figure 12.



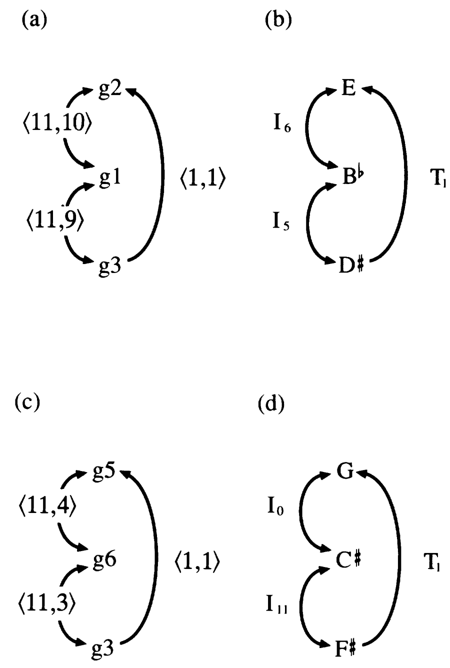
the fundamental bass. The foregoing oversimplifies the matter: for "chord" there one should substitute "chord-interpretation" throughout. (One should in Rameau, too, but that is another matter.)

The isography of 12a with 12b is particularly impressive visually on the figure because wherever "I_j" appears on 12b, "⟨11,j⟩" appears at the corresponding place on 12a, using the same number j. That, however, is a coincidence caused by having labeled pitch class C = 0 to begin with. The reader may verify this by going over Figure 8 and relabeling everything using the convention that E = 0, rather than C = 0. Figure 12 will then look like Figure 13.

The new Network 13b, which corresponds to 12b, has I_6 and I_5 arrows rather than I_2 and I_1 arrows. And 13a, the new network corresponding to 12a, has ⟨11,10⟩ and ⟨11,9⟩ arrows, rather than ⟨11,2⟩ and ⟨11,1⟩ arrows. What remains the case is that 13b is isographic to 13a. Adding 4 to the I-numbers of Figure 13b yields the corresponding 11-numbers of Figure 13a.

Similarly, adding 4 to the I-numbers of 13d yields the corresponding 11-numbers of 13c. The "positive isography" which maps 13b to 13a is the same as the "positive isography" which maps 13d to 13c. Here "positive isography" appears within quotation marks, for the basic isomorphism of the two groups involved has not yet been formally established. Once this has been done, it will be possible to state formally that the relation of 13a to 13c is essentially the same as that of 13b to 13d. The latter relation is by way of ⟨1,6⟩: to get from 13b to 13d, leave the T-numbers alone and add 6 to each I-number. Under the desired isomorphism, the relation of 13a to 13c is also "by way of ⟨1,6⟩": to get from 13a to 13c, leave the 1-numbers alone and add 6 to each 11-number. In this way, an interpreted relation of opening progression to closing progression reproduces, at a higher hierarchic level, the lower-level interpreted relation from the opening pcset to the closing pcset of the phrase. Here, the ⟨1,6⟩-ness of the relationships just discussed does not depend on the initial choice of C = 0 or E = 0 or any other such

Figure 13.



labeling convention: for example, 12a to 12c is a "⟨1,6⟩ relation" in exactly the same sense as is 13a to 13c.

IV. MATHEMATICAL BACKGROUND (2); RECURSIVE STRUCTURING

This is the place to establish the formal isomorphism required, an isomorphism between certain F-operations on graphs and the (T/I)-operations on pitch classes. The first step is to compute formally the way in which the F⟨u,j⟩ automorphisms combine, one with another, in all generality.

$$\text{FORMULA 1: } F\langle u,j\rangle F\langle v,k\rangle = F\langle uv, uk+j\rangle$$

A proof of Formula 1 follows; readers who are not interested may safely omit the rest of this paragraph. We simply compute the effect of F⟨u,j⟩F⟨v,k⟩ on each sample T_n and on each sample I_n, according to the definitions given earlier:

$$F\langle u,j\rangle(F\langle v,k\rangle(T_n)) = F\langle u,j\rangle(T_{vn}) = T_{uvn}.$$
$$F\langle u,j\rangle(F\langle v,k\rangle(I_n)) = F\langle u,j\rangle(I_{vn+k}) = I_{uvn+uk+j}.$$

So, if F⟨u,j⟩F⟨v,k⟩ is denoted by G, then G maps T_n into $T_{(uv)n}$, and G maps I_n into $I_{(uv)n\,+\,(uk+j)}$. It follows that G operates precisely as F⟨uv,uk+j⟩. And that proves the formula.

From here on, the symbols ⟨u,j⟩ (and so forth) will be used as contractions of F⟨u,j⟩ (and so forth). Formula 1 can then be compacted into Formula 1a.

$$\text{FORMULA 1a: } \langle u,j\rangle\langle v,k\rangle = \langle uv, uk+j\rangle$$

The automorphisms F⟨u,j⟩ form a mathematical group of operations on the (T/I)-group: an automorphism following an automorphism is an automorphism; the inverse map to an automorphism is an automorphism. Formula 1a shows how the ⟨u,j⟩ symbols, labeling the members of the F-group, combine under these circumstances. The ⟨u,j⟩ symbols are a group under the combination law given by Formula 1a. The identity is ⟨1,0⟩; the group inverse of ⟨u,j⟩ is ⟨u,−uj⟩. (Set v = u and k = −uj in Formula 1a; then uv = uu = 1, and uk + j = −uuj + j = −j + j = 0.)

These symbols combine, when we restrict the values of u and v to be specifically either 1 or 11, as follows.

FORMULAS 2:
(a) $\langle 1,j\rangle \langle 1,k\rangle = \langle 1,j+k\rangle$
(b) $\langle 1,j\rangle \langle 11,k\rangle = \langle 11,j+k\rangle$
(c) $\langle 11,j\rangle \langle 1,k\rangle = \langle 11,j-k\rangle$
(d) $\langle 11,j\rangle \langle 11,k\rangle = \langle 1,j-k\rangle$

The four equations of Formulas 2 are easily derived by substituting the appropriate values for u and v into Formula 1a.

Formulas 2 demonstrate that the elements ⟨u,j⟩ with u = either 1 or 11 form a group amongst themselves. The group is naturally isomorphic to the T/I group, under the identification of ⟨l,j⟩ with T_j, ⟨11,j⟩ with I_j. To see that, one need only compare Formulas 2 with the familiar T/I group table below:

(a) $T_j\,T_k = T_{j+k}$
(b) $T_j\,I_k = I_{j+k}$
(c) $I_j\,T_k = I_{j-k}$
(d) $I_j\,I_k = T_{j-k}$

The gist of this work is that the ⟨1,j⟩'s and ⟨11,k⟩'s, with arrows labeled on networks-of-graphs, can be treated exactly as if they were T_j's and I_k's, with arrows labeled on Klumpenhouwer Networks that interpret pcsets.

Among other things, it is now possible to speak precisely of an "isography" between a Klumpenhouwer Network on the one hand, and a (⟨1/⟨11)-network of graphs on the other. For each ⟨u,j⟩, in fact, there is a corresponding possible isography: replace each T_n of the (T/I)-network by ⟨1,un⟩ in the (⟨1/⟨11)-network; replace each I_n of the (T/I)-network by ⟨11,un + j⟩ in the (⟨1/⟨11)-network. Such an isography obtains, for example, between the (T/I)-network of Figure 13b and the (⟨1/⟨11)-network of Figure 13a. The isography is of type "⟨1,4⟩": to get from 13b to 13a, replace T_n-arrows of 13b by ⟨1,n⟩-arrows in 13a, and replace I_n-arrows of 13b by ⟨11,n + 4⟩-arrows in 13a.

Perhaps the most powerful fallout from the isomorphism of the (⟨1/⟨11)-group with the (T/I)-group is the possibility it opens up for hierarchic recursion in our network-structuring. Consider Figure 13 yet once more. Figure 13b is a network of pitch classes. Figure 13a is a network of network-graphs, a hierarchic level higher. So is Figure 13c. Figure 13c is "⟨1,6⟩-isographic to" 13a, because the ⟨1 and ⟨11 arrows of 13a and 13c behave just like T and I arrows. And now 13a and 13c can *themselves* be arranged in a simple (⟨1/⟨11)-network, shown as Figure 14.

Here, the right-pointing ⟨1,6⟩ arrow means: to get from the graph of 13a to the graph of 13c, replace each ⟨1,j⟩-arrow of 13a by a ⟨1,j⟩-arrow in 13c, and replace each ⟨11,j⟩-arrow of 13a by an ⟨11,j + 6⟩-arrow in 13c. The left-pointing arrow signifies that one can get from 13c to 13a by exactly the same replacement procedure.

Figure 14 is thus a network of networks-of-network-graphs. And so forth, through networks of networks-of-networks-of-networks-of . . . networks of network-graphs. The possibilities are literally limitless, so long as ⟨1,j⟩ and ⟨11,j⟩ relations are the only types applied at each level.[6] Furthermore, these relations allow isographies between any two levels of the system. For example, the ⟨1,6⟩-relation of Figure 14 can be regarded as a high-level "tritone relation" between the "opening progression" Figure 13a and the "cadential progression" Figure 13b. That is because the high-level "⟨1,6⟩" of Figure 14 can be identified with its natural isomorphic image T_6 in the ground-level (T/I)-group.

An earlier figure (Figure 6) and its accompanying discussion (p. 89) showed that two trichords are bound to have isographic network-interpretations, provided only that they share at least one common dyad up to transposition. Given that fact, and given the identification of the T/I group with the ⟨1/⟨11 group just developed, one might ask how significant it really is, that Figure 13a is isographic with Figure 13b. Does this not simply mean that the opening chord of the "Wäscherin" passage includes a dyad of interval class 1, and that the positive isography between graphs g3 and g2 is by way of ⟨1,1⟩-relationship? Does not everything else follow automatically from that?

No. First of all, we had to *construct* the graphs g1, g2, and g3 as interpretations; they were not given automatically by the chords of the music. Figures 1 and 2 will remind the reader how many possible choices there are in selecting a particular network to interpret a particular trichord. (And the 12 graphs of Figures 1 and 2 are by no means exhaustive.) In particular, g3 and g2 had to be positively isomorphic before there could be any "⟨1,1⟩-relation" between them. Then, given that these graphs have "one T and two I's," g1 had to be *negatively* isomorphic with g2 and g3, so that there would be an ⟨11-relation between g1 and g2, and an ⟨11-relation between g1 and g3, two ⟨11-relations to go with the one ⟨1-relation involving g3 and g2. Finally, all this interpretation had to be arranged so that g1 itself, as a graph for a particular interpretation of the opening chord, would have two I-arrows and one T-arrow that would be specifically T_1, corresponding to the ⟨1,1⟩-relation between g3 and g2. Configuring all these interpretations so as to make Figure 13a and b "work" was by no means an automatic affair; it was a combination of art and will.

Figure 14.

(graph of) Fig. 13(a) ←———— ⟨1,6⟩ ————→ (graph of) Fig. 13(c)

[6] The whole theory could be developed to allow multiplication by 5 and by 7 at every level, including that of the pitch classes themselves. Interested readers will find the mathematical work developed in Appendix B.

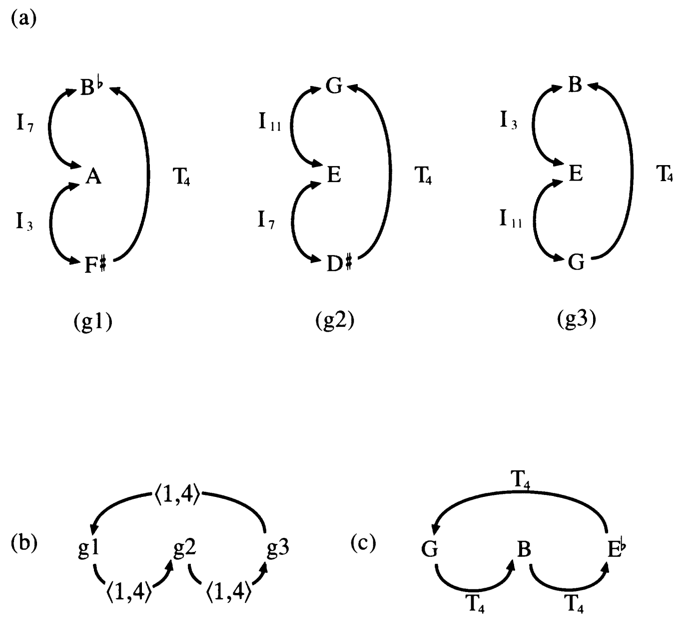
V. ANALYTIC EXERCISES (2)

To illustrate the methodological point, consider the musical phrase that accompanies the first textual utterance in the same piece, "Eine blasse Wäscherin wäscht zur Nachtzeit bleiche Tücher." The musical phrase ends together with the word "bleiche" on an augmented triad comprising a G, a B, and an E♭. This pcset is recognizable as an ending harmony, from earlier in the piece.

Listening over the phrase as a whole, paying particular attention to interval 4, one becomes aware that the first three chords of the passage contain dyads of interval class 4, while the fifth and sixth do not. Figure 15a collates isographic Klumpenhouwer Networks for the opening three chords, making the shared 4-dyads the basis for the isographies. The graphs for the three network-interpretations are labeled g1, g2, and g3. Figure 15b arranges the three graphs in a ⟨1-network: from g1 to g2 is ⟨1,4⟩, from g2 to g3 is also ⟨1,4⟩, and from g3 back to g1 is also ⟨1,4⟩. The ⟨1-arrangement of 15b is thus isographic to the T-arrangement of Figure 15c, which is of course a characteristic network-interpretation for the augmented triad that closes the phrase. In other words, the augmented-triad interpretation of 15c is "elaborated" by the three opening harmonies of the phrase, a grouping of harmonies that contain ic4 dyads, under the interpretation of 15a.

Some methodological points deserve special attention here as matters of "art and will." First, to make Figure 15b isographic with 15c, it is necessary that graph g1 be *positively* isomorphic with graph g2, in order to assert the desired ⟨1,4⟩ arrow between them. The network-interpretations are made with that desideratum in mind, even though the *pcsets* of the first and second chords are *inversions*, each of the other. The inversional relation might suggest something like the interpretations of Figure 16a instead; that figure makes hypothetical network g1′ *negatively* isographic to g2. But such interpretation is resisted

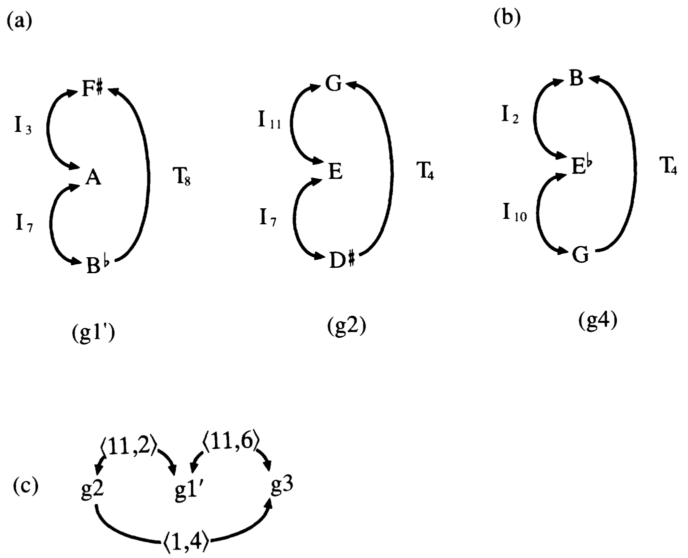
Figure 15.



here, to make the isography work out between Figure 15b and 15c.

Another methodological point is that Figure 15c, as a Klumpenhouwer Network interpreting the augmented triad, is *not* isographic with g1, g2, and g3 in Figure 15a. Of course, nothing would be easier than to set up a different interpretation for the augmented triad which *would* be isographic with the three g-networks here: Figure 16b displays such an interpretation, labeled "g4." Then, using the interpretations of 16a, one could assert a new arrangement of graph g1′, g2, and g3, the arrange-

Figure 16.



ment shown in Figure 16c, which would itself be isographic with the new interpretation of the augmented triad. That is, the graph of 16c is isomorphic to the graph g4 of 16b.

There is nothing wrong with all this, but it asserts something quite different from Figure 15. For the graph g4 of Figure 16b is not particularly characteristic of the augmented triad. Indeed, g4 is isomorphic to g1′, g2, g3, and g1 as well, graphs for networks that interpret (any and all) "trichords which contain major thirds." The augmented triad is something characteristically more than another trichord which contains a major third. And the graph of Figure 15c, which is also the graph of Figure 15b, shows the characteristic at issue: among pcsets, *only* the augmented triad can have the graph of Figure 15c. Thus the interpretations of Figure 15a, including g1 in particular, show a characteristic elaboration of an "augmented triad" structure in the progression as interpreted; in contrast, the progression of Figure 16c shows a structure isomorphic indifferently to g1, g1′, g2, g3, or g4. Figure 16c does thereby manifest the characteristic recursive property we discussed before, and that makes it worth studying. It shows how three naturally grouped Klumpenhouwer Networks, each including one T_4 or T_8-arrow, and each containing two I-arrows, can be arranged in an isography-structure that includes one ⟨1,4⟩ arrow and two ⟨11-arrows. But Figure 15 manifests something else, namely the special structuring influence of the augmented triad on the progression of g1 through g2 to g3.

VI. ANALYTIC EXERCISES (3); MODES OF NETWORK-INTERPRETATION

To illustrate at greater length some techniques for using these ideas in analysis, let us spend some time with the opening phrase of No. 9 from *Pierrot Lunaire,* "Gebet an Pierrot," whose concomitant text is "Pierrot! mein Lachen hab' ich verlernt!" Figure 17 displays the pitch classes.

The clarinet part is on the top line of the figure; the rest is played by the piano. The vertical ordering of the piano part in the figure projects the registral order of the pitches in that instrument. The clarinet's music is grouped by rhythm and contour into the two tetrachords A♭-F-D-C and G-E-E♭-C♯, followed by the isolated pitch class D; Figure 17 projects that grouping spatially. The clarinet rests for a half-measure pickup, and for all of m. 1; after poising itself for a bit on its A♭ entrance, in m. 2, it then plays in rapid succession during the remainder of that measure all of the pitches portrayed in Figure 17, except for the final D. The D is attacked at the bar line of m.

Figure 17.

A♭ F D C   G E E♭ C♯   D

B♭   B C

G♭   G F♯

C   E

F   G

D♭   C♯

A

3, repeated twice, and sustained for about a measure; the instrument then rests for over two measures. As a result the motivic grouping of the two tetrachords, discussed above, becomes a particularly salient feature of the solo instrument's musical characterization during the opening section of the piece.

Now the two unordered tetrachords (A♭,F,D,C) and (G,E,E♭,C♯), motivically grouped by rhythm and contour, are not of the same set class. However, they *can* be interpreted by isographic Klumpenhouwer Networks, and that seems like a good point of departure for analytic exploration. Actually, the two tetrachords can be interpreted by isographic networks in several basically different ways. Each will be investigated in turn and called a "mode" for interpretation.

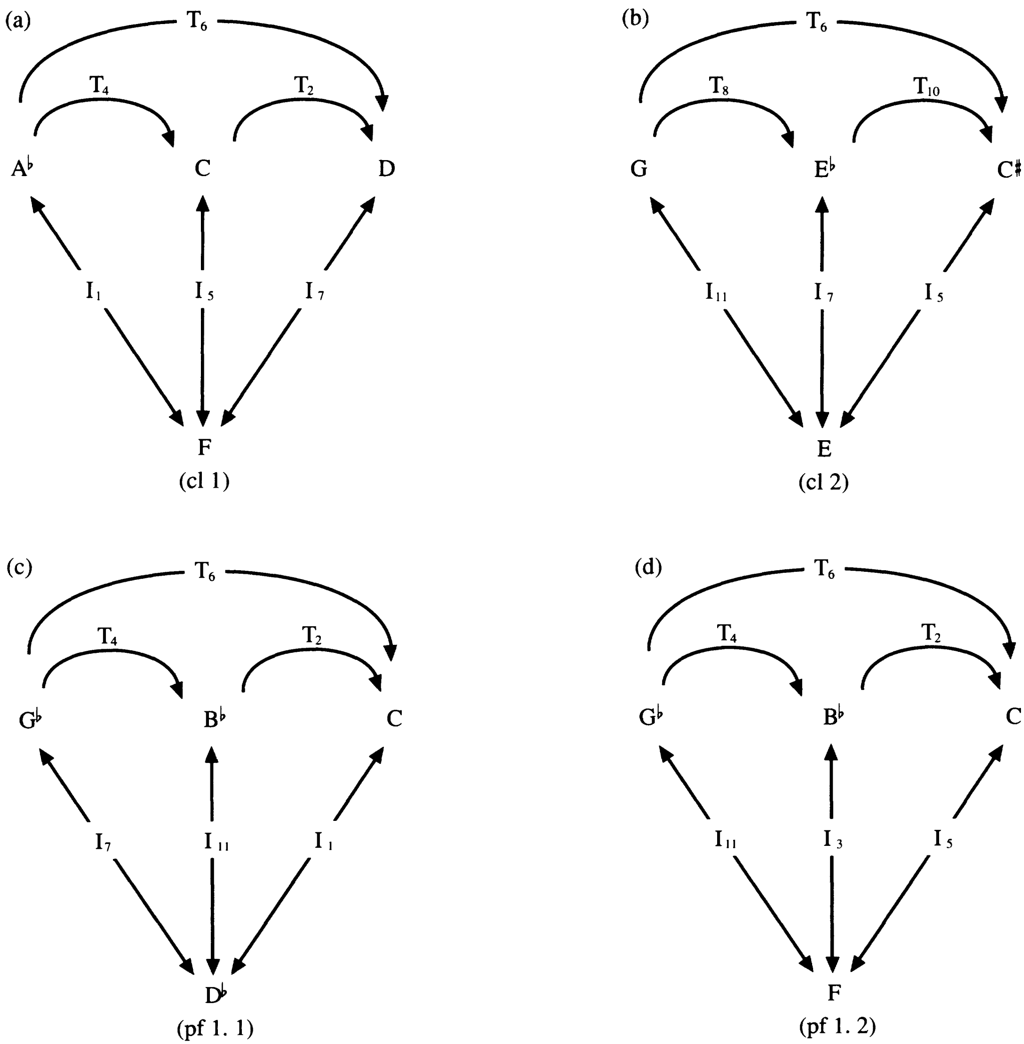
MODE I: The tetrachords both contain trichords of type 3-8. It follows that they will be interpreted isographically by networks that lay out the 3-8 trichords with T-arrows in a corresponding way, provided that the networks also connect with I-arrows the notes of the 3-8 trichords to the odd-notes-out of the respective tetrachords. Figure 18a and Figure 18b show isographic interpretations of this sort, whose graphs are labeled "cl1" and cl2" for "clarinet 1" and "clarinet 2."

Since the 3-8 forms are inversions of each other, the two networks are negatively isographic. The next step is to search for other networks isographic to these that interpret salient features of the music. To implement the search only requires listening for salient forms of set class 3-8, and exploring ways in which these can be interpreted, along with plausible "odd notes out," into networks resembling 18a and 18b. The presence of a registrally contiguous 3-8 set, (G♭,B♭,C), on top of the first five-note harmony in the piano makes plausible the assertion of Networks 18c and 18d, networks that interpret this set along with each of the lower two notes of the harmony, D♭ and F. The networks are labeled "pf1.1" and "pf1.2."

Now, networks pf1.1, pf1.2, and cl1 are all positive isographs of each other, while network cl2 is a negative isograph of the three. The analyst who examines the ⟨1-relations among pf1.1, pf1.2, and cl1 will find it possible to arrange the three graphs by means of ⟨1,4⟩, ⟨1,2⟩, and ⟨1,6⟩ relations, in a manner corresponding to the T-relations within each individual graph. Since the negative isograph cl2 will be related to each of the three others by means of ⟨11-relations, it follows that we can construct an isomorph of the cl1-graph at a higher structural level. Figure 19 shows this accomplished. Thus manifest is an interesting relation binding the opening piano harmony to the later clarinet figuration in a complex of relationships that itself embodies the "motive" of the clarinet figuration, as that motive is interpreted in MODE I.

A detail here is worth noting: the networks pf1.1, pf1.2, cl1, and cl2 all have *odd* I-numbers, while the isographic higher-level network of Figure 19 has *even* ⟨11-numbers. This structural feature of the situation does not depend on having taken the pitch class C = 0 in calculating I-labels. A change of referential pitch-class (say to F = 0) would have the effect of adding an *even* constant to each I-number of Figure 18; the new I-

Figure 18.

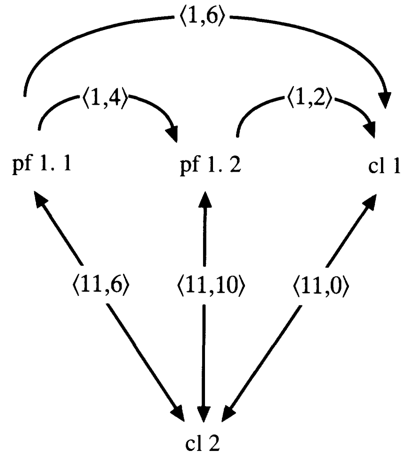

(a)
T6
T4
T2
A♭
C
D
I1
I5
I7
F
(cl 1)
(b)
T6
T8
T10
G
E♭
C♯
I11
I7
I5
E
(cl 2)
(c)
T6
T4
T2
G♭
B♭
C
I7
I11
I1
D♭
(pf 1. 1)
(d)
T6
T4
T2
G♭
B♭
C
I11
I3
I5
F
(pf 1. 2)

numbers would all remain odd, while the ⟨11-numbers of Figure 19 would remain even. (The reader may verify this by trial, or prove it as an exercise.) The graphs of Figure 19 will be called *odd-I* forms of the motivic graph, and the graph of Figure 19 an *even-⟨11* form.

The third full harmony in the piano—a six-note chord A-C♯-G-E-F♯-C, reading from the bass up—launches the clarinet into its solo. In the present context, the articulation of the piano's chord into two forms of 3-8, namely A-C♯-G and E-F♯-C, is clearly audible. The articulation is supported by registral contiguity, and by the division of left and right hands. This hearing of the chord interprets its harmony as saturated with MODE I networks. Figure 20 organizes them for inspection. The top halves of networks 20a, 20b, and 20c are all exactly the same, structuring the pcset (C,E,F♯) by T-arrows. The I-numbers of 20a, 20b, and 20c are the label sums of the respective odd-notes-out C♯, A, and G with the labels of the invariant trichord (C,E,F♯). When the odd-note-out C♯ in Network 20a becomes the odd-note-out A in Network 20b, all the I-labels of 20b increase by 8 over the I-labels of 20a, since the label of A is 8 more than the label of C♯. Likewise, when the odd-note-out A in Network 20b becomes the odd-note-out G in Network 20c, all the I-labels of 20c increase by 10 over the I-labels of 20b, since the label of G is 10 more than the label of A. But the odd-notes-out C♯, A, and G, themselves form a 3-8 set. By the very nature of the six-note chord, it thus follows that the graphs for 20a,b,c can be arranged to form a network whose ⟨1-arrows correspond isomorphically with the T-arrows of the basic cl1-graph. Figure 21a shows the arrangement, and Figure 21b shows the like arrangement for the graphs of 20d,e,f.

The graphs pf3.1, pf3.2, and pf3.3 are all positive isomorphs one of another (as seen in Figure 20a,b,c). Suppose that NEG is any negative isograph of the three. Then NEG can be connected by ⟨11-relations to each of pf3.1, pf3.2, and pf3.3. And then the structure of Figure 21a, with NEG and the three ⟨11-arrows adjoined, will produce a MODE I network. Since the network will have ⟨1,8⟩ and ⟨1,10⟩ arrows, rather than ⟨1,4⟩ and

Figure 19.



⟨1,2⟩ arrows, it will be positively isographic to cl2 (and so forth), negatively isographic to cl1 (and so forth).

Of particular analytic interest in this connection is the choice of cl2 itself for NEG. Figure 22a shows pf3.1, pf3.2, pf3.3 and cl2 arranged in a network that is isographic to cl2. Below Figure 22a, Figure 22c lays out once more the second clarinet tetrachord, as interpreted by the cl2 network. One sees the positive isography of 22a with 22c.

The same work applies, *mutatis mutandis,* for the graphs pf3.3, 3.4, and 3.5. Figure 22b shows those graphs arranged with cl1, in a network that is positively isographic to cl1 itself. Figure 22d, below Figure 22b, lays out once more the first clarinet tetrachord, as interpreted by the cl1 network.

In colloquial language, this work can be summarized as follows. Given the six-note chord that launches the clarinet solo, and given the two motivic tetrachords of that solo, the right hand piano trichord forms a MODE I network in conjunction with each note of the left hand; furthermore, these three

Figure 20.

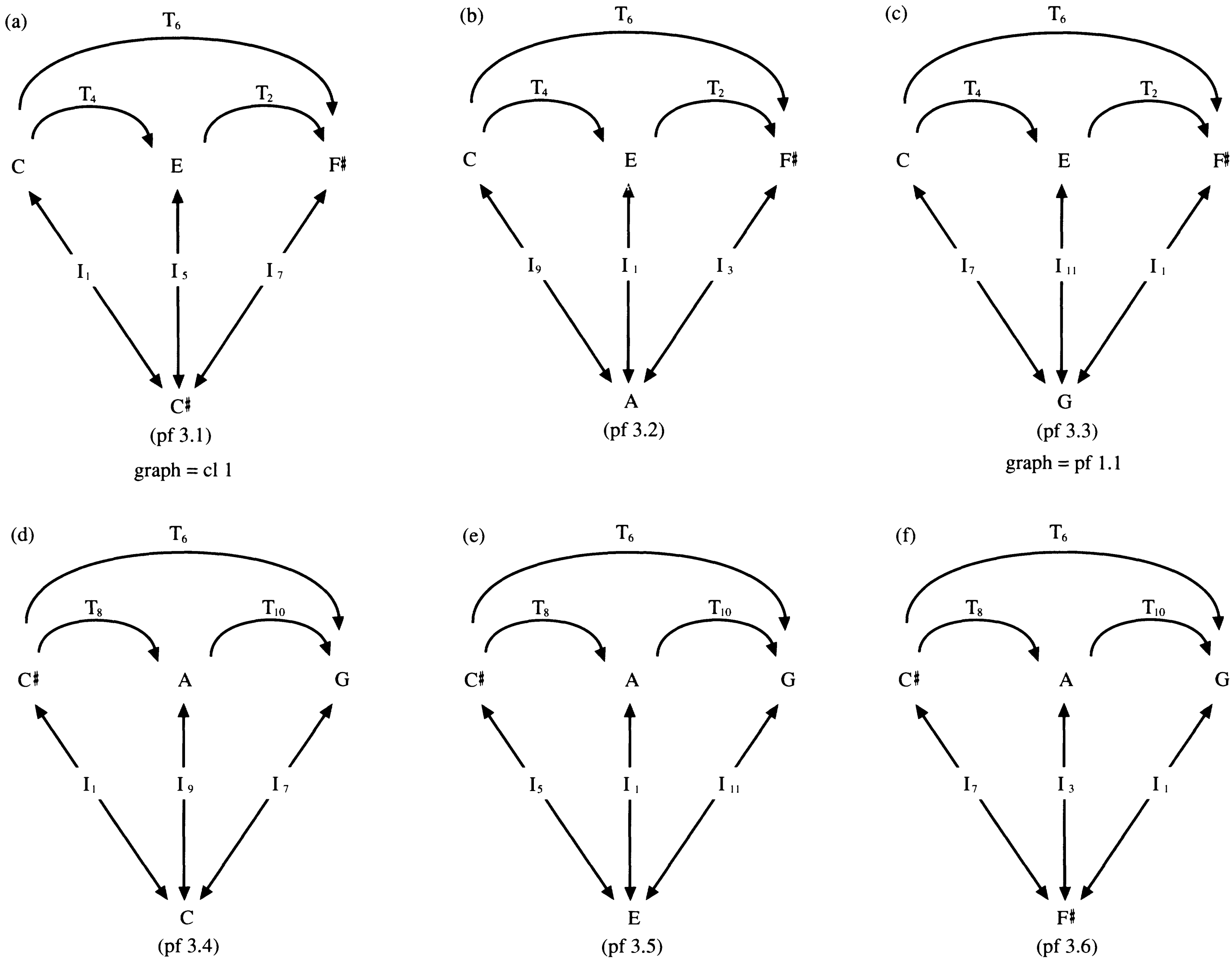
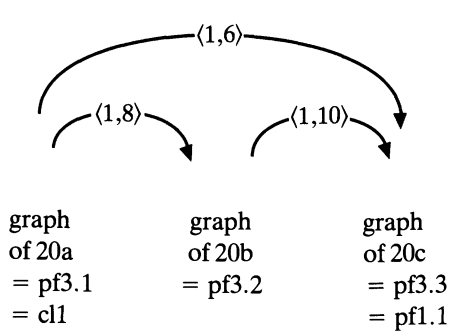
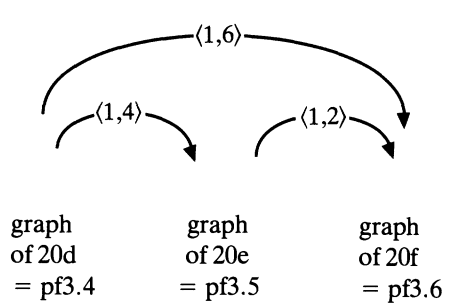

(a)
T6
T4
T2
C
E
F♯
I1
I5
I7
C♯
(pf 3.1)
graph = cl 1
(b)
T6
T4
T2
C
E
F♯
I9
I1
I3
A
(pf 3.2)
(c)
T6
T4
T2
C
E
F♯
I7
I11
I1
G
(pf 3.3)
graph = pf 1.1
(d)
T6
T8
T10
C♯
A
G
I1
I9
I7
C
(pf 3.4)
(e)
T6
T8
T10
C♯
A
G
I5
I1
I11
E
(pf 3.5)
(f)
T6
T8
T10
C♯
A
G
I7
I3
I1
F♯
(pf 3.6)

Figure 21.

(a)



(b)

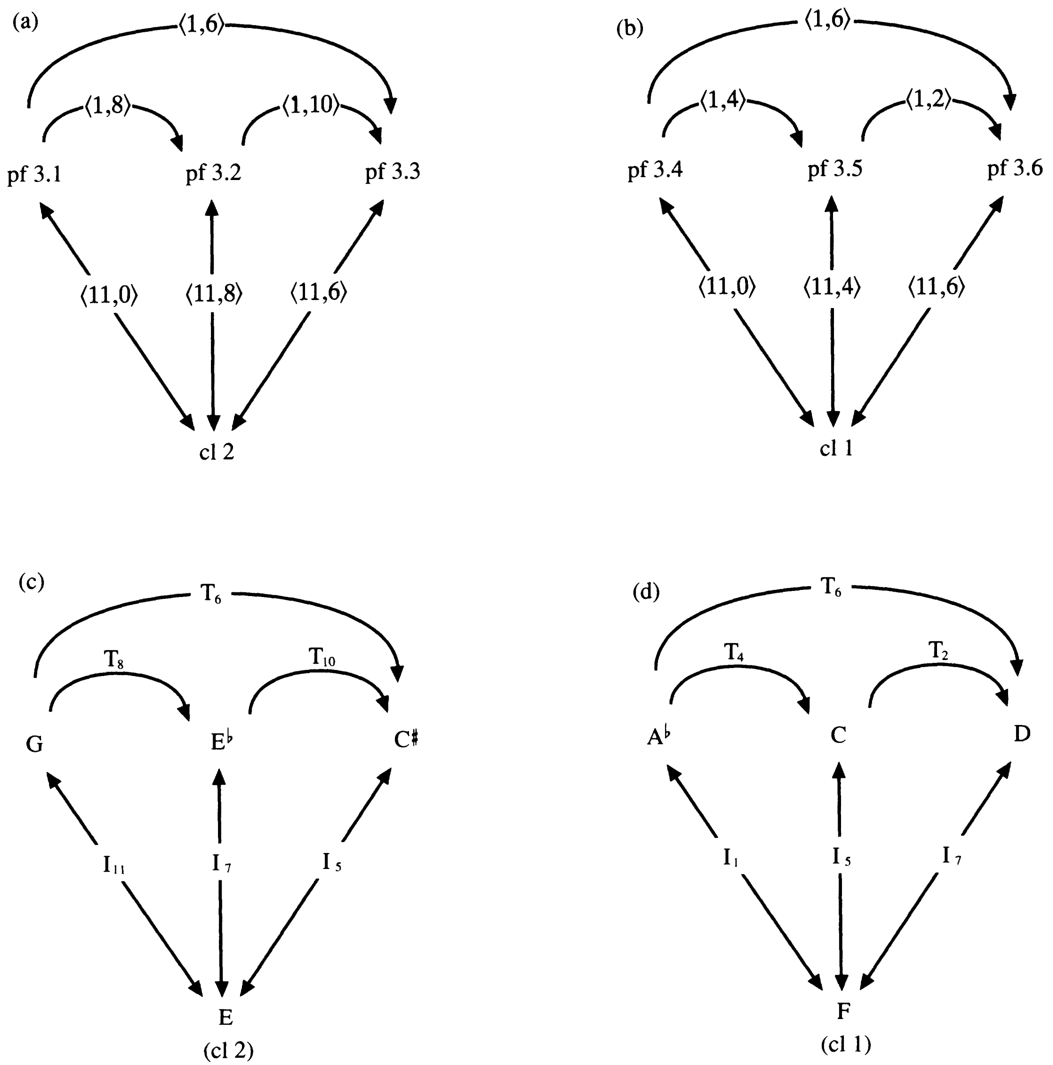


MODE I networks have graphs which combine with the MODE I graph of the second clarinet tetrachord to form a MODE I network-of-graphs at a higher hierarchic level. The higher-level network is positively isographic to the network for the second clarinet tetrachord. Also, given that six-note chord, and given the clarinet tetrachords, the left hand piano trichord forms a MODE I network in conjunction with each note of the right hand; furthermore, these three MODE I networks have graphs which combine with the MODE I graph of the first clarinet tetrachord to form a MODE I network-of-graphs at a higher hierarchic level. The higher-level network here is positively isographic to the network for the first clarinet tetrachord.

Regarding the general situation observed in connection with Figure 22: if NEG is the graph for any Klumpenhouwer Network that is negatively isomorphic to cl1, then NEG, together with Figure 22a and the three ⟨11-arrows connecting NEG with the pf-graphs, will form a higher level network-of-graphs that is positively isographic to NEG itself (or to cl2). The previous discussion explored the implications of taking NEG = cl2. But any of the networks 20d,e,f—that is, any MODE I network involving the left-hand trichord—could just as well be taken as NEG. In colloquial language: The three MODE I networks involving the right-hand trichord, together with any single MODE I network involving the left-hand trichord, naturally form a MODE I network at a higher hierarchic level. The higher-level network is positively isographic to the network interpreting the second clarinet tetrachord. Dually, the three MODE I networks involving the left-hand trichord, together with any single MODE I network involving the right-hand trichord, naturally form a MODE I network at a higher hierarchic level. The higher-level network here is positively isographic to the network interpreting the first clarinet tetrachord. As observed earlier, the networks of pitch classes, throughout this discussion, all have odd I-numbers, while the higher-level networks-of-graphs all have even ⟨11-numbers.

It is time now to bring into this analysis the second complete harmony in the piano, the harmony comprising D♭, F, C, G, and B, reading from the lowest pitch up. The harmony contains four instances of the characteristic MODE I trichord 3-8, and each 3-8 trichord will form a MODE I Klumpenhouwer Net-

Figure 22.


(a)
⟨1,6⟩
⟨1,8⟩
⟨1,10⟩
pf 3.1
pf 3.2
pf 3.3
⟨11,0⟩
⟨11,8⟩
⟨11,6⟩
cl 2
(b)
⟨1,6⟩
⟨1,4⟩
⟨1,2⟩
pf 3.4
pf 3.5
pf 3.6
⟨11,0⟩
⟨11,4⟩
⟨11,6⟩
cl 1
(c)
T_6
T_8
T_{10}
G
E♭
C♯
I_{11}
I_7
I_5
E
(cl 2)
(d)
T_6
T_4
T_2
A♭
C
D
I_1
I_5
I_7
F
(cl 1)

work when suitably interpreted in conjunction with either of the other two notes in the harmony. There are then eight new networks to be brought into the current context, which can be done in a quite organized way, as follows.

There are four odd-I networks among the eight. These are the networks that interpret the four trichord-forms (G,B,D♭), (D♭,F,G), (B,G,F), and (F,D♭,B), each in conjunction with the note C of the harmony. Each trichord as cited in the preceding sentence is ordered by means of intervals 4 and 2, or by means of intervals 8 and 10—in "MODE I order," so to speak. This ordering yields the I-numbers for the MODE I Klumpenhouwer Network that interprets (G,B,D♭) and C: the I-numbers will be "G + C, B + C, and D♭ + C," under the labeling system that makes C = 0. That is, the I-numbers will be 7, 11, and 1. The ordering of (G,B,D♭) shows that the MODE I T-numbers will be 4, 2, and 6. Therefore, the network is a "positive" form (like cl1) with I-numbers 7,11,1. The network is strongly isographic to the networks for pf1.1 and pf3.3—all three graphs have exactly the same T-numbers and exactly the same I-numbers.

The numbers for the MODE I network interpreting (D♭,F,G) plus C can be computed in similar fashion. The T-numbers will be 4,2, and 6; the I-numbers will be 1,5, and 7. The network is strongly isographic to the network for cl1. In the MODE I network interpreting (B,G,F) plus C, the T-numbers will be 8,10, and 6; the I-numbers will be 11,7, and 5. The network is strongly isographic to the network for cl2. Finally, in the MODE I network interpreting (F,D♭,B) plus C, the T-numbers will be 8,10, and 6; the I-numbers will be 5,1, and 11. The network is strongly isographic to the network for pf3.5.

In sum, the four odd-I networks interpreting aspects of this harmony do not introduce any new graphs into the analysis. Specifically, all their graphs can be found as aspects of the third harmony in the piano, or as aspects of the clarinet solo. This befits the way the chord sounds, as "passing" to the bar line of m. 2, where the third harmony is attacked and the clarinet enters.[7] The implied analysis is that the chord "anticipates" a few odd-I networks, so to speak; those odd-I networks, together with others, are then put into strict order by the MODE I structures of measure 2.

So much for the odd-I MODE I networks of the second harmony. The harmony also contains a tetrachord which can be interpreted by *even*-I networks of MODE I, and that is something new in our study of the passage. So far, all the "foreground" networks of the MODE have been odd-I networks; even-I networks have appeared only "in the middleground." Indeed, *every* higher-level MODE I network has been of even-⟨11 species. This suggests a special function for the second piano harmony, namely, to introduce even-I networks of MODE I "in the foreground," that is, as Klumpenhouwer Networks of pitch classes.

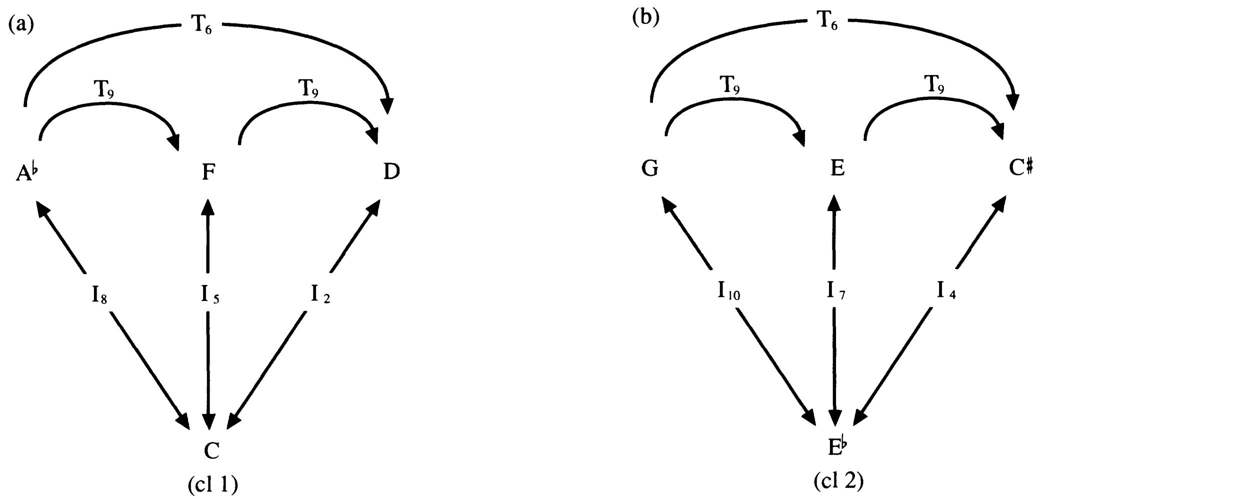
The even-I networks are all interpretations of the single tetrachord (D♭,F,G,B). The ordering (D♭,F,G) plus B gives rise to a network with T-numbers 4,2, and 6; the I-numbers are 0,4, and 6. The ordering (G,B,D♭) plus F gives rise to a strongly isographic network: the T-numbers are again 4,2, and 6; the I-numbers are again 0,4, and 6. The ordering (F,G,B) plus D♭ gives rise to a network with T-numbers 8,10, and 6; the I-numbers are 6,8, and 0. The ordering (B,D♭,F) plus G gives rise to a strongly isographic network: the T-numbers are again 8,10, and 6; the I-numbers are again 6,8, and 0.

One last observation worth making about MODE I is that the structures of the clarinet's tetrachords under this interpretation are to a certain extent projected by the rhythm and contour of the motive, and by the many 3-8 trichords in the preceding piano music. That is, one can hear the clarinet's motives, to a reasonable extent, as A♭-(and)-D-C / G-(and)-E♭-C♯.

MODE II: This interpretation, for the clarinet solo and for

[7]The "passing" harmony is omitted at the reprise in m. 13.

Figure 23.



works that lay out the 3-10 trichords with T-arrows in a corresponding way, provided that the networks also connect with I-arrows the notes of the 3-10 trichords to the odd-notes-out of the respective tetrachords. Figure 23a and Figure 23b show isographic interpretations of this sort, whose graphs are labeled "cl1" and "cl2" in the new mode. To implement the search for other networks, isographic to these, that interpret salient features of the music only requires listening for salient forms of set class 3-10 and exploring ways in which these can be interpreted, along with plausible "odd notes out," into networks resembling 23a and 23b—that is, into MODE II networks.

Forms of 3-10 are not much in evidence until the third harmony of the piano. This chord can be articulated in the present the third harmony in the piano, is less easy to hear than MODE I. However, it is still worth exploring.

The clarinet's tetrachords both contain trichords of type 3-10. It follows that they will be interpreted isographically by net-context, into two 3-10 trichords, A-( )-F♯-C and C♯-G-E, which provide the basis for six MODE II networks in a manner analogous to that of Figure 20 earlier. Figure 24 displays the analogous catalogue. The graphs for 24d,e,f duplicate exactly the graphs for 24a,b,c respectively, and the corresponding pairs of networks are strongly isographic. Let us refer to the three graphs as pf.1 = 4, pf.2 = 5, and pf.3 = 6. These three graphs are themselves related by ⟨1,9⟩-relations and ⟨1,6⟩-relations. (This continues the analogy with the MODE I structure of the chord.) Hence the three graphs, together with any fourth MODE II Klumpenhouwer Network considered in ⟨11-relations with the three of them, will perforce form a higher-level network-of-graphs that is itself a MODE II network. Fig-

Figure 24.

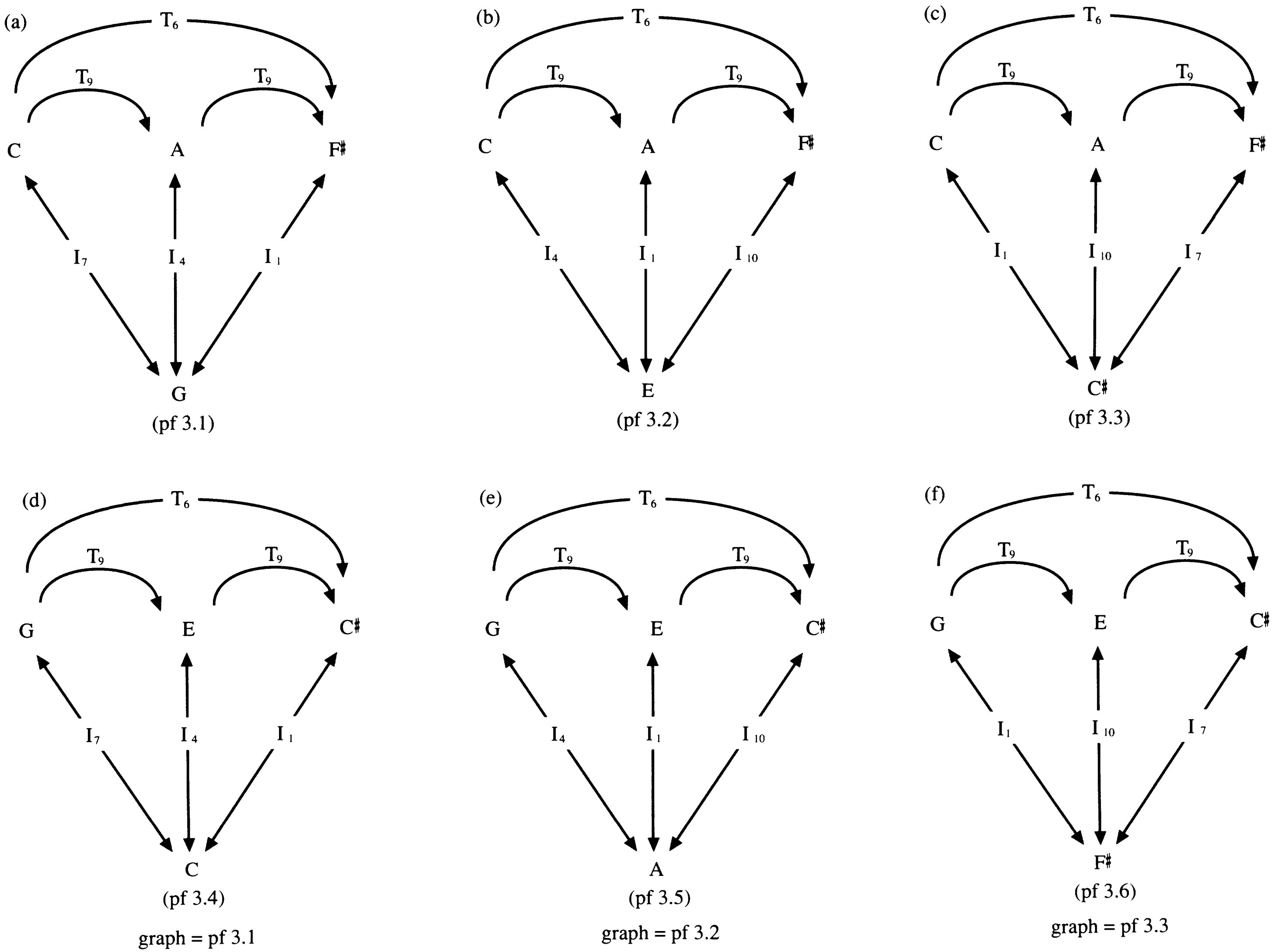
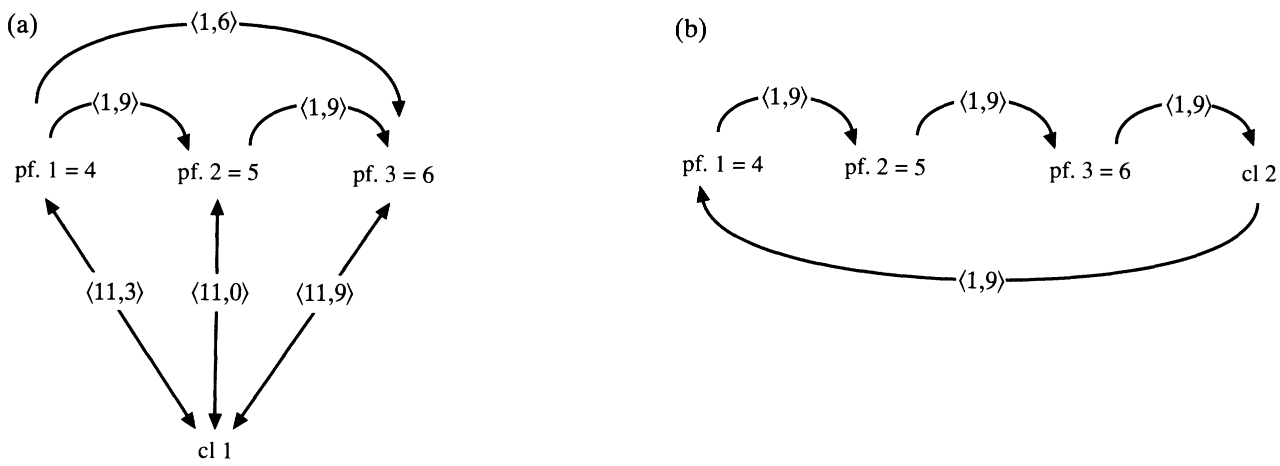

(a)
T6
T9
T9
C
A
F♯
I7
I4
I1
G
(pf 3.1)
(b)
T6
T9
T9
C
A
F♯
I4
I1
I10
E
(pf 3.2)
(c)
T6
T9
T9
C
A
F♯
I1
I10
I7
C♯
(pf 3.3)
(d)
T6
T9
T9
G
E
C♯
I7
I4
I1
C
(pf 3.4)
graph = pf 3.1
(e)
T6
T9
T9
G
E
C♯
I4
I1
I10
A
(pf 3.5)
graph = pf 3.2
(f)
T6
T9
T9
G
E
C♯
I1
I10
I7
F♯
(pf 3.6)
graph = pf 3.3

Figure 25.



ure 25a exemplifies this, using the MODE II graph of cl1 for the purpose.

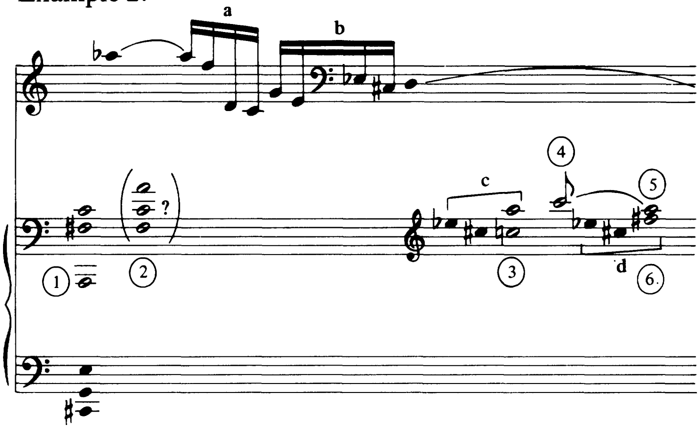
Any other MODE II network instead of cl1 would have done just as well for the fourth constituent of Figure 25a (adjusting the I-numbers accordingly), but it is particularly nice for analytic purposes to bind up the three piano networks here with a clarinet network: the clarinet tetrachords, after all, were the point of departure for the entire exercise concerning "Gebet." (The two tetrachords, in particular, determine the various MODES.) The cl2 network binds up with the three piano networks in the manner of Figure 25a, but it also enjoys another interesting higher-level relationship with the piano networks. That is shown in Figure 25b: the cl2 network completes the ⟨1,9⟩-cycle suggested by the three piano networks.

Example 2 will help explore some implications of MODE II structuring a bit further into the piece. The figure graphs aspects of the music from the bar line of m. 2 through the middle of m. 4. Numbers and letters on the example key to the commentary that follows.

At (1), the registration of the piano chord puts the A on the bottom and the C on top. The MODE II structure of the harmony would be much more audible if the A were moved up to be on top of the chord, as at (2); that would articulate the two 3-10 trichords in the registral ordering. The suggestion is that A "belongs on top" of the piano in MODE II. When the piano next enters, at (3), the A has indeed moved to be "on top" of the entire piece so far.[8] And at (3), where the A gets on top, the

[8]One hears very strongly that this A is even higher than the A♭ on which the clarinet entered. I do not think it is a coincidence that the clarinet is a clarinet in A. The piano here plays the note which, for the clarinet, is its "high C." The clarinet, at its entrance, fell a half-step flat of its high C, which it has "verlernt."

Example 2.



C is now on the bottom. The exchange of outer voices in the piano, from (1) to (3), is easily heard once noticed. At (4) the C attempts to regain its supremacy, but it quickly falls back, an eighth note later at (5), to the high A. F♯ reappears here as well ((6)), to bind the C and the A together into the same 3-10 trichord that bound them earlier, at (1).[9]

On the example, the first and second tetrachords of the clarinet have been labeled *a* and *b*. The example annotates two tetrachords that are naturally articulated by the piano in the music that follows, tetrachords *c* and *d*. One recognizes that the pcset *c* inverts pcset *b* by pitch-class inversion about the cadential note D of the clarinet. And one recognizes that pcset *d* is a T_1-transpose of pcset *a*. Tetrachords *c* and *d* thereby are primed to enter into more MODE II structuring with earlier events in the piano and the clarinet. However, the specifics of that matter, if pursued further, would exfoliate too rapidly and voluminously into the rest of the piece; it will not be practical to consider them here. The diminished-seventh chord (C,A,F♯,E♭), audible in the music corresponding to the end of Example 2, presents in the foreground of the piece the complete 9-cycling observed earlier in the network of MODE II graphs given by Figure 25b.

MODE III: The clarinet tetrachords can each be articulated into a dyad of ic4 plus a dyad of ic3: A♭-F-D-C can articulate as (A♭,C) plus (F,D); G-E-E♭-C♯ can articulate as (G,E♭) plus (E,C♯). This feature can yield isographic Klumpenhouwer Networks to interpret the two tetrachords. Figure 26a and Figure 26b illustrate such a pair of interpretations. This articulation of the clarinet tetrachords is hard to hear in the music, but it is nevertheless worth considering partly for the sake of being completely systematic and thorough (lest we ignore something that might later be interesting to hear) and partly because a certain interesting higher-level MODE III structure will emerge later, in connection with the (very audible) MODE IV articulation of the clarinet tetrachords.

To hear further MODE III structures in the opening phrase means listening for other places with suitable collations of 4-dyads and 3-dyads. These are not forthcoming in the first two complete harmonies of the piano. But such structures are audible in the third harmony of the piano. Within A-C♯-G-E-F♯-C, specifically, the lowest 4 notes can be heard as a 4-dyad plus a 3-dyad; Figure 26c interprets this tetrachord by a MODE III network isographic to 26a and 26b. Also within A-C♯-G-E-F♯-C, the top three notes plus the bass can be heard as a 4-dyad (E,C) plus a 3-dyad (A,F♯); Figure 26d interprets the tetrachord by a MODE III network isographic to 26a,b, and c.

The search for higher-level structuring in the four graphs of Figure 26 reveals a ⟨1,3⟩ relation between 26a and 26c, but no ⟨1,4⟩ or ⟨1,8⟩ relation between 26b and 26d. However, the ⟨1,4⟩ relation between 26d and 26a strikes fire: there is also a ⟨1,3⟩ relation between 26c and 26b.

[9]The sorts of registral play going on in (1) through (6), in the context of network-structuring, are very much the sorts of things which Klumpenhouwer is addressing in his dissertation.

Figure 26.

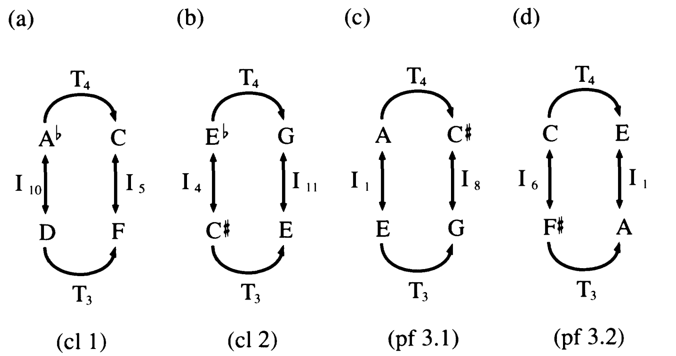
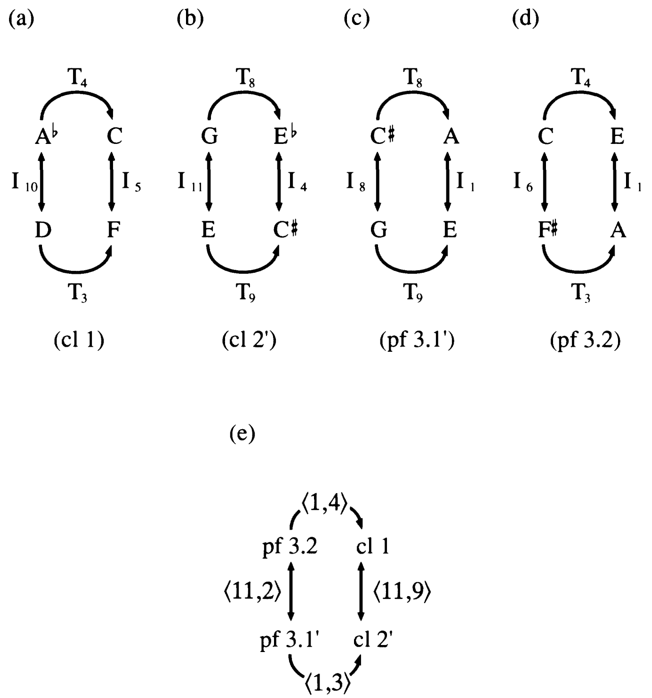


Figure 27.

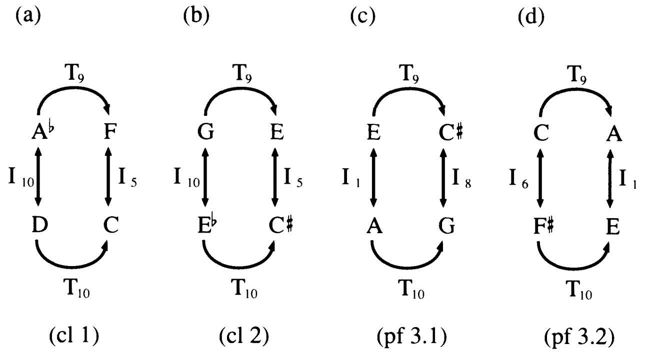


The next step is to analyze ⟨11-relations between the ⟨1,3⟩-related graphs and the ⟨1,4⟩-related graphs, analogous to the I-relations within 26a itself. Figures 26c and 26b, the ⟨1,3⟩-related graphs, are not in fact ⟨11-related to 26d and 26a, the ⟨1,4⟩-related graphs; however, "negative forms" of 26c and 26b will interpret the same respective tetrachords. A suitable network of 26d, 26a, negative 26b, and negative 26c will then provide a network-of-graphs that reproduces the MODE III form at a higher level. Figure 27 carries through this program. Figures 27a and d reproduce the interpretations of Figures 26a and d. Figure 27b is an interpretation of the second clarinet tetrachord whose graph is a negative isomorph to that of Figure 26b; the graph of 27a will be called "cl2′." Likewise, Figure 27c reinterprets its piano tetrachord via a Klumpenhouwer Network negatively isomorphic to Figure 26c; the new graph is "pf3.1′." Figure 27e, finally, arranges the four graphs of 27a,b,c, and d into a higher-level network isographic to Figure 27a itself.

MODE IV: One hears each clarinet tetrachord articulated into a 9-dyad plus a 10-dyad: A♭-F-D-C is articulated as (A♭,F) plus (D,C); G-E-E♭-C♯ is articulated as (G,E) plus (E♭,C♯). This articulation is strongly supported not only by serial ordering but also by register (in several ways); it is surely the most natural way to hear the two tetrachords fitting the rhythm-and-contour motive "in a similar fashion," and it is therefore the most natural way to hear the tetrachords interpreted by isographic networks. Figures 28a and 28b present such networks.

The "similar fashion" in which the tetrachords fit the motive translates here into the strong isography between 28a and 28b.

Figure 28.



One hears the corresponding "wedging in" of the dyadic groups by way of I_{10} and I_5 in the music: the I_{10}-dyad (A♭,D) of the first motive-form in the clarinet wedges into the I_{10}-dyad (G,E♭) of the second motive-form; the I_5-dyad (F,C) of the first form similarly wedges into the I_5-dyad (E,C♯) of the second form.

The tetrachords A-C♯-G-E-( ) and A-( )-E-F♯-C of the third piano harmony, tetrachords earlier subjected to MODE III analysis, can also be analyzed in MODE IV. That is, each tetrachord contains one 9-dyad and one 10-dyad. Figures 28c and 28d give this analysis, using suitable Klumpenhouwer Networks isographic to 28a and 28b.

It is not possible to arrange the graphs of Figure 28a,b,c,d and/or their corresponding "negatives" into a MODE IV supernetwork. However, it *is* possible to arrange the graphs in a MODE III supernetwork. Figure 29 shows how. Here the interpretations of Figures 28b and 28c have been replaced by the "negative" interpretations of Figure 29b and 29c, labeled as "cl2′ " and "pf3.1′ " (in MODE IV). Figure 29e displays the graphs of 29a,b,c,d, arranged in a MODE III network-of-(MODE IV) graphs.

Figure 29e involves both cl1 and cl2′ in its structure. The phenomenon deserves particular attention because, simply as a graph, cl2 = cl1. That is, the corresponding T and I arrows of Figure 28a and Figure 28b have exactly the same numerical labels (in exactly the same configuration of nodes and arrows). If the graph of cl1 = cl2 is simply called "cl" for the moment, then 29e involves both cl in its "positive" aspect (as cl1) and also cl again in its "negative" aspect (as cl2′). The phenomenology of this gives few difficulties, because "the first clarinet motive-form" is clearly in a different phenomenological place from "the second clarinet motive-form," and the double duty of the graph cl, on Figure 29e, is clearly intended to model interpretations that go with those two different "places." Hence it seems quite right that cl should manifest itself in two different "places" on the visual display for Figure 29e. The reader should mark the foregoing discussion, because it will return shortly in a context where the phenomenology is more difficult.

A search conducted in the usual way for MODE IV Klumpenhouwer Networks to interpret aspects of the first or second harmonies in the piano, D♭-F-C-G♭-B♭ or D♭-F-C-G-B, is unproductive, for in neither of these pentachords are there a 9-dyad and a 10-dyad embedded disjointly. The first pentachord contains only one 9-dyad, the dyad involving the D♭ and the B♭ of its outer voices. This observation leads us to note that the second pentachord spans a 10-dyad D♭-( )-B between *its* outer voices. And the third piano harmony, A-C♯-G-E-F♯-C, spans a 3-dyad A-( )-C between *its* outer voices, a 3-dyad which later turns into a 9-dyad when the A "gets on top of the C" and the outer voices exchange, as studied earlier in connection with Example 2. Figure 30 arranges these outer-voice dyads into two new MODE IV networks, 30e and 30f. Figure 30e, labeled as "ov1," assembles the outer voices of the first harmony, D♭ and B♭, together with the outer voices of the second harmony, D♭ and B, into a suitable network. Figure 30f, labeled as "ov2," does the same for the outer voices of the second harmony, D♭ and B, together with the outer voices of the third harmony, A

Figure 29.

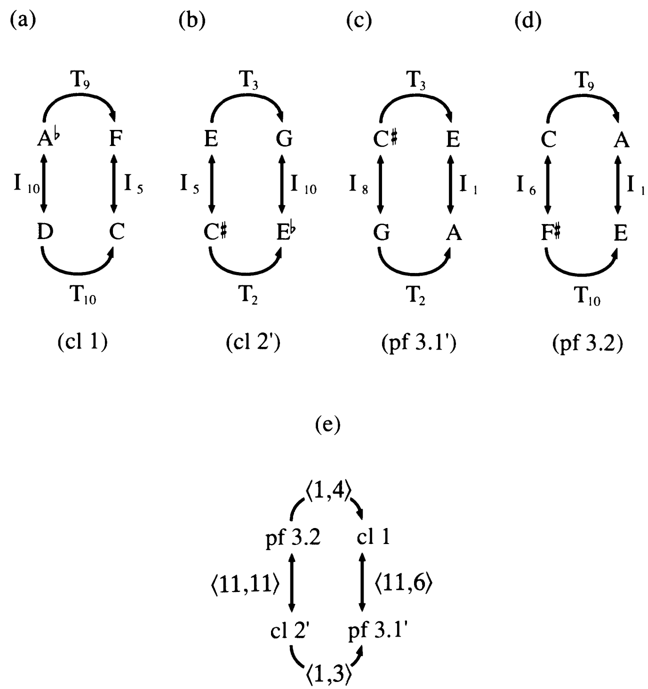
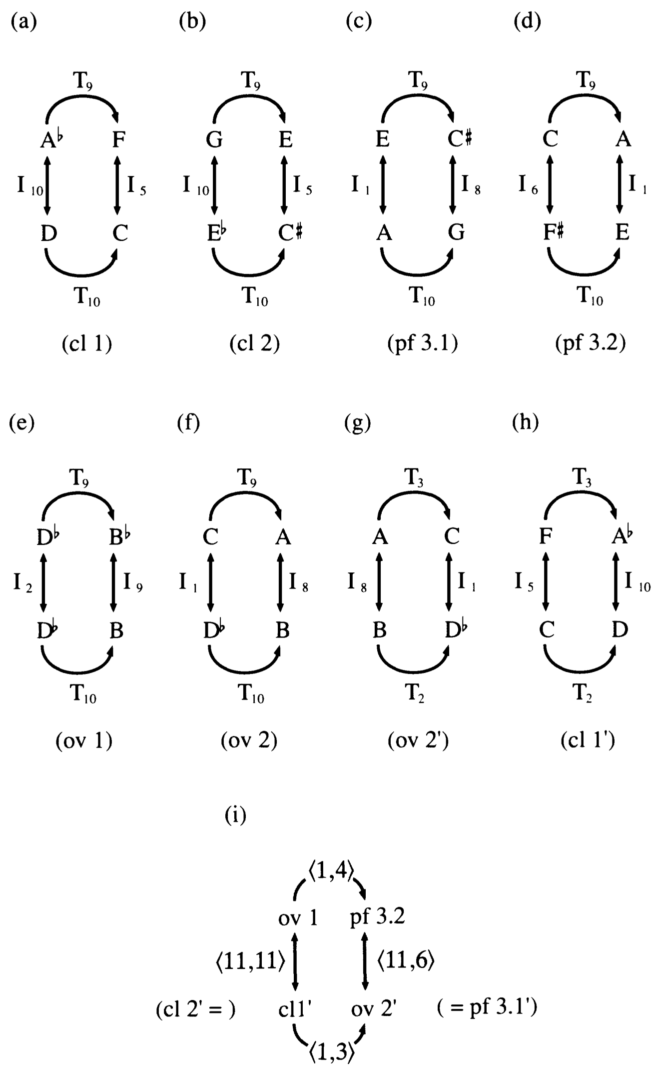


Figure 30.



and C. Figures 30a,b,c,d simply reproduce Figures 28a,b,c,d for comparison. Figure 30g is the negative network-interpretation of 30f, and Figure 30h is the negative network-interpretation of 30a.

Figure 30i shows the graphs of 30e,d,h, and g arranged in a MODE III network-of-(MODE IV) graphs. The arrangement really involves all of the graphs cl1, cl2, pf3.1, pf3.2, ov1, and ov2, since cl1 = cl2 as a graph and ov2 = pf3.1 as a graph. (Networks 30a and b are strongly isographic; so are networks 30c and f.) Being addressed by the supernetwork of 30i, then, is a complex of structures involving the progression of outer voices from the first piano harmony to the second, the progression of outer voices from the second piano harmony to the third, the internal structure of the third piano harmony, and the two motive-forms of the clarinet, each as interpreted in MODE IV. The interrelation of these structures at a higher level exhibits organization in MODE III.

This concludes the analysis of "Gebet an Pierrot." One final observation of special interest, though, pertains to Figure 30e. The hitherto unusual feature of this network is, of course, that the pitch class D♭ appears at two different places within it. Here the earlier phenomenological excursion will stand the reader in good stead. Figure 30e is not interpreting "a trichord" or "a pcset of cardinality 3" or "a three-note harmony." It is, rather, interpreting a progression of outer voices from one harmony to the next. In the phenomenology of this interpretation there are four characteristically marked "places," namely the bottom of the first harmony, the top of the first harmony, the bottom of the second harmony, and the top of the second harmony. Those places are interpreted by the four nodes of Network 30e, namely the upper-left node, the upper-right node, the lower-left node, and the lower-right node respectively. So the D♭ that appears as the CONTENTS of the upper-left node represents the low note of the first harmony, in one "place," while the D♭ that appears as the CONTENTS of the lower-left node represents the low note of the second harmony, in another "place."

Any reader initially feeling uncomfortable with this aspect of Figure 30e is suffering from a phenomenological malady that bears special watching, a malady which encourages the identification of networks-of-pcs with pcsets. Even where a pcset has the same cardinality as the number of nodes in a Klumpenhouwer Network that is interpreting it, the identification is invalid and potentially confusing: as witnessed repeatedly in this paper, one pcset may be interpreted in different MODES by a great variety of Klumpenhouwer Networks. To be sure, Klumpenhouwer Networks are particularly interesting when used to interpret pcsets. But Figure 30e testifies that Klumpenhouwer Networks have many other possible uses as well. Exploring other uses, indeed, seems one logical plan for continuing these explorations.[10]

VII. AFTERWORD

This postscript provides information on Klumpenhouwer and an account of the circumstances under which he and I made our respective contributions to the foregoing work.

Henry Klumpenhouwer is a doctoral student at Harvard, working on a dissertation under my supervision. The dissertation studies Generalized Interval Systems whose "intervals" involve permutations in the registral orderings of suitably interrelated pitch-class structures. Analytic applications center on the standard atonal literature, also referring back to *Tristan* and forward to classical serial music.

At first, Klumpenhouwer assumed the suitably interrelated pitch-class structures to be T- or I-related pcsets. But after a time, he came to suspect that his machinery could be adapted to address (registral orderings of) pcsets that were not T- or I-related. Using the terminology just developed in this article,

[10]I believe, for example, that Klumpenhouwer Networks could prove very suggestive in connection with the theories and compositional methods of George Perle, which he expounds in *Twelve-Tone Tonality* (Berkeley: University of California Press, 1977). I suspect that isographies of the Networks, and networks-of-Networks, could reveal aspects of large-scale structuring in Perle's music beyond the extent of the formal analyses which Perle himself presents. In this connection it might be worth observing that the transformation $F\langle u,j\rangle$ of Part III still defines a mathematical homomorphism, satisfying the law $F(XY) = F(X)F(Y)$, even if u is not 1, 5, 7, or 11. *GMIT* contains pertinent material on "homomorphisms of graphs," to enable the interested reader to follow up the point in Perle's contexts.

one can now put his intuition in the following form: Certain pairs of pcsets, even if not T/I related, can be interpreted by Klumpenhouwer Networks that *are* "T/I related" in some rigorous extended sense; this being done, a basis is laid for comparing the registral orderings of nodes within the Networks. Remarks concluding the discussion of Example 1, the reader will recall, showed how the pertinent network structures can engage registral ordering. Here the "suitably interrelated pitch-class structures" are in fact isographic Klumpenhouwer Networks.

Working on these intuitions, Klumpenhouwer developed his ideas about "strong isography" and "network isomorphism," ideas exposed in the article. With their help, he was able to show that certain Klumpenhouwer Networks enjoyed a special relationship which now can be described as an "extended T_k relation,"—that is, a ⟨1,k⟩ relation—where k is an even number modulo 12. Using the ideas quoted above, he proved that such networks are formally isographic. He conjectured that network-pairs exhibiting analogous structural features would also be isographic, even if the number k that related them were odd.

At this point, he wanted advice from me. His intuitions were not as yet so neatly concretized as the foregoing synopsis makes them out to be; he sought my reactions in general to the work summarized above, and he thought in particular that I might be able to respond very quickly as to the truth or falsehood of his conjecture.

Initially skeptical about the analytic significance of Klumpenhouwer's Networks, I soon became drawn into their study myself, and enthusiastic about their potential. To some extent my conversion involved working out in my own terms a phenomenology to support the implicit analytic assertions of the Networks. The beginnings of such an agenda are suggested by the discussion of Example 1, where my phenomenological remarks supplement Klumpenhouwer's analytic observations. Largely, however, my enthusiasm developed as a result of my response to Klumpenhouwer's conjecture. I sketched out "the automorphisms of the T/I group" as an abstract mathematical exercise. The exercise proves the conjecture true. Beyond that, it also establishes a rigorous sense in which certain Klumpenhouwer Networks can be considered "I-related" (or "M-related") as well as "T-related." Klumpenhouwer had not yet consciously formulated any intuitions that involved "I-related" Networks, though as I look back now I find such intuitions strongly implicit in his original *Einfall.*

Not yet realizing the implications of the situation myself, I began analytic work on passages from *Pierrot,* thinking to evaluate by this means the extent to which I found the Networks either suggestive or limited as analytic tools. In the course of this work I began to notice the recursive potentialities of the theoretical apparatus. When a lower-level Klumpenhouwer Network is interpreting a chord, and a higher-level network-of-Networks is interpreting a progression of chords (more precisely, of chord-interpretations), I noted that one could conceive of the higher-level network as "prolonging" the lower-level one, particularly when the given chord is part of the given progression. This potentiality of the system, observed again and again in the article, can afford an especially compelling rationale (albeit a non-phenomenological one) for asserting one particular Klumpenhouwer Network rather than another, to interpret a given chord. I found it suggestively comparable, methodologically, to the ways in which a choice among foreground readings in a Schenkerian analysis can be influenced by middleground considerations.

I became fully aware only at this point that the isographies at issue among Klumpenhouwer Networks were in fact "T/I relations in a rigorously extended sense"—that is, that the ⟨1,j⟩/⟨11,j⟩ group is isomorphic to the T/I group, as discussed in connection with Formulas 2. Only at this point, then, did it become possible for me to describe Klumpenhouwer's original intuition in the way I described it above.

Finally, as I was writing up the article, I began to notice that

Klumpenhouwer Networks can be used for analytic purposes quite other than interpreting chords, or even pcsets. Some of the possibilities are hinted at in the discussion of Figure 30e, and in footnote 10.

ABSTRACT

Networks involving T and I operations are useful for interpreting pc-sets, and for other purposes. Certain groups of isographies among such networks, being isomorphic to the T/I group itself, are particularly interesting. A T/I network can interpret one chord within a progression, while an isographic network can interpret an interrelation among the several (interpreted) chords of that progression as a whole. A variety of interpretations is possible at each level, posing challenges for analysis.

In the process of considering this article for publication, readers raised the possibility that the work described herein should be credited to Lewin and Klumpenhouwer as coauthors. In my subsequent, separate conversations with them, it became clear to me that both Lewin and Klumpenhouwer had been sensitive to this issue all along. I am satisfied that the present format adequately acknowledges Klumpenhouwer's contribution, and that the listing of Lewin as sole author is both accurate and appropriate.—*Ed.*

Appendix A

Here is the proof for the theorem stated in the main text (p. 88) concerning the automorphisms of the T/I group.

Given u = 1,5,7, or 11; given j any number mod 12. Let $F\langle u,j\rangle$ be defined:

$F\langle u,j\rangle(T_n) = T_{un}$

$F\langle u,j\rangle(I_n) = I_{un+j}$.

Over this section of the proof, "F" stands for "$F\langle u,j\rangle$" to save space. To show that F is an automorphism of the T/I group, it must first be shown that for all X and Y in that group, F(XY) = F(X)F(Y). There are four possible generic cases, each to be considered separately:

Case 1: $X = T_m$ and $Y = T_n$. Then $F(XY) = F(T_mT_n) = F(T_{m+n}) = T_{u(m+n)} = T_{um+un} = T_{um}T_{un} = F(T_m)F(T_n) = F(X)F(Y)$ as desired.

Case 2: $X = T_m$ and $Y = I_n$. Then $F(XY) = F(T_mI_n) = F(I_{m+n}) = I_{u(m+n)+j} = I_{um+(un+j)} = T_{um}\quad I_{un+j} = F(T_m)F(I_n) = F(X)F(Y)$ as desired.

Case 3: $X = I_m$ and $Y = T_m$. Then $F(XY) = F(I_mT_n) = F(I_{m-n}) = I_{u(m-n)+j} = I_{(um+j)-un} = I_{um+j}\ T_{un} = F(I_m)F(T_n) = F(X)F(Y)$ as desired.

Case 4: $X = I_m$ and $Y = I_n$. Then $F(XY) = F(I_mI_n) = F(T_{m-n}) = T_{u(m-n)} = T_{um-un} = T_{um+j-un-j} = I_{um+j}\ I_{un+j} = F(I_m)F(I_n) = F(X)F(Y)$ as desired.

The above work shows that F interacts correctly with the group structure of the Ts and Is. To show that F is an automorphism only requires showing that F is one-to-one and onto, as a map of the T/I group into itself. By standard considerations of finite group theory, it suffices to show that F maps no operation X, other than T_0, into the identity operation T_0. And that is clearly the case here: if $F(X) = T_0$, then X cannot be an inversion, so X must be some transposition T_n such that un = 0 mod

12. But, because of the restriction on the value of u, that can only happen if n = 0; hence $X = T_0$ as desired. Q.E.D.

Now we pass on to the second assertion of the main text. Let G be any automorphism of the T/I group. We have to show that there exist u and j such that $G = F\langle u,j\rangle$.

The operator T_1 is of order 12 in the T/I group. Since G is an automorphism, $G(T_1)$ must be of order 12. Then $G(T_1)$ must be either T_1, or T_5, or T_7, or T_{11}, since those are the only elements of order 12 in the group. Set $G(T_1) = T_u$, where u = 1, 5, 7, or 11. Then $G(T_2) = G(T_1)G(T_1) = T_{2u}$. Then $G(T_3) = G(T_2)G(T_1) = T_{2u}T_u = T_{3u}$. And so on by induction: for any n, $G(T_n) = T_{nu} = T_{un}$.

Since I_0 is of order 2 in the group, $G(I_0)$ must be of order 2. Hence either $G(I_0)$ is an inversion, or else $G(I_0) = T6$. But, since 6u = 6, $G(T_6) = T6$. Since G is one-to-one, it cannot map both I_0 and T_6 to T_6. Hence $G(I_0)$ is some inversion. Set $G(I_0) = I_j$. Then for any n, $G(I_n) = G(T_n)G(I_0) = T_{un}\ I_j = I_{un+j}$. In sum, $G(T_n) = T_{un}$, and $G(I_n) = I_{un+j}$. Hence $G = F\langle u,j\rangle$ as asserted. Q.E.D.

The inner automorphisms of the T/I group are precisely those $F\langle u,j\rangle$ with u = 1 or 11, and j = 2k even. To see this, we compute the effect of the inner automorphism G determined by T_k: $G(X) = T_k\ X\ T_{-k}$. For each n, we have $G(T_n) = T_k\ T_n\ T_{-k} = T_n$. Furthermore, $G(I_n) = T_k\ I_n\ T_{-k} = T_k\ T_k\ I_n = T_{2k}\ I_n = I_{n+2k}$. So the effect of the inner automorphism G is $G(T_n) = T_n$; $G(I_n) = I_{n+2k}$. G is therefore the automorphism $F\langle 1,2k\rangle$. Next we compute the effect of the inner automorphism H determined by I_k: $H(X) = I_k\ X\ I_k$. For each n, we have $H(T_n) = I_k\ T_n\ I_k = I_k\ I_k\ T_{-n} = T_{-n}$. Furthermore, $H(I_n) = I_k\ I_n\ I_k = T_{k-n}\ I_k = I_{2k-n} = I_{-n+2k}$. So the effect of the inner automorphism H is $H(T_n) = T_{-n}$; $H(I_n) = I_{-n+2k}$. H is therefore the automorphism $F\langle 11,2k\rangle$.

It follows that when j is odd, the automorphisms $F\langle 1,j\rangle$ and $F\langle 11,j\rangle$ cannot be found as inner automorphisms. Adding on the multiplicative operations M_5 and M_7 for pitch classes does not change matters in this respect. The mapping that takes T_n to $(M_5\ T_n\ M_5)$ and I_n to $(M_5\ I_n\ M_5)$ is in fact $F\langle 5,0\rangle$, taking T_n to T_{5n} and I_n to I_{5n}; the mapping does not help generate any "odd j's." Of course this mapping is not an inner automorphism of the T/I group, since M_5 is not a member of that group.

Appendix B

As promised in footnote 6, here are developed the mathematics needed to include M-operations in a recursive system at all levels.

B1.DEF: By the T/M group are meant the 48 pc operations of form T_nM_a, where n ranges from 0 through 11 mod 12 and a takes on the values 1, 5, 7, and 11 mod 12. The operation T_nM_a performed on the pitch class labeled x yields the pitch class labeled $T_nM_a(x) = n + ax$.

M_1 is the identity operation, so the operations T_nM_1 are the transpositions. $M_{11}(x) = 11x = -x$ mod 12, so the operations T_nM_{11} are the inversions.

The T/M operations combine according to the rule

$$(T_mM_a)\ (T_nM_b) = T_{m+an}\ M_{ab}.$$

The following facts will be needed later on.

B2. LEMMAS: A member X of the T/M group

(a) has order 12 if and only if $X = T_1, T_5, T_7$, or T_{11}.

(b) has order 4 if and only if $X = T_3$ or T_9, or $X = T_nM_5$ where n is odd.

(c) has order 6 if and only if $X = T_2$ or T_{10}, or $X = T_nM_7$ where n is not divisible by 3.

(d) has order 3 if and only if $X = T_4$ or T_8.

(e) has order 1 if and only if $X = T_0$ (which $= M_1 = T_0M_1$).

(f) has order 2 otherwise.

Proof: We know the various orders of the transpositions $X = T_n = T_nM_1$; we also know that every inversion operation T_nM_{11} has order 2. To prove the lemmas, then, it will suffice to prove (g) and (h) following.

(g) T_nM_5 has order 2 if n is even, order 4 if n is odd.
(h) T_nM_7 has order 2 if n is a multiple of 3, order 6 otherwise.

Proof of (g): $(T_nM_5)(T_nM_5) = T_nT_{5n}$ [by the rule of combination], which $= T_{6n}$. If n is even, $T_{6n} = T_0$ and so T_nM_5 has order 2 as asserted. If n is odd, $T_{6n} = T_6$; it follows that T_nM_5 has order 4 as asserted.

The proof of (h) is analogous.

B3. THEOREM: For each u = 1,5,7, or 11 mod 12,
for each *even* j mod 12,
for each p mod 12 *divisible by 3*, a corresponding automorphism F of the T/M group is determined by the following formulas:

$$F(T_n) = T_{un}$$
$$F(T_nM_5) = T_{j+un}M_5$$
$$F(T_nM_7) = T_{p+un}M_7$$
$$F(T_nM_{11}) = T_{j+p+un}M_{11}.$$

Proof: One verifies the rule $F(XY) = F(X)F(Y)$ for each of the 16 possible cases $X = T_mM_a$, $Y = T_nM_b$, as a and b each assume any of the four values 1,5,7, or 11.

For example, in case $X = T_mM_7$ and $Y = T_nM_7$, then $F(XY) = F(T_mM_7T_nM_7) = F(T_{m+7n})$ [by the combination rule]; this, according to the pertinent F-formula of the theorem, $= T_{u(m+7n)}$. In sum, F(XY) here $= T_{u(m+7n)}$.

We compute now the value for F(X)F(Y) in this specific case $(X = T_mM_7, Y = T_nM_7)$. According to the pertinent F-formula of the theorem, $F(X) = T_{p+um}M_7$, while $F(Y) = T_{p+un}M_7$. So F(X)F(Y) here $= T_{p+um}M_7T_{p+un}M_7$; by the rule of combination for the group, this is $T_{p+um+7p+7un} = T_{8p+u(m+7n)}$. And this is $T_{8p}T_{u(m+7n)}$. Since p is divisible by 3, 8p = 0 mod 12; hence $F(X)F(Y) = T_{u(m+7n)}$. The last equation can be compared to the equation at the end of the preceding paragraph; that verifies the desired relation F(XY) = F(X)F(Y) in this particular case $(X = T_mM_7, Y = T_nM_7)$.

One verifies each of the other 15 possible generic cases in similar fashion. The exercise establishes that F is a homomorphism: F(XY) = F(X)F(Y) for all members X and Y of the given group. Clearly F(X), as given by the formulas of the theorem, is not the identity operation $T_0 = M_1 = T_0M_1$ unless X is itself the identity operation. It follows, by way of the basic theory of finite groups, that F is an isomorphism (mapping the group one-to-one onto itself). Q.E.D.

B4. THEOREM: Let G be any automorphism of the T/M group. Then there exist a number u = 1,5,7, or 11 mod 12, and an even number j mod 12, and a number p mod 12 divisible by 3, such that G is exactly the mapping F of Theorem B3 above.

Proof: Since G is an automorphism, $G(T_1)$ must have order 12. By Lemma B2(a), there is a u as required such that $G(T_1) = T_u$. Then $G(T_2) = G(T_1)G(T_1) = T_uT_u = T_{2u}$; then $G(T_3) = G(T_2)G(T_1) = T_{2u}T_u = T_{3u}$; and so forth: for every n, $G(T_n) = T_{nu} = T_{un}$.

Since T_1M_5 has order 4 [B2(b)], $G(T_1M_5)$ must also have order 4. Furthermore, $G(T_1M_5)$ cannot be a transposition: the mapping G, as just shown, maps transpositions into transpositions, and since G is an automorphism, G maps *only* transpositions into transpositions. It follows [B2(b)] that $G(T_1M_5) = T_kM_5$ for some odd k. Set j = k − u; then j is even and $G(M_5) = (G(T_1)\text{-inverse})(G(T_1M_5)) = (T_{-u})(T_kM_5) = T_jM_5$. It follows: for every n, $G(T_nM_5) = G(T_n)G(M_5) = T_{un}T_jM_5 = T_{j+un}M_5$.

Since T_1M_7 has order 6 [B2(c)], $G(T_1M_7)$ must also have order 6. Furthermore, $G(T_1M_7)$ cannot be a transposition. Hence $G(T_1M_7) = T_hM_7$ for some h not divisible by 3 [B2(c)]. Set p = h − u; then—following the method outlined in the preceding paragraph—$G(M_7) = T_pM_7$, and $G(T_nM_7) = T_{p+un}M_7$.

The number p must be divisible by 3. This is shown as follows: $G(T_0) = G(M_7M_7) = G(M_7)G(M_7) = T_pM_7T_pM_7 =$

$T_pT_{7p}M_7M_7 = T_{8p}$. Hence 8p = 0 mod 12; p is divisible by 3 as asserted.

Now $G(T_nM_{11}) = G(T_nM_5M_7) = G(T_nM_5)G(M_7) = (T_{j+un}M_5)(T_pM_7) = T_{j+un}T_{5p}M_5M_7$. Since p is divisible by 3, 5p = 4p + p = p mod 12. Then $G(T_nM_{11})$ above = $T_{j+p+un}M_{11}$.

In sum, u = 1,5,7, or 11, j is even, and p is divisible by 3, such that $G(T_n) = T_{un}$, and $G(T_nM_5) = T_{j+un}M_5$, and $G(T_nM_7) = T_{p+un}M_7$, and $G(T_nM_{11}) = T_{j+p+un}M_{11}$. Hence the given automorphism G is precisely the map "F" of theorem B3 which is determined by that selection of u, j, and p. Q.E.D.

B5. THEOREM: The inner automorphisms of the T/M group are exactly those F of B3 such that j is divisible by 4 and p is divisible by 6.

Proof: If k is a fixed number mod 12 and v is fixed as 1, 5, 7, or 11, then a generic inner automorphism of the T/M group is given by the formula.

$$F(X) = (T_kM_v)X((T_kM_v)\text{-inverse}).$$

The inverse of T_kM_v is $T_{-kv}M_v$ [by way of the combination rule for the group], so

$$F(X) = (T_kM_v)X(T_{-kv}M_v).$$

Now the j-value for the automorphism F, according to B3, is determined by the equation $F(M_5) = T_jM_5$. For the inner automorphism F as given above, compute $F(M_5) = (T_kM_v)M_5(T_{-kv}M_v) = T_kM_{5v}T_{-kv}M_v = T_kT_{-5k}M_{5v}M_v = T_{8k}M_5$. Hence j = 8k; j must be divisible by 4.

The p-value for F, according to B3, is determined by the equation $F(M_7) = T_pM_7$. For the inner automorphism F as given, compute $F(M_7) = T_kM_vM_7T_{-kv}M_v = T_kM_{7v}T_{-kv}M_v = T_kT_{-7k}M_{7v}M_v = T_{6k}M_7$. Hence p = 6k; p is divisible by 6.

The u-value for F, according to B3, is determined by the equation $F(T_1) = T_u$. For the inner automorphism F above compute $F(T_1) = T_kM_vT_1T_{-kv}M_v = T_kT_vM_vT_{-kv}M_v = T_kT_v T_{-k}M_vM_v = T_v$. Hence u = v.

In summary: Given any inner automorphism F, then the corresponding j-value and p-value for F, determined by B3, are divisible respectively by 4 and by 6. That proves half of theorem B5.

The other half can be verified by running the preceding calculations backwards. Given u and j and p, with u = 1,5,7, or 11, and j divisible by 4, and p divisible by 6, we show first that there is a value for k mod 12 which satisfies the pair of equations j = 8k, p = 6k. If j and p are respectively 0 and 0, take k = 0 or 6; if j and p are respectively 0 and 6, take k = 3 or 9; if j and p are respectively 4 and 0, take k = 2 or 8; if j and p are respectively 4 and 6, take k = 5 or 11; if j and p are respectively 8 and 0, take k = 4 or 10; finally, if j and p are respectively 8 and 6, take k = 1 or 7.

Equipped with a pertinent such k, take v = u; the calculations above now show that the automorphism F given by the rule $F(X) = (T_kM_v)X((T_kM_v)\text{-inverse})$ is exactly the F determined, by B3, through the choice of the given u, j, and p. Q.E.D.

B6. NOTE: The automorphism labeled ⟨u,j,p⟩ by B3 combines with the automorphism labeled ⟨v,k,q⟩, to form the automorphism labeled ⟨uv, j + uk, p + uq⟩.

The proof is by straightforward calculation, from the F-formulas of B3. If F is the first automorphism of B6 and G is the second automorphism of B6, then

$FG(T_1) = F(T_v) = T_{uv}$; and
$FG(M_5) = F(T_kM_5)$
$= F(T_k)F(M_5)$
$= T_{uk}T_jM_5 = T_{j+uk}M_5$; and
$FG(M_7)$ similarly $= T_{p+uq}M_7$.

B7: The automorphisms ⟨u,j,p⟩ such that j is divisible by 4 (while p is divisible by 3) form a subgroup RECURSE of the group of automorphisms. RECURSE contains all the inner automorphisms [B5]. RECURSE is naturally isomorphic to the T/M group itself, under the correspondence of ⟨u,j,p⟩ with $T_{j+p}M_u$.

The point is that each n mod 12 can be uniquely written as n = j + p, where j is some number mod 12 divisible by 4 and p is some number mod 12 divisible by 3. Hence the correspondence of the Theorem is one-to-one between the members $\langle u,j,p \rangle$ of the group RECURSE as defined, and the members T_nM_u of the T/M group. Under the correspondence as given,

$\langle u,j,p \rangle \langle v,k,q \rangle = \langle uv, j + uk, p + uq \rangle$ [B6], while

$(T_{j+p}M_u)(T_{k+q}M_v) = T_{j+p}T_{u(k+q)}M_uM_v$ [combination rule in the group]

$= T_{(j+uk)+(p+uq)}M_{uv}$.

The results above may be collated into a big structure theorem.

B8. STRUCTURE THEOREM: The following groups are of interest, identifying the automorphism F with the number-triple $\langle u,j,p \rangle$, using the conventions of B3 and B6:

AUTOMORPH: Those number-triples $\langle u,j,p \rangle$ such that $u = 1,5,7$, or 11, and j is even, and p is divisible by 3.

RECURSE: Those number-triples $\langle u,j,p \rangle$ such that $u = 1,5,7$, or 11, and j is divisible by 4, and p is divisible by 3.

INNER: Those number-triples $\langle u,j,p \rangle$ such that $u = 1,5,7$, or 11, and j is divisible by 4, and p is divisible by 6.

INNER is a subgroup of RECURSE, which is in turn a subgroup of AUTOMORPH. AUTOMORPH corresponds to all possible automorphisms of the T/M group. INNER corresponds to all possible inner automorphisms of that group. RECURSE is isomorphic to the T/M group itself, under the identification of $\langle u,j,p \rangle$ in RECURSE with $T_{j+p}M_u$ in the T/M group.