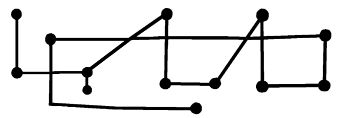
Constructing Rhythmic Canons



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1. Introduction

Recently a number of advanced mathematical models for music analysis and composition have appeared (e.g., Mazzola 2002). In particular, Vuza (1991–3, 1995) has developed a pioneering model for finding rhythms with special properties. The requirements for these rhythms were formulated in part by Vieru (1993).

Vieru's and Vuza's goal was to transfer Messiaen's (1944) *modes of limited transposition* to the domain of rhythm.[1] Recall that Messiaen considered a set of disjoint pitch classes with the same interval content which covers the twelve-tone tempered scale. For instance, four pitch classes {C, E♭, F♯, A} and two transpositions, by one and by two semitones, cover the twelve-tone scale and, consequently, meet this requirement. This is similar to what is called in mathematics *tiling*, that is, covering an area, e.g., a square, by disjoint equal figures.

Instead of the tempered scale, Vieru and Vuza considered a regular pulse train. By analogy with covering the scale by a few pitch classes and their transpositions, the pulse train was covered by a certain rhythmic pattern with different delays. The disjointedness of pitch classes implied no common beats in different instances of the rhythmic pattern. The circularity of pitch classes (= octave periodicity) corresponded to circular time (= beats in a measure).

Vieru and Vuza intended such "rhythms of limited transposition," or, better, "rhythms of limited delay," for constructing unending (= infinite, periodic) canons. Recall that a *canon* is a polyphonic piece whose voices lead the same melody with different delays. A *rhythmic canon* is one whose tone onsets result in a regular pulse train with no simultaneous tone onsets at a time. In that sense, a rhythmic canon tiles time, covering a regular pulse train by disjoint equal rhythms from different voices. Note that the established term "rhythmic canon" is somewhat misleading, and "disjoint rhythm canon" might be more exact.

From a musical standpoint, time-tiling is a technique of making polyphonic pieces from a single rhythmic/melodic pattern. It meets the principle of economy in both classical and twentieth-century music: recall long phrases built from the opening four-note motive in Beethoven's Fifth Symphony, twelve-tone composition, etc. On the other hand, in rhythmic canons the independence of the voices is maximal, since no two tones occur simultaneously, which is much appreciated in polyphony.

It is not surprising that time-tiling attracted the attention of music theorists (Amiot 2002–3), Andreatta et al. 2001, Fripertinger 2002, 2003). It turned out, however, that solutions to the time-tiling problem are mainly trivial and musically not interesting. A typical solution is a metronome rhythm entering with equal delays, e.g., a sequence of every fourth beat, entering at the first, at the second, and at the third beat, which is a rhythm analogy of the transpositions of pitch class {C, E♭, F♯, A}. Non-trivial solutions have been found by Vuza for a circular time with periods 72, 108, 120, . . ., meeting some factorization requirements.

As one can imagine, these solutions result in overcomplicated musical structures which are hard to hear as such. The effect is similar to the one in serial music, as described by Xenakis (1963):

> Linear polyphony destroys itself by its very complexity; what one hears is in reality nothing but a mass of notes in various registers. The enormous complexity prevents the audience from following the intertwining of the lines and has as its macroscopic effect an irrational and fortuitous dispersion of sounds over the whole extent of the

> sonic spectrum. There is consequently a contradiction between the polyphonic linear system and the heard result, which is surface or mass.[2]

Johnson (2001) considered the time-tiling problem in a less strict way. In addition to a given rhythmic pattern he also used its augmentation, that is, the pattern with double durations, like in Bach's *The Art of the Fugue*. He heuristically constructed a simple finite canon (as opposed to unending canon) and asked for the existence of other solutions. Vuza's method was, however, adaptable neither to using augmentations of the theme, nor to linear time (as opposed to circular time).

This paper provides a numerical solution to the general problem. It introduces an algorithm for constructing rhythmic canons from several rhythmic patterns, in particular, from successive augmentations of the theme. As for an analytical solution, it is shown that the problem is equivalent to solving Diophantine equations in special polynomials (the dates of Diophante's life are not known exactly and are estimated 325–409 AD). For this purpose an isomorphism between rhythmic canons and these polynomials is established. Finally, an application of the method to algorithmic composition is described.

In Section 2, "Problem formulation," basic assumptions are introduced and illustrated with an example.

In Section 3, "Polynomial representation of rhythmic canons and some implications," an isomorphism is established between rhythms and 0–1 polynomials, that is, whose coefficients are zeros and ones, the same as for representing the structure of sound spectra (Tangian, formerly spelled Tanguiane, 1993, 1995, 2001). Then the problem of constructing rhythmic canons is reformulated as finding sums of products of 0–1 polynomials, which is analogous to Diophantine equations in 0–1 polynomials. Since no general solution is known for Diophantine equations already in integers, there is little hope to solve them in polynomials (polynomials generalize integers, containing them as polynomials of degree 0). Respectively, the question of analytically constructing rhythmic canons remains open.

Section 4, "Algorithm for constructing rhythmic canons," introduces a coding convention for rhythmic canons with no redundancy, and an enumeration algorithm. Its idea is similar to that of the sieve of Eratosthene (284–192 BC) for finding prime numbers. Some details on the algorithm implementation and processing are provided.

Section 5, "Example of application," describes the use of computer output for making the musical piece *Eine kleine Mathmusik*.

In Section 6, "Generalizations," some further extensions of the model are outlined, as using several basic patterns instead of one, fitting the patterns to a user-defined pulse train, or allowing simultaneous tone onsets.

In Section 7, "Summary," the main results of the paper are recapitulated.

The Appendix contains the rhythmic scores computed which were used in the composition of *Eine kleine Mathmusik*.

2. Problem Formulation

Consider Johnson's (2001) rhythm and its coding by zeros and ones with respect to a pulse train of sixteenths:

1 1 0 0 1

We are going to build rhythmic canons from this pattern and its augmentations shown in Example 1.

Pattern number	Musical Meaning	Progression of tone onsets and empty beats
1	Theme	11001
2	Theme in augmentation	101000001
3	Theme in double augmentation	10001000000000001

EXAMPLE 1: THREE RHYTHMIC PATTERNS CODED BY ONES AND ZEROS

To provide a homogeneous pulse train required in rhythmic canons, assume the following:

Assumption 1 (no gap). *The composite tone onsets result in a regular pulse (= no simultaneous zeros in all the voices).*

Assumption 2 (no double beat). *No tone onset occurs simultaneously in two or more voices (= no simultaneous ones in any of two voices).*

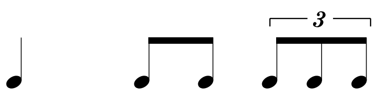
Example 2 depicts the score of a rhythmic canon (that is, the one which satisfies both assumptions).

Voice number	Pattern number	Beat number														
		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	1	1	0	0	1	.	.	.	.	.	.	.	.	.	.
2	1	.	.	1	1	0	0	1	.	.	.	.	.	.	.	.
3	2	.	.	.	.	.	1	0	1	0	0	0	0	0	1	0
4	1	.	.	.	.	.	.	.	.	1	1	0	0	1	.	.
5	1	.	.	.	.	.	.	.	.	.	.	1	1	0	0	1
Simultaneous onsets (pulse train)		1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

Example 2: A score of rhythmic canon 11211

The canon code "11211" is the succession of patterns as they enter in the canon given in the second column of the table in Example 2. In the score, ones are tone onsets, zeros denote sustained tones (tied notes), or, if the composer elects, sixteenth rests within the pattern, and periods denote sixteenth rests outside the pattern.

Coding a rhythm by a sequence of zeros and ones is feasible for all notationable rhythms, provided the reference pulse train is sufficiently dense, being a common divisor of the durations considered. For instance, a quarter note, two eighths, and three eighth triplets can be coded as follows.



1 0 0 0 0 0 1 0 0 1 0 0 1 0 1 0 1 0

3. Polynomial Representation of Rhythmic Canons and Some Implications

Define an isomorphism between rhythms and polynomials with coefficients 0 or 1. To be specific, represent the first pattern from Example 1 as follows:

$$P = 1\ 1\ 0\ 0\ 1 \longleftrightarrow p(x) = 1 + 1x + 0x^2 + 0x^3 + 1x^4.$$

If pattern P delays by 2 beats as in the second voice in Example 2, multiply $p(x)$ by x^2:

$$P_2 = 0\ 0\ 1\ 1\ 0\ 0\ 1 \longleftrightarrow p(x)x^2 = 0 + 0x + 1x^2 + 1x^3 + 0x^4 + 0x^5 + 1x^6.$$

No shift corresponds to the multiplication of $p(x)$ by the polynomial unit 1.

Generally speaking, if P is a rhythmic pattern represented by polynomial $p(x)$ then its delay P_k by k beats is represented by polynomial $p(x)x^k$:

$$P_k \longleftrightarrow p(x)x^k.$$

A superposition of rhythmic patterns corresponds to the sum of the associated polynomials. For instance, the superposition of P and P_2:

$$P + P_2 = 1\ 1\ 1\ 1\ 1\ 0\ 1 \longleftrightarrow p(x) + p(x)x^2 = p(x)(1 + x^2).$$

A double beat results in a coefficient 2 instead of 1 for a single beat:

$$P + P_3 = 1\ 1\ 0\ 1\ 2\ 0\ 0\ 1 \longleftrightarrow p(x) + p(x)x^3 = p(x)(1 + x^3).$$

Multiple superpositions of $P \leftrightarrow p(x)$ with delays correspond to polynomial products $p(x)q(x)$, where $q(x)$ represents multiple time delays. For instance, the superposition of P with delays by 2, 8, and 10 beats (sum of voices 1, 2, 4, and 5 in Example 2) corresponds to

$$p(x)q(x), \text{ where } q(x) = 1 + x^2 + x^8 + x^{10}.$$

Let voice delays in a rhythmic canon generated by pattern $P \leftrightarrow p(x)$ be represented by polynomial $q(x)$. Assumptions 1–2 mean that

$$p(x)q(x) = I_n(x) = \sum_{i=0}^{n} x^n, \qquad (1)$$

where n is the sum of degrees of $p(x)$ and $q(x)$. In this case, the length of the canon is $n + 1$ beats.

PROPOSITION 1 (Existence and Uniqueness of a Rhythmic Canon). *A rhythmic canon generated by pattern* $\mathrm{P} \leftrightarrow \mathrm{p(x)}$ *can be* $\mathrm{n} + 1$ *beats long if and only if there exists a polynomial* $\mathrm{q(x)}$ *with coefficients* 0 *or* 1, *satisfying formula* (1). *If such a canon exists, it is unique to within permutation and union of voices.*

Proof. Indeed, if such a canon exists, the polynomial $I_n(x)$ from formula (1) is divisible by $p(x)$, and the result of the division, that is, some polynomial $q(x)$, is unique (Van der Waerden 1931). It means that the beats of entries of pattern P are uniquely determined, and the only freedom left is how to assign the patterns to voices. Q.E.D.

The reservation "unique to within permutation and union of voices" in Proposition 1 means that canons are considered equivalent if we (a) renumber the voices, or (b) reduce the number of voices by putting disjoint rhythmic patterns into the same voice. For instance, five voices in Example 2 can be reduced to three voices by uniting the voices 1 with 3 and 2 with 5.

Now note that the j^{th} augmentation $P^{(j)}$ of pattern P corresponds to the polynomial

$$P^{(j)} \longleftrightarrow p(x^{2^j}).$$

For instance, the augmentations from Example 1 correspond to the polynomials

$$\text{First augmentation} \longleftrightarrow p(x^2) = 1 + x^2 + x^8$$
$$\text{Second augmentation} \longleftrightarrow p(x^4) = 1 + x^4 + x^{16}.$$

Consequently, a rhythmic canon built from the rhythmic "theme" P and its two successive augmentations must satisfy the polynomial equation

$$p(x)q(x) + p(x^2)q_1(x) + p(x^4)q_2(x) = I_n(x), \qquad (2)$$

where polynomial $q_j(x)$ is associated with entry delays of the j^{th} augmentation. For example, the canon in Example 2 satisfies equation (2) for the following polynomials:

$$q(x) = 1 + x^2 + x^8 + x^{10}$$

$$q_1(x) = x^5$$
$$q_2(x) = 0$$
$$I_n(x) = 1 + x + \ldots + x^{14}.$$

Unlike (1), where the uniqueness of polynomial factorization implies the uniqueness of solution $q(x)$ (if it exists), we expect no uniqueness of a solution $q(x)$, $q_1(x)$, $q_2(x)$ to (2).

The isomorphism between rhythms and 0–1 polynomials is useful in analyzing properties of rhythmic canons. In particular, it enables to estimate the difficulties in finding a general analytical solution of the problem considered.

Note that polynomial classes inherit some properties of the number classes used for their coefficients (one can consider polynomials with integer coefficients, or rational coefficients, or real coefficients, etc.):

- Polynomials include numbers as polynomials of degree 0.
- Addition, subtraction, multiplication, and division are defined for polynomials.
- The division properties of polynomials are similar to those of real numbers, with the unique factorization into irreducible polynomials, which are analogous to primes.

From this standpoint, equation (2) is a polynomial version of the Diophantine equation

$$pq + p_1q_1 + p_2q_2 = I$$

with positive integer coefficients p, p_1, p_2, I to be solved in positive integers q, q_1, q_2. For instance, the Diophantine equation

$$5q + 7q_1 = 100 \quad (3)$$

has two solutions, (6, 10), and (13, 5).

The existence of a general analytical solution (with a formula) to (2) would mean the existence of an analytical solution to much more simple Diophantine equations in integers. Since no general solution to Diophantine equations is known, there is little hope to solve more general Diophantine equations for polynomials.[3]

4. Algorithm for Constructing Rhythmic Canons

An appropriate coding convention is often a ladder to success in combinations. Such a coding convention can imply an enumeration algorithm with the fewest parameters.

Proposition 2 (Coding Convention). *Under Assumptions* 1 *and* 2, *a rhythmic canon coded by a succession of entering rhythmic patterns is unique to within permutation and union of voices.*

Proof. The succession of rhythmic patterns, for instance, {11211} in Example 2, uniquely determines the instances of pattern entries, namely, each next pattern enters at the first common rest of previous patterns. Otherwise there would be a gap, against Assumption 1, or a double beat, against Assumption 2. Q.E.D.

Proposition 2 implies that a canon C can be unambiguously coded by a succession of entering patterns

$$C = \{\pi_1\pi_2...\pi_i\}, \text{ where } \pi_i = 1,2,3,$$

where 1 stands for the pattern P, 2 for its augmentation, and 3 for its double augmentation. Now rhythmic canons can be constructed by enumerating successions of numbers 1,2,3 as candidates for canons and sorting out inappropriate ones. More specifically, do the following:

0. Initialize the list C of *C*andidates for canon with $C[1] = \{1\}$ (a trivial sequence of entering patterns which consists of the single pattern P). Initialize the list S of *S*elected canons to be the empty list.
1. Append $\pi = 1,2$, or 3 to $C[1]$. This means that the pattern P (respectively, its augmentation, or double augmentation) enters at the first gap, i.e., at the first 0 of pulse train of $C[1]$.

   There are three possibilities:

   (a) The new succession $\{C[1],\pi\}$ is a rhythmic canon (= no gaps and no double beats). In such a case the new succession is appended to the list S of selected canons. This implies removing the whole branch of its descendants from further considerations. In our case, the first selected canon is $S[1] = \{11211\}$.

   (b) The new succession $\{C[1],\pi\}$ is not a candidate for canon, because the new pattern π entering at the first gap results in a double beat. By this reason the new succession is left out. Thereby all its descendants, containing the double beat, are removed from further considerations.

(c) The new succession $\{C[1],\pi\}$ is a candidate for canon, because the new pattern entering at the first gap results in no double beats. Then the new succession is appended to the bottom of the list *C* of candidates for canon.

2. After having performed all three trials with $\pi = 1,2,3$, delete the currently considered (first) candidate $C[1]$ from list *C* as unnecessary. Return to Item 1, considering the first remaining candidate in the list *C*.

Thus *C* is destroyed from the top, appended to the bottom, and some selected elements of *C* are moved to *S*.

This sorting algorithm resembles the famous sieve of Eratosthene (284–192 BC) for finding primes:

- If we remove an element (in our case, a candidate for canon) then we delete the branch with all its descendants that stems from this element.
- We always start with the first remaining element (in our case, a candidate for canon).

The list of selected canons has no repeats in the sense that no smaller canon is a part of a larger canon. Indeed, if a canon is accomplished then it is moved from the list of candidates to the selected list, leaving no descendants in list *C*. Thus, each selected canon is continuous, with the end of a rhythmic pattern in one voice occurring in the middle of a rhythmic pattern of some other voice.

The algorithm does not miss any canon, because it is based on enumerating all successions of numbers 1, 2, 3. Due to restrictions imposed by Assumptions 1–2, the number of branches retained remains within operational limits, enabling us to perform computations in reasonable time.

The implementation of the algorithm includes several important devices. First of all, the list *C* of candidates for canon should be stored and processed by portions to avoid a long processing time and running out of memory. In my implementation, the list *C* is stored in a series of temporary files, while keeping in memory only the first file (to be destroyed from the top) and the last file (to be appended from the bottom up to a certain size, after which a new file should be opened).

Moreover, for each candidate for canon, its pulse train after the first gap should be saved. It prevents from reconstructing the pulse train while appending a rhythmic pattern to the current candidate for canon.

The program has been written in the MATLAB (= MATrix LABoratory) C++-based computer programming environment for matrix and vector operations. The program output is a LATEX text file with rhythmic scores of canons as in Example 2. A typical processing report for a program run on a PC with a Pentium II 300 MHz processor (note that MATLAB) is not a compiler but an interpreter) is given in Example 3.

These reports are helpful for composition. One can see that the number of canons found is large, so it is not convenient to examine all of them. For compositional tasks, some additional selection requirements

Totally tested combinations (candidates for canon)	1260234		
Maximal number of voices in preselection/selection		6	6
Maximal mean pattern number in preselection/selection		1.8	1.7
Periodicity in the preselection/selection		No	Yes
Found/preselected/selected canons of length 5	1	1	0
Found/preselected/selected canons of length 10	6	3	0
Found/preselected/selected canons of length 15	20	0	0
Found/preselected/selected canons of length 20	93	21	1
Found/preselected/selected canons of length 25	348	0	0
Found/preselected/selected canons of length 30	1460	0	0
Found/preselected/selected canons of length 35	5759	0	0
Found/preselected/selected canons of length 40	23502	961	15
Totally found/preselected/selected canons	31189	986	16
Maximal number of files on disk	120		
Maximum/average number of candidates for canon in memory	1000	296	
Time for computing/selection/making LATEX file, in seconds	1856	10	7

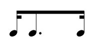
EXAMPLE 3: PROCESSING REPORT ON COMPUTING RHYTHMIC CANONS

may be formulated. Besides finding canons the program also classifies and selects canons according to several useful criteria:

- length
- maximal number of simultaneous voices, which indicates the number of instruments with which the canon can be performed
- prevailing pattern (the pattern, its augmentation, or double augmentation) which characterizes the relative rhythmic density
- uniformity of using all the patterns, which characterizes the variety of rhythms used, and
- periodicity in the canon structure, which is practical for making harmonic sequences

5. Example of Application

Example 4 contains the opening of *Eine kleine Mathmusik*.[4] The piece has been written partially algorithmically, partially heuristically. It was performed at IRCAM, Paris, on February 9, 2002. The Appendix contains the scores of rhythmic canons used in the piece.

Eine kleine Mathmusik is a piece centered on G for a woodwind sextet which uses a selection of eight computed rhythmic canons. All eight canons are built from the *basic rhythmic pattern* 11001 = , its augmentation, and its double augmentation. In keeping with the length of the basic pattern, the basic *time signature* is $\frac{5}{16}$.

The reverse of the basic code 11001—that is, 10011—determines *basic melodic intervals* in the theme which are thirds and seconds. Thirds and seconds can be either minor or major. The principal *theme* motive is g_1, b_1, c_2.

In order to reduce the number of performers, non-overlapping rhythmic patterns are grouped into a few *composite voices* which are performed by the same instrument. For instance, the canon 11211 shown in Example 2 has five entering patterns which can be reduced to three composite voices. This is done heuristically in order to construct more developed *melodies from successive basic motives.*

Since the piece consists of a series of canons, they are separated by additional 1/16–3/16 rests inserted manually, which are rhythmically to be perceived as stops and harmonically emphasized as *cadences*. The basic $\frac{5}{16}$ measures are therefore extended, which is the cause of variable meter in the piece.

The piece suggests a loose analogy with formal procedures in tonal music. It was intended to be *neo-baroque* in a *sonata form* with two canons as contrasting "themes" implemented by "major-minor harmonies" and "tonal" development. The "harmony" is articulated in arpeggiations, not as score verticals, but rather operating in a time slot of several beats.

EXAMPLE 4: THE OPENING OF *EINE KLEINE MATHMUSIK*

The development is based on a certain *variation* principle. A canon is assumed to be a variation of some other canon if it has the same beginning but a new ending, e.g.

$$1121\boxed{1} \rightarrow 1121\boxed{331121}$$

Due to particularities of the algorithm, the list of canons selected is ordered with respect to their size, from shorter to longer, and within every size canons are ordered lexicographically, e.g., canons beginning with 112... come before the canons beginning with 113... . That means the closest variations of a given canon are its neighbors in the list.

The musical form of the piece is displayed in Example 5. As one can see, the harmonic plan of the piece is an analogy to Western tonal music. The first entry of the second theme is at the fifth (in analogy to the dominant), the development begins with the first theme in the "dominant," and the return to the main tonality passes through the "subdominant."

The selection of a particular canon for a particular purpose is motivated by several reasons:

1. For Theme 1, the shortest available canon (No. 1) is selected and used twice with harmonic modification, so that the rhythmic structure of Theme 1 is 1 + 1.
2. The closest variations of Theme 1, Canons Nos. 2–4 (the latter taken twice), are used to build a transition to Theme 2. The resulting rhythmic structure of the transition is 2 + 2 + 2 + 2.
3. Theme 2 (Canon No. 29, the first of relative length 4, i.e., four times longer than the theme, with fewer than six physical voices) is "slower" due to prevailing patterns of augmentation and second augmentation.
4. The "Variation of Theme 2" is a quite distant variation (Canon No. 55), but it is the only canon of the same length as Canon No. 29 with only four physical voices. The economy of physical voices is quite important to preserve harmonic transparency.
5. The Development contains the longest canon in the piece, with forty entries of the basic motive, which gives 120 beats. It has been selected due to its periodicity (which enables making harmonic sequences that are the norm in a classical development section) and economy of physical voices (six).

   Another selection criterion is the *mean pattern number* of the patterns used in the canon. For instance, the first canon 11211 has mean pattern number (1 + 1 + 2 + 1 + 1)/5 = 6/5 = 1.2, indicating that the basic rhythmic pattern with number 1 prevails over the augmentations coded by 2 or 3 (double augmentation). A low mean pattern number implies shorter durations, an easier recognizability of the theme, and a more vivid melodic development. Conversely, for slower sections a high mean pattern number can be desirable. But in the given piece (with a relatively high rhythmic density), a low mean was always preferred.

Section	Material	Measures	Description
Exposition	Theme 1	1–6	Canon No. 1, twice 11211 + 11211 G→D C→G
	Transition 1	7–18	Canons No. 2 and 3 1121 [331121] + 1121 [332222] C→C^7 F→$F^{6/9}$
	Transition 2	19–30	Canon no. 4, twice 11 [31211211] + 11 [31211211] Dm→A^7 Dm→F^6
	Theme 2	31–42	Canon No. 29 112 [222332111312112 11] D→F♯^{7}
	Var. Theme 2	43–54	Canon No. 55 11 [3121331123321112 11] E→A^+
Development	Theme 1 Var. Trans. 1	55–60 61–84	Canon No. 1, twice, D A, G D Canon No. 8005 with 3 periods 1 1222233211 1222233211 1222233211 121332222 G D,B^7,Am^6,E^7 G,E^7,Dm^6,A^7 C,A^7,Gm^6,D^7 F→ G^7
	Var. Trans. 2	85–96	Canon No. 49 1131211 [312113121 1211] Cm→A♭
	Theme 2 Var. Theme 2	97–108 109–20	Canon No. 29, C→E^7 Canon no. 55, D→G^+
Recapitulation	Theme 1 Trans. 2	121–6 127–38	Canon No. 1, twice, G→D C→G Canon No. 4, twice, Gm→D^7, Gm→E♭
Coda	Theme 1	151–62	Canon No. 1, four times Gm→D, B♭→F, Fm→Cm, D^{-9}→G

EXAMPLE 5: THE FORM OF *EINE KLEINE MATHMUSIK*

6. Generalizations

6.1 using several thematic elements

Instead of augmentations of the theme, the model can operate with some other arbitrary rhythmic patterns.

In fact, the algorithm fits several rhythmic patterns to a given pulse train. In our specific model these rhythmic patterns are restricted to the "theme," its augmentation, and double augmentation. However nothing prevents the model from using some other building blocks, e.g., two themes and/or some of their derivatives. The sorting algorithm will just operate on some other basic elements.

Thus besides rhythmic canons restricted to a single theme, one can construct, for instance, "rhythmic fugues" with several themes and counter-subjects.

6.2 producing an irregular pulse train

Assumption 1 (no gaps) is not obligatory for the model, and it can be replaced by a more general one. Instead of a regular pulse train which must be covered (tiled) by a restricted set of basic rhythmic patterns, an arbitrary pulse train can be considered:

For instance, consider the task of tiling the pulse train

110110...

with our three patterns from Example 1. A candidate for such a canon solution is shown in Example 6.

Voice number	Pattern number	Beat number											
		1	2	3	4	5	6	7	8	9	10	11	12
1	3	1	0	0	0	1	0	0	0	0	0	0	0
2	2	.	1	0	1	0	0	0	0	0	1	.	.
3	1	.	.	.	.	.	.	1	1	0	0	1	.
Simultaneous onsets		1	1	0	1	1	0	1	1	0	1	1	0

example 6: tiling the pulse train 110110...

The difference is that the sorting algorithm must fit given patterns not to the regular rhythmic grid but rather to a user-defined one. Then the resulting polynomial in equation (2) is

$$I_n(x) = 1 + x + 0x^2 + x^3 + x^4 + 0x^5 + \dots .$$

6.3 ALLOWING SIMULTANEOUS TONE ONSETS

Assumption 2 (no double beats) is not obligatory either. Double beats can be prohibited at certain beats, or allowed; triple beats can be required for certain accents, etc. Again, the performance of the model is determined by the resulting pulse train which can look, for instance, like

310121013101...

which means in particular that every fourth beat must be accentuated either with triple or double tone onset. Then the resulting polynomial in equation (2) is

$$I_n(x) = 3 + x + 0x^2 + x^3 + 2x^4 + x^5 + 0x^6 + 3x^7 + x^8 + 0x^9 + x^{10} + \dots .$$

Then the model will fit the basic elements to such a grid.

It should be mentioned that the increasing degree of freedom will require more memory and will reduce the search speed.

6.4 CONSTRUCTING CANONS FROM AGGREGATE BLOCKS

Consider the original problem of constructing rhythmic canons, that is, with no double beats and no rhythmic gaps. Note that each candidate for canon has a regular pulse train until the first gap. Let us call the irregular pulse train after the first gap a *junction*. For instance, the candidate for canon 1121 has the following pulse train (see Example 2):

$$11111111110\ \ \underbrace{011}_{\textit{junction}}$$

The candidate for canon 1121 can be completed to a canon either by pattern 1 (as in Canon No. 1), or by patterns 331121 (as in Canon No. 2), or by patterns 332222 (as in Canon No. 3), or by some other sequence, all meeting the same junction.

Note that the length of a junction cannot be longer than the longest pattern considered minus two beats, in our case 15 beats. Not all junctions are possible. There is a finite number of junctions, and for every two junctions there must be a bridge (= a sequence of patterns providing a transition from one junction to another). Knowing all the bridges enables the constructing of rhythmic canons of arbitrary length. Then constructing canons can be reduced to manipulating a finite number of building blocks (like in puzzle games).

6.5 MOTIVES AS VECTOR NOTES

Motives can be regarded as vector notes. Similarly to the use of single notes restricted by certain rules of harmony in the Western tonal syntax, the use of vector notes in our model is restricted by the "no gap" and "no double beat" assumptions. The compatibility of vector notes can be developed into a theory similar to harmony for single notes. The difference is that this theory is rhythm-based, and therefore contributes to a theory of rhythm as well.

7. SUMMARY

Let us recapitulate the main results of the paper. We suggested an algorithmic solution to the problem of finding finite rhythmic canons with augmentations. The application of the model for practical composition was illustrated with an example of the piece *Eine kleine Mathmusik*. The model can be adapted for more general tasks which are outlined briefly: using several basic patterns to make rhythmic fugues, producing a user-defined pulse train, and/or allowing simultaneous notes for making special accents.

Appendix: Scores of Rhythmic Canons Used in *Eine Kleine Mathmusik*

Voice	Pattern	Score														
1	1	1	1	0	0	1	.	.	.	.	.	.	.	.	.	.
2	1	.	.	1	1	0	0	1	.	.	.	.	.	.	.	.
3	2	.	.	.	.	.	1	0	1	0	0	0	0	0	1	.
4	1	.	.	.	.	.	.	.	.	1	1	0	0	1	.	.
5	1	.	.	.	.	.	.	.	.	.	.	1	1	0	0	1

Canon no. 1 of length 15 beats with 3 simultaneous voices and mean pattern no. 1.2

Voice	Pattern	Score					
1	1	1 1 0 0 1	. . . . .	. . . . .	. . . . .	. . . . .	. . . . .
2	1	. . 1 1 0	0 1 . . .	. . . . .	. . . . .	. . . . .	. . . . .
3	2	. . . . .	1 0 1 0 0	0 0 0 1 .	. . . . .	. . . . .	. . . . .
4	1	. . . . .	. . . 1 1	0 0 1 . .	. . . . .	. . . . .	. . . . .
5	3	. . . . .	. . . . .	1 0 0 0 1	0 0 0 0 0	0 0 0 0 0	0 1 . . .
6	3	. . . . .	. . . . .	. 1 0 0 0	1 0 0 0 0	0 0 0 0 0	0 0 1 . .
7	1	. . . . .	. . . . .	. . . . .	. 1 1 0 0	1	. . . . .
8	1	. . . . .	. . . . .	. . . . .	. . . 1 1	0 0 1 . .	. . . . .
9	2	. . . . .	. . . . .	. . . . .	. . . . .	. 1 0 1 0	0 0 0 0 1
10	1	. . . . .	. . . . .	. . . . .	. . . . .	 1	1 0 0 1 .

CANON NO. 2 OF LENGTH 30 BEATS WITH 4 SIMULTANEOUS VOICES AND MEAN PATTERN NO. 1.6

Voice	Pattern	Score																													
1	1	1	1	0	0	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
2	1	.	.	1	1	0	0	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
3	2	.	.	.	.	.	1	0	1	0	0	0	0	0	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
4	1	.	.	.	.	.	.	.	.	1	1	0	0	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
5	3	.	.	.	.	.	.	.	.	.	.	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	.	.	.
6	3	.	.	.	.	.	.	.	.	.	.	.	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	.	.
7	2	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	0	1	0	0	0	0	0	1	.	.	.	.	.
8	2	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	0	1	0	0	0	0	0	1	.	.	.	.
9	2	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	0	1	0	0	0	0	0	1	.
10	2	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	0	1	0	0	0	0	0	1

CANON NO. 3 OF LENGTH 30 BEATS WITH 6 SIMULTANEOUS VOICES AND MEAN PATTERN NO. 1.9

Voice	Pattern	Score																													
1	1	1	1	0	0	1	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
2	1	·	·	1	1	0	0	1	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
3	3	·	·	·	·	·	1	0	0	0	1	0	0	0	0	0	0	1	·	·	·	·	·	·	·	·	·	·	·	·	·
4	1	·	·	·	·	·	·	·	1	1	0	0	1	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
5	2	·	·	·	·	·	·	·	·	·	·	1	0	1	0	0	0	0	0	1	·	·	·	·	·	·	·	·	·	·	·
6	1	·	·	·	·	·	·	·	·	·	·	·	·	·	1	1	0	0	1	·	·	·	·	·	·	·	·	·	·	·	·
7	1	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	1	1	0	0	1	·	·	·	·	·	·	·	·	·	·
8	2	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	1	0	1	0	0	0	0	0	1	·
9	1	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	1	1	0	0	1	·	·
10	1	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	1	1	0	0	1

Canon no. 4 of length 30 beats with 4 simultaneous voices and mean pattern no. 1.4

Voice	Pattern	Score											
1	1	11001	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
2	1	..110	01...	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
3	2	.....	10100	0001.	.....	.....	.....	.....	.....	.....	.....	.....	.....
4	2	.....	...10	10000	01...	.....	.....	.....	.....	.....	.....	.....	.....
5	2	.....	1	01000	001..	.....	.....	.....	.....	.....	.....	.....	.....
6	2	.....	.....	..101	00000	1....	.....	.....	.....	.....	.....	.....	.....
7	3	.....	.....	.....	10001	00000	00000	01...	.....	.....	.....	.....	.....
8	3	.....	.....	.....	...10	00100	00000	00001	.....	.....	.....	.....	.....
9	2	.....	.....	.....	.....	.1010	00001	.....	.....	.....	.....	.....	.....
10	1	.....	.....	.....	.....	1	1001.	.....	.....	.....	.....	.....	.....
11	1	.....	.....	.....	.....	.....	.1100	1....	.....	.....	.....	.....	.....
12	1	.....	.....	.....	.....	.....	.....	..110	01...	.....	.....	.....	.....
13	2	.....	.....	.....	.....	.....	.....	.....	10100	0001.	.....	.....	.....
14	1	.....	.....	.....	.....	.....	.....	.....	...11	001..	.....	.....	.....
15	3	.....	.....	.....	.....	.....	.....	.....	.....	10001	00000	00000	01...
16	3	.....	.....	.....	.....	.....	.....	.....	.....	.1000	10000	00000	001..
17	1	.....	.....	.....	.....	.....	.....	.....	.....	.....	.1100	1....	.....
18	1	.....	.....	.....	.....	.....	.....	.....	.....	.....	...11	001..	.....
19	2	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.1010	00001
20	1	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	1	1001.

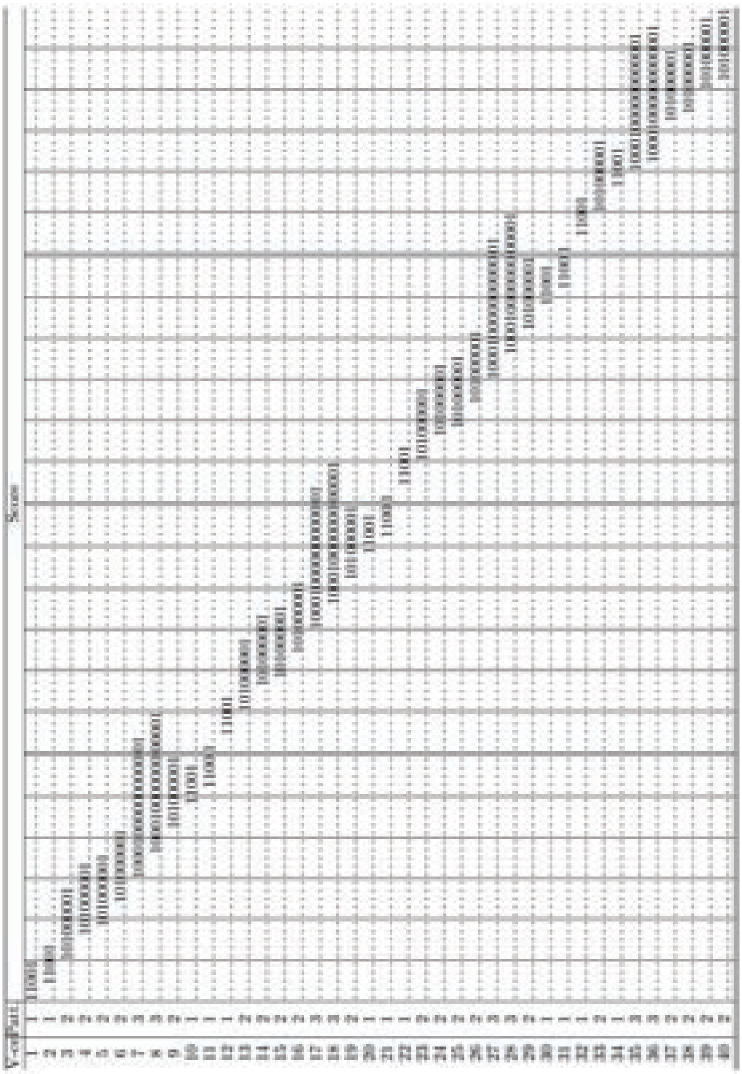
CANON NO. 29 OF LENGTH 60 BEATS WITH 5 SIMULTANEOUS VOICES AND MEAN PATTERN NO. 1.8

Voice	Pattern	Score											
1	1	11001	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
2	1	..110	01...	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
3	3	.....	10001	00000	00000	01...	.....	.....	.....	.....	.....	.....	.....
4	1	.....	..110	01...	.....	.....	.....	.....	.....	.....	.....	.....	.....
5	2	.....	.....	10100	0001.	.....	.....	.....	.....	.....	.....	.....	.....
6	1	.....	.....	...11	001..	.....	.....	.....	.....	.....	.....	.....	.....
7	1	.....	.....	.....	11001	.....	.....	.....	.....	.....	.....	.....	.....
8	3	.....	.....	.....	.....	10001	00000	00000	01...	.....	.....	.....	.....
9	1	.....	.....	.....	.....	..110	01...	.....	.....	.....	.....	.....	.....
10	2	.....	.....	.....	.....	.....	10100	0001.	.....	.....	.....	.....	.....
11	1	.....	.....	.....	.....	.....	...11	001..	.....	.....	.....	.....	.....
12	1	.....	.....	.....	.....	.....	.....	11001	.....	.....	.....	.....	.....
13	3	.....	.....	.....	.....	.....	.....	.....	10001	00000	00000	01...	.....
14	1	.....	.....	.....	.....	.....	.....	.....	..110	01...	.....	.....	.....
15	2	.....	.....	.....	.....	.....	.....	.....	.....	10100	0001.	.....	.....
16	1	.....	.....	.....	.....	.....	.....	.....	.....	...11	001..	.....	.....
17	1	.....	.....	.....	.....	.....	.....	.....	.....	.....	11001	.....	.....
18	2	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	10100	0001.
19	1	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	...11	001..
20	1	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	11001

CANON NO. 49 OF LENGTH 60 BEATS WITH 4 SIMULTANEOUS VOICES AND MEAN PATTERN NO. 1.5

Voice	Pattern	Score											
1	1	11001	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
2	1	..110	01...	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
3	3	.....	10001	00000	00000	01...	.....	.....	.....	.....	.....	.....	.....
4	1	.....	..110	01...	.....	.....	.....	.....	.....	.....	.....	.....	.....
5	2	.....	.....	10100	0001.	.....	.....	.....	.....	.....	.....	.....	.....
6	1	.....	.....	...11	001..	.....	.....	.....	.....	.....	.....	.....	.....
7	3	.....	.....	.....	10001	00000	00000	01...	.....	.....	.....	.....	.....
8	3	.....	.....	.....	.1000	10000	00000	001..		.....	.....	.....	.....
9	1	.....	.....	.....	.....	..110	01...	.....	.....	.....	.....	.....	.....
10	1	.....	.....	.....	.....	1	1001.	.....	.....	.....	.....	.....	.....
11	2	.....	.....	.....	.....	.....	..101	00000	1....	.....	.....	.....	.....
12	3	.....	.....	.....	.....	.....	.....	10001	00000	00000	01...	.....	.....
13	3	.....	.....	.....	.....	.....	.....	...10	00100	00000	00001	.....	.....
14	2	.....	.....	.....	.....	.....	.....	.....	.1010	00001	.....	.....	.....
15	1	.....	.....	.....	.....	.....	.....	.....	1	1001.	.....	.....	.....
16	1	.....	.....	.....	.....	.....	.....	.....	.....	.1100	1....	.....	.....
17	1	.....	.....	.....	.....	.....	.....	.....	.....	.....	..110	01...	.....
18	2	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	10100	0001.
19	1	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	...11	001..
20	1	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	11001

CANON NO. 55 OF LENGTH 60 BEATS WITH 5 SIMULTANEOUS VOICES AND MEAN PATTERN NO. 1.7



CANON NO. 8005 OF LENGTH 120 BEATS WITH PERIOD 10
WITH 6 SIMULTANEOUS VOICES AND MEAN PATTERN NO. 1.9

Notes

The author thanks Tom Johnson for fruitful discussions and thoughtful reading of the draft of the paper and Professor Robert Morris for numerous suggestions which improved both the content and the style.

1. As noted by Robert Morris, Messiaen's "non-retrogradable rhythms" are already a transference of pitch to rhythm, except that the "modes" are invariant under shift of pitch (transposition) and the non-retrogradable rhythms are invariant under (retrograde) inversion.
2. The English translation is given by Xenakis (1971, 8).
3. Recall that Fermat (1601–1665) stated his Last Theorem as a marginal note in Diophante's Arithmetic as a step towards the unsolvable general case.
4. The full score is available from the author of the paper.

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