TILING THE INTEGERS WITH APERIODIC TILES

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Abstract. A finite subset A of integers tiles the discrete line $\mathbb{Z}$ if the integers can be written as a disjoint union of translates of A. In some cases, necessary and sufficient conditions for A to tile the integers are known. We extend this result to a large class of nonperiodic tilings and give a new formulation of the Coven-Meyerowitz reciprocity conjecture which is equivalent to the Flugede conjecture in one dimension.

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1. Introduction

A *tiling* of the finite abelian group G (written additively) is a pair (A, C) of subsets of G such that 0 is in both A and C and every x of G can be uniquely written as $x = a + c$ with $a \in A$ and $c \in C$. When G is the integers $\mathbb{Z}$, we said that the subset A $(0 \in A)$ *tiles the integers* if there is a set C $(0 \in C)$ such that $\mathbb{Z} = A \oplus C$. The set A is called a *tile* and C the *translation set*. This decomposition of a finite abelian group in two subsets had been studied by Hajós [8], Rédei [13], Sands [15, 16], and others.

It is well known that any tiling (A, C) of the integers must be periodic. There is a finite subset B of $\mathbb{Z}$, and a nonnegative integer n such that $C = B + n\mathbb{Z}$ with $|A| \cdot |B| = n$. This leads to focuse on the factorisation of the abelian group $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ for some nonnegative integer n in two factors

$$A \oplus B = \mathbb{Z}_n$$

The factorization of finite abelian groups was introduced by Hajós in 1941 [7] in order to solve the Minkowski's conjecture (see [11]) on homogeneous linear forms. A subset $A \subset G$ is *periodic* if there is some nonidentity element $k \in G\backslash\{0\}$ such that $A + k = A$. The factorization $G = A \oplus B$ is periodic if one of the subsets A or B is periodic, otherwise the sets and the factorization are *aperiodic*. Hajós asked for which groups the factorization is periodic and this leads to a new definition. A finite abelian group is a *Hajós group* (also called a "good group" in the literature, or has the *2-Hajós property*) if in each factorization of G into two subsets $G = A \oplus B$ at least one factor is periodic. Otherwise, G is a *non-Hajós group* (also called a "bad group"). Restricted to $G = \mathbb{Z}_n$, the Hajós question was to determine the values of n for which the factorization is periodic. The problem was solved by N.G. de Bruijn [2] and A. Sands. In 1962, A. Sands [16] gave a complete classification of all Hajós groups.

Theorem 1. *$\mathbb{Z}_n$ is a Hajós group if and only if n is of the form: p^k for $k \geq 0$, $p^k q$ for $k \geq 1$, p^2q^2, pqr, p^2qr or $pqrs$ for distinct primes p, q, r, s.*

Corollary 2. *$\mathbb{Z}_n$ is a non-Hajós group if and only if n can be expressed in the form $p_1p_2n_1n_2n_3$ where p_1, p_2 are primes, $p_in_i \geq 2$, $i = 1, 2, 3$ and $gcd(n_1p_1, n_2p_2) = 1$.*

The smallest values of n for which $\mathbb{Z}_n$ is a non-Hajós group are: 72, 108, 120, 144, 168, etc.

In 1977, Newman [12] found which sets of prime power size ($|A| = p^k$, for p prime and $k \geq 1$) tile the integers. Later, Coven and Meyerowitz [4] found necessary and sufficient conditions for A to tile the integers when $|A|$ has at most two prime factors. To review these conditions, we introduce for each finite set A of nonnegative integers, the characteristic polynomial

$$A(x) = \sum_{a\in A} x^a$$

such that $A(1) = |A|$. The factorization of $\mathbb{Z}_n = A \oplus B$ is thus equivalent to the algebraic expression

$$A(x)B(x) \equiv 1 + x + x^2 + ... + x^{n-1} \qquad \mod\ x^n - 1$$

Let $\Phi_s(x)$ be the s-th cyclotomic polynomial and S_A be the set of prime powers s such that the s-th cyclotomic polynomial $\Phi_s(x)$ divides $A(x)$. The pair (A, B) is a factorization of $\mathbb{Z}_n$ if and only if $|A|\,|{\cdot}B| = n$ and $\Phi_s(x)$ divides $A(x)B(x)$ for all $s \neq 1$ factors of n. The Coven-Meyerowitz conditions are
(T1) $A(1) = \prod_{s\in S_A} \Phi_s(1)$
(T2) If $s_1, ..., s_m \in S_A$ are powers of different primes then $\Phi_{s_1...s_m}(x)$ divides $A(x)$.

Coven and Meyerowitz [4] proved that (we summarize their results in one theorem):

Theorem 3. *(1) If $A(x)$ satisfies (T1) and (T2), then A tiles the integers.*
(2) If A tiles the integers, then $A(x)$ satisfies (T1).
(3) If A tiles the integers, and $|A|$ has at most two prime factors then (T2) holds.

Granville, Laba and Wang [6] extend this result to certain set A. More precisely, if A and B are two sets of integers such that $|A| = p^\alpha q^\beta r^\gamma$ and $|B| = pqr$ with p, q, r distincts primes that tile $\mathbb{Z}_n$, they show that if $\Phi_p(x)$, $\Phi_q(x)$, $\Phi_r(x)$ divide $A(x)$ then so do $\Phi_{pq}(x)$, $\Phi_{pr}(x)$, $\Phi_{qr}(x)$ and $\Phi_{pqr}(x)$. But in general, it is not known if (T1) and (T2) are necessary and sufficient conditions for A to tiles the integers. We will show that it is true for periodic canons and for some aperiodic canons.

2. Aperiodic canons

Each factorization of a non-Hajós group $\mathbb{Z}_n = A \oplus B$ in two aperiodic subsets (A, B) is called an *aperiodic canon*. The problem arises in music theory and had been studied by Vuza [20, 21], Fripertinger [5], Amiot [1] and others. The set A describes the time location of the events of the canon or the *ground voice* (also called the *inner rhythm*) and the set B corresponds to the attack times of the voices of the canon (also called *onsets* or *outer rhythm*). All voices are just copies of the ground voice translated in time. Vuza gave an algorithm to determine what he called *regular complementary canons of maximal category* (here aperiodic canons), but it has been shown that not all aperiodic canons are given by this method. At the present time, a suitable algorithm to determine all aperiodic canons is not known for all values of n.

For nonnegative integers $r, n > 1$, we denote $\overline{n}$ the set $\overline{n} = \{0, 1, ..., n-1\}$ and $r \cdot \overline{n} = \{0, r, ..., r(n-1)\}$ the set obtained by multiplying each element of $\overline{n}$ by r. The following result shows that it exists an infinity of aperiodic canons and gives the expression of one canon for each non-Hajós group.

Theorem 4. *Let $\mathbb{Z}_n$ be a non-Hajós group with $n = p_1p_2n_1n_2n_3$. Denote K_1 and K_2 the sets*

$$\begin{aligned} K_1 &= n_2n_3(\overline{p_2} \oplus p_2n_1 \cdot \overline{p_1}) \\ K_2 &= n_1n_3(\overline{p_1} \oplus p_1n_2 \cdot \overline{p_2}) \end{aligned}$$

and let $T_j(B_2) = \{j\} \oplus K_2$ be the translation of j. The canon (A, B) of lengths $|A| = n_1n_2$ and $|B| = p_1p_2n_3$, defined by

$$\begin{aligned} A &= n_3(p_2n_2 \cdot \overline{n_1} \oplus p_1n_1 \cdot \overline{n_2}) \\ B &= K_1 \cup T_1(K_2) \cup ... \cup T_{n_3-1}(K_2) \end{aligned}$$

is an aperiodic canon, called the standard aperiodic canon.

Proof. The proof uses the following properties

$$\begin{aligned} \overline{a} \oplus a \cdot \overline{b} &= \overline{ab} \\ c \cdot \overline{a} \oplus a \cdot \overline{bc} &= \overline{abc} \qquad \bmod\ abc \quad \gcd(a, c) = 1 \end{aligned}$$

Compute the direct sum of A with K_1

$$\begin{aligned} A \oplus K_1 &= p_2n_2n_3 \cdot \overline{n_1} \oplus p_1n_1n_3 \cdot \overline{n_2} \oplus n_2n_3 \cdot \overline{p_2} \oplus p_2n_1n_2n_3 \cdot \overline{p_1} \\ &= n_2n_3(p_2 \cdot \overline{n_1} \oplus \cdot \overline{p_2}) \oplus n_1n_3(p_1 \cdot \overline{n_2} \oplus p_2n_2 \cdot \overline{p_1}) \\ &= n_2n_3(\overline{n_1p_2}) \oplus n_1n_3(p_1 \cdot \overline{n_2} \oplus p_2n_2 \cdot \overline{p_1}) \\ &= n_2n_3(\overline{n_1p_2} \oplus p_2n_1 \cdot \overline{p_1}) \oplus n_1n_3p_1 \cdot \overline{n_2} \\ &= n_2n_3 \cdot \overline{p_1p_2n_1} \oplus n_1n_3p_1 \cdot \overline{n_2} \\ &= n_3(n_1p_1 \cdot \overline{n_2} \oplus n_2 \cdot \overline{p_1p_2n_1}) \\ &= n_3 \cdot \overline{p_1p_2n_1n_2} \end{aligned}$$

The direct sum of A with K_2,

$$\begin{aligned} A \oplus K_2 &= p_2n_2n_3 \cdot \overline{n_1} \oplus p_1n_1n_3 \cdot \overline{n_2} \oplus n_1n_3 \cdot \overline{p_1} \oplus p_1n_1n_2n_3 \cdot \overline{p_2} \\ &= p_1n_1n_3(\overline{n_2} \oplus n_2 \cdot \overline{p_2}) \oplus n_3(p_2n_2 \cdot \overline{n_1} \oplus n_1 \cdot \overline{p_1}) \\ &= n_1n_3(\overline{p_1} \oplus p_1 \cdot \overline{n_2p_2}) \oplus n_2n_3p_2 \cdot \overline{n_1} \\ &= n_3(n_1 \cdot \overline{p_1p_2n_2} \oplus p_2n_2 \cdot \overline{n_1}) \\ &= n_3 \cdot \overline{p_1p_2n_1n_2} \end{aligned}$$

Lastly, the set $A \oplus B$ tiles the set $\overline{n}$

$$\begin{aligned} A \oplus B &= n_3 \cdot \overline{p_1p_2n_1n_2} \cup \{1\} \oplus n_3 \cdot \overline{p_1p_2n_1n_2} \cup .... \cup \{n_3 - 1\} \oplus n_3 \cdot \overline{p_1p_2n_1n_2} \\ &= \overline{p_1p_2n_1n_2n_3} \\ &= \overline{n} \end{aligned}$$

□

In the following, we define non-isomorphic canons and show that the (T2) property is preserved by isomorphisms. The cyclic group C_n is the group generated by the translation T which maps i to $i + 1 \bmod n$. This translation acts in a natural way on $\mathbb{Z}_n$ and on the set of all subsets of $\mathbb{Z}_n$ by $T(A) = \{T(i), i \in A\}$. Every subset

A of $\mathbb{Z}_n$ is identified with its characteristic function $\chi_A : \mathbb{Z}_n \to \{0,1\}$, $\chi_A(i) = 1$ if $i \in A$ and 0 otherwise. The cyclic group $\mathcal{C}_n = \langle T \rangle$ acts on the set $\{0,1\}^{\mathbb{Z}_n}$ of all functions from $\mathbb{Z}_n$ to $\{0,1\}$ by the action

$$C_n \times \{0,1\}^{\mathbb{Z}_n} \to \{0,1\}^{\mathbb{Z}_n}, \quad (T^j, f) \to f \circ T^{-j}$$

Two canons (A, B) and (A', B') of $\mathbb{Z}_n$ are C_m-isomorphic if they have the same lengths $|A| = |A'|$, $|B| = |B'|$ and if it exists a translation T^m and a permutation σ such that

$$T_m(A + b_i) = A' + b'_{\sigma(i)}$$

for $i = 1, .., |B|$. Let $I : \mathbb{Z}_n \to \mathbb{Z}_n$ be the inversion $I(i) = -i \bmod n$. The dihedral group $\mathcal{D}_n = \langle T, I \rangle$ generated by T and I also acts on the subsets of $\mathbb{Z}_n$ in the same way as $\mathcal{C}_n$ does. Two canons (A, B) and (A', B') of $\mathbb{Z}_n$ are D_m-isomorphic if they have the same lengths and if it exists a translation T^m and a permutation σ such that

$$I_m(A + b_i) = A' + b'_{\sigma(i)}$$

where $I_m = T^m \circ I$ is the commutative composition of the translation T^m with the inversion I. The affine group $\mathcal{A}_n$ generated by the operators $T_{a,b}(i) = ai + b$ acts on the subsets of $\mathbb{Z}_n$. Two canons (A, B) and (A', B') of $\mathbb{Z}_n$ are A_m-isomorphic if they have the same lengths and if the set A' is an affine transformation of A ($A' = aA + b$). Two canons are isomorphic if they are C_m, D_m or A_m-isomorphic.

Example. For $n = 72$, Sands-de Bruijn decomposition is $p_1 = n_1 = n_3 = 2$ and $p_2 = n_2 = 3$. The previous result leads to $K_1 = 6 \cdot \overline{3} \oplus 36 \cdot \overline{2} = \{0, 6, 12, 36, 42, 48\}$, $K_2 = 4 \cdot \overline{2} \oplus 24 \cdot \overline{3} = \{0, 4, 24, 28, 48, 52\}$. The computation of

$$B_1 = K_1 \cup T_1(K_2) = \{0, 1, 5, 6, 12, 25, 29, 36, 42, 48, 49, 53\}$$

and

$$A_1 = 18 \cdot \overline{2} \oplus 8 \cdot \overline{3} = \{0, 8, 16, 18, 26, 34\}$$

gives the standard aperiodic canon (A_1, B_1) for the non-Hajós group $\mathbb{Z}_{72}$. There are three non-isomorphic solutions for the inner rhythm under C_m or D_m-equivalence, namely $A_1 = \{0, 18\} \oplus \{0, 8, 16\}$, $A_2 = \{0, 18\} \oplus \{0, 8, 40\}$, $A_3 = \{0, 18\} \oplus \{0, 8, 64\}$, but there is only one non-isomorphic solution since $A_2 = 5A_1$ and $A_3 = 7A_1$. In the other hand, there are six solutions for the outer rhythm under C_m, and three solutions under D_m, namely $B_1 = \{0, 1, 5, 6, 12, 25, 29, 36, 42, 48, 49, 53\}$, $B_2 = K_1 \cup T_5(K_2) = \{0, 5, 6, 9, 12, 29, 33, 36, 42, 48, 53, 57\}$ and $B_3 = K_1 \cup T_{11}(K_2) = \{0, 6, 11, 12, 15, 35, 36, 39, 42, 48, 59, 63\}$, but there are only two non-isomorphic solutions because $B_3 = 7B_2$. Thus, we found two non-isomorphic aperiodic canons (A_1, B_1) and (A_1, B_2) for $n = 72$. The characteristic polynomial $A_1(x)$ can be expressed with cyclotomic polynomials $A_1(x) = 1 + x^8 + x^{16} + x^{26} + x^{34} = \Phi_3\Phi_4\Phi_6\Phi_{12}^2$ $\Phi_{24}\Phi_{36}(x)$. The set of indices of these factors is denoted by the multiset $T_{A_1} = \{3, 4, 6, 12, 12, 24, 36\}$. For each set B_j with $j = 1, 2, 3$, the cyclotomic factorization leads to $T_{B_j} = \{2, 8, 9, 18, 72\}$ such that $T_{A_1} \cup T_{B_j} = \mathfrak{D}_{72}$ where $\mathfrak{D}_{72}$ is the set of divisors of 72 minus $\{1\}$. For an isomorphic aperiodic canon, we have only an inclusion. The set $B' = K_1 \cup T_3(K_2) = \{0, 3, 6, 7, 12, 27, 31, 36, 42, 48, 51, 55\}$ forms an aperiodic canon (A_1, B') isomorphic to (A_1, B_2), but $\Phi_{10}(x)$ divise $B'(x)$ and $T_{B'} = T_{B_j} \cup \{10\} = \{2, 8, 9, 10, 18, 72\}$.

Proposition 5. *Let (A, B) be a canon of $\mathbb{Z}_n$. Then $A(x)$ satisfies (T2) if and only if $B(x)$ satisfies (T2).*

Proposition 6. *Let (A,B) and (A',B') two non-isomorphic canons of $\mathbb{Z}_n$, then $A(x)$ satisfies (T2) if and only if $A'(x)$ satisfies (T2).*

Proposition 7. *If (A,B) is an aperiodic canon of $\mathbb{Z}_n$, and k a nonnegative integer $k>1$, then*

$$A' = kA \oplus \{0,1,...,k-1\} \text{ and } B' = kB$$

is an aperiodic canon of $\mathbb{Z}_{kn}$.

Example. For $n=72$ and $k=2$, the sets $A=\{0,18\}\oplus\{0,8,16\}$ and $B=\{0,1,5,6,12,25,29,36,42,48,49,53\}$ form the standard aperiodic canon. Consequently, the pair $(2A,2B)$ is an aperiodic canon for $n=144$. The cyclotomic polynomials $A(x)=\Phi_3\Phi_4\Phi_6\Phi_{12}^2\Phi_{24}\Phi_{36}(x)$ and $B(x)=\Phi_2\Phi_8\Phi_9\Phi_{18}\Phi_{72}\psi(x)$ of the aperiodic canon for $n=72$ give some information on the cyclotomic polynomials for $n=144$. Using the property,

$$\Phi_s(x^p) = \begin{cases} \Phi_{ps}(x) & \text{if } p \text{ is a factor of } s \\ \Phi_s(x)\Phi_{ps}(x) & \text{if } p \text{ is not a factor of } s \end{cases} \tag{1}$$

we compute the cyclotomic polynomials of $A'(x)=A(x^2)=\Phi_3\Phi_6\Phi_8\Phi_{12}\Phi_{24}^2\Phi_{48}(x)$ and some of $B'(x)=B(x^2)=\Phi_4\Phi_{16}\Phi_9\Phi_{18}\Phi_{36}\Phi_{144}\psi(x^2)$.

Theorem 8. *Let (A,B) be a canon of $\mathbb{Z}_n$. If A is periodic then $A(x)$ satisfies (T2).*

Proof. If A is periodic, then $\mathbb{Z}_n$ is a group of Hajós. Consequently, n is of the form: p^k for $k\geq 0$, p^kq for $k\geq 1$, p^2q^2, pqr, p^2qr or $pqrs$ for distinct primes p,q,r,s. The cardinality of the set A (or B) has either two prime factors and $A(x)$ satisfies (T2) by Coven-Meyerowitz theorem, either it has three prime factors and satisfies also (T2) by Granville-Laba-Wang results. □

Theorem 9. *Let (A,B) be a canon and (A',B') an isomorphic canon. Then $A(x)$ satisfies $(T2)$ if and only if $A'(x)$ satisfies $(T2)$.*

Proof. The result is obvious if A' is a translation of A or if it is an inversion. The lemma 1.4 of [4] shows that it is also true for $A'=kA$, with $k>1$. □

Theorem 10. *If (A,B) denotes the standard aperiodic canon then $A(x)$ and $B(x)$ satisfies (T2).*

Proof. The proof is a consequence of the results given in the next section. □

Let (U,V) be an aperiodic canon and (A,B) the standard aperiodic canon. If we are able to show that

$$U(x) \text{ satisfies (T2) if and only if } A(x) \text{ satisfies (T2)}$$

then the conjecture of Coven-Meyerowitz will be true, and consequently the Flugede conjecture in one dimension.

3. Cyclotomic structure of the standard canon

From now on, we set

$$p_1 = p, \quad p_2 = q, \quad n_1 = r^{\alpha+1}, \quad n_2 = s^{\beta+1}, \quad n_3 = t^{\gamma+1}$$

We write the characteristic polynomial of the set A :

$$A(x) = \prod_{u|r^{\alpha+1}} \Phi_u\left(x^{t^{\gamma+1}s^{\beta+1}q}\right) \prod_{u|s^{\beta+1}} \Phi_u\left(x^{t^{\gamma+1}r^{\alpha+1}p}\right)$$

$$\begin{aligned} A(x) &= \Phi_r\left(x^{t^{\gamma+1}s^{\beta+1}q}\right)\Phi_{r^2}\left(x^{t^{\gamma+1}s^{\beta+1}q}\right)...\Phi_{r^{\alpha+1}}\left(x^{t^{\gamma+1}s^{\beta+1}q}\right)\cdot \\ &\quad \Phi_s\left(x^{t^{\gamma+1}r^{\alpha+1}p}\right)\Phi_{s^2}\left(x^{t^{\gamma+1}r^{\alpha+1}p}\right)...\Phi_{s^{\beta+1}}\left(x^{t^{\gamma+1}r^{\alpha+1}p}\right) \end{aligned}$$

and the characteristic polynomial $K_1(x)$ and $K_2(x)$:

$$\begin{aligned} K_1(x) &= \Phi_q\left(x^{t^{\gamma+1}s^{\beta+1}}\right)\Phi_p\left(x^{qr^{\alpha+1}s^{\beta+1}t^{\gamma+1}}\right) \\ K_2(x) &= \Phi_p\left(x^{r^{\alpha+1}t^{\gamma+1}}\right)\Phi_q\left(x^{pr^{\alpha+1}s^{\beta+1}t^{\gamma+1}}\right) \end{aligned}$$

The multisets $T_A = \{s,\, \Phi_s \mid A(x)\}$ and $T_B = \{s,\, \Phi_s \mid B(x)\}$, or

$$T_B = T_{K_1} \cap T_{K_2} \cup \{t, t^2, ..., t^{\gamma+1}\}$$

will be obtain using the property (1). Since $gcd(n_1p, n_2q) = 1$, we have necessarily

$$\begin{array}{ll} p \neq q & p \neq s \\ q \neq r & r \neq s \end{array}$$

It remains 14 cases given by (if we denote only the distinct prime numbers)

1	pqrst	8	pqrqq
2	pqrsr	9	pqpst
3	pqrsp	10	pqpsp
4	pqrsq	11	pqpsq
5	pqrqt	12	pqpqt
6	pqrqr	13	pqpqp
7	pqrqp	14	pqpqq

The verification of (T2) by each of the 14 cases leads to the previous theorem, because the result is unchanged if we replace each prime power of the form r^α by a multi-index $r_1^{\alpha_1} r_2^{\alpha_2} ... r_m^{\alpha_m}$.

• **Case 1: $\mathbf{p, q, r, s, t}$ distincts**

Proposition 11. *Let n be $n = pqn_1n_2n_3$. If $n_1 = r^{\alpha+1}$, $n_2 = s^{\beta+1}$ and $n_3 = t^{\gamma+1}$ where p,q,r,s,t are distinct primes ≥ 2, then the sets of cyclotomic polynomial indices of the standard aperiodic canon are*

$$\begin{aligned} T_A &= r\{1,q\}\{1,r,...,r^\alpha\}\{1,s,...,s^{\beta+1}\}\{1,t,...,t^{\gamma+1}\} \cup \\ &\quad s\{1,p\}\{1,r,...,r^{\alpha+1}\}\{1,s,...,s^\beta\}\{1,t,...,t^{\gamma+1}\} \end{aligned}$$

$$\begin{aligned} T_B &= \{t,t^2,...,t^{\gamma+1}\} \cup q\{1,s,...,s^{\beta+1}\}\{1,t,...,t^{\gamma+1}\} \\ &\quad \cup p\{1,r,...,r^{\alpha+1}\}\{1,t,...,t^{\gamma+1}\} \\ &\quad \cup pq\{1,r,...,r^{\alpha+1}\}\{1,s,...,s^{\beta+1}\}\{1,t,...,t^{\gamma+1}\} \end{aligned}$$

and the sets of prime powers

$$\begin{aligned} S_A &= \{r,...,r^{\alpha+1},s,...,s^{\beta+1}\} \\ S_B &= \{t,t^2,...,t^{\gamma+1},p,q\} \end{aligned}$$

satisfy(T2).

Proof. For the polynomial $K_1(x)$

$$K_1(x) = \Phi_q\left(x^{t^{\gamma+1}s^{\beta+1}}\right)\Phi_p\left(x^{qr^{\alpha+1}s^{\beta+1}t^{\gamma+1}}\right)$$

We get

$$\begin{aligned} T_{K_1} &= q\{1,s,...,s^{\beta+1}\}\{1,t,...,t^{\gamma+1}\} \cup p\{1,q\}\{1,r,...,r^{\alpha+1}\} \cdot \\ & \quad \{1,s,...,s^{\beta+1}\}\{1,t,...,t^{\gamma+1}\} \end{aligned}$$

In the same way, for $K_2(x)$

$$K_2(x) = \Phi_p\left(x^{r^{\alpha+1}t^{\gamma+1}}\right)\Phi_q\left(x^{pr^{\alpha+1}s^{\beta+1}t^{\gamma+1}}\right)$$

We have

$$\begin{aligned} T_{K_2} &= q\{1,p\}\{1,r,...,r^{\alpha+1}\}\{1,s,...,s^{\beta+1}\}\{1,t,...,t^{\gamma+1}\} \cup \\ & \quad p\{1,r,...,r^{\alpha+1}\}\{1,t,...,t^{\gamma+1}\} \end{aligned}$$

Consequently,

$$\begin{aligned} T_B &= T_{K_1} \cap T_{K_2} \cup \{t,t^2,...,t^{\gamma+1}\} \\ &= \{t,t^2,...,t^{\gamma+1}\} \cup q\{1,s,...,s^{\beta+1}\}\{1,t,...,t^{\gamma+1}\} \\ & \quad \cup p\{1,r,...,r^{\alpha+1}\}\{1,t,...,t^{\gamma+1}\} \\ & \quad \cup pq\{1,r,...,r^{\alpha+1}\}\{1,s,...,s^{\beta+1}\}\{1,t,...,t^{\gamma+1}\} \end{aligned}$$

□

• **Case 2: $\mathbf{t = r}$, and $\mathbf{p, q, r, s}$ distincts**

Proposition 12. *Let n be $n = pqn_1n_2n_3$. If $n_1 = r^{\alpha+1}$, $n_2 = s^{\beta+1}$ and $n_3 = r^{\gamma+1}$ where p, q, r, s are distinct primes ≥ 2, then the sets of cyclotomic polynomial indices of the standard aperiodic canon are*

$$\begin{aligned} T_A &= r^{\gamma+2}\{1,q\}\{1,r,...,r^{\alpha}\}\{1,s,...,s^{\beta+1}\} \cup \\ & \quad s\{1,p\}\{1,r,...,r^{\alpha+\gamma+2}\}\{1,s,...,s^{\beta}\} \end{aligned}$$

$$\begin{aligned} T_B &= \{r,r^2,...,r^{\gamma+1}\} \ \cup \ q\{1,s,...,s^{\beta+1}\}\{1,r,...,r^{\gamma+1}\} \\ & \quad \cup p\{1,r,...,r^{\alpha+1}\} \ \cup \ pq\{1,r,...,r^{\alpha+1}\}\{1,s,...,s^{\beta+1}\} \end{aligned}$$

and the sets of prime powers

$$\begin{aligned} S_A &= \{r^{\gamma+2},...,r^{\alpha+\gamma+2},s,...,s^{\beta+1}\} \\ S_B &= \{r,r^2,...,r^{\gamma+1},p,q\} \end{aligned}$$

satisfy (T2)

Proof. Replacing t by r in $A(x)$, we have

$$\begin{aligned} A(x) &= \Phi_r\left(x^{r^{\gamma+1}s^{\beta+1}q}\right)\Phi_{r^2}\left(x^{r^{\gamma+1}s^{\beta+1}q}\right)...\Phi_{r^{\alpha+1}}\left(x^{r^{\gamma+1}s^{\beta+1}q}\right) \cdot \\ & \quad \Phi_s\left(x^{r^{\gamma+1}r^{\alpha+1}p}\right)\Phi_{s^2}\left(x^{r^{\gamma+1}r^{\alpha+1}p}\right)...\Phi_{s^{\beta+1}}\left(x^{r^{\gamma+1}r^{\alpha+1}p}\right) \end{aligned}$$

Using the property (1)

$$\begin{aligned} A(x) &= \Phi_{r^{\gamma+2}}\left(x^{s^{\beta+1}q}\right)\Phi_{r^{\gamma+3}}\left(x^{s^{\beta+1}q}\right)...\Phi_{r^{\alpha+\gamma+2}}\left(x^{s^{\beta+1}q}\right) \cdot \\ & \quad \Phi_s\left(x^{r^{\alpha+\gamma+2}p}\right)\Phi_{s^2}\left(x^{r^{\alpha+\gamma+2}p}\right)...\Phi_{s^{\beta+1}}\left(x^{r^{\alpha+\gamma+2}p}\right) \end{aligned}$$

We get

$$\begin{aligned} A(x) &= \prod_{u|qs^{\beta+1}} \Phi_{ur^{\gamma+2}}(x) \prod_{u|qs^{\beta+1}} \Phi_{ur^{\gamma+3}}(x) ... \prod_{u|qs^{\beta+1}} \Phi_{ur^{\alpha+\gamma+2}}(x) \\ & \prod_{v|pr^{\alpha+\gamma+2}} \Phi_{vs}(x) \prod_{v|pr^{\alpha+\gamma+2}} \Phi_{vs^2}(x) ... \prod_{v|pr^{\alpha+\gamma+2}} \Phi_{vs^{\beta+1}}(x) \end{aligned}$$

Consequently,

$$\begin{aligned} T_A &= r^{\gamma+2}\{u,\ u \mid qr^{\alpha}s^{\beta+1}\} \cup s\{v,\ v \mid pr^{\alpha+\gamma+2}s^{\beta}\} \\ &= r^{\gamma+2}\{1,q\}\{1,r,...,r^{\alpha}\}\{1,s,...,s^{\beta+1}\} \cup \\ & \quad s\{1,p\}\{1,r,...,r^{\alpha+\gamma+2}\}\{1,s,...,s^{\beta}\} \end{aligned}$$

and

$$\begin{aligned} T_B &= \{r,r^2,...,r^{\gamma+1}\} \ \cup \ q\{1,s,...,s^{\beta+1}\}\{1,r,...,r^{\gamma+1}\} \\ & \cup p\{1,r,...,r^{\alpha+1}\} \ \cup \ pq\{1,r,...,r^{\alpha+1}\}\{1,s,...,s^{\beta+1}\} \end{aligned}$$

□

• Case 3 : t = p, and p,q,r,s distincts

Proposition 13. *Let n be $n = pqn_1n_2n_3$. If $n_1 = r^{\alpha+1}$, $n_2 = s^{\beta+1}$ and $n_3 = p^{\gamma+1}$ where p,q,r,s are distinct primes ≥ 2, then the sets of cyclotomic polynomial indices of the standard aperiodic canon are*

$$\begin{aligned} T_A &= r\{1,q\}\{1,r,...,r^{\alpha}\}\{1,s,...,s^{\beta+1}\}\{1,p,...,p^{\gamma+1}\} \ \cup \\ & \quad s\{1,p\}\{1,r,...,r^{\alpha+1}\}\{1,s,...,s^{\beta}\}\{1,p,...,p^{\gamma+1}\} \end{aligned}$$

$$\begin{aligned} T_B &= p^{\gamma+2}\{1,r,...,r^{\alpha+1}\} \cup q\{1,s,...,s^{\beta+1}\}\{1,p,...,p^{\gamma+1}\} \\ & \cup\ p^{\gamma+2}q\{1,r,...,r^{\alpha+1}\}\{1,s,...,s^{\beta+1}\} \cup \{p,...,p^{\gamma+1}\} \end{aligned}$$

and the sets of prime powers

$$\begin{aligned} S_A &= \{r,...,r^{\alpha+1},s,...,s^{\beta+1}\} \\ S_B &= \{p,...,p^{\gamma+2},q\} \end{aligned}$$

satisfy (T2)

Proof. We get

$$K_1(x) = \Phi_q\left(x^{s^{\beta+1}p^{\gamma+1}}\right)\Phi_{p^{\gamma+2}}\left(x^{qr^{\alpha+1}s^{\beta+1}}\right)$$

and

$$K_2(x) = \Phi_{p^{\gamma+2}}\left(x^{s^{\beta+1}p^{\gamma+1}}\right)\Phi_{p^{\gamma+2}}\left(x^{qr^{\alpha+1}s^{\beta+1}}\right)$$

Thus

$$T_{K_1} = q\{1,...,s^{\beta+1}\}\{1,p,...,p^{\gamma+1}\} \cup p^{\gamma+2}\{1,q\}\{1,r,...,r^{\alpha+1}\}\{1,...,s^{\beta+1}\}$$

and

$$T_{K_2} = p^{\gamma+2}\{1,r,...,r^{\alpha+1}\} \cup q\{1,p,...,p^{\gamma+2}\}\{1,r,...,r^{\alpha+1}\}\{1,...,s^{\beta+1}\}$$

Consequently,

$$\begin{aligned} T_B &= p^{\gamma+2}\{1,r,...,r^{\alpha+1}\} \cup q\{1,s,...,s^{\beta+1}\} \cup \{p,...,p^{\gamma+1}\} \\ & \cup\ p^{\gamma+2}q\{1,r,...,r^{\alpha+1}\}\{1,s,...,s^{\beta+1}\} \end{aligned}$$

□

• **Case 4: $\mathbf{t = q}$, and $\mathbf{p, q, r, s}$ distincts**

Proposition 14. *Let n be $n = pqn_1n_2n_3$. If $n_1 = r^{\alpha+1}$, $n_2 = s^{\beta+1}$ and $n_3 = q^{\gamma+1}$ where p,q,r,s are distinct primes ≥ 2, then the sets of cyclotomic polynomial indices of the standard aperiodic canon are*

$$\begin{aligned} T_A &= r\{1,q\}\{1,r,...,r^{\alpha}\}\{1,s,...,s^{\beta+1}\}\{1,q,...,q^{\gamma+1}\} \ \cup \\ & \quad s\{1,p\}\{1,r,...,r^{\alpha+1}\}\{1,s,...,s^{\beta}\}\{1,q,...,q^{\gamma+1}\} \end{aligned}$$

$$\begin{aligned} T_B &= q^{\gamma+2}\{1,s,...,s^{\beta+1}\} \cup p\{1,r,...,r^{\alpha+1}\} \cup \{q,...,q^{\gamma+1}\} \\ & \quad \cup \ pq^{\gamma+2}\{1,r,...,r^{\alpha+1}\}\{1,s,...,s^{\beta+1}\} \end{aligned}$$

and the sets of prime powers

$$\begin{aligned} S_A &= \{r,...,r^{\alpha+1},s,...,s^{\beta+1}\} \\ S_B &= \{q,...,q^{\gamma+2},p\} \end{aligned}$$

satisfy (T2)

Proof. We have

$$K_1(x) = \Phi_{q^{\gamma+2}}\left(x^{s^{\beta+1}}\right)\Phi_p\left(x^{r^{\alpha+1}s^{\beta+1}q^{\gamma+2}}\right)$$

and

$$K_2(x) = \Phi_p\left(x^{r^{\alpha+1}p^{\gamma+1}}\right)\Phi_{q^{\gamma+2}}\left(x^{pr^{\alpha+1}s^{\beta+1}}\right)$$

Thus

$$T_{K_1} = q^{\gamma+2}\{1,...,s^{\beta+1}\} \cup p\{1,r,...,r^{\alpha+1}\}\{1,...,s^{\beta+1}\}\{1,q,...,q^{\gamma+2}\}$$

and

$$T_{K_2} = p\{1,p,...,p^{\gamma+1}\}\{1,r,...,r^{\alpha+1}\} \cup q^{\gamma+2}\{1,p\}\{1,r,...,r^{\alpha+1}\}\{1,...,s^{\beta+1}\}$$

Consequently,

$$\begin{aligned} T_B &= q^{\gamma+2}\{1,s,...,s^{\beta+1}\} \cup p\{1,r,...,r^{\alpha+1}\} \cup \{q,...,q^{\gamma+1}\} \\ & \quad \cup \ pq^{\gamma+2}\{1,r,...,r^{\alpha+1}\}\{1,s,...,s^{\beta+1}\} \end{aligned}$$

□

• **Case 5: $\mathbf{s = q}$ and $\mathbf{p, q, r, t}$ distincts**

Proposition 15. *Let n be $n = pqn_1n_2n_3$. If $n_1 = r^{\alpha+1}$, $n_2 = q^{\beta+1}$ and $n_3 = t^{\gamma+1}$ where p,q,r,t are distinct primes ≥ 2, then the sets of cyclotomic polynomial indices of the standard aperiodic canon are*

$$\begin{aligned} T_A &= r\{1,r,...,r^{\alpha}\}\{1,q,...,q^{\beta+2}\}\{1,t,...,t^{\gamma+1}\} \ \cup \\ & \quad q\{1,q,...,q^{\beta}\}\{1,p\}\{1,r,...,r^{\alpha+1}\}\{1,t,...,t^{\gamma+1}\} \end{aligned}$$

$$\begin{aligned} T_B &= \{t,...,t^{\gamma+1}\} \ \cup \ q^{\beta+2}\{1,t,...,t^{\gamma+1}\} \ \cup \ p\{1,r,...,r^{\alpha+1}\}\{1,t,...,t^{\gamma+1}\} \\ & \quad \cup \ pq^{\beta+2}\{1,r,...,r^{\alpha+1}\}\{1,t,...,t^{\gamma+1}\} \end{aligned}$$

and the sets of prime powers

$$\begin{aligned} S_A &= \{r,...,r^{\alpha+1},q,...,q^{\beta+1}\} \\ S_B &= \{t,...,t^{\gamma+1},p,q^{\beta+2}\} \end{aligned}$$

satisfy (T2)

Proof. We have

$$K_1(x) = \Phi_{q^{\beta+2}}\left(x^{t^{\gamma+1}}\right)\Phi_p\left(x^{q^{\beta+2}r^{\alpha+1}t^{\gamma+1}}\right)$$

and

$$K_2(x) = \Phi_p\left(x^{r^{\alpha+1}t^{\gamma+1}}\right)\Phi_{q^{\beta+2}}\left(x^{pr^{\alpha+1}t^{\gamma+1}}\right)$$

Thus

$$T_{K_1} = q^{\beta+2}\{1,...,t^{\gamma+1}\} \ \cup \ p\{1,q,...,q^{\beta+2}\}\{1,r,...,r^{\alpha+1}\}\{1,...,t^{\gamma+1}\}$$

and

$$T_{K_2} = p\{1,r,...,r^{\alpha+1}\}\{1,...,t^{\gamma+1}\} \cup q^{\beta+2}\{1,p\}\{1,r,...,r^{\alpha+1}\}\{1,...,t^{\gamma+1}\}$$

Consequently,

$$\begin{aligned} T_B &= \{t,...,t^{\gamma+1}\} \cup q^{\beta+2}\{1,...,t^{\gamma+1}\} \ \cup \ p\{1,r,...,r^{\alpha+1}\}\{1,...,t^{\gamma+1}\} \\ &\quad \cup pq^{\beta+2}\{1,r,...,r^{\alpha+1}\}\{1,...,t^{\gamma+1}\} \end{aligned}$$

□

• Case 6: $\mathbf{t = r}$, $\mathbf{s = q}$, and $\mathbf{p, q, r}$ distincts

Proposition 16. *Let n be $n = pqn_1n_2n_3$. If $n_1 = r^{\alpha+1}$, $n_2 = q^{\beta+1}$ and $n_3 = r^{\gamma+1}$ where p,q,r are distinct primes ≥ 2, then the sets of cyclotomic polynomial indices of the standard aperiodic canon are*

$$\begin{aligned} T_A &= r^{\gamma+2}\{1,r,...,r^{\alpha}\}\{1,q,...,q^{\beta+2}\} \ \cup \\ &\quad q\{1,p\}\{1,r,...,r^{\alpha+\gamma+2}\}\{1,q,...,q^{\beta}\} \\ T_B &= q^{\beta+2}\{1,r,...,r^{\gamma+1}\} \cup p\{1,r,...,r^{\alpha+\gamma+2}\} \\ &\quad \cup pq^{\beta+2}\{1,r,...,r^{\alpha+\gamma+2}\} \cup \{r,...,r^{\gamma+1}\} \end{aligned}$$

and the sets of prime powers

$$\begin{aligned} S_A &= \{r,...,r^{\alpha+\gamma+2},q,...,q^{\beta+1}\} \\ S_B &= \{r,...,r^{\gamma+1},p,q^{\beta+2}\} \end{aligned}$$

satisfy (T2)

Proof. We have

$$K_1(x) = \Phi_{q^{\beta+2}}\left(x^{r^{\gamma+1}}\right)\Phi_p\left(x^{r^{\alpha+\gamma+2}q^{\beta+2}}\right)$$

and

$$K_2(x) = \Phi_p\left(x^{r^{\alpha+\gamma+2}}\right)\Phi_{q^{\beta+2}}\left(x^{pr^{\alpha+\gamma+2}}\right)$$

Thus

$$T_{K_1} = q^{\beta+2}\{1,...,r^{\gamma+1}\} \cup p\{1,q,...,q^{\beta+2}\}\{1,r,...,r^{\alpha+\gamma+2}\}$$

and

$$T_{K_2} = p\{1,r,...,r^{\alpha+\gamma+2}\} \cup q^{\beta+2}\{1,p\}\{1,r,...,r^{\alpha+\gamma+2}\}$$

Consequently,

$$\begin{aligned} T_B &= q^{\beta+2}\{1,r,...,r^{\gamma+1}\} \cup p\{1,r,...,r^{\alpha+\gamma+2}\} \\ &\quad \cup pq^{\beta+2}\{1,r,...,r^{\alpha+\gamma+2}\} \cup \{r,...,r^{\gamma+1}\} \end{aligned}$$

□

• **Case 7:** $\mathbf{s = q,\ t = p}$ **and** $\mathbf{p, q, r}$ **distincts**

Proposition 17. *Let* n *be* $n = pqn_1n_2n_3$*. If* $n_1 = r^{\alpha+1}$*,* $n_2 = q^{\beta+1}$ *and* $n_3 = p^{\gamma+1}$ *where p,q,r are distinct primes* ≥ 2*, then the sets of cyclotomic polynomial indices of the standard aperiodic canon are*

$$\begin{aligned} T_A &= r\{1, r, ..., r^{\alpha}\}\{1, q, ..., q^{\beta+2}\}\{1, p, ..., p^{\gamma+1}\} \ \cup \\ & \quad q\{1, q, ..., q^{\beta}\}\{1, p, ..., p^{\gamma+1}\}\{1, r, ..., r^{\alpha+1}\} \end{aligned}$$

$$\begin{aligned} T_B &= \{p, ..., p^{\gamma+1}\} \ \cup \ q^{\beta+2}\{1, p, ..., p^{\gamma+2}\} \ \cup \ p^{\gamma+2}\{1, r, ..., r^{\alpha+1}\} \\ & \quad \cup\, p^{\gamma+2}q^{\beta+2}\{1, r, ..., r^{\alpha+1}\} \end{aligned}$$

and the sets of prime powers

$$\begin{aligned} S_A &= \{r, ..., r^{\alpha+1}, q, ..., q^{\beta+1}\} \\ S_B &= \{p, ..., p^{\gamma+2}, q^{\beta+2}\} \end{aligned}$$

satisfy (T2)

Proof. We have

$$K_1(x) = \Phi_{q^{\beta+2}}\left(x^{p^{\gamma+1}}\right)\Phi_{p^{\gamma+2}}\left(x^{q^{\beta+2}r^{\alpha+1}}\right)$$

and

$$K_2(x) = \Phi_{p^{\gamma+2}}\left(x^{r^{\alpha+1}}\right)\Phi_{q^{\beta+2}}\left(x^{r^{\alpha+1}p^{\gamma+2}}\right)$$

Thus

$$T_{K_1} = q^{\beta+2}\{1, ..., p^{\gamma+1}\} \ \cup \ p^{\gamma+2}\{1, q, ..., q^{\beta+2}\}\{1, r, ..., r^{\alpha+1}\}$$

and

$$T_{K_2} = p^{\gamma+2}\{1, r, ..., r^{\alpha+1}\} \cup q^{\beta+2}\{1, r, ..., r^{\alpha+1}\}\{1, ..., p^{\gamma+2}\}$$

Consequently,

$$\begin{aligned} T_B &= \{p, ..., p^{\gamma+1}\} \ \cup \ q^{\beta+2}\{1, p, ..., p^{\gamma+2}\} \ \cup \ p^{\gamma+2}\{1, r, ..., r^{\alpha+1}\} \\ & \quad \cup\, p^{\gamma+2}q^{\beta+2}\{1, r, ..., r^{\alpha+1}\} \end{aligned}$$

□

• **Case 8:** $\mathbf{s = t = q}$**, and** $\mathbf{p, q, r}$ **distincts**

Proposition 18. *Let* n *be* $n = pqn_1n_2n_3$*. If* $n_1 = r^{\alpha+1}$*,* $n_2 = q^{\beta+1}$ *and* $n_3 = q^{\gamma+1}$ *where p,q,r are distinct primes* ≥ 2*, then the sets of cyclotomic polynomial indices of the standard aperiodic canon are*

$$\begin{aligned} T_A &= r\{1, r, ..., r^{\alpha}\}\{1, q, ..., q^{\beta+\gamma+3}\} \ \cup \\ & \quad q^{\gamma+2}\{1, q, ..., q^{\beta}\}\{1, p\}\{1, r, ..., r^{\alpha+1}\} \end{aligned}$$

$$\begin{aligned} T_B &= \{q, ..., q^{\gamma+1}\} \ \cup \ \{q^{\beta+\gamma+3}\} \ \cup \ p\{1, r, ..., r^{\alpha+1}\}\{1, q, ..., q^{\gamma+1}\} \\ & \quad \cup\, pq^{\beta+\gamma+3}\{1, r, ..., r^{\alpha+1}\} \end{aligned}$$

and the sets of prime powers

$$\begin{aligned} S_A &= \{r, ..., r^{\alpha+1}, q^{\gamma+2}, ..., q^{\beta+\gamma+2}\} \\ S_B &= \{q, ..., q^{\gamma+1}, q^{\beta+\gamma+3}, p\} \end{aligned}$$

satisfy (T2)

Proof. We have

$$K_1(x) = \Phi_{q^{\beta+\gamma+3}}(x)\,\Phi_p\left(x^{q^{\beta+\gamma+3}r^{\alpha+1}}\right)$$

and

$$K_2(x) = \Phi_p\left(x^{r^{\alpha+1}q^{\gamma+1}}\right)\Phi_{q^{\beta+\gamma+3}}\left(x^{pr^{\alpha+1}}\right)$$

Thus

$$T_{K_1} = \{q^{\beta+\gamma+3}\} \ \cup\ p\{1, q, ..., q^{\beta+\gamma+3}\}\{1, r, ..., r^{\alpha+1}\}$$

and

$$T_{K_2} = p\{1, r, ..., r^{\alpha+1}\}\{1, q, ..., q^{\gamma+1}\} \cup q^{\beta+\gamma+3}\{1, p\}\{1, r, ..., r^{\alpha+1}\}$$

Consequently,

$$\begin{aligned} T_B &= \{q, ..., q^{\gamma+1}\} \ \cup\ \{q^{\beta+\gamma+3}\} \ \cup\ p\{1, r, ..., r^{\alpha+1}\}\{1, q, ..., q^{\gamma+1}\} \\ &\quad \cup pq^{\beta+\gamma+3}\{1, r, ..., r^{\alpha+1}\} \end{aligned}$$

□

• **Case 9: $\mathbf{r = p}$, and $\mathbf{p, q, s, t}$ distincts**

Proposition 19. *Let n be $n = pqn_1n_2n_3$. If $n_1 = p^{\alpha+1}$, $n_2 = s^{\beta+1}$ and $n_3 = t^{\gamma+1}$ where p,q,s,t are distinct primes ≥ 2, then the sets of cyclotomic polynomial indices of the standard aperiodic canon are*

$$\begin{aligned} T_A &= p\{1, q\}\{1, p, ..., p^{\alpha}\}\{1, s, ..., s^{\beta+1}\}\{1, t, ..., t^{\gamma+1}\} \ \cup \\ &\quad s\{1, p, ..., p^{\alpha+2}\}\{1, s, ..., s^{\beta}\}\{1, t, ..., t^{\gamma+1}\} \end{aligned}$$

$$\begin{aligned} T_B &= \{t, ..., t^{\gamma+1}\} \ \cup\ q\{1, s, ..., s^{\beta+1}\}\{1, t, ..., t^{\gamma+1}\} \\ &\quad \cup p^{\alpha+2}\{1, t, ..., t^{\alpha+1}\} \cup p^{\alpha+2}q\{1, s, ..., s^{\beta+1}\}\{1, t, ..., t^{\gamma+1}\} \end{aligned}$$

and the sets of prime powers

$$\begin{aligned} S_A &= \{p, ..., p^{\alpha+1}, s, ..., s^{\beta+1}\} \\ S_B &= \{t, ..., t^{\gamma+1}, p^{\alpha+2}, q\} \end{aligned}$$

satisfy (T2)

Proof. We have

$$K_1(x) = \Phi_q\left(x^{s^{\beta+1}t^{\gamma+1}}\right)\Phi_{p^{\alpha+2}}\left(x^{qs^{\beta+1}t^{\gamma+1}}\right)$$

and

$$K_2(x) = \Phi_{p^{\alpha+2}}\left(x^{t^{\gamma+1}}\right)\Phi_q\left(x^{p^{\alpha+2}s^{\beta+1}t^{\gamma+1}}\right)$$

Thus

$$T_{K_1} = q\{1, ..., s^{\beta+1}\}\{1, t, ..., t^{\gamma+1}\} \cup p^{\alpha+2}\{1, q\}\{1, ..., s^{\beta+1}\}\{1, t, ..., t^{\gamma+1}\}$$

and

$$T_{K_2} = p^{\alpha+2}\{1, t, ..., t^{\gamma+1}\} \cup q\{1, p, ..., p^{\alpha+2}\}\{1, ..., s^{\beta+1}\}\{1, t, ..., t^{\gamma+1}\}$$

Consequently,

$$\begin{aligned} T_B &= \{t, ..., t^{\gamma+1}\} \ \cup\ q\{1, s, ..., s^{\beta+1}\}\{1, t, ..., t^{\gamma+1}\} \\ &\quad \cup p^{\alpha+2}\{1, t, ..., t^{\alpha+1}\} \cup p^{\alpha+2}q\{1, s, ..., s^{\beta+1}\}\{1, t, ..., t^{\gamma+1}\} \end{aligned}$$

□

• **Case 10: $\mathbf{r = t = p}$, and $\mathbf{p, q, s}$ distincts**

Proposition 20. *Let n be $n = pqn_1n_2n_3$. If $n_1 = p^{\alpha+1}$, $n_2 = s^{\beta+1}$ and $n_3 = p^{\gamma+1}$ where p,q,s,t are distinct primes ≥ 2, then the sets of cyclotomic polynomial indices of the standard aperiodic canon are*

$$\begin{aligned} T_A &= p^{\gamma+2}\{1, q\}\{1, p, ..., p^{\alpha}\}\{1, s, ..., s^{\beta+1}\} \cup \\ & \quad s\{1, p, ..., p^{\alpha+\gamma+3}\}\{1, s, ..., s^{\beta}\} \end{aligned}$$

$$\begin{aligned} T_B &= \{p, ..., p^{\gamma+1}\} \cup \{p^{\alpha+\gamma+3}\} \cup q\{1, s, ..., s^{\beta+1}\}\{1, p, ..., p^{\gamma+1}\} \\ & \cup p^{\alpha+\gamma+3}q\{1, s, ..., s^{\beta+1}\} \end{aligned}$$

and the sets of prime powers

$$\begin{aligned} S_A &= \{p^{\gamma+2}, ..., p^{\alpha+\gamma+2}, s, ..., s^{\beta+1}\} \\ S_B &= \{p, ..., p^{\gamma+1}, p^{\alpha+\gamma+3}, q\} \end{aligned}$$

satisfy (T2)

Proof. We have

$$K_1(x) = \Phi_q\left(x^{s^{\beta+1}p^{\gamma+1}}\right)\Phi_{p^{\alpha+\gamma+3}}\left(x^{qs^{\beta+1}}\right)$$

and

$$K_2(x) = \Phi_{p^{\alpha+\gamma+3}}(x)\,\Phi_q\left(x^{p^{\alpha+\gamma+3}s^{\beta+1}}\right)$$

Thus

$$T_{K_1} = q\{1, ..., s^{\beta+1}\}\{1, p, ..., p^{\gamma+1}\} \cup p^{\alpha+\gamma+3}\{1, q\}\{1, ..., s^{\beta+1}\}$$

and

$$T_{K_2} = \{p^{\alpha+\gamma+3}\} \cup q\{1, p, ..., p^{\alpha+\gamma+3}\}\{1, ..., s^{\beta+1}\}$$

Consequently,

$$\begin{aligned} T_B &= \{p, ..., p^{\gamma+1}\} \cup \{p^{\alpha+\gamma+3}\} \cup q\{1, s, ..., s^{\beta+1}\}\{1, p, ..., p^{\gamma+1}\} \\ & \cup p^{\alpha+\gamma+3}q\{1, s, ..., s^{\beta+1}\} \end{aligned}$$

□

• **Case 11: $\mathbf{r = p}$, $\mathbf{t = q}$ and $\mathbf{p, q, s}$ distincts**

Proposition 21. *Let n be $n = pqn_1n_2n_3$. If $n_1 = p^{\alpha+1}$, $n_2 = s^{\beta+1}$ and $n_3 = q^{\gamma+1}$ where p,q,s are distinct primes ≥ 2, then the sets of cyclotomic polynomial indices of the standard aperiodic canon are*

$$\begin{aligned} T_A &= p\{1, p, ..., p^{\alpha}\}\{1, q, ..., q^{\gamma+2}\}\{1, s, ..., s^{\beta+1}\} \cup \\ & \quad s\{1, p, ..., p^{\alpha+2}\}\{1, s, ..., s^{\beta}\}\{1, q, ..., q^{\gamma+1}\} \end{aligned}$$

$$\begin{aligned} T_B &= \{q, ..., q^{\gamma+1}\} \cup q^{\gamma+2}\{1, s, ..., s^{\beta+1}\} \cup p^{\alpha+2}\{1, q, ..., q^{\gamma+1}\} \\ & \cup p^{\alpha+2}q^{\gamma+2}\{1, s, ..., s^{\beta+1}\} \end{aligned}$$

and the sets of prime powers

$$\begin{aligned} S_A &= \{p, ..., p^{\alpha+1}, s, ..., s^{\beta+1}\} \\ S_B &= \{q, ..., q^{\gamma+2}, p^{\alpha+2}\} \end{aligned}$$

satisfy (T2)

Proof. We have

$$K_1(x) = \Phi_{q^{\gamma+2}}\left(x^{s^{\beta+1}}\right)\Phi_{p^{\alpha+2}}\left(x^{s^{\beta+1}q^{\gamma+2}}\right)$$

and

$$K_2(x) = \Phi_{p^{\alpha+2}}\left(x^{q^{\gamma+1}}\right)\Phi_{q^{\gamma+2}}\left(x^{p^{\alpha+2}s^{\beta+1}}\right)$$

Thus

$$T_{K_1} = q^{\gamma+2}\{1,...,s^{\beta+1}\} \cup p^{\alpha+2}\{1,...,s^{\beta+1}\}\{1,q,...,q^{\gamma+2}\}$$

and

$$T_{K_2} = p^{\alpha+2}\{1,q,...,q^{\gamma+1}\} \cup q^{\gamma+2}\{1,p,...,p^{\alpha+2}\}\{1,...,s^{\beta+1}\}$$

Consequently,

$$\begin{aligned} T_B \;=\; & \{q,...,q^{\gamma+1}\} \;\cup\; q^{\gamma+2}\{1,s,...,s^{\beta+1}\} \;\cup\; p^{\alpha+2}\{1,q,...,q^{\gamma+1}\} \\ & \cup\, p^{\alpha+2}q^{\gamma+2}\{1,s,...,s^{\beta+1}\} \end{aligned}$$

□

- **Case 12: $\mathbf{r = p}$, $\mathbf{s = q}$ and $\mathbf{p, q, t}$ distincts**

Proposition 22. *Let n be $n = pqn_1n_2n_3$. If $n_1 = p^{\alpha+1}$, $n_2 = q^{\beta+1}$ and $n_3 = t^{\gamma+1}$ where p,q,t are distinct primes ≥ 2, then the sets of cyclotomic polynomial indices of the standard aperiodic canon are*

$$\begin{aligned} T_A \;=\; & p\{1,p,...,p^{\alpha}\}\{1,q,...,q^{\beta+2}\}\{1,t,...,t^{\gamma+1}\} \;\cup \\ & q\{1,p,...,p^{\alpha+2}\}\{1,q,...,q^{\beta}\}\{1,t,...,t^{\gamma+1}\} \end{aligned}$$

$$\begin{aligned} T_B \;=\; & \{t,...,t^{\gamma+1}\} \;\cup\; q^{\beta+2}\{1,t,...,t^{\gamma+1}\} \;\cup \\ & \cup\, p^{\alpha+2}\{1,t,...,t^{\gamma+1}\} \cup p^{\alpha+2}q^{\beta+2}\{1,t,...,t^{\gamma+1}\} \end{aligned}$$

and the sets of prime powers

$$\begin{aligned} S_A &= \{p,...,p^{\alpha+1},q,...,q^{\beta+1}\} \\ S_B &= \{t,...,t^{\gamma+1},p^{\alpha+2},q^{\beta+2}\} \end{aligned}$$

satisfy (T2)

Proof. We have

$$K_1(x) = \Phi_{q^{\beta+2}}\left(x^{t^{\gamma+1}}\right)\Phi_{p^{\alpha+2}}\left(x^{q^{\beta+2}t^{\gamma+1}}\right)$$

and

$$K_2(x) = \Phi_{p^{\alpha+2}}\left(x^{t^{\gamma+1}}\right)\Phi_{q^{\beta+2}}\left(x^{p^{\alpha+2}t^{\gamma+1}}\right)$$

Consequently,

$$T_{K_1} = q^{\beta+2}\{1,t,...,t^{\gamma+1}\} \cup p^{\alpha+2}\{1,...,q^{\beta+2}\}\{1,t,...,t^{\gamma+1}\}$$

and

$$T_{K_2} = p^{\alpha+2}\{1,t,...,t^{\gamma+1}\} \cup q^{\beta+2}\{1,p,...,p^{\alpha+2}\}\{1,t,...,t^{\gamma+1}\}$$

Thus

$$\begin{aligned} T_B \;=\; & \{t,...,t^{\gamma+1}\} \;\cup\; q^{\beta+2}\{1,t,...,t^{\gamma+1}\} \;\cup \\ & \cup\, p^{\alpha+2}\{1,t,...,t^{\gamma+1}\} \cup p^{\alpha+2}q^{\beta+2}\{1,t,...,t^{\gamma+1}\} \end{aligned}$$

□

• **Case 13: $\mathbf{r = p}$, $\mathbf{s = q}$, $\mathbf{t = p}$, and $\mathbf{p, q}$ distincts**

Proposition 23. *Let n be $n = pqn_1n_2n_3$. If $n_1 = p^{\alpha+1}$, $n_2 = q^{\beta+1}$ and $n_3 = p^{\gamma+1}$ where p,q are distinct primes ≥ 2, then the sets of cyclotomic polynomial indices of the standard aperiodic canon are*

$$\begin{array}{rcl} T_A & = & p^{\gamma+2}\{1,...,p^{\alpha}\}\{1,q,...,q^{\beta+2}\} \ \cup \ q\{1,q,...,q^{\beta}\}\{1,...,p^{\alpha+\gamma+3}\} \\ T_B & = & \{p,...,p^{\gamma+1}\} \ \cup \ \{p^{\alpha+\gamma+3}\} \ \cup \ q^{\beta+2}\{1,p,...,p^{\gamma+1}\} \ \cup \ \{q^{\beta+2}p^{\alpha+\gamma+3}\} \end{array}$$

and the sets of prime powers

$$\begin{array}{rcl} S_A & = & \{p^{\gamma+2},...,p^{\alpha+\gamma+2},q,...,q^{\beta+1}\} \\ S_B & = & \{p,...,p^{\gamma+1},p^{\alpha+\gamma+3},q^{\beta+2}\} \end{array}$$

satisfy (T2)

Proof. We have

$$K_1(x) = \Phi_{q^{\beta+2}}\left(x^{p^{\gamma+1}}\right)\Phi_{p^{\alpha+\gamma+3}}\left(x^{q^{\beta+2}}\right)$$

and

$$K_2(x) = \Phi_{p^{\alpha+\gamma+3}}(x)\,\Phi_{q^{\beta+2}}\left(x^{p^{\alpha+\gamma+3}}\right)$$

Thus

$$T_{K_1} = q^{\beta+2}\{1,p,...,p^{\gamma+1}\} \cup p^{\alpha+\gamma+3}\{1,...,q^{\beta+2}\}$$

and

$$T_{K_2} = \{p^{\alpha+\gamma+3}\} \cup q^{\beta+2}\{1,p,...,p^{\alpha+\gamma+3}\}$$

□

Remark. For $\alpha = \beta = \gamma = 0$, we find the sets $T_A = p^2\{1,q,q^2\} \ \cup \ q\{1,p,p^2,p^3\}$ and $T_B = p \ \cup \ p^3 \ \cup \ q^2\{1,p\} \ \cup \ q^2p^3$. For $n = 72$, $p = 2$, $q = 3$ we recover the example of the previous section.

• **Case 14: $\mathbf{r = p}$, $\mathbf{s = t = q}$, and $\mathbf{p, q}$ distincts**

Proposition 24. *Let n be $n = pqn_1n_2n_3$. If $n_1 = p^{\alpha+1}$, $n_2 = q^{\beta+1}$ and $n_3 = q^{\gamma+1}$ where p,q are distinct primes ≥ 2, then the sets of cyclotomic polynomial indices of the standard aperiodic canon are*

$$\begin{array}{rcl} T_A & = & p\{1,...,p^{\alpha}\}\{1,q,...,q^{\beta+\gamma+3}\} \ \cup \ q^{\gamma+2}\{1,q,...,q^{\beta}\}\{1,...,p^{\alpha+2}\} \\ T_B & = & \{q,...,q^{\gamma+1}\} \ \cup \ \{q^{\beta+\gamma+3}\} \ \cup \ p^{\alpha+2}\{1,q,...,q^{\gamma+1}\} \ \cup \ \{q^{\beta+\gamma+3}p^{\alpha++2}\} \end{array}$$

and the sets of prime powers

$$\begin{array}{rcl} S_A & = & \{p,...,p^{\alpha+1},q^{\gamma+2},...,q^{\beta+\gamma+2}\} \\ S_B & = & \{q,...,q^{\gamma+1},p^{\alpha+2},q^{\beta+\gamma+3}\} \end{array}$$

satisfy (T2)

Proof. We have

$$K_1(x) = \Phi_{q^{\beta+\gamma+3}}(x)\,\Phi_{p^{\alpha+2}}\left(x^{q^{\beta+\gamma+3}}\right)$$

and

$$K_2(x) = \Phi_{p^{\alpha+2}}\left(x^{q^{\gamma+1}}\right)\Phi_{q^{\beta+\gamma+3}}\left(x^{p^{\alpha+2}}\right)$$

Thus

$$T_{K_1} = \{q^{\beta+\gamma+3}\} \cup p^{\alpha+2}\{1,...,q^{\beta+\gamma+3}\}$$

and

$$T_{K_2} = p^{\alpha+2}\{1, ..., q^{\gamma+1}\} \cup q^{\beta+\gamma+3}\{1, p, ..., p^{\alpha+2}\}$$

□

4. Some other tiles

Lagarias and Szabó [10] proposed the following aperiodic canon as a counterexample to Tijdeman's conjecture.

$$\begin{aligned} U &= 36\{0,1,..,4\} \oplus 100\{0,1,2\} \oplus 225\{0,1\} \\ V &= \{0,30,60,126,180,210,220,240,300,306, \\ &\quad 330,360,375,390,480,486,510,520,540,570, \\ &\quad 660,666,690,750,780,820,825,840,846,870\} \end{aligned}$$

The indices of the cyclotomic polynomials are

$$\begin{aligned} T_U &= \{2,3,5,6,6,10,10,12,15,15,18,20,30,30,30,45,50, \\ &\quad 60,60,75,90,90,150,150,180,300,450\} \\ T_V &= \{4,9,25,36,100,225,900\} \end{aligned}$$

The polynomial $V(x) = \Phi_4\Phi_9\Phi_{25}\Phi_{36}\Phi_{100}\Phi_{225}\Phi_{900}\psi(x)$ has a complicated remainder $\psi(x) = 1 - x^2 - x^3 + ... - x^{427} - x^{428} + x^{430}$. The standard aperiodic canon corresponds to the case 12 ($pqpqt$) with $\alpha = \beta = 0$, $\gamma = 1$, $t = 5$, $p = 2$, $q = 3$ (or $p = 3$, $q = 2$), $A = 100\{0,1,2\} \oplus 225\{0,1\}$, $B = K_1 \cup T_1(K_2) \cup ... \cup T_{24}(K_2)$, $K_1 = 75\{0,1,2\} \oplus 450\{0,1\}$ and $K_2 = 36\{0,1,2\} \oplus 6\{0,1\}$. For $K_3 = 36\{0,1,...,4\}$, the characteristic polynomial leads to the following set $T_{K_3} = \{5,10,15,20,30,45,60,90,180\} = 5\{1,2,4\}\{1,3,9\}$. The set T_A and T_B are linked to the sets T_U and T_V by

$$\begin{aligned} T_A &= \{2,3,6,6,10,12,15,18,30,30,50,60,75,90,150,150,300,450\} \\ &= T_U \setminus T_{K_3} \end{aligned}$$

and

$$\begin{aligned} T_B &= \{4,\underline{5},9,\underline{20},25,36,\underline{45},100,\underline{180},225,900\} \\ &= T_V \ \cup \ \{5,20,45,180\} = T_V \ \cup \ 5\{1,4\}\{1,9\} \end{aligned}$$

The cyclotomic structure of the solution of Lagarias-Szabó is different from the cyclotomic structure of the standard aperiodic canon, but closely linked to it.

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