Musical Works are Maximal Memory Stores

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Abstract

The book *A Generative Theory of Shape* (Michael Leyton, Springer-Verlag, 2001) develops new foundations to geometry in which *shape is equivalent to memory storage*. With respect to this, the argument is given that *art-works are maximal memory stores.* The present paper reviews some of the basic principles concerning our claim that, in particular, musical works are maximal memory stores. The argument is that *maximizing memory storage explains the structure of musical works.* We first review the basic geometric theory of the book: A generative theory of shape is developed that has two properties regarded as fundamental to *intelligence* – maximization of transfer and maximization of recoverability. *Aesthetic structuration is taken to be equivalent to intelligence.* Thus aesthetics is brought into the very foundations of the new theory of geometry. A mathematical theory of transfer and recoverability is developed, using symmetry-breaking wreath products. From this, it becomes possible to develop a theory of musical composition, as follows: Musical works are complex shapes. A theory of *complex-shape generation* is presented, in which any structure is described as *unfolded* from a maximally collapsed version of that structure, called an *alignment kernel*. This process is formalized by proposing a new class of groups called *unfolding groups*. The alignment kernel is a subgroup of that structure, consisting of symmetry ground-states which are themselves formalized by a new class of groups called *iso-regular groups.* In music, the iso-regular groups represent the anticipation hierarchies, for example the regular meters of the work. The process of musical composition is then described by an unfolding group, which "unfolds" the work, by successively breaking the iso-regular groups of the alignment kernel.

1 Introduction

The book, *A Generative Theory of Shape* (Michael Leyton, Springer-Verlag, 2001) develops new foundations to geometry in which *shape is equivalent to memory storage*. With respect to this, the argument is given that *art-works are maximal memory stores.* The present paper reviews some of the basic principles concerning our claim that, in particular, musical works are maximal memory stores. The argument is that maximizing memory storage explains the structure of musical works.

We begin by looking at the nature of geometry, generally. The book develops new foundations to geometry that are diametrically opposed to the foundations that have existed for almost 3000 years. The conventional foundations are strongly related to Klein's invariants program, which was formally stated in the late 19th century, but whose origins can be traced back to Euclid's fundamental concern with congruence. Euclid's congruence program was later *generalized* as Klein's invariants program, and the latter, in turn, became the basis of 20th century geometry and physics – for example, Cartan's moving-frame classification of curves and surfaces, Einstein's special and general principles of relativity, Wigner's method of classifying quantum-mechanical particles, etc.

Our book argues that the invariants program is fundamentally destructive of the needs of modern computation. The reason is that invariants are those aspects that are *memoryless* with respect to applied action: i.e., no action can be recovered from an object which it leaves invariant, i.e., from a geometric object in Klein's definition. Since computing systems are required to increase memory, the invariants program defeats this fundamental purpose. A related way of saying this is that invariants defeat generativity, which is basic to computation; i.e., an object that is invariant under a generative operation will not alter under the action of the operation, and thus negate the purpose of the operation.

In contrast, the theory of geometry developed in our book is concerned with the needs of computation. For this, we elaborate a theory in which geometric objects store the effects of actions, i.e., act as memory stores for action. In fact, our basic claim is this:

$$\textbf{Shape} \quad \equiv \quad \textbf{Memory Storage.}$$

This directly opposes the Klein program. As an example, consider the shape of the human body. There is very little that is congruent or invariant between the developed body and the original spherical egg from which it arose. Thus Euclid's or Klein's program has almost nothing to say about this situation. However, notice that, from the developed body, one can recover a considerable amount of the history of embryological development and subsequent growth, that the body underwent. Therefore, it is much more valuable to argue that the shape of the body is equivalent to the history that it underwent, instead of invariants that should have survived in the situation. In fact, observe that it is the absence of invariants that allows the history to be recovered. Thus we argue that geometry should not be the study of invariants, or equivalently, memoryless-ness; but rather it should be the study of memory storage. The book shows that this new theory of geometry allows one to develop powerful analyses of *perception, robotics,* and *design.*

This paper presents the theory of music that arises from the book. The purpose of the paper is to show that *music is a particular example of the theory of geometry as memory storage.*

2 Two Basic Requirements

Let us begin by looking at the general theory of geometry, developed in the book. The basic argument is that one substantially increases the power of geometry by establishing a generative theory of shape founded on the following two criteria which we regard as fundamental to *intelligent* and *insightful* behavior:

(1) Maximization of Transfer. Any agent is regarded as displaying intelligence and insight when it is able to *transfer* actions used in previous situations to new situations. For example, a robot might need to transfer a task developed for one region of a work-space onto another region of a work-space. The ability to transfer past solutions to new problems is at the very core of what it means to have knowledge.

Also transfer can be regarded as equivalent to *re-usability*, which is a major issue driving software-development, e.g., it was the major stimulus in the development of object-oriented technology.

The book argues that transfer is basic to aesthetics. For example, a symphonic movement by Beethoven has remarkably few basic elements. The entire movement is generated by the transfer of these elements into different pitches, major and minor forms, overlapping positions in counterpoint, and so on. We argue:

<div align="center">

Aesthetics is the maximization of transfer.

</div>

That is, transfer is the basis of all intelligent behavior, and we argue that *intelligence is equivalent to aesthetic structuration*. Furthermore, those objects in which transfer is maximized are actually those objects that people call art-works.

(2) Maximization of Recoverability. A basic factor of intelligence is the ability to give explanations. An agent must be able to infer the causes of its own current state, in order to identify why it failed or succeeded – and edit its behavior. This is basic, for example, to design, where one might need to go back to a previous stage, and proceed in a different direction.

But note also that, with respect to an apparently very different set of issues, recoverability is basic to computer vision which requires recovering, from the retinal image, the environmental processes that produced that image – an inference often referred to as inverse optics. With respect to a related set of issues, all *science* is about the recoverability of those causal processes that lead to the results on the measuring instruments.

We shall see that a basic aspect of a musical work is that it is organized to ensure recoverability.

This section presented our two fundamental criteria of intelligence: maximization of transfer and maximization of recoverability. It will be seen that, if generativity satisfies these two criteria, then it has a powerful mathematical structure. Essentially, this involves giving a new approach to geometry that incorporates intelligence (aesthetic structuration) into the very definition of shape.

3   Complex Shape Generation

The primary goal of the book is to handle *complex* shape. Such a shape might be a highly complex design, such as a Beethoven symphony, or an assembly in mechanical CAD – or it might be a complex real-world scene that confronts the visual system. The remarkable fact about the human cognitive system is that, when presented with a highly complex structure, such as a real-world scene, it is able to convert the complexity into an entirely understandable form. This exemplifies the general problem that we will investigate:

(1) The conversion of complexity into understandability: *The basic purpose is to give a generative theory of complex shape such that the complexity is entirely accounted for, and yet the structure is completely understandable.*

(2) Understandability and intelligence: *Deep consideration reveals that understandability of a structure is achieved by maximizing transfer and recoverability.*

(3) The mathematics of understandability: *A significant portion of the book is the development of a mathematical theory of how understandability is created in a structure.*

When putting together the statement in (2) and the theory of aesthetics in section 2, one sees that, according to our theory of geometry, aesthetics is basic to the conversion of complexity to understandability. Thus, for example, the computer-vision problem, for complex scenes, is solved by aesthetics, rather than the techniques currently developed in the research literature. Furthermore, a consequence of the statement in (3) is that we will be giving a mathematic theory of aesthetics.

4   Object-Oriented Theory of Geometry

Geometry of the last 3,000 years is not object-oriented. A principal reason is that object-oriented programming allows the identification and tracking of objects through histories of complex modification (i.e., allows for recoverability or memory storage of action) and the congruence/invariance program defeats this; e.g., the adult body and egg could not be the same geometric object in Klein's sense of object. One of the purposes of our book is the development of an *object-oriented formalization of geometry*. The result is a theory of geometry that has fundamentally opposite characteristics from previous geometry. The object-orientedness is formulated in a rather novel use of group theory: Groups are seen as descriptions of asymmetries rather than symmetries. One important consequence is that the theory provides an entirely new formulation of the meaning of symmetry-breaking, in which the group expands, on symmetry-breaking, rather than reduces, as it does in modern physics. This increases the descriptive power of the theory, because an expanding group provides greater number of algebraic operators.
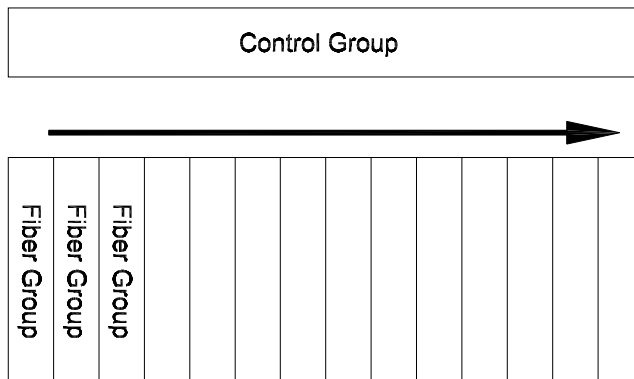
Figure 1: The control group transferring the fiber group.

5   Transfer

A generative theory of shape characterizes a shape by a sequence of actions needed to generate it. According to our theory, aesthetics is increased by transferring actions within the sequence. Indeed we argue that an artwork, e.g., a symphonic movement by Beethoven, is a structure that has maximal complexity while simultaneously maximizing transfer along the generative sequence. That is, the principle of the maximization of transfer can be stated thus:

MAXIMIZATION OF TRANSFER. *Make one part of the generative sequence a transfer of another part of the generative sequence, whenever possible.*

It will be argued that the appropriate formulation of this is as follows: A situation of transfer (see Fig 1) involves two levels: a *fiber group*, which is the group of actions to be transferred; and a *control group*, which is the group of actions that will transfer the fiber group. The justification for these structures algebraically being groups will be given later, but the theory of transfer will work equally for semi-groups, which is the most general case one would need to consider for generativity.

Now, one can think of transfer as the control group moving the fiber group around some space; i.e., *transferring* it. The transferred versions of the fiber group are shown as the vertical copies in Fig 1, and will be called the *fiber-group copies*. The control group acts from above, and transfers the fiber-group copies onto each other, as indicated by the arrow. This will often be referred to as a structure of *nested control*.

A basic part of our approach is this:

   (1) Give an algebraic theory of transfer.

(2) Reduce complex situations down to structures of transfer.

Transfer will be modeled by a group-theoretic construct called a *wreath product*. This is a group that will be notated in the following way:

$$\text{Fiber Group} \quad \text{ⓦ} \quad \text{Control Group.}$$

Intuitively, a wreath-product is a group that contains the entire structure shown in Fig 1; that is, it has an upper subgroup that will be called a *control group*, and a system of lower subgroups that will be called the *fiber-group copies*. The control group sends the fiber-group copies onto each other. It does so simply by conjugating them onto each other.

6   Wreath Products

This section describes wreath products in detail. Consider a group, $G(C)$, called the **control group**, acting on a set, C, called the **control set**. This action, called the **control action**, is given thus:

$$\begin{cases} G(C) & \times & C & \longrightarrow & C \\ \\ (\quad g & , & c_i\ ) & \longmapsto & gc_i. \end{cases} \tag{1}$$

Consider also another group, $G(F)$, called the **fiber group**, acting on a set, F, called the **fiber set**. This action, called the **fiber action**, is given thus:

$$\begin{cases} G(F) & \times & F & \longrightarrow & F \\ \\ (\quad T & , & f\ ) & \longmapsto & Tf. \end{cases} \tag{2}$$

For each member c of the control set C, make a copy of the fiber action (2), thus:

$$\begin{cases} G(F)_c & \times & F_c & \longrightarrow & F_c \\ \\ (\quad T_c & , & f_c\ ) & \longmapsto & T_c f_c. \end{cases} \tag{3}$$

Notice that there will now be a set of copies F_c of the fiber set, called the **fiber-set copies**, indexed in the control set C. Also, there will be a set of copies $G(F)_c$ of the fiber group, called the **fiber-group copies**, also indexed in the control set C. The fiber-group copies correspond to the columns in Fig 1. Each such column acts on its own "personal" copy of the fiber set.

The following point is crucial: If we think of the control action, given at (1) above, as a permutational action by the control group on the elements of the control set, then this same action induces a permutational action by the control group on the copies of the fiber. The latter permutational action is indicated by the arrow in Fig 1.

In most cases, in this paper, the control set C will be the control group $G(C)$ itself. Thus the fiber copies will be indexed in the control group. This is called a *regular wreath product*. In Fig 1, this would mean that there is one column (fiber-group copy) for each element in the control group above. However, the present section defines the most general type of wreath product – where the set C is some general set on which the control group has an action.

Now take the direct product of the fiber-group copies. This will be called the **fiber-group product**, given thus:

$$\prod_{c \in C} G(F)_c. \tag{4}$$

The entire bottom block in Fig 1 can be considered to illustrate this direct product. Notice that the control group action of $G(C)$ on C induces an action of $G(C)$ on the set of indexes c within the direct product in (4). Most crucially, this action of $G(C)$ is an *automorphic* action on the direct product.

Next, take the semi-direct product of the entire lower block and the control group above, thus:

$$\{\prod_{c \in C} G(F)_c\} \, ⓢ \, G(C). \tag{5}$$

The lower block $\prod_{c \in C} G(F)_c$ is the normal subgroup of the semi-direct product. In any semi-direct product, the upper group acts as an automorphism group of the normal subgroup (here the lower block); and in this case the chosen automorphic action will be the one defined in the previous paragraph.

Next consider the set $F \times C$, which we will call the **data set**. Notice that this set decomposes into the fiber-set copies F_c. The data set is shown as the entire block in Fig 2 (ignore the arrows for the moment). The columns in this figure are the fiber-set copies. This block corresponds, column for column, to the lower block in Fig 1 where the columns are the fiber-group copies. The fiber-group copies in Fig 1 act on their corresponding fiber-set copies in Fig 2.

Now for the final fundamental point concerning wreath products: There is a group action of the wreath product $G(F)ⓦG(C)$ on the data set $F \times C$. That is, in terms of the figures we have given, there is an action of the entire Fig 1 (upper control group and lower block) on the block in Fig 2. To define this action, let us assume, only for the purposes of notation, that the control set C is finite, of cardinality n. Observe that, by the semi-direct product structure of the wreath product (as shown in expression (5)), a single element from the wreath product must be of the form:

$$\langle \, (\, T_{c_1} \, , \, T_{c_2} \, , \, \ldots \, , \, T_{c_n} \,) \mid g \, \rangle \tag{6}$$

where each T_{c_i} is an element taken from its fiber-group copy $G(F)_{c_i}$ (column in Fig 1); and g is an element taken from the control group (upper level in Fig 1).

Now the full element shown as expression (6) acts on the data set (the block in Fig 2) in the following way: Each element T_{c_i} in expression (6) acts only on its own fiber-set copy F_{c_i} (its corresponding column in Fig 2). Then the control element g in expression (6) permutes the fiber-set copies (columns in Fig 2).

Let us therefore see the effect of the full element in expression (6) on a single point in the data set. This point will be shown as the first dot (in the sequence of dots) in Fig
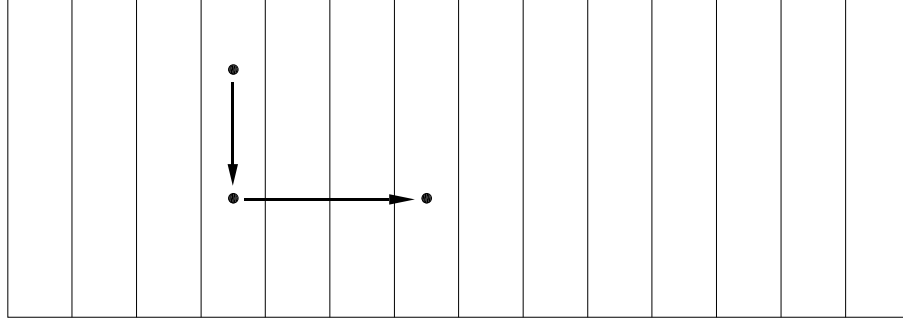
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Figure 2: Illustrating the action of the wreath product on the data set.

2. This is located in the fiber-set copy F_{c_i} (its column in Fig 2). Notice that the dot can be expressed as the ordered pair $(f,\ c_i)$ in the data set $F \times C$ (the full block).

Now apply the full group element, expression (6), to this point. Notice, since each T_{c_j} in expression (6) acts only on its personal fiber-set copy, only the element T_{c_i} will act on the point $(f,\ c_i)$. It will move it to the point $(T_{c_i}f,\ c_i)$ which is the second dot in Fig 2. The arrow from the first to the second dot corresponds to the action of T_{c_i}. Finally, the element g in expression (6) moves the second dot to its corresponding position in the column indexed by gc_i. So the final dot is given by the ordered pair $(T_{c_i}f,\ gc_i)$ in the data set $F \times C$.

The above therefore defines the group action of wreath product $G(F)\textcircled{w}G(C)$ on the data set $F \times C$. It will be called the **full wreath action**, and we have seen that it is given thus:

$$
\begin{cases}
\qquad G(F)\textcircled{w}G(C) \qquad\quad \times \quad [F \times C] \quad \longrightarrow \quad [F \times C] \\[2ex]
(\ \langle\,(\,T_{c_1}\,,\ T_{c_2}\,,\ \ldots\,,\ T_{c_n}\,)\,|\,g\,\rangle \quad,\quad (f,c_i)\,) \quad \longmapsto \quad (T_{c_i}f, gc_i).
\end{cases}
\tag{7}
$$

7 Mathematical Theory of Transfer

We are now ready to give our rigorous theory of *transfer*, as follows: Each copy $G(F)_c$, of the fiber group, acts on its own copy of the fiber set F_c. One can view the control group as transferring the fiber-group copies around the fiber-set copies. In fact, this action is achieved by the automorphic action of the control group within the wreath product, as given by the map τ. This action sends the fiber-group copies onto each other via *conjugation*. Therefore:

MATHEMATICAL STRUCTURE OF TRANSFER. *Transfer will be modelled by*

the conjugacy action of the control group on the fiber-group copies within a wreath product.

8 Theory of Gestalt

A musical work is structured by perceptual organization – often referred to as Gestalt organization. In Leyton [16], [17], [18], [19], [20], [21], I put forward several hundred pages of psychological evidence that lead to the following theory of Gestalt:

THEORY OF PERCEPTUAL ORGANIZATION (GESTALT)

The human perceptual system forms organizations by maximizing transfer; i.e., the structural cohesion is formed by making one part of the perceptual input the transfer of another part of the perceptual input, where-ever possible. The mathematical consequence is that perceptual organizations are structured as n-fold wreath products, $G_1 ⓦ G_2 ⓦ \ldots ⓦ G_n$.[1]

We will later give several extended examples of this with musical meter, modulation, and melodic form. However, it is best to initially illustrate the above with an example from the visual domain, to show that the musical examples are merely instances of the general process of perception – which in turn is merely an instance of our general theory of geometry.

As an initial example, consider how the human visual system structures a square. In a sequence of psychological experiments, Leyton [18] [19], we showed that human vision represents a square *generatively*, in much the same way that one draws it on a sheet of paper – i.e., drawing the sides sequentially around the square. Notice that this in fact involves a crucial transfer structure thus: The first side is generated by starting with a corner point, and applying translations to trace out the side, as shown in Fig 3.

Next, this translational structure is *transferred* from one side to the next – rotationally around the square. In other words, there is transfer of translations by rotations. This is illustrated in Fig 4.

Therefore, the transfer structure is defined as the wreath product:

$$\text{Translations} \ ⓦ \ \text{Rotations}$$

where Translations is the fiber group (corresponding to the side) and Rotations is the control group (transferring the side). This will now be defined rigorously, as follows:

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[1]Throughout this paper, the term and notation n-fold wreath product $G_1 ⓦ G_2 ⓦ \ldots ⓦ G_n$ will mean that the hierarchy of control groups were added successively from left-to-right, thus: $(\ldots (((G_1 ⓦ G_2) ⓦ G_3) ⓦ G_4) ⓦ \ldots) ⓦ G_n$.

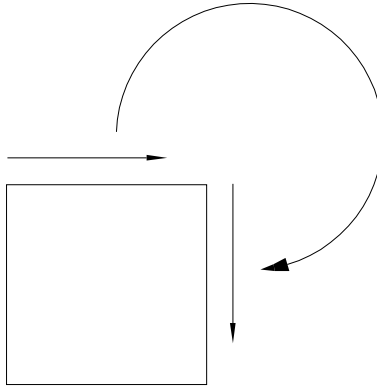Figure 3: The generation of a side, using translations.

Figure 4: Transfer of translation by rotation.

The translation group (generating the side) will be denoted by the additive group $\mathbb{R}$. The rotation group is $\mathbb{Z}_4$, the cyclic group of order 4, which will be represented as

$$\mathbb{Z}_4 = \{ \; e, \quad r_{90}, \quad r_{180}, \quad r_{270} \; \}$$

where r_θ means clockwise rotation by θ degrees. We now construct a regular wreath product of these two groups. The construction will use the terminology of section 6.

The group $\mathbb{Z}_4$ will be the control group, $G(C)$, and the control set will be the set C of four side-positions around the square:

$$c_1 = \text{top}, \quad c_2 = \text{right}, \quad c_3 = \text{bottom}, \quad c_4 = \text{left}. \tag{8}$$

The control action of $\mathbb{Z}_4$ on the set $\{c_1, c_2, c_3, c_4\}$ will correspond to the clockwise rotation of the four side-positions onto each other.

The translation group $\mathbb{R}$ will be the fiber group, $G(F)$, and the fiber set will be the infinite line F containing the finite side as a subset. This is mathematically and psychologically an important concept, as will be observed shortly. The fiber action of $\mathbb{R}$ on the fiber set F will be the obvious translation of the infinite line along itself.
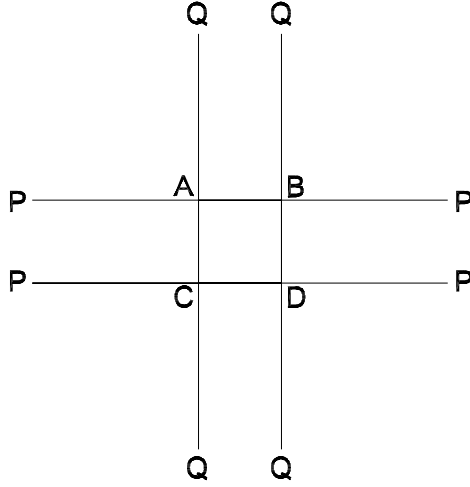
Figure 5: The square on the projective plane.

For each of the four members c of the control set C, make a copy of the fiber action. Thus there will now be a set of four copies $\{F_{c_1}, F_{c_2}, F_{c_3}, F_{c_4}\}$ of the fiber set, called the fiber-set copies, indexed in the control set C. These will be the four infinite lines that contain the four finite sides as subsets. This structure is illustrated in Fig 5, where P and Q represent the points at infinity. Clearly the structure has considerable mathematical significance because it corresponds to what is called the *complete quadrilateral* in projective geometry. It also has considerable psychological significance. For example, in Leyton [23], we have shown that it allows one to solve long-standing Gestalt problems such as the orientation-and-form problem. Particularly important is the fact that it psychologically acts as the *Gestalt completion* of the square because it allows the location of vanishing points in perspective projections.

Corresponding to the four fiber-set copies, there will be four copies $\{\mathbb{R}_{c_1}, \mathbb{R}_{c_2}, \mathbb{R}_{c_3}, \mathbb{R}_{c_4}\}$ of the fiber group, called the fiber-group copies, also indexed in the control set C. Each fiber-group copy (translation group) will act on its own "personal" copy of the fiber set (infinite line).

One can now define the regular wreath product:

$$\mathbb{R} \, \textcircled{w} \, \mathbb{Z}_4 \tag{9}$$

where this group is the semi-direct product:

$$[\mathbb{R}_{c_1} \times \mathbb{R}_{c_2} \times \mathbb{R}_{c_3} \times \mathbb{R}_{c_4}] \, \textcircled{s} \, \mathbb{Z}_4. \tag{10}$$

The automorphic action of the control group $\mathbb{Z}_4$, on the fiber-group product $\mathbb{R}_{c_1} \times \mathbb{R}_{c_2} \times \mathbb{R}_{c_3} \times \mathbb{R}_{c_4}$ corresponds to the action of $\mathbb{Z}_4$ on the control set $\{c_1, c_2, c_3, c_4\}$, whose elements now appear as the indexes on the four fiber-group copies $\mathbb{R}_{c_i}$. This means that the fiber-group copies are rotated around the square. In fact, in accord with

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the structure of a semi-direct product, $\mathbb{Z}_4$ carries out this action by *conjugating* the fiber-group copies onto each other.

The data set $F \times C$, in this example, is the Gestalt completion of the square – i.e., given by the four infinite lines containing the four finite sides, as indicated in Fig 5. We can think of this as *four infinite wires* overlapping each other.[2]

The wreath product $\mathbb{R}\widehat{w}\mathbb{Z}_4$ acts on the data set (the Gestalt completion), in the following way: By inspection of the semi-direct product form (10) of the wreath product, an individual element from the wreath product is of this form:

$$\langle\ (\ T_{c_1}\ ,\ T_{c_2}\ ,\ T_{c_3}\ ,\ T_{c_4}\ )\ \mid\ r_\theta\ \rangle \tag{11}$$

where $T_{c_i} \in \mathbb{R}_{c_i}$ and $r_\theta \in \mathbb{Z}_4$. The action of the group element (11) is then interpreted as follows: Each translation T_{c_i} moves its own infinite wire along itself by the amount indicated by that translation, and then the remaining component r_θ rotates the four wires by the amount θ. Notice therefore that the group element (11) maps the Gestalt completion of the square to itself, and that consequently the wreath product at (10) is a symmetry group of the Gestalt completion.

In our generative theory, the Gestalt completion is cut down to its visible portion, the finite square, by placing what we call an **occupancy group**, $\mathbb{Z}_2$ (a cyclic group of order 2), at each point along the infinite line containing a side. The group switches between two states, "occupied" and "non-occupied," and is wreath sub-appended below the above group thus

$$\mathbb{Z}_2 \ \widehat{w} \ \mathbb{R} \ \widehat{w} \ \mathbb{Z}_4.$$

Notice the power of this wreath product is that it is a regular one. That is, going from left to right, there is one copy of $\mathbb{Z}_2$ for each element in the group $\mathbb{R}$ above it, and there is one copy of $\mathbb{R}$ for each element of the group $\mathbb{Z}_4$ above that. For ease of exposition, the occupancy level will be ignored in the present paper.

Now observe that the group at (9) gives **generative coordinates** to the square, as follows. Since the wreath product is regular, we can identify the members c_i of the control set with the members r_θ of the control group. Thus, any fiber-group copy can be labelled $\mathbb{R}_{r_\theta}$, and its elements can be labelled t_{r_θ}. Therefore, any point on the square can be described by a pair of coordinates:

$$(t, r_\theta) \ = \ t_{r_\theta} \ \in \ \mathbb{R}_{r_\theta}.$$

The first coordinate gives the generative (translational) distance along a side, and the second coordinate gives the generative (rotational) distance of a side from the first generated side. Therefore a point is given a complete generative description from the origin. (This relies on the fact that the fiber-action is transitive.)

Fig 6 illustrates this by giving the coordinates of four of the points.

The crucial thing to observe is that the coordinates *maximize transfer*. Fig 7 illustrates this by showing that the coordinates on one side are a transfer of the coordinates of another side. Notice that, given an individual point (t, r_θ) on a side, its four transfer-equivalent copies (on each of the four sides), are now given by the diagonal embedding of the fiber group into the fiber-group product, thus: $t \longrightarrow (t, t, t, t)$.

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[2]Note that, since, generally, fiber-set copies are independent sets, the four infinite lines to not intersect but *overlap*.
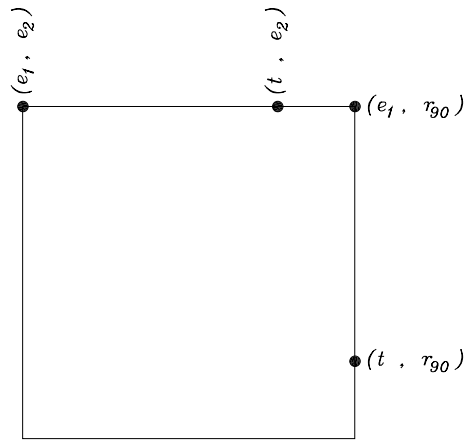
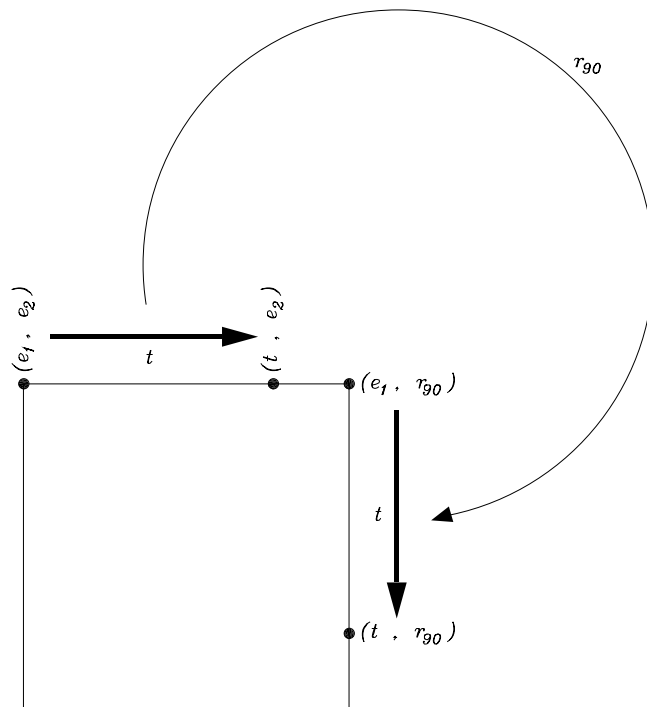Figure 6: The coordinates of four points.

Figure 7: The control-nested structure of those coordinates.

Now, *deformed* shapes are handled in our system by adding extra layers of transfer. For example, to obtain a parallelogram, one adds the general linear group $GL(2, \mathbb{R})$ onto the two-level group of the square thus:

$$\mathbb{R} \, \textcircled{w} \, \mathbb{Z}_4 \, \textcircled{w} \, GL(2, \mathbb{R}). \tag{12}$$

Notice that the operation used to add $GL(2, \mathbb{R})$ on to the lower structure $\mathbb{R}\textcircled{w}\mathbb{Z}_4$ is, once again, the wreath-product $\textcircled{w}$ which means that $GL(2, \mathbb{R})$ acts by *transferring* $\mathbb{R}\textcircled{w}\mathbb{Z}_4$, as follows: Since the fiber group $\mathbb{R}\textcircled{w}\mathbb{Z}_4$ represents the structure of the square, this means that $GL(2, \mathbb{R})$ transfers the structure of the square onto the parallelogram. In particular, it transfers the *generative coordinates* of the square onto the parallelogram. For example, $GL(2, \mathbb{R})$ transfers the four points on the square in Fig 6 onto the corresponding four points on the parallelogram, as shown in Fig 8.

More deeply still, the fiber group $\mathbb{R}\textcircled{w}\mathbb{Z}_4$ in expression (12) is itself a transfer structure, as seen in Fig 7, where rotation transferred the translation process from the top side onto the right side. This transfer structure is itself transferred, by $GL(2, \mathbb{R})$, onto the parallelogram, as shown in Fig 9. That is, we have **transfer of transfer**. This recursive transfer is encoded by the successive $\textcircled{w}$ operations in expression (12).

What has been illustrated here is our principle of the maximization of transfer: The parallelogram is given a generative description, all the way up from a point, that maximizes transfer. That is, the point is transferred by translations to create a side, the side is transferred by rotations to create a square, and the square is transferred by the general linear group to create a parallelogram. Everything is *re-used*. This is the basis of our theory of aesthetics. For example this is the basis of a symphonic movement by Beethoven. We are giving here a simple first illustration of the mathematical principles involved.

In our theory of music, the concept of *anticipation hierarchies* will be crucial. We shall argue that, for deep mathematical reasons, such hierarchies have a role corresponding to the 3D shape primitives of mechanical CAD. Therefore it will be useful here to illustrate our theory of geometry with three-dimensional shape. For example, consider the structure of a cylinder. The standard group-theoretic description of a cylinder is

$$SO(2) \, \times \, \mathbb{R} \tag{13}$$

where $SO(2)$, the group of planar rotations around a fixed point, gives the rotational symmetry of the cross-section, and $\mathbb{R}$ gives the translational symmetry along the axis. Notice that in (13) the operation linking the two groups is the direct product operation $\times$.

For us, the problem with this expression is that it does not give a generative description of the cylinder. In computer vision and graphics, cylinders are described generatively as the sweeping of the circular cross-section along the axis, as shown in Fig 10. To our knowledge, the group of this sweeping structure has never been given. We propose that the appropriate group is:

$$SO(2) \, \textcircled{w} \, \mathbb{R}. \tag{14}$$

Notice that it uses the wreath-product operation $\textcircled{w}$ rather than the direct product $\times$, and therefore the group has a very different structure from that in expression (13). The

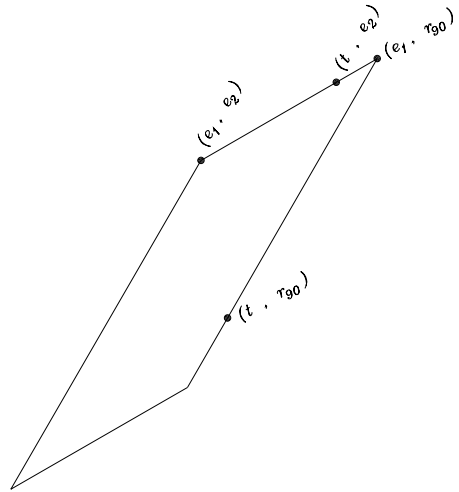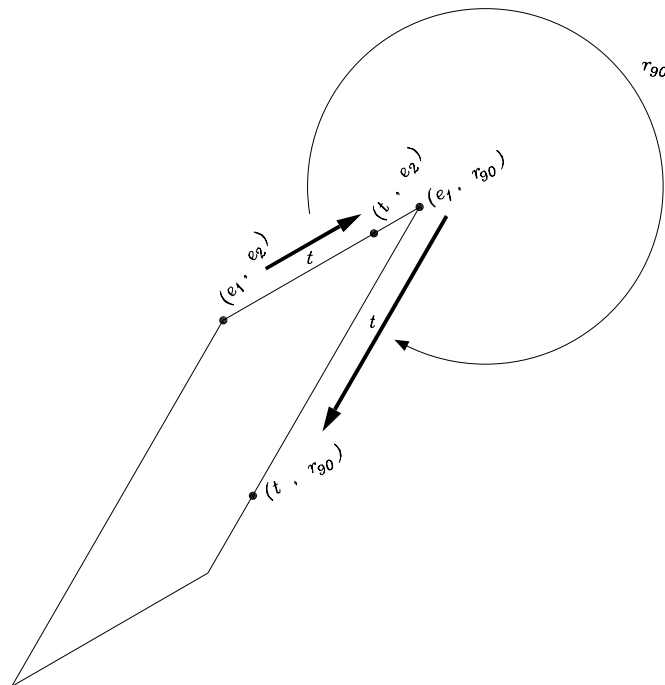Figure 8: The transferred coordinates from a square.

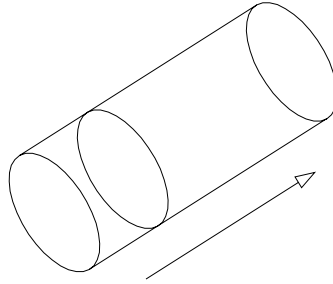Figure 9: The transfer of transfer.

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Figure 10: The sweep structure of a cylinder.

operation ⓦ means that this new group has a *fiber-control* structure, in which $SO(2)$ is the fiber group and $\mathbb{R}$ is the control group. This is exactly what is seen in the sweeping structure shown in Fig 10. The cross-section is generated first as a fiber, and then its position is controlled by translation.

We conclude this section by stating, more precisely, the principle of the maximization of transfer. It says two things: Given a data set: (1) Generate the set by maximizing re-use of the parts of the generative sequence; i.e., maximize the height of the wreath product. (2) However, make the height *non-spurious*; i.e., do not introduce levels where there are no detectable distinguishabilities in the data set. This second condition relates to the theory of recoverability (section 14), which says that generative operations are introduced to account only for asymmetries (distinguishabilties), not for symmetries (indistinguishabilities). This can be illustrated as follows: A set of equally-spaced points along a line can be generated by the group of integers, constituting one level. Alternatively, this set can be generated by a two-level structure, in which one particular fiber copy corresponds to a particular pair of adjacent points, and the control corresponds to the group of even numbers that moves this pair onto the next pair, and then the next pair, ..., successively along the line. However, since the points are equally spaced, this would be a spurious decomposition into levels, because there is no distinguishability, along the line of points, that justifies this decomposition.

9   Shape Generation by Group Extensions

One can see from the above discussion that the concept of *group extension* is basic to our generative theory. A group extension takes a group G_1 and adds to it a second group G_2 to produce a third, more encompassing, group G, thus:

$$G_1 \text{ (E) } G_2 \;\; = \;\; G$$

where (E) is the extension operation. (For introduction to group extensions, see Rotman [34].)

It is clear, looking back over the examples given so far, that according to our theory:

Shape generation proceeds by a sequence of group extensions.

That is, shape generation starts with a base group and successively adds groups obtaining a structure of this form:

$$G_1 \text{ (E) } G_2 \text{ (E) } \ldots \text{ (E) } G_n.$$

This approach to shape-generation differs substantially from standard shape-grammar approaches, e.g., that of Stiny and Gips [10], [36], which are based on the application of production rules. In our approach, structural elements correspond to groups, and the addition of structural elements corresponds to group extensions.

Structural elements $\longrightarrow$ Groups.

Addition of structural elements $\longrightarrow$ Group extensions.

Furthermore, imposing the condition of *maximization of transfer* demands that the structure $G_1\text{(E)}G_2\text{(E)}\ldots\text{(E)}G_n$ be actually of the form $G_1\text{(W)}G_2\text{(W)}\ldots\text{(W)}G_n$. In other words, the extension operation (E) is the control-nesting operation (W).

10 Algebraic Theory of Inheritance

Our theory of music is inherently object-oriented, in the sense of object-oriented programming. Indeed we argue that object-oriented *inheritance* is a fundamental part of musical perception and composition. A central component of our generative theory of shape is a mathematical theory of inheritance, which will now be described, and which becomes essential to every part of the music theory.

The term *inheritance*, in object-oriented programming, refers to the passing of properties from a parent to a child, [27]. The child incorporates these parent properties, but also adds its own. This kind of structure covers two types of situation. The first is class inheritance, which is a static software concept, and the second is a type of dynamic linking created at run-time. The book gives an algebraic theory of both types of inheritance, in the geometrical domain; i.e., related to shape. Notice that, since our theory of shape is *generative*, spatial movement and deformation are understood as part of the specification of shape. Thus the command operations in shape classes are understood as part of the specification of higher order shapes (e.g., configurations). Notice that, in shape classes, the command operations – which include spatial movements and deformations – form groups.

In this paper, we will have time to deal with only the dynamic type of inheritance created at run-time. This is fundamental to all computer-aided design, assembly, robotics, animation, etc. A typical example is a child object inheriting the transform of a parent

object, and adding its own. It is instructive for the music theorist to consider the following example in architectural CAD: Here a door is defined as a child of a wall, and moves with the wall if the designer decides to change the position of the wall. However, the door can also open and close with respect to its attached position in the wall. This means that the door inherits the movement of the wall, but adds its personal movement with respect to the latter. Clearly, examples of this type are profuse in music. For instance, we shall later show that modulation has exactly this structure.

Now for the basic statement of our algebraic theory of inheritance:

ALGEBRAIC THEORY OF INHERITANCE. *Inheritance arises from a wreath product:*

> *Parent* $\longleftrightarrow$ *Control group*
> *Child* $\longleftrightarrow$ *Fiber group.*

This can be illustrated by returning to the door/wall example. Let us suppose that the command group of motions for the wall is the Euclidean group $E(2)$ on the plane (i.e., the base-line of the wall can be translated and/or rotated within the plane of the floor plan – which is a typical operation in architectural CAD). Let us also suppose that the command group of motions for the door is the rotation group $SO(2)$, since the door can rotate about its fixed hinge in the wall. Then, our claim is that the combined transform structure of the door and wall is given by a regular wreath product of the two command groups thus:

$$SO(2) \text{ \textcircled{w} } E(2).$$

The reason is easy to see as follows: Let us move the wall by a command operation $g \in E(2)$. Then, because the door moves with the wall, g moves a copy of the door's rotation group $SO(2)$ together with the wall, i.e., sends the copy of the door's rotation group within the first wall position onto the copy of the door's rotation group within the second wall position. In fact, g achieves this by *conjugating* the first copy of the rotation group onto the second copy. Thus the fiber-group copies of $SO(2)$, in the above wreath product, correspond to the copies of the doors's rotation group in each of the different wall positions.
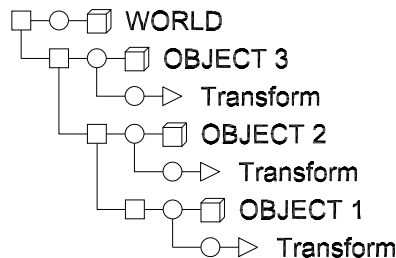

Figure 11: The representation of parent-child relations in *3D Studio Max*.

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It will be useful, for later discussion in this paper, to consider here diagrammatic aspects of current design programs. Because run-time inheritance is created by the designer, it is usually represented by diagrams that the designer can view. It will be useful for us to show how these diagrams can be converted into algebra. A good diagrammatic representation is used by *3D Studio Max*, as illustrated in Fig 11. Here, inheritance is represented by indentation – i.e., an indented object is a child of the next object above with respect to which it is indented. Each object, except the World object, has a transform shown just below it. The transform relates the coordinate frame of the object to the coordinate frame of its parent. This transform is the "personal" transform of the object. In addition, the object inherits the transform of its parent. The object therefore adds its personal transform to its inherited transform. This means, of course, that via its parent, it inherits the transform of its parent's parent, and so on. By the above Algebraic Theory of Inheritance, such diagrams can be converted into algebra in the following way:

GROUP OF ENTIRE TRANSFORM STRUCTURE. *Consider a set of $n + 1$ objects: Object 1 to n, and the World. Suppose that they are linked such that Object i is the child of Object $i + 1$, and Object n is the child of the World. Let Object i have personal transform G_i. Then the group of the entire transform structure is the wreath product:*

$$G_1 \, \textcircled{w} \, G_2 \, \textcircled{w} \, \ldots \, \textcircled{w} \, G_n.$$

11   Theory of Relative Motion

We shall soon argue the following:

Musical works are relative motion systems.

This will allow us to give a detailed algebraic theory of music, because relative motion is an inheritance phenomenon, and thus it will be possible to use the algebraic theory of inheritance from the previous section.

Relative motion is a powerful organizing force in cognitive representation. For example, it is a classic Gestalt result that the visual system organizes motion into hierarchies of relative motion; furthermore, basic decomposition theorems in classical and quantum mechanics allow momentum to be organized into hierarchies of relative momenta – a basic tool for problem-solving in physics.

Computer animators know that relative motion is an inheritance phenomenon. Thus, using our algebraic theory of inheritance, it is now possible to give an algebraic theory of relative motion:

ALGEBRAIC THEORY OF RELATIVE MOTION. *A relative motion system corresponds to a wreath product in which the relative motion is given by the fiber group*
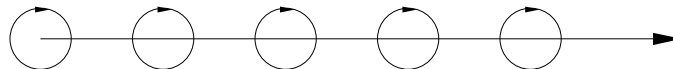
Figure 12: A relative motion system.

and the absolute motion, to which it is judged, is given by the control group:

$$relative\ motion\ ⓦ\ absolute\ motion.$$

This theory will be used to explain both melodic and rhythmic organization in music. However, it is instructive to first look at a visual example. Fig 12 shows a wheel moving along the ground. If one follows a single point on the wheel, it makes a complex curve called a cycloid. However, the human eye does not organize the movement in this way. Instead it decomposes the motion into a relative motion hierarchy, in which the point is seen as executing circular motion around the wheel-center, and the wheel is seen as moving as whole in a straight line. Our algebraic theory explains this as follows: The visual system organizes the motion into a wreath product in which the fiber is the rotation group $SO(2)$, and the control is the translation group $\mathbb{R}$:

$$SO(2)\ ⓦ\ \mathbb{R}.$$

Generally, our fundamental rule of relative motion is this:

> **Decompose the motion into two symmetry groups, such that one group** *transfers* **the other.**

This gives a wreath product where the transferring symmetry group is the control group and the transferred symmetry group is the fiber group.

As shown in Leyton [23], the above theory explains relative motion in human perception, classical and quantum mechanics, robotics, and computer animation.

12   Serial-Link Manipulators

It will now be argued that there is a profound relation between serial-link manipulators in robotics, and modulation in music. Both are decompositional means of reaching a point by hierarchical transfer. Therefore their mathematical structure is identical. This section considers serial-link manipulators, and the next deals with musical modulation. Both illustrate our theory of inheritance, and in particular our theory of relative motion.

Standardly in a serial-link manipulator (such as the human arm), one says that the frames of two successive links are related by a special Euclidean transformation A_i, and

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thus the overall relationship between the hand coordinate frame and the base coordinate frame is given by the product of matrices

$$A_1 A_2 \ldots A_n \tag{15}$$

corresponding to the succession of links. (see [2], [31]).

Now, in setting up the object-oriented structure of such manipulators, one usually stipulates that a distal link is a child of the next proximal link, and so on, successively along the manipulator. Our argument is that this arises from the *transfer* structure: The distal link has a space of actions that is transferred through the environment by the next proximal link. This exemplifies our claim that *the basis of inheritance is the deeper notion of transfer*. It is this that allows us to formulate inheritance algebraically in terms of wreath products. Thus, we argue that the group of a serial-link manipulator has the following wreath-product structure:

$$SE(3)_1 \, \textcircled{w} \, SE(3)_2 \, \textcircled{w} \, \ldots \textcircled{w} \, SE(3)_n \tag{16}$$

where each level $SE(3)_i$ is isomorphic to the special Euclidean group $SE(3)$, and the succession from left to right corresponds to the succession from hand to base (distal to proximal).

The entire group we have given in (16) for the serial-link manipulator, is very different from the group that is normally given in robotics for serial-link manipulators. Standardly, it is assumed that, because one is multiplying the matrices in (15) together, and therefore producing an overall Euclidean motion T between hand and base, the group of such motions T is simply $SE(3)$. However, we argue that this is not the case. The group is the much more complicated group given in expression (16). The conventional group $SE(3)$ necessarily models the arm as a rigid structure, whereas the wreath product (16) models the arm as a structure we call *semi-rigid*: a group where rigidity breaks down at a discrete set of points. Most crucially the wreath product models the object-oriented structure, which is basic to all computation concerning the kinematics.

13   Musical Modulation

With the above concepts, it is now possible to understand deeply modulation in music. The first thing to observe is that modulation, rather than being simply a translation system, is actually a *relative motion system*. For example, when one talks about a musical piece has having modulated to the dominant, one means that movement is now judged as within the dominant key, yet the dominant key is judged, as a whole, relative to the tonic key. Notice that the same movement *could* be judged as within the tonic key. However, its position is instead judged through a hierarchy of relative motion.

The fact that modulation is a relative motion system allows us to see that it necessarily involves an inheritance hierarchy. For example, movement within the tonic key is the transform belonging the parent, and movement within the dominant key is the transform belonging to the child. The latter inherits the former in the hierarchical manner described

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above. Thus, let the symbol $\mathbb{S}$ be the group of movements within a scale. Then, the ability to move the scale to any position within the scale is given by the following wreath product:

$$\mathbb{S} \textcircled{w} \mathbb{S}.$$

The control group represents the action of modulation, and the fiber group represents the key to which one modulates. This will be now be explained using the full detail of section 6.

For the purposes of illustration, we will assume that the group $\mathbb{S}$ of scale movements is given by $\mathbb{Z}_{12}$ acting along the semitone scale. Thus both the control group and fiber group will have the structure of $\mathbb{Z}_{12}$, and, to distinguish between these two roles, we will let the control group be denoted by $G(C)$, and the fiber group by $G(F)$. Furthermore, both the control set and fiber set will be the set of twelve semitones, and, again, to distinguish between these two roles, the control set will be denoted by C and the fiber set by F. Members of these two sets will be indicated by c_i and f_i respectively.

Now, for each of the 12 members c_i of the control semitone set C, make a copy of the fiber action. Thus there will now be 12 copies $\{F_{c_1}, F_{c_2}, \ldots, F_{c_{12}}\}$ of the fiber set, indexed in the control semitone scale C. Each of these copies will itself be the semitone scale (as fiber) rooted at a different tonic c_i, within the control semitone scale. That is, we now understand the control semitone set to be the set of available tonics for modulation.

Corresponding to the 12 fiber-set copies, there will be 12 copies $\{G(F)_{c_1}, G(F)_{c_2}, \ldots, G(F)_{c_{12}}\}$ of the fiber group, also indexed in the control set C. Each fiber-group copy (copy of the group of scale movements) will act on its own "personal" copy of the fiber-set; i.e., its own personal scale.

One can now define the regular wreath product:

$$G(F) \textcircled{w} G(C) \quad = \quad \mathbb{Z}_{12} \textcircled{w} \mathbb{Z}_{12} \tag{17}$$

where this group is the semi-direct product:

$$[G(F)_{c_1} \times G(F)_{c_2} \times \cdots \times G(F)_{c_{12}}] \textcircled{s} G(C). \tag{18}$$

The automorphic action of the control group $G(C)$, on the fiber-group product $G(F)_{c_1} \times G(F)_{c_2} \times \cdots \times G(F)_{c_{12}}$, corresponds to the action of $G(C)$ on the control set $\{c_1, c_2, \ldots, c_{12}\}$, whose elements are the tonics and now appear as the indexes on the 12 fiber-group copies $G(C)_{c_i}$. This means that the fiber-group copies are moved up and down the control scale; i.e., there is modulation. In fact, in accord with the structure of a semi-direct product, $G(C) = \mathbb{Z}_{12}$ carries out this action by *conjugating* the fiber-scale groups onto each other. The following observations are crucial:

(1) The above discussion clearly illustrates our claim that modulation is a *relative motion system*. That is, the control group $G(C) = \mathbb{Z}_{12}$, which represents the modulation movement, corresponds to the absolute motion, with respect to which any fiber-group copy represents the relative motion within the "frame" that has been moved by the absolute motion.

(2) The above discussion also illustrates our claim that modulation is an object-oriented inheritance system. In accord with our algebraic theory of inheritance (section 10), the parent corresponds to the control group and the child corresponds to the fiber group. Thus, any movement by the parent is inherited by the child.

The wreath product gives the complete symmetry of the scale structure, as follows: The data set $F \times C$, in this example, is decomposable into the set of fiber-scale sets, rooted at the different tonics. It will be called the *scale system*. The wreath product $G(F) \widehat{w} G(C) = \mathbb{Z}_{12} \widehat{w} \mathbb{Z}_{12}$ acts on the scale system, in the following way: By inspection of the semi-direct product form (18) of the wreath product, an individual element from the wreath product is of the form

$$\langle \ (\ T_{c_1} \ , \ T_{c_2} \ , \ \ldots \ , \ T_{c_{12}} \ ) \ | \ M \ \rangle \tag{19}$$

where $T_{c_i} \in G(F)_{c_i}$ and $M \in G(C)$. The action of the group element (19) is then interpreted as follows: Each scale movement T_{c_i} shifts the notes of its own scale F_{c_i}, and then the remaining component M performs a modulation across scales. Notice therefore that the group element (19) maps the scale system to itself, and that consequently the wreath product at (18) is a symmetry group of the scale system.

The following profound point should be observed:

The object-oriented structure arises from the symmetry structure.

That is, in the symmetry structure we have developed, the fiber of the symmetry corresponds to a child object whose group of command operations consists of movements within a scale, and the control level of the symmetry corresponds to a parent object whose command group is modulation.

The reader can see that successive modulation from the home key is given by an iterated wreath product:

$$\ldots \ \widehat{w} \ \mathbb{S} \ \widehat{w} \ \mathbb{S} \ \widehat{w} \ \mathbb{S} \ \widehat{w} \ \mathbb{S}.$$

Notice that this is structurally equivalent to the type of group we gave for the serial-link manipulator in expression (16); that is, the recursive substitution of a group action within itself. The reason is simple but profound: In both cases, the hierarchy represents a hierarchy of *workspaces*. Furthermore, in both cases, the workspace on any level has the same structure as the workspace on any other level. This means that a workspace moves an identical workspace about itself.

14 Recoverability

According to our theory, people call a structure, an art-work, if it is maximally complex, but allows the *maximal conversion of the complexity into understandability*. Our theory says that this conversion is achieved by maximizing transfer and recoverability. We have begun to examine the algebraic structure of transfer. It is now necessary to bring in the factor of recoverability. By *recovery*, we mean the following problem:

> **Given a data set, recover or infer a sequence of operations that generate the set.**

Our first book [21] was a 600-page analysis of this problem, and one of the main conclusions of this analysis was the following:

ASYMMETRY PRINCIPLE. *The only recoverable operations are symmetry-breaking ones. That is, a generative program is recoverable only if it is symmetry-breaking on each of the successively generated states.*

Now it is clear that there are many processes in the world that a not symmetry-breaking, but are symmetry-increasing; e.g., a tank of gas settling to equilibrium under the standard entropy-increasing process. Our theory says this:

SYMMETRY-INCREASING PROCESSES. *A symmetry-increasing process is recoverable only if it is symmetry-decreasing on successive data sets.*

So, for example, you can recover the fact that the tank of gas was entropy-increasing over time, if you kept a set of records (e.g., photographs) and the records are linearly ordered, e.g., they are laid out from left to right on a table, in which case the sequence of photographs breaks the left-right symmetry of the table. In other words, the increase in spatial symmetry in the tank of gas corresponded to a decrease in spatial symmetry of the record structure.

15   Theory of Symmetry-Breaking

A basic factor emerges from the above discussion: In order to ensure recoverability, the control group must be symmetry-breaking on its fiber. Thus, the following should be observed: The transfer component of our theory leads to wreath products, and the recoverability component adds the construct that the wreath products are symmetry-breaking.

Close examination reveals that this gives a far more powerful theory of symmetry-breaking than the conventional one that underlies physics and chemistry.

CONVENTIONAL VIEW OF SYMMETRY-BREAKING. *Symmetry-breaking is a reduction of symmetry group.*

Thus the transition from a square to a parallelogram is conventionally given by the following reduction in symmetry group:

$$D_4 \quad \longrightarrow \quad \mathbb{Z}_2.$$

That is, the eight operations in D_4 are reduced to the two operations in $\mathbb{Z}_2$.

However, according to our view, this is inherently weak because it means a loss of algebraic structure.

In contrast, our approach to symmetry-breaking can be illustrated with the example given in section 8 of the transition from a square to a parallelogram: This transition was modelled by adding, to the symmetry group of the square, the general linear linear group, via a wreath product. Thus, in our approach, symmetry-breaking actually preserves the original group. The breaking of a symmetry group G_1 is carried out by extending G_1 by another symmetry group G_2 via a wreath product thus: $G_1 \circledw G_2$. The original symmetry is given by the fiber copy of G_1 which corresponds to the identity element in the control group G_2. Non-identity elements in G_2 break the symmetry of the fiber group. Wreath products of this kind will be called **symmetry-breaking wreath products.**

NEW VIEW OF SYMMETRY-BREAKING. *Symmetry-breaking is extension via a wreath product. The extending group is the symmetry group of the asymmetrizing action.*

Most crucially, in our view, symmetry-breaking corresponds to an increase in symmetry group! More deeply, this undermines the standard notion that groups represent symmetry. Rather, we argue that they are maximally compact descriptions of asymmetry.

16 New Foundations to Geometry

Essentially, the recoverability of generative operations from the data set means that the shape acts as a memory store for the operations. More strongly, we argue in the book that all memory storage takes place via geometry. In fact, a fundamental proposal of our theory is this:

$$\text{Geometry} \quad \equiv \quad \text{Memory Storage.}$$

This theory of geometry is fundamentally opposite to that of Klein's in which geometric objects are defined as invariant under actions. If an object is invariant under actions, the actions are not recoverable from the object. Therefore Klein's theory of geometry concerns *memorylessness*, and ours concerns *memory retention*.

17 Rigorous Theory of Aesthetics

Near the beginning of this paper, we proposed that aesthetics is the maximization of transfer. It will be useful for us, at this moment, to consider this in the area of physics. The term aesthetics is used in physics with respect to symmetries of the dynamic equation (law) of any particular branch of physics. This in fact means that the symmetry group

of the equation is *transferring* flow lines of the dynamic equation onto each other. So the attempt to identify symmetries of the equation is an example of our principle of the maximization of transfer.

In fact, the term aesthetics is used also with respect to one other phenomenon in physics: origin states. For example, current models explaining the physical constitution of the universe argue for a succession of symmetry-breakings from the underlying starting state (first to hypercharge, isospin, and color, and then to the electromagnetic guage group). There is considerable puzzlement in physics as to why backward symmetrization from the present data set should be the case. However, according to our theory, backward symmetrization is logically necessary. Our Asymmetry Principle states that a generative sequence is *recoverable* only if present asymmetries go back to past symmetries in the generative sequence.

The above considerations therefore show that there are two uses for the term aesthetics in physics: (1) the characterization of transfer, and (2) the characterization of recovered states. In fact, one can see this in all aspects of quantum mechanics.

The question therefore is this: To what extent are these two situations of aesthetic judgement separate from each other? Our theory says that they are not separate. In section 15, it was seen that each level of the wreath hierarchy necessarily takes on simultaneously the roles of transfer and recoverability. To use physics as an example: The symmetry group acts both as the past state and as the operational structure that transfers flow lines of the Schrödinger equation onto each other. This is clearly evidenced for example in spectroscopy. Thus our complete definition of aesthetics can now be stated:

Aesthetics is the maximization of transfer and recoverability.

In this way, transfer becomes closely linked to memory storage. This allows us also to give a theory of art-works:

Art-works are maximal memory stores.
The rules of aesthetics are therefore the rules of memory storage.

18   Inferred Starting States

We will now give a rule that turns out to be fundamental to the entire process of recovering generative history:

INFERRED STARTING STATES PRINCIPLE.   *To maximize recoverability, the inferred starting state of a generative sequence must be structured by an iso-regular group.*

where we define:

LEVEL-CONTINUOUS

Plane	$\mathbb{R}$ ⓦ $\mathbb{R}$
Sphere	$SO(2)$ ⓦ $SO(2)$
Cross-Section Cylinder	$SO(2)$ ⓦ $\mathbb{R}$
Ruled Cylinder	$\mathbb{R}$ ⓦ $SO(2)$

LEVEL-DISCRETE

Cube	$\mathbb{R}$ ⓦ $\mathbb{R}$ ⓦ $\mathbb{Z}_2$ ⓦ $\mathbb{Z}_3$
Cross-Section Block	$\mathbb{R}$ ⓦ $\mathbb{Z}_n$ ⓦ $\mathbb{R}$
Ruled or Planar-Face Block	$\mathbb{R}$ ⓦ $\mathbb{R}$ ⓦ $\mathbb{Z}_n$

Table 1: Classification of surface primitives by maximizing transfer and recoverability

ISO-REGULAR GROUP. *This is a group satisfying the following three conditions:*

(1) It is an n-fold wreath product, G_1ⓦG_2ⓦ$\ldots$ⓦG_n.
(2) Each level G_i is either a cyclic group or a connected 1-parameter Lie group.
(3) Each level G_i is represented as an isometry group.

To illustrate the concept of an iso-regular group, notice carefully that two of the groups given earlier are examples of iso-regular groups:

Square: $\mathbb{R}$ ⓦ $\mathbb{Z}_4$

Cylinder: $SO(2)$ ⓦ $\mathbb{R}$.

Now turn to the Inferred Starting States Principle given above. We will soon see deep illustrations of this in music, but as an initial intuitive example in the visual domain, consider a bent pipe that you might see lying in the road. It is clear that, merely by describing this as bent, you understand the generative origin to have been a straight pipe, i.e., a cylinder. But a cylinder is given by an iso-regular group. So the generative origin is characterized by an iso-regular group, which is what is predicted by the Inferred Starting States Principle.

One powerful advantage of this principle is that it allows us to give a systematic classification of the surface primitives (starting states) of visual perception and computer-aided design. This is shown in Table 1.

Not only do iso-regular groups characterize the starting states in human perception and computer-aided design, but they characterize the starting states of physics (e.g., flat-space time universes in relativity, and sets of commuting observables in quantum

mechanics), as shown by us in Leyton [23]. The next section will demonstrate their crucial importance to music.

Our claim is that the fundamental power of structuring origin states by iso-regular groups is that this allows maximal recoverability.

19 Musical Meter

With the above proposal, that origin states are iso-regular groups, it is now possible to understand deep aspects of musical structure. We will first examine musical meter. Standardly, one says that the beat stream is divided into a number of levels of groupings:

(1) Primary accent grouping.
(2) Secondary accent grouping.
(3) Division into beats.
(4) Division of beats.
(5) Subdivision of beats.

The first level corresponds to the bars. The second is the major division of the bar, that occurs in the case of bars with more than three beats. For example, 4/4 time is usually perceived as divided into two successive subgroupings of two beats. The third level is the division into the beat itself. And the fourth and fifth levels are successive divisions of the beat. Now for our algebraic theory of meter:

ALGEBRAIC THEORY OF METRICAL STRUCTURE. *Given a metrical unit (e.g., a bar, a subgrouping, a beat), its occurrence within the next higher unit is given by a cyclic group $\mathbb{Z}_i$, and its division is given by a cyclic group $\mathbb{Z}_j$. The upper group $\mathbb{Z}_i$ transfers copies of the lower group $\mathbb{Z}_j$ as fiber, along the musical work. Therefore, the relation between the upper and lower group is that of a regular wreath product:*

$$\mathbb{Z}_j \, \textcircled{w} \, \mathbb{Z}_i.$$

The full metrical hierarchy, corresponding to the accent hierarchy of the bar structure, is therefore given by an n-fold wreath product

$$\mathbb{Z}_{m_1} \, \textcircled{w} \, \mathbb{Z}_{m_2} \, \textcircled{w} \, \ldots \, \textcircled{w} \, \mathbb{Z}_{m_n}.$$

If one defines the standard invariant metric on time, then this wreath product is an iso-regular group.

A particular aspect of this statement can be given as follows:

THEORY OF DIVISION. *Division by j is wreath sub-appendment by $\mathbb{Z}_j$.*
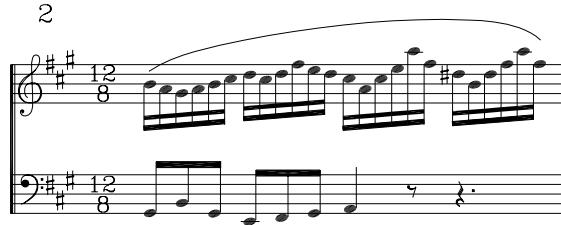
Figure 13: Bach: Two-Part Inventions, No. 12.

As an illustration, consider an excerpt from Bach's Two-Part Invention No. 12, shown in Fig 13. The time signature 12/8 has a subgrouping structure that divides the bar into two halves each of which has two beats. The right keyboard hand further divides the beat by three, and the left hand creates an additional division by two. Thus the full metrical structure is given by the following wreath product.

$$beatsubdivision \wr beatdivision \wr beat \wr subgrouping$$

$$= \quad \mathbb{Z}_2 \wr \mathbb{Z}_3 \wr \mathbb{Z}_2 \wr \mathbb{Z}_2.$$

The following should be noticed: Generally, a group of the form $\mathbb{Z}_2 \wr S_n$, where there are n copies of the fiber $\mathbb{Z}_2$, and the control group S_n acts as a permutation group on those n copies, is called the hyperoctahedral group of degree n. When n is 2, the hyperoctahedral group is $\mathbb{Z}_2 \wr S_2$, which is actually the dihedral group of order 8 (the standard symmetry group of the square). As illustrated in the above example, the top two levels of the 12/8 signature comprise this hyperocahedral group. In the general case, therefore, the time signature 12/8 is the wreath-subappendment of $\mathbb{Z}_3$ to the hyperoctahedral group, thus:

$$\mathbb{Z}_3 \wr \mathbb{Z}_2 \wr \mathbb{Z}_2.$$

Now let us give a theory of *simultaneous* division. For example, some bars can have double and triple division occurring simultaneously:

SIMULTANEOUS DIVISION. *Simultaneous division of an interval by different numbers $D_1, D_2, \ldots D_n$, will be given by wreath sub-appendment by the direct product $\mathbb{Z}_{D_1} \times \mathbb{Z}_{D_2} \times \ldots \times \mathbb{Z}_{D_n}$.*

This can be illustrated with the second movement of Brahms 1st Piano Concerto, as shown in Fig 14. The time signature is 6/4, which is interpreted here as a sextuple
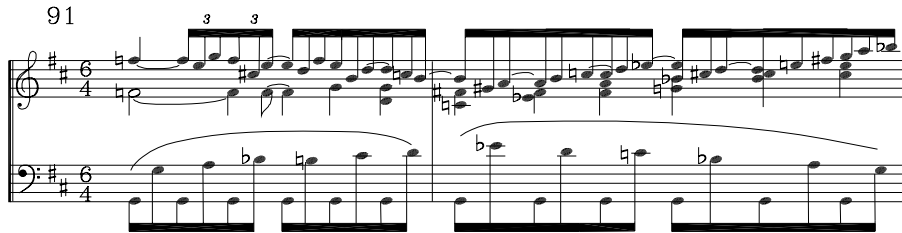
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Figure 14: Brahms: Piano Concerto No. 1, second movement.

meter due to the slowness of the tempo. Such 6/4 meter decomposes the bar into two subgroupings, each of three quarter notes. Then the simultaneous use of simple and compound meter divides the beat into two (in the right hand) and into three (in the left hand). Therefore, the full group of the metrical structure is this:

$$[divisionA \times divisionB] \ \textcircled{w} \ beat \ \textcircled{w} \ subgrouping$$

$$= \ [\mathbb{Z}_2 \times \mathbb{Z}_3] \ \textcircled{w} \ \mathbb{Z}_3 \ \textcircled{w} \ \mathbb{Z}_2.$$

As a further illustration, consider Fig 15 from the second movement of Bartok's String Quartet No. 4. The second violin and viola are in 2/4; and the first violin and cello are in 6/8. Both 2/4 and 6/8 are duple meters. Thus the beats coincide in all four instruments. Nevertheless, the divisions do not. In the second violin and viola, the division is into two; whereas in the first violin and cello, the division is into three. Notice that there is no division between the level of the bar and the level of the beat. That is, the full group of the metrical structure is:

$$[divisionA \times divisionB] \ \textcircled{w} \ beat$$

$$= \ [\mathbb{Z}_2 \times \mathbb{Z}_3] \ \textcircled{w} \ \mathbb{Z}_2.$$

Notice that this means that the simultaneous divisions are each children of the beat – in accord with our algebraic theory of object-oriented inheritance.

20 Complex Shape

Using the geometric concepts developed above, it becomes possible to give a powerful theory of musical composition. To do so, it is necessary to examine some of the main theory of complex shape developed in the book Leyton [23]. In this section, the theory

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Figure 15: Bartok: String Quartet No. 4, second movement.

will be illustrated using the visual domain, because, we will argue, in the next section, that there are deep abstract relationships between the visual domain and musical composition. This particularly concerns a profound correspondence between the iso-regular groups of the visual and musical domains.

Consider the main problem for establishing a generative theory of complex structure. According to section 14, recoverability is possible only if the generative operations are symmetry-breaking. But this means that, as one proceeds forward in the generative sequence, the symmetry group of the structure quickly reduces to nothing. This means that there is a loss of algebriac information, which means a loss of generativity. This problem will now be solved, using the theory of symmetry-breaking of section 15.

What we will do here is develop a symmetry group for a complex environment. This will be a powerful structure because it will contain all the required information for usability, navigation, manipulation, etc. The theory will become fundamental to the theory of musical composition presented in the next section.

It is necessary to solve the fundamental problem of *concatenation*. Consider Fig 16. Each of the two objects *individually* has a high-degree of symmetry. However, the *combined* structure, shown, looses much of this symmetry; i.e., causes a severe reduction in symmetry group. We want to develop a group theory that encodes exactly what the eye can see. In particular, in the combined situation, one can *still* see the individual objects. Therefore, we want to develop a symmetry group of the concatenated structure in which the symmetry groups of the individual objects are preserved, and yet there is the extra information of concatenation.

The solution to be proposed is this: The generative history starts out with the two independent objects, and therefore the symmetry of this starting situation is given thus:

$$G_{cylinder} \times G_{cube}$$

which is the direct product of the groups of the two independent objects. The reader
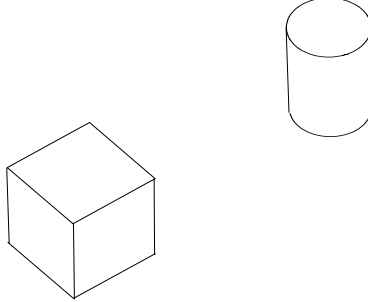
Figure 16: Concatenation of cylinder and cube.

should carefully notice the following: The direct product symbol here should not be regarded as representing a direct product *between* fibers, as previously. It will be *within* a single fiber.

Now, by the maximization of transfer, the starting group, i.e., this direct product group, must be transferred onto subsequent states in the generative history, and therefore it must be the fiber of the wreath product in which the control group creates the subsequent generative process.

Let us take the control group to be the *affine group* $AGL(3, \mathbb{R})$ on three-dimensional real space.[3] The full structure, fiber plus control, is therefore the following wreath product:

$$[G_{cylinder} \times G_{cube}] \circledW AGL(3, \mathbb{R}).$$

Now, it is necessary to fix the *group representation* of this wreath product. First, by our theory of recoverability, the control group must have an asymmetrizing action. Thus proceed as follows: The particular fiber-group copy

$$[G_{cylinder} \times G_{cube}]_e$$

corresponding to the identity element e in the affine control group, must be the most symmetrical configuration possible. This exists only when the cube and the cylinder are coincident, with their symmetry structures *maximally aligned*. It will be called the **alignment kernel**.

Next, choose one of the two objects to be a reference object. This will remain fixed at the origin of the coordinate system. Let us choose the cube as the referent. Given this, now describe the action of the affine control group as providing an affine motion of the cylinder *relative* to the cube. Each fiber-group copy

$$[G_{cylinder} \times G_{cube}]_g$$

---

[3]An element of this group is a linear transformation composed with a translation. AGL means Affine General Linear.

for some member g, of the control affine group, is therefore an arrangement of this system. In fact, any fiber copy will be called a **configuration** of the system. For example, Fig 16 corresponds to a configuration. The crucial concept is this: *The role of the affine control group is to* **transfer configurations onto configurations**.

The wreath product we have presented:

$$[G_{cylinder} \times G_{cube}] \textcircled{w} AGL(3, \mathbb{R})$$

gives the *complete* symmetry group of the concatenated situation. It has all the internal symmetries of the objects individually, as well as their relationships.

Let us now understand how to add a further object, for example a sphere. First of all, the fiber becomes the following, with the added sphere group:

$$G_{sphere} \times G_{cylinder} \times G_{cube}.$$

In such expressions, our rule will be that each object, encoded along this sequence, provides the reference for its left-subsequence of objects. Thus the cube is the referent for the cylinder-sphere pair, and the cylinder is the referent for the sphere.

Accordingly, there are now two levels of control, each of which is the affine group $AGL(3, \mathbb{R})$, and each of which is added via a wreath product. Thus we obtain the 3-level wreath product:

$$[G_{sphere} \times G_{cylinder} \times G_{cube}] \textcircled{w} AGL(3, \mathbb{R}) \textcircled{w} AGL(3, \mathbb{R}).$$

This is interpreted in the following way: Initially, the three objects (cube, cylinder, sphere) are coincident with their symmetry structures maximally aligned. This is the fiber-group copy called the *alignment kernel* above. The higher affine group moves the cylinder-sphere pair in relation to the cube. The lower affine group moves the sphere in relation to the cylinder.

The above discussion has been illustrating a class of groups we call **telescope groups**, which were proposed by us in Leyton [23]. To get an intuitive sense of a telescope group, think of an ordinary telescope. In an ordinary telescope, you have a set of rings that are initially maximally coincident. Then you pull them successively out of alignment with respect to each other. A telescope group is a group structured like this.

In fact, it is part of a still larger class of groups we call **unfolding groups**, which which were also proposed by us in Leyton [23]. Unfolding groups are the most important class of algebraic structures introduced in that book. The basic idea is that any complex structure such as a design in CAD is *unfolded* from a maximally collapsed form which we call the alignment kernel. Two main properties characterize unfolding groups:

Selection: The control group acts selectively on only part of its fiber.

Misalignment. The control group is symmetry-breaking by misalignment.

Now, in order to establish a group theory of CAD, our procedure was this: We spent several years working through every single operation in each of several of the main CAD, solid modeling, assembly, and animation programs, including several releases of
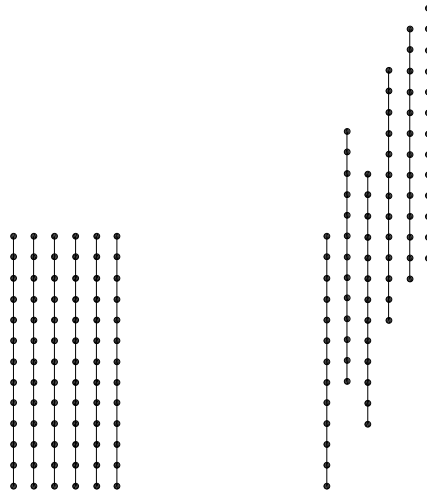
Figure 17: Modulation has an opening telescope structure.

AutoCAD, ProEngineer, 3D Studio Max and Viz, Architectural Desktop, Mechanical Desktop, etc., as well as all the major manuals on each of the programs – approximately 15,000 pages of text.

Each individual situation was characterized by a group, and a new class of groups was invented for any situation that could not be formalized in terms of a previously created class of groups. Proceeding in this manner, we eventually found that three classes of groups could handle any newly created situation. They were called:

(1) Telescope groups.

(2) Super-local unfoldings.

(3) Sub-local unfoldings.

The above has looked so far (intuitively) at the structure of telescope groups. The book shows that serial-link manipulators are examples of telescope groups. Therefore:

Musical modulation is an example of a telescope group.

That is, intuitively, modulation involves a set of initially coincident scales that successively slide against each other, out of alignment – like an opening telescope. This is illustrated in Fig 17.

Super-local unfolding groups are structured by adding a control level *above* the existing wreath hierarchy, but such that it acts selectively on only part of the existing hierarchy. Such groups model situations, for example, in AutoCAD, where one freezes part of the existing structure and manipulates some unfrozen cross-hierarchy selection of elements; or conversely, situations, for example, in 3D Studio Max, where the cross-hierarchy selection is locked and manipulated over a sequence of steps.

Now consider *sub-local unfoldings*. Here, an extra fiber level is attached *below only some part* of the existing hierarchy. We shall soon see that this is the major basis of

musical composition. However, in order to fully understand this, it is best to first examine an apparently different design process: mechanical CAD. This is design in mechanical engineering – forming the basis, for example, of the aerospace and automotive industries. It is generally accepted that mechanical CAD proceeds by a process called *feature attachment*. The remainder of the paper will do the following:

1. Give an algebraic theory of feature attachment in mechanical CAD.

2. Show that this is equivalent to musical composition.

To accomplish this, it is necessary to give more of the algebraic theory of object-oriented software, that we developed in Leyton [23]. First of all, within that theory, there is an analysis of class structure which says that each geometric class consists of an *internal symmetry group*, specified often in the invariants clauses of the software text for the class – and an *external group* consisting of command operations, such as deformations, specified in the feature clauses of the class text.

A principle claim of the theory is that the relation between the internal symmetry group and command structure, in the software text, is a wreath product, thus:

$$G_{sym} \, \textcircled{w} \, G(C)$$

where G_{sym} is the internal symmetry group and $G(C)$ is the group of command operations. This is no more than an algebraic formulation of the phenomenon of *re-usability*, within the specification text of the class. That is, according to our theory, re-usability is transfer and transfer is modelled by wreath products.

Now let us turn to *cloning* in object-oriented programming. It is important to notice that, when one clones an object, one is producing a copy with the same instance values. This means that one is essentially creating a copy that is *aligned* with the original, as can be seen in such programs such as 3D Studio Max and Viz. This copy can then be manipulated via its command operations, which will then pull the clone out of alignment, i.e., break the symmetry of the object-clone pair.

We are now ready to turn to *object creation and feature attachment.* Feature attachment is the term used in mechanical design for the successive addition of structural units and components. It is, of course, the main process in any design. Our basic proposal is this:

THEORY OF FEATURE ATTACHMENT

When one creates objects and attaches them in the design structure, one is entering new instances into the alignment kernel, and positioning the command group for each new instance in the appropriate wreath position within the unfolding group corresponding to the inheritance hierarchy of the structure.

21 Theory of Musical Composition

With these concepts, it is now possible to give a theory of musical composition. Recall that our basic principle of aesthetics says:

Aesthetics is the maximization of transfer.

This was seen in section 13 with respect to the structure of modulation which was modelled by an n-fold wreath product, i.e., hierarchical transfer, of the scale structure; and it was seen also in section 19 with respect to musical meter which was modelled by an iso-regular group – again, a structure of hierarchical transfer. It will now be seen with respect to *melodic form.*

Significant progress has been made in understanding sequential organization by psychologists working on the generation of serial patterns. Herbert Simon, himself an outstanding musician, together with collegues, was the first to consider rule-systems for psychological sequence generation, Simon & Kotovsky [35], Kotovsky & Simon [14]. A further advance was made by Restle [32], who used hierarchies of rules. Fig 18 shows an example typical of one of Restle's hierarchies. Three generative rules are used in this hierarchy: T = transpose by one unit upwards in the scale; R = repeat; and M = mirror about the scale center. Here the scale is assumed to consist of 12 notes. Each of the operators takes the entire subsequence that it dominates via its left node and maps it to the entire subsequence dominated by its right node. Notice that the tree is *strictly nested*, a term used by Greeno & Simon [12], meaning that all the operators within a level are exactly the same. The condition of strict nesting is equivalent to the fact that the tree can be represented by a recursive formula. In the example shown, the formula is:

$$M(T(R(T(1)))). \tag{20}$$

The symbol 1, in this formula, is the left most 1 in Fig 18; and the formula generates the remainder of the sequence.

An additional advance came when a number of researchers independently started to use groups to structure the rules; Babbit [1], Leyton [15], Greeno & Simon [12], Jones [13]. The major school for the use of group theory in music has become that of Guerino Mazzola in Switzerland: Mazzola [24], [25], [26]. See also the work of Thomas Noll [29], [30]. Further group-theoretic work has been done by Economou, [3], [4], [5]; and also Gollin [11].

We argue that, when one examines the hierarchical theory of Restle, one must conclude that the human mind is maximizing transfer. That is,

The process of sequence comprehension or generation is a process of transferring previous structure onto future structure.

The fact that the mind tries to maximize this can be seen by the psychological studies carried out by Restle to support his hierarchical rules – e.g., profiles of *anticipation errors* showed that subjects were mapping previous structure onto the anticipated structure, Restle & Brown [33].

Our generative theory of shape says that this is best modelled by wreath products. We proceed as follows: Let us call a group generated by a set of compositional operators,
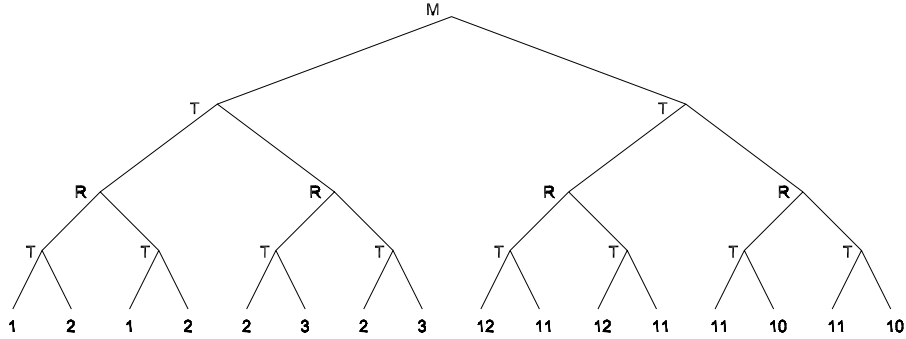
Figure 18: An example of one of Restle's rule hierarchies.

a **rule group** $\Re_i$. Given a hierarchy of the type shown in Fig 18, the levels will be numbered upward from 1 to n. Now assign a rule group to each node. Within any level i, the nodes should each receive the *same* rule group $\Re_i$. We argue that the rule-structure of the hierarchy is given by taking the wreath product of the groups $\Re_i$, thus:

$$\Re_1 \,\text{\textcircled{w}}\, \Re_2 \,\text{\textcircled{w}}\, \ldots \text{\textcircled{w}}\, \Re_n.$$

Observe that, with the three operators defined by Restle, the group is *iso-regular*. Therefore, we now see that both the metrical and melodic structures, in their basic forms, are given by iso-regular groups. This supports our claim that origin states are given by iso-regular groups and that the subsequent generative process is symmetry-breaking by breaking the iso-regularity.

Now according to our generative theory of shape, complex structure is created by *unfolding*, which is selection plus misalignment. One loads a set of iso-regular groups into the alignment kernel. These will represent the strict metrical or melodic anticipation hierarchies of Restle.

According to our theory, the anticipation hierarchies are iso-regular groups and complexity in a work will break the iso-regularity, and thus break the anticipated structure. This is a fundamentally important concept: Breaking the anticipation hierarchy is equivalent to breaking the iso-regular group:

Breaking the anticipation hierarchy.
$\updownarrow$
Breaking the iso-regular symmetry.

Most crucially, our theory says that breaking the iso-regular group must *itself* be achieved by *transfer*; i.e., broken iso-regularity must be perceived as the transfer of iso-regularity. Thus, since the iso-regular components are loaded into the alignment
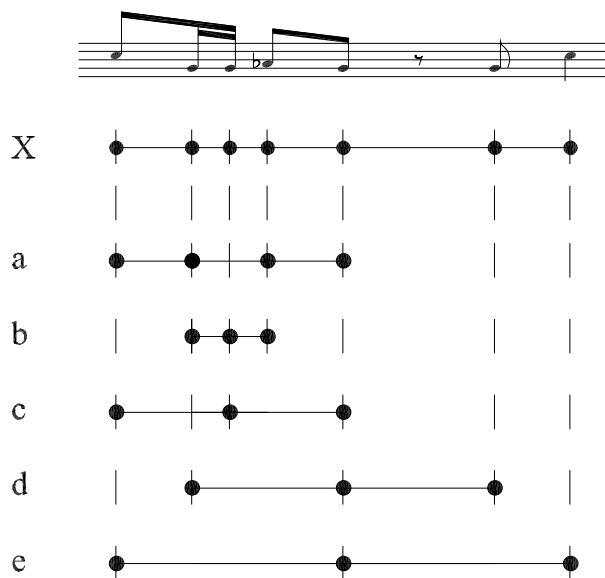
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Figure 19: An illustration of *local meter atlas*, from [8]

kernel, the breaking is carried out by adding control groups above the alignment kernel which will selectively deform and move the iso-regular components.

To illustrate, let us return to meter. Mazzola [25], [26] invented the important concept of a *local meter atlas*. This is the covering of an irregular pattern by local regular meters. To illustrate, observe that, in Fig 19, the melody has an irregular structure of onsets, as shown in line X of the diagram. Below this, the lines marked a-e are each regular meters that cover the onsets in the above irregular pattern. These local meters are maximal in the sense that they extend the furthest distance allowed by the onsets in the irregular pattern. Nestke, & Noll [28] and Fleischer & Noll [9] call these, *inner local meters*, to distinguish them from the meter structure determined by the time-signature of the score; i.e., corresponding to the conventional hierarchy of beat accents associated with the bar-lines, etc. The accent structure corresponding to the time-signature is called the *outer meter structure*. Using this theory, extensive and insightful analyses have been developed by Mazzola [26], Fleischer, Mazzola, Noll, [8], Fleischer [6], [7], Fleischer & Noll [9].

What we do now is propose a *generative* theory of the local meter atlas. To illustrate, return to Fig 19. Recall that lines a-e in that diagram give the inner local meters. Notice that they all correspond to iso-regular groups (each with an occupancy subgroup). What we argue is that the atlas, i.e., arrangement of these groups, was generated from a starting state in which these iso-regular groups were maximally aligned, and that the subsequent generative process *misaligned* those groups. That is, the atlas is the misaligned version of the alignment kernel. The misalignments were created by wreath-appending control groups above the alignment kernel that selectively deformed and moved the iso-regular

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groups that comprised the alignment kernel. Notice that one can regard the ultimate reference object within the alignment kernel – i.e., the ultimate parent in the inheritance hierarchy defined by the control groups – as the metric structure given by the time signature.

Our musical theory is therefore mathematically equivalent to our theory of mechanical CAD. To see this crucial similarity, consider the following example of an unfolding group:

$$[G_1 \times G_2 \times G_3] \, ⓦ \, AGL(3, \mathbb{R}) \, ⓦ \, AGL(3, \mathbb{R}).$$

The fiber $[G_1 \times G_2 \times G_3]$ is the direct product of iso-regular groups; and the copy of this fiber, corresponding to the identity element in the full control group, is the alignment kernel. The successive control groups $AGL(3, \mathbb{R})$ create successive misalignments of the iso-regular groups in the alignment kernel. The profound point is this:

> **Mechanical CAD:** the iso-regular groups, G_i, loaded into the alignment kernel, are the shape primitives (cylinder, sphere, cube, etc.).

> **Music:** the iso-regular groups, G_i, loaded into the alignment kernel, are the anticipation hierarchies.

In other words, what have been called *features* in mechanical CAD (e.g., the cylindrical hole or rectangular block) correspond to the metrical and melodic *anticipation hierarchies* in music.

Therefore our theory of feature attachment in mechanical CAD, becomes equivalent to our theory of composition in music. That is:

THEORY OF MUSICAL COMPOSITION

Musical composition proceeds by successively adding new iso-regular groups (anticipation hierarchies) into the alignment kernel, and positioning the command group for each new instance in the appropriate wreath position within the unfolding group corresponding to the inheritance hierarchy of the structure.

References

[1] Babbit, M. (1961). Set structure as a compositional determinant. *Journal of Music Theory*, **5**, 72-94.

[2] Denavit, J., & Hartenberg, R.S. (1955). A kinematic notation for lower-pair mechanisms based on matrices. *Journal of Applied Mechanics*, ASME, **22**, 215-221.

[3] Economou, A. (1998). The symmetry of the Equal Temperament Scale. *Mathematics and Design 98: Proceedings of the Second International Conference.* Ed. J Barrallo, The University of the Basque Country, San Sebastian, Spain, pp. 557-566.

[4] Economou, A. (2000). Spatial Canons and Fugues. In *Proceedings of Greenwich 2000:Digital Creativity Symposium: Architecture, Landscape, Design.* Ed. C Teeling, London, UK: University of Greenwich, pp 53-60.

[5] Economou, A. (2001). C2C2C2: Pythagorean Structures in Design. In *Proceedings of Mathematics and Design 2001: The Third International Conference: Digital. Hand. Eye. Ear. Mind.* eds. M. Burry, S. Datta, A. Dawson, J. Rollo. Geelong, Australia: Deakin University, pp 128-138.

[6] Fleischer, A. (2002). *Die analytische Interpretation. Schritte zurErschliessung eines Forschungsfeldes am Beispiel der Metrik.*. PhD Thesis, Humboldt-Universitat zu Berline.

[7] Fleischer, A. (2002). A Model of Metric Coherence. In *Procedings of the 2nd Conference, Understanding and Creating Music.*

[8] Fleischer, A., Mazzola, G., & Noll, T. (2000). Computergestütze Musiktheorie. *Musiktheorie,* Heft 4, 314-325

[9] Fleischer, A. & Noll, T. (2002). Analytic coherence and performance regulation. In: *Human Supervision and Control in Engineering and Music.* Special Issue of *Journal of New Music Research.*

[10] Gips, J. & Stiny, G. (1980). Production systems and grammars: a uniform characterization. *Environment and Planning B,* **7**, 399-408.

[11] Gollin, E. (2000) Representations of space and conceptions of distance in transformational music theories. PhD Thesis. Havard University.

[12] Greeno, J.G. & Simon, H.A. (1974). Processes for sequence production. *Psychological Review*, **74**, 187-198.

[13] Jones, M.R. (1974). Cognitive representations of serial patterns. In B. Kantowitz (Editor). *Human information processing: Tutorials in performance cognition.* Potamac, MD: Erlbaum.

[14] Kotovsky, K. & Simon, H.A. (1973). Empirical tests of a theory of human acquisition of concepts for sequential events. *Cognitive Psychology*, **4**, 399-424.

[15] Leyton, M., (1974). *Mathematical-logical postulates at the foundations of art.* Tech Report, Mathematics Department, University of Warwick.

[16] Leyton, M. (1984). Perceptual organization as nested control. *Biological Cybernetics,* **51**, 141-153.

[17] Leyton, M. (1986a). Principles of information structure common to six levels of the human cognitive system. *Information Sciences,* **38**, 1-120. Entire journal issue.

[18] Leyton, M. (1986b). A theory of information structure I: General principles. *Journal of Mathematical Psychology,* **30**, 103-160.

[19] Leyton, M. (1986c). A theory of information structure II: A theory of perceptual organization *Journal of Mathematical Psychology,* **30**, 257-305.

[20] Leyton, M. (1987a). Nested structures of control: An intuitive view. *Computer Vision, Graphics, and Image Processing,* **37**, 20-53.

[21] Leyton, M. (1992). *Symmetry, Causality, Mind.* Cambridge, Mass: MIT Press.

[22] Leyton, M. (1999). New foundations for perception. In Lepore, E. (Editor). *Invitation to Cognitive Science.* Blackwell, Oxford. p121 - 171.

[23] Leyton, M. (2001). *A Generative Theory of Shape.* Berlin: Springer-Verlag.

[24] Mazzola, G. (1990). *Geometrie der Töne - Elemente der Mathematischen Musiktheorie.* Basel: Birkhäuser.

[25] Mazzola, G. (1993-1996). *Geometry and Logic of Musical Performance I-III.* Reports for Schweiz. Nationalfonds, Univ. Zürich.

[26] Mazzola, G. (2002). *The Topos of Music.* Basel: Birkhäuser.

[27] Meyer, B. (1997). *Object-Oriented Software Construction.* New Jersey: Prentice Hall.

[28] Nestke, A. & Noll, T. (2001). Inner Metric Analysis. In: Jan Haluska (ed.) *Harmonic Analysis and Tone Systems.* Tatra Mountains Mathematical Publications, Volume 23, Bratislava.

[29] Noll, T. (1995). Fractal Depth Structure of Tonal Harmony. In R. Bidlack (Editor). *ICMC-Proceedings.* Banff: The Banff Centre for the Arts.

[30] Noll, T. (1997). *Morphologische Grundlagen der abendländischen Harmonik.* Musikometrika 7, Bochum: Brockmeyer.

[31] Paul, R.P. (1981). *Robot manipulators.* Cambridge, Mass: MIT Press.

[32] Restle, F. (1970). Theory of serial pattern learning: Structural trees. *Psychological Review*, **77**, 481-495.

[33] Restle, F. & Brown, E.R. (1996). Organization of serial pattern learning. In G.H. Bower (Editor) (eds.) *The psychology of learning and motivation: Advances in research and theory (Vol 4)..* New York: Academic Press.

[34] Rotman, J.J. (1995). *An Introduction to the Theory of Groups.* Berlin: Springer-Verlag.

[35] Simon, H.A. & Kotovsky, K. (1963). Human acquisition of concepts for sequential patterns. *Psychological Review*, **70**, 534-546.

[36] Stiny, G. (1980). Introduction to shape and shape grammars. *Environment and Planning B,* **7**, 343-351.