Constructing rhythmic fugues

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Abstract

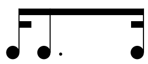
A fugue is a polyphonic piece whose voices lead a few melodies (subjects and counter-subjects) with different delays. A rhythmic fugue is one whose tone onsets result in a regular pulse train with no simultaneous tone onsets at a time.

Tiling in geometry is covering an area by disjoint equal figures, e.g., a rectanle by triangles. In that sense, a rhythmic fugue tiles the time, providing a covering of a regular pulse train by a few disjoint rhythmic patterns.

The paper suggests a general computational method for constructing rgythmic fugues. It further develops the previous approach of the author based on an isomorphism of rhythms and special polynomials. The method is used to make a composition *Eine kleine Mathmusik 2*.

1 Introduction

My involvement in musical tiling[1] goes back to the conference *Journées de l'informatique musicale, Bourges, 7–9 June 2001,* where I got acquainted with minimalist composer Tom Johnson. Having learned that I was a mathematician he showed me a peculiar rhythmic structure which he called rhythmic canon. The theme (or subject) was a rhythm which beats were coded by 1's at the scale of sixteenths:



1 1 0 0 1 (1)

The complete structure — the rhythmic canon — was built of five instances of the theme, with one instance having been in augmentation (in twice slower tempo). The sum of all beats constituted a regular puls train with no two onsets at a time as shown in Table **??**:

This example illustrated the idea of rhythmic tiling. Regular pulse train was covered by instances of a rhythmic pattern and of its augmentation with no beat overlaps. From the musical viewpoint the result was a canon, since it was generated by one theme entering with different delays.

The mathematical question was the existence of other rhythmic canons. We have agreed that if I manage to find a solution then Tom Johnson uses it for a composition.

[1] *Tiling* in geometry is covering an area by disjoint equal figures, e.g., a rectangle by triangles.

Table 1: Johnson's rhythmic canon

Voice	Pattern	Pattern type	
1	1	Theme	1 1 0 0 1 . . . .
2	1	Theme	. . 1 1 0 0 1 . . . .
3	2	Theme in augmentation	. . . . . 1 0 1 0 0 0 0 0 1 .
4	1	Theme	. . . . 1 1 0 0 1 . .
5	1	Theme	. . . . 1 1 0 0 1
		Pulse train (sum of onsets at a time)	1 1 1 1 1 1 1 1 1 1 1 1 1 1 1

In a few weeks I developed a computational model and obtained rhythmic canons of different lengths. Surprisingly Tom Johnson lost any interest in the project. As far as I understood, he hoped that the rhythmic canon he had discovered was a unique wunder, a kind of philosopher's stone in rhythmic tiling. Numerous solutions, lacking any minimalistic elegance, were most disappointing.

It was however a pity to forward weeks of work into the waste-basket. Therefore, I made a three-note motive with the Johnson's rhythm (**??**), used it with transpositions throughout several rhythmic canons computed, and recorded the score playback with synthesized woodwinds. The solution to the tiling problem (Tangian 2001b) and the musical piece entitled *Eine kleine Mathmusik* were presented at the MaMuX seminar, IRCAM, Paris, on February 9, 2002.

The seminar was comprehensive and illuminating. The time-tiling problem turned out to have profound roots in music theory (Andreatta et al. 2002). The prototypes were Messiaen's (1944) *modes of limited transposition*, that is, scales with specific interval relations which transpositions disjointly covered the 12-tone tempered scale. For instance, this is the case of the mode with piches $\{c, e\flat, f\sharp, a\}$ and its two transpositions, by one and by two semitones.

That were Vuza (1991–1993, 1995) and Vieru (1993) who transferred Messian's ideas from the domain of pitch to the domain of rhythm. By analogy with covering the 12-tone tempered scale by a mode and its transpositions, the regular pulse train was covered by a rhythmic pattern with shifts. The disjointedness of pitch classes implied the prohibition of beat overlaps, and the circularity of pitch (= octave periodicity) corresponded to circular time (= periodicity of rhythmic structure). Vieru and Vuza intended such "rhythms of limited transposition", or, better, "rhythms of limited delay", for constructing unending (= infinite, periodic) canons.

The musical motivation was making polyphonic pieces from a single rhythmic/melodic pattern. It meets the principle of economy in both classical and 20th century music: recall long phrases built from the opening four-note motive in Beethoven's Fifth Symphony, 12-tone composition, etc. On the other hand, in rhythmic canons the independence of voices is maximal, since no two tones occur simultaneously, which is much appreciated in polyphony.

After contributions of Vieru and Vuza, time-tiling attracted attention of several music theorists (Amiot 2002a, Andreatta et al. 2001, Fripertinger 2002, 2003). However, solutions to the time-tiling problem appeared to be trivial and musically not interesting. A typical solution was a metronome rhythm entering with equal delays, e.g., a sequence of every fourth beat, entering at the first, at the second, and at the third beat. Non-trivial solutions were found by Vuza for a circular time with periods $72, 108, 120, \ldots$

These long complex solutions were difficult for perception. The effect was similar to the one in serial music, as described by Xenakis (1963):

> Linear polyphony destroys itself by its very complexity; what one hears is in reality nothing but a mass of notes in various registers. The enormous complexity prevents the audience from following the intertwining of the lines and has as its macroscopic effect an irrational and fortuitous dispersion of sounds over the whole extent of the sonic spectrum. There is consequently a contradiction between the polyphonic linear system and the heard result, which is surface or mass.
>
> The English translation is given by (Xenakis 1971, p. 8).

To find simple rhythmic canons, Johnson (2001) relaxed the time-tiling constraints in two directions. Additionally to the theme, he authorized the use of its augmentation and double augmentation, like in Bach's *The Art of the Fugue*. Additionally to unending canons with circular time, he also considered finite canons with linear time. The Vuza's method was however adaptable neither to using augmentations of the theme, nor to linear time. Having no general method, Johnson nevertheless heuristically constructed the non-trivial finite rhythmic canon (Table **??**) and initiated a new field of studies.

My approach to finding finite rhythmic canons used an isomorphism between binary structures and polynomials introduced in some earlier publications (Tangian 1993–1995, 2001a). This isomorphism enabled to construct rhythmic canons by factorizing special polynomials. The algorithm for revealing irreducible ones resembled the sieve of Erathosthene (284–192 BC) for finding prime numbers; for details see Tangian (2001b, 2003). This isomorphism intended mainly for computational purposes was used by other researches for theoretical purposes as well; see Amiot (2002–2005), Gilbert (2007) and some other authors cited at the dedicated webpage of Seminar MaMuX (2002–2007); for a recent survey see Amiot, Andreatta, and Agon (2005).

The given paper generalizes the computational model (Tangian 2001b, 2003) for providing more compositional freedom. First, tiling can be performed with a few rhythmic patterns. For instance, one can use two different patterns, "theme" and "counterpoint", as in fugues. Drawing analogy to tiling in geometry, one can cover an area with figures of two different shapes, say, with squares and triangles.

Second, to avoid toccata-like rhythmic homogeniety, patterns can be fitted to an irregular pulse train. For example, the sum of onsets can look like 1 1 0 0 1 1 0 0 0 1 ... Drawing analogy to geometry, the area to be covered can have holes, and their location and size can be irregular.

A particular application of irregular pulse trains is constructing unending rhythmic canons by the technique of finite canons. For instance, we construct a rhythmic canon for a pulse train segment with "matching ends" like

Pulse train   1 0 1 1 1 1 1 1 1 1 1 1 1 1 1 1 0 1

Then the "matching ends" are connected, transforming the linear canon into a loop with a regular pulse train.

Third, the irregular pulse train can be made even "more irregular" by allowing overlaps to produce accents of variable strength. For instance, rhythmic patterns can be arranged to produce the sum of onsets like

$$\text{Pulse train} \quad 1\ 1\ 0\ 0\ 2\ 1\ 2\ 1\ 1\ 1\ 1\ 1\ 2\ 1 \tag{2}$$

The goal is constructing *rhythmic fugues,* that is, rhythmic structures generated by a few partterns fitted to a given pulse train. In other words, the task is formulated as tiling an arbitrary sequence of time events with a few given patterns. For example, rhythmic pattern (**??**), its augmentation, and its retrograde version can be used to obtain a rhythmic fugue with the sum of onsets (**??**) as shown in Table **??**.

Table 2: Rhythmic fugue with three patterns and irregular pulse train

Voice	Pattern	Pattern type	
1	1	Theme	1 1 0 0 1 . . . .
2	2	Theme in augmentation	. . . . 1 0 1 0 0 0 0 0 1 .
3	2	Theme in augmentation	. . . . . 1 0 1 0 0 0 0 0 1
4	3	Retrograde theme	. . . . . . 1 0 0 1 1 . . .
5	3	Retrograde theme	. . . . 1 0 0 1 1 .
		Pulse train	1 1 0 0 2 1 2 1 1 1 1 1 2 1

All of these is implemented in the piece *Eine kleine Mathmusik 2* based on a number of rhythmic fugues, similarly to *Eine kleine Mathmusik* which was based on a number of rhythmic canons.

2 Isomorphism between rhythms and polynomials

Associate rhythmic patterns P with 0–1 polynomials $P(x)$, that is, with coefficients 0, 1:

$$P = \delta_1 \ \ldots \ \delta_n \quad \text{where} \quad \delta_i = 0, 1 \quad \longleftrightarrow \quad P(x) = \sum_{i=0}^{n} \delta_i x^i$$

For example, Johnson's rhythmic pattern (**??**) is represented by a polynomial as follows:

$$J = 1\ 1\ 0\ 0\ 1 \quad \longleftrightarrow \quad J(x) = 1 + 1x + 0x^2 + 0x^3 + 1x^4 \ .$$

Delay. A delay of a rhythmic pattern by k beats corresponds to multiplying the associated polynomial by x^k. For instance, the delay of J by two beats implies

$$J_2 = \ 0\ 0\ 1\ 1\ 0\ 0\ 1 \quad \longleftrightarrow \quad J(x)x^2 = 0 + 0x + 1x^2 + 1x^3 + 0x^4 + 0x^5 + 1x^6 \ .$$

Augmentation. The augmentation of a rhythmic pattern corresponds to taking the associated polynomial with the argument x^2. For instance,

$$\begin{aligned} \text{Augmentation of } J \quad &\longleftrightarrow \quad J(x^2) = 1 + x^2 + x^8 \\ \text{Double augmentation of } J \quad &\longleftrightarrow \quad J\left((x^2)^2\right) = J(x^4) = 1 + x^4 + x^{16} \ . \end{aligned}$$

Superposition. A superposition of rhythmic patterns corresponds to the sum of the associated polynomials. For instance, the superposition of Johnson's pattern with itself delayed by three beats implies

$$J + J_3 = \begin{array}{cccccccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ \hline 1 & 1 & 0 & 1 & 2 & 0 & 0 & 1 \end{array} \quad \longleftrightarrow \quad J(x) + J(x)x^3 = J(x)(1 + x^3) \ .$$

Pulse train. The pulse train constituted by the sum of onsets at a time is associated with the polynomial

$$S(x) = \sum_{i=0}^{n} s_i x^i \quad \text{where} \quad s_i = \text{the number of overlaps at the } i\text{th beat.}$$

The regular pulse train corresponds to the case when all the coefficients $s_i = 1$.

Rhythmic fugues as polynomial equations. Rhythmic fugues are associated with polynomial equations. For instance, fugue in Table **??** is associated with the equation

$$J(x)U(x) + J(x^2)V(x) + J^R(x)W(x) = S(x) \ , \tag{3}$$

where

$J(x)$ is the 0–1 polynomial of pattern J

$J(x^2)$ is the 0–1 polynomial of augmented pattern J

$J^R(x)$ is the 0–1 polynomial of retrograde pattern J

$S(x)$ is the polynomial with non-negative integer coefficients of the pulse train in (**??**), and

$U(x)$, $V(x)$, and $W(x)$ are unknown 0–1 polynomials which determine the entries of pattern J, of augmented J, and of retrograde J, respectively.

Note that equations like (**??**) can have numerous solutions, a unique solution, or none.

3 Algorithm for constructing rhythmic fugues

We shall describe the algorithm for solving equations like (**??**) in terms of rhythmic patterns. First of all define and enumerate the patterns to be used, for example

1	Theme	1 1 0 0 1
2	Theme in augmentation	1 0 1 0 0 0 0 0 1
3	Retrograde theme	1 0 0 1 1
4	Retrograde theme in augmentation	1 0 0 0 0 0 1 0 1
	Pulse train	1 1 0 0 ‖:2 1 2 1 1 1 1 1 2 1:‖

where ‖: :‖ are repeat signs.

Note that a fugue is completely determined by given pulse train and succession of entering patterns. For example, consider the fugue in Table **??**. Successively fitting patterns

1, 2, 2, 3, 3 to the given pulse train is only possible at beats 1, 5, 6, 7, 9, respectively. Thus, given a pulse train, a rhythmic fugue can be labelled with the succession of entering patterns. For instance, fugue (??) is labelled "12233".

This notation enables to construct rhythmic fugues by building labels. Each such a sequences of pattern numbers is regarded as a *candidates for fugue*. A new number is appended to the label after the matching test, that is, if the tail of the resulting succession of patterns is compatible with the given pulse train. The mismatching candidates are deleted from further consideration.

More specifically, do the following.

1. Initialize the list C of candidates for fugue with label "1" (a sequence with a single pattern — the theme). Initialize the list F of fugues to be the empty list.

2. Append number $\pi = 1, 2, 3$, or 4 to the label of the first candidate in list C, making four new candidates from one root. As for the resulting score, the pattern with number π must enter at the very first possibility, where the root candidate's sum of onsets is less than in the pulse train.

   For every new candidate, perform the matching test with three outcomes:

   (a) The matching test reveals a complete fugue, that is, a perfect fit of the new candidate's sum of onsets along its full length to the given pulse train. Then append the new candidate to the list F of fugues.

   (b) The matching test fails, that is, the new candidate's sum of onsets surpasses the given pulse train at some beat. Then the new candidate is left out.

   (c) The matching test does not reveal a complete fugue and does not fail, "leaving a chance" for the new candidate. Then append it to the end of list C.

3. After having tested four new candidates delete the root candidate (the first in C) and return to Item 2.

Thus the list of candidates C is destroyed from the top and appended from the bottom with a "new generation" of candidates. The "successful candidates", that is, complete fugues, are moved from C to F. This sorting algorithm resembles the sieve of Eratosthene (284–192 BC) for finding primes:

- If we remove an element (in our case, a candidate for fugue) then we delete the whole branch with all its descendants which stem from this element.

- We always start with the first retained element (in our case, a candidate for fugue).

The list of selected fugues has no repeats in the sense that no smaller fugue is a part of a larger fugue. Indeed, if a fugue is accomplished then it is moved from the list of candidates to the selected list, leaving no descendants in list C. Thus, each selected fugue is "continuous", with the end of a rhythmic pattern in one voice occurring in the middle of a rhythmic pattern of some other voice. The algorithm does not miss any fugue, because it is based on enumerating all successions of numerals 1,2,3,4.

The implementation of the algorithm includes several technicalities. First of all, the list C of candidates is stored and processed by portions to avoid runs out of memory and long disk exchanges. In my implementation the list C is stored in a series of temporary files, while keeping in memory only the first and the last one (the list C is destroyed from the top and appended from the bottom). If the last file surpasses a certain size, a new file is opened.

Second, to simplify the matching test, each candidate for fugue is saved together with the "tail of its score", that is, with the sum of onsets starting at the first mismatch with the pulse train.

The computer program has been written in MATLAB. It outputs a LaTeX text file, containing rhythmic scores like Table ?? together with specifications of each fugue:

- first pattern, e.g., the theme,
- length of the fugue in beats,
- number of entering patterns,
- maximal number of simultaneous voices, i.e., sufficient size of performing ensemble,
- prevailing patterns (the pattern, its augmentation, or double augmentation) to characterize the relative rhythmic density,
- uniformity of using the patterns to characterizes the variety of rhythms used, and
- periodicity in the fugue structure which is practical for making harmonic sequences.

The program can also preselect fugues with respect to their particular specifications.

4 Assembled and unending rhythmic fugues

Assemble one pulse train from two pulse trains with "matching ends":

Pulse train I	1	1	1	1	1	0	1	.	.	.	.	.	.	.
Pulse train II	.	.	.	.	.	1	0	1	1	1	1	1	1	1
Total pulse train	1	1	1	1	1	1	1	1	1	1	1	1	1	1

Fitting rhythmic pattern (??) and its augmentation to Pulse trains I and II results in two parts of the rhythmic canon in Table ??: Voices 1–2 for Pulse train I, and Voices 3–5 for Pulse train II. This way the rhythmic cannon can be assembled of two sections.

Of course, "matching ends" can be longer and with different "profiles". However, their variety is limited, in particular, by the length of the rhythmic patterns used in the construction. Therefore, short blocks with different ends on both sides exhaust all elements with which rhythmic fugues of arbitrary length can be assembled.

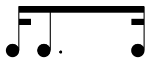
A pulse train with both ends matching to each other can be transformed into a pulse train loop. This device is practical for constructing unending rhythmic fugues and canons. An example is shown in Table ??.

Table 3: Unending rhythmic canon

Voice	Pattern	Pattern type	
1	3	Theme in double augmentation	1 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 1
2	1	Theme	. . 1 1 0 0 1 . . . .
3	2	Theme in augmentation	. . . . . 1 0 1 0 0 0 0 0 1 . . .
4	1	Theme	. . . . 1 1 0 0 1 . . . .
5	1	Theme	. . . . 1 1 0 0 1 . .
		Pulse train with matching ends (loop)	1 0 1 1 1 1 1 1 1 1 1 1 1 1 1 0 1

5 Application to composition

Figures ??–?? show the score of *Eine kleine Mathmusik 2*, a piece in C for woodwind sextet. It is based on seven rhythmic fugues computed with the model which scores are given in Annex. The pitches are "manually" assigned to time events, and the tonal development is performed as a walk on the tonal map described in the next section. The first electroacoustic performance took place at the MaMuX seminar, IRCAM, Paris, on January 25, 2003.

All the seven fugues are built from the basic rhythmic pattern 11001 =  , its augmentation, its retrograde version, and the retrograde version in augmentation.

The retrograde basic code 11001, that is, 10011 determines melodic intervals of the theme, third and second. Thirds and seconds can be either minor or major. The principal thematic motive is c_1, a, g.

The non-overlapping rhythmic patterns are grouped into a few physical voices, each being played by one instrument. For instance, five patterns of the first fugue are assigned to four physical voices. This is done not only to reduce the number of performers, but also to construct musically more interesting longer motives.

To embed the fugues into the metric structure of the piece, some fugues are separated by additional rests. They are perceived as stops and are harmonically emphasized as cadences. The harmony is articulated in arpeggiations, not as score verticals but rather as operating in a time-slot of several beats.

The development is based on a certain variation principle. A fugue is assumed to be a variation of some other fugue if it has the same beginning but a new ending. For instance, the second and the third fugues are variations of the first one. It can be seen in their labels which indicate entering patterns:

$$\text{Fugue 1: } 12\boxed{233} \quad \rightarrow \quad \text{Fugue 2: } 12\ \boxed{411}$$
$$\text{Fugue 1: } 1223\boxed{3} \quad \rightarrow \quad \text{Fugue 3: } 1223\boxed{41234331}$$

The musical form of the piece is summarized in Table ??. As one can see, the harmonic plan of the piece is in analogy to Western tonal music. The development begins with the theme at the fifth ("dominant"), and the return to the main tonality passes through the "subdominant".

The selection of a particular fugue for a particular purpose is motivated by several reasons:

Figure 1: Andranik Tangian. *Eine kleine Mathmusik 2*

Figure 2: Andranik Tangian. *Eine kleine Mathmusik 2* (continued)

Table 4: The form of *Eine kleine Mathmusik 2*

Section	Material	Bars	Description
Exposition (repeated)	Theme	0–4	Fugues 1 & 2 $\underbrace{12\boxed{233}}_{C\to G^7}+\underbrace{12\boxed{411}}_{g_m^6\to d_m}$
	Variation 1	4–8	Fugue 3 $\underbrace{1223\boxed{4}12\boxed{34331}}_{F\to C}$
Development (repeated)	Variation 2	9–13	Fugues 4 & 5 $\underbrace{12\boxed{3441}+1\boxed{31431}}_{g_m\to A\flat}$
	Variation 3	13–17	Fugue 6 $\underbrace{12\boxed{3431}1\boxed{43411}}_{A\flat\to f_m}$
Slow trio (repeated)	Variation 4	18–34	Fugue 7 $\underbrace{12\boxed{31313411343114331143411}}_{F\to A\flat}$
Recapitulation	Theme	35–39	Fugues 1 & 2 $\underbrace{12\boxed{233}}_{C\to G^7}+\underbrace{12\boxed{411}}_{g_m^6\to d_m}$
	Variation 1m (minor subdominant)	39–43	Fugue 3 $\underbrace{1223\boxed{4}12\boxed{34331}}_{F\to C}$

1. For the theme, the two shortest fugues of equal length are selected, so that the form of the theme is $1 + 1$.

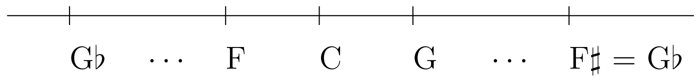
2. Variation 1 is twice longer than the theme. Thereby the exposition (Theme and Variation 1) has the form $1 + 1 + 2$.

3. The development has the same form as the exposition. For continuity, the fugues selected are somewhat longer, so that there are no gaps between them.

4. Trio is a 60-beats long rhythmic fugue with at most three simultaneous voices. To make it sound even longer, its tempo is made twice slower, so that it actually takes 120 beats. Thereby trio provides a counterbalance to the exposition and development. As usual, trio with its "thin harmony" due to few voices is put before the recapitulation.

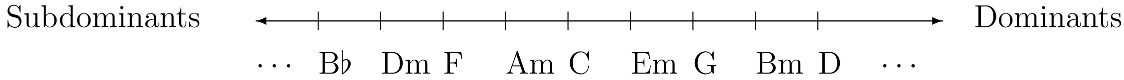
6 Tonal map in *Eine kleine Mathmusic 2*

There exist a number of maps for visualizing relationships between tonalities and chords; see Krumhansl (2002) for a survey. The best known is the line of fifths often rolled into the enharmonic circle



Here, the distance between two chords/tonalities is the number of fifths between their roots. A similar representation exists for minor chords.

Subdominant–dominant axis It is however possible to visualize the tonal proximity of both major and minor chords by putting them on one axis



and defining the distance between the neighboring chords to be 0.5. (At the line of fifths, each chord differs from its neighbor in two notes, whereas here in one.)

The left neighbors of a chord are its subdominants of different degree. For instance, if C is tonic then Am is the half-subdominant, F is the 1st subdominant, Dm is the 1.5-subdominant, *B*♭ is the 2nd subdominant, and so on. The common subdominant nature of all left-hand neighbors of a chord is illustrated by perceptual similarity of the three chord progressions

C	Am	F	G
C	Am	Dm	G
C	F	Dm	G

Here, the subdominant-dominant direction of change is more important than the leaps between the chord roots. Similarly, in melodic variations, the melody remains recognizable if the ascending/descending direction of melodic intervals at metrical accents is preserved, while the interval values being less important (Zaripov 1983).

The right neighbors of a chord are its dominants ranked in the same way. For instance, if Am is tonic, C is the half-dominant. Recall that the parallel major is often used in sonatas in minor for the second theme or for the second movement, thereby playing the same role as dominant in sonatas in major.

Tonal function of mediant chords Recall that mediant chords (e.g. Am, Em and Dm in C major) have been qualified by Weber (1817–1821), Tchaikovski (1872), and Rimski-Korsakov (1886) as auxiliary to three main harmonic functions of tonic (C), dominant (G) and subdominant (F). In our arrangement all the chords are considered together and distinguished by numerical grades.

Riemann (1893) has qualified mediants as parallel tonic, parallel dominant and parallel subdominant. In our case, the chord system is not split into parallel classes but is located on one axis with a unique tonic.

Catuar (1924), Schenker (1906–1935) and Tulin-Privano (1965) have proposed a context-dependent functional interpretation of mediant chords. For instance, Am in C major can be regarded either as tonic, or subdominant. Such an ambiguity is surmounted by considering half-steps in the discrimination of tonal functions. For instance, Am is separable both from tonic C and from subdominant F.

Visualizing the proximity of major and minor chords of the same root The proximity of major and minor chords of the same root can be reflected by the two-dimensional map in Figure **??**.

						Dominants
	⋮	⋮	⋮	⋮	⋮	↗
⋯	F♯m	A	C♯m	E	G♯m	⋯
⋯	F	Am	C	Em	G	⋯
⋯	Fm	A♭	Cm	E♭	Gm	⋯
↙	⋮	⋮	⋮	⋮	⋮	
Subdominants						

Figure 3: Map of major and minor tonalities/chords used in *Eine kleine Mathmusic 2*

The difference between horizontally neighboring chords is one note, and between vertically neighboring chords — one or two notes (respectively between C and Cm, or between C and C♯m). Therefore, the distance between vertically neighboring chords can be considered, depending on the case, as 0.5, or 1. rThe proximity of the chords with the same root can be also visualized by rolling the subdominant–dominant axis into a coil shown in Figure **??**.

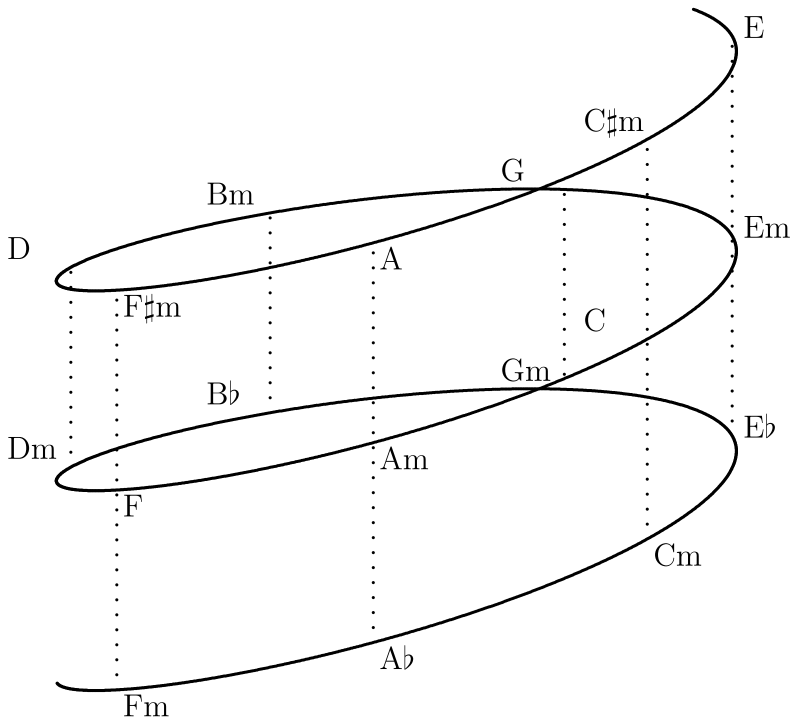


Figure 4: Subdominant-dominant coil

The chord map with enharmonic equivalence Recall that the enharmonic circle is obtained from identifying pitch classes which are 12 fifths apart. Then the cylindrical coil is rolled into a toroidal coil. It should be however emphasized that the toroidal model should not be used for finding modulation paths. For example, the enharmonic tonic, appearing in distant modulations, sounds different from the true tonic. Therefore, the return to the tonic should be done through successive back-steps on the plane map in Figure **??** rather than by enharmonic shortcuts on the toroid.

7 Summary

Let us recapitulate the main results.

The paper describes a general computational method for tiling musical events with a few rhythmic patterns. It enables constructing finite and infinite rhythmic canons and rhythmic fugues.

The method is based on the isomorphism of rhythmic structures with polynomial equations. It is implemented in a computer program which outputs rhythmic scores.

The model applications to practical composition is illustrated with a piece *Eine kleine Mathmusik 2*. It is based on rhythmic fugues computed, and the harmonic development is designed with a special tonal map.

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9 Annex: Scores of rhythmic fugues used in *Eine kleine Mathmusik 2*

Table 5: *Eine kleine Mathmusik 2.* Fugue No. 1 with 4 voices, 14 beats long, and mean pattern No. 2.2

Voice	Pattern	Measures 0	Measures 1	Measures 2
1	1	1 1 0 0 1	. . . .	.
2	2	 1	0 1 0 0 0 0 0 1	.
3	2	. . . . .	1 0 1 0 0 0 0 0	1
4	3	. . . . .	. 1 0 0 1 1 . .	.
5	3	. . . . .	. . . 1 0 0 1 1	.
Simultaneous voices		1 1 1 1 2	2 3 3 4 4 4 3 3	1
Pulse train		1 1 0 0 2	1 2 1 1 1 1 1 2	1

Table 6: *Eine kleine Mathmusik 2.* Fugue No. 2 with 4 voices, 14 beats long, and mean pattern No. 1.8

Voice	Pattern	Measures 0	Measures 1	Measures 2
1	1	1 1 0 0 1	. . . .	.
2	2	 1	0 1 0 0 0 0 0 1	.
3	4	. . . . .	1 0 0 0 0 0 1 0	1
4	1	. . . . .	. 1 1 0 0 1 . .	.
5	1	. . . . .	. . . 1 1 0 0 1	.
Simultaneous voices		1 1 1 1 2	2 3 3 4 4 4 3 3	1
Pulse train		1 1 0 0 2	1 2 1 1 1 1 1 2	1

Table 7: *Eine kleine Mathmusik 2.* Fugue No. 3 with 4 voices, 30 beats long, and mean pattern No. 2.42

Voice	Pattern	Measures 0	Measures 1	Measures 2	Measures 3	Measures 4
1	1	1 1 0 0 1	. . . .	. . . .	. . . .	.
2	2	 1	0 1 0 0 0 0 0 1	. . . .	. . . .	.
3	2	. . . . .	1 0 1 0 0 0 0 0	1	. . . .	.
4	3	. . . . .	. 1 0 0 1 1 . .	. . . .	. . . .	.
5	4	. . . . .	. . . 1 0 0 0 0	0 1 0 1	. . . .	.
6	1	. . . . .	. . . . . . 1 1	0 0 1	. . . .	.
7	2	. . . . .	. . . .	. 1 0 1 0 0 0 0	0 1	.
8	3	. . . . .	. . . .	. . . . 1 0 0 1	1	.
9	4	. . . . .	. . . .	. . . . . 1 0 0	0 0 0 1 0 1 . .	.
10	3	. . . . .	. . . .	. . . . . . 1 0	0 1 1	.
11	3	. . . . .	. . . .	. . . .	. . . 1 0 0 1 1	.
12	1	. . . . .	. . . .	. . . .	. . . . 1 1 0 0	1
Simultaneous voices		1 1 1 1 2	2 3 3 4 4 4 4 4	3 3 3 2 2 3 4 4	4 3 2 2 3 3 2 2	1
Pulse train		1 1 0 0 2	1 2 1 1 1 1 1 2	1 2 1 2 1 1 1 1	1 2 1 2 1 2 1 1	1

Table 8: *Eine kleine Mathmusik 2.* Fugue No. 4 with 4 voices, 16 beats long, and mean pattern No. 2.5

Voice	Pattern	Measures 0	Measures 1	Measures 2
1	1	1 1 0 0 1	. . . .	. . .
2	2	 1	0 1 0 0 0 0 0 1	. . .
3	3	. . . . .	1 0 0 1 1 . . .	. . .
4	4	. . . . .	. 1 0 0 0 0 0 1	0 1 .
5	4	. . . . .	. . 1 0 0 0 0 0	1 0 1
6	1	. . . . .	. . . . . 1 1 0	0 1 .
Simultaneous voices		1 1 1 1 2	2 3 4 4 4 4 4 4	3 3 1
Pulse train		1 1 0 0 2	1 2 1 1 1 1 1 2	1 2 1

Table 9: *Eine kleine Mathmusik 2.* Fugue No. 5 with 3 voices, 16 beats long, and mean pattern No. 2.17

Voice	Pattern	Measures 0	Measures 1	Measures 2
1	1	1 1 0 0 1	. . . .	. . .
2	3	 1	0 0 1 1	. . .
3	1	. . . . .	1 1 0 0 1 . . .	. . .
4	4	. . . . .	. 1 0 0 0 0 0 1	0 1 .
5	3	. . . . .	. . . . . 1 0 0	1 1 .
6	1	. . . . .	. . . . . . 1 1	0 0 1
Simultaneous voices		1 1 1 1 2	2 3 3 3 2 2 3 3	3 3 1
Pulse train		1 1 0 0 2	1 2 1 1 1 1 1 2	1 2 1

Table 10: *Eine kleine Mathmusik 2.* Fugue No. 6 with 4 voices, 30 beats long, and mean pattern No. 2.33

Voice	Pattern	Measures 0	1	2	3	4
1	1	1 1 0 0 1	. . . .	. . . .	. . . .	.
2	2	 1	0 1 0 0 0 0 0 1	. . . .	. . . .	.
3	3	. . . . .	1 0 0 1 1 . . .	. . . .	. . . .	.
4	4	. . . . .	. 1 0 0 0 0 0 1	0 1	. . . .	.
5	3	. . . . .	. . 1 0 0 1 1 .	. . . .	. . . .	.
6	1	. . . . .	. . . .	1 1 0 0 1 . . .	. . . .	.
7	1	. . . . .	. . . .	. . 1 1 0 0 1 .	. . . .	.
8	4	. . . . .	. . . .	. . . 1 0 0 0 0	0 1 0 1	.
9	3	. . . . .	. . . .	. . . . . 1 0 0	1 1	.
10	4	. . . . .	. . . .	 1	0 0 0 0 0 1 0 1	.
11	1	. . . . .	. . . .	. . . .	. . 1 1 0 0 1 .	.
12	1	. . . . .	. . . .	. . . .	. . . . 1 1 0 0	1
Simultaneous voices		1 1 1 1 2	2 3 4 4 4 3 3 2	2 2 2 3 3 3 3 3	3 3 3 3 3 3 3 2	1
Pulse train		1 1 0 0 2	1 2 1 1 1 1 1 2	1 2 1 2 1 1 1 1	1 2 1 2 1 2 1 1	1

Table 11: *Eine kleine Mathmusik 2.* Fugue No. 7 with 3 voices, 60 beats long, and mean pattern No. 2.28

Voice	Pattern	Measures 0	1	2	3	4	5	6	7	8
1	1	1	1 0 0 1	. . . .	. . . .	. . . .	. . . .	. . . .	. . . .	. . .
2	2	.	. . . 1 0 1 0 0	0 0 0 1	. . . .	. . . .	. . . .	. . . .	. . . .	. . .
3	3	.	. . . . 1 0 0 1	1	. . . .	. . . .	. . . .	. . . .	. . . .	. . .
4	1	.	. . . . . 1 1 0	0 1	. . . .	. . . .	. . . .	. . . .	. . . .	. . .
5	3	.	. . . .	. . 1 0 0 1 1 .	. . . .	. . . .	. . . .	. . . .	. . . .	. . .
6	1	.	. . . .	. . . 1 1 0 0 1	. . . .	. . . .	. . . .	. . . .	. . . .	. . .
7	3	.	. . . .	. . . . . 1 0 0	1 1	. . . .	. . . .	. . . .	. . . .	. . .
8	4	.	. . . .	 1	0 0 0 0 0 1 0 1	. . . .	. . . .	. . . .	. . . .	. . .
9	1	.	. . . .	. . . .	. . 1 1 0 0 1 .	. . . .	. . . .	. . . .	. . . .	. . .
10	1	.	. . . .	. . . .	. . . . 1 1 0 0	1	. . . .	. . . .	. . . .	. . .
11	3	.	. . . .	. . . .	 1	0 0 1 1	. . . .	. . . .	. . . .	. . .
12	4	.	. . . .	. . . .	. . . .	. 1 0 0 0 0 0 1	0 1	. . . .	. . . .	. . .
13	3	.	. . . .	. . . .	. . . .	. 1 0 0 1 1 . .	. . . .	. . . .	. . . .	. . .
14	1	.	. . . .	. . . .	. . . .	. . . . . . 1 1	0 0 1	. . . .	. . . .	. . .
15	1	.	. . . .	. . . .	. . . .	. . . .	1 1 0 0 1 . . .	. . . .	. . . .	. . .
16	4	.	. . . .	. . . .	. . . .	. . . .	. . . 1 0 0 0 0	0 1 0 1	. . . .	. . .
17	3	.	. . . .	. . . .	. . . .	. . . .	. . . 1 0 0 1 1	. . . .	. . . .	. . .
18	3	.	. . . .	. . . .	. . . .	. . . .	. . . . . 1 0 0	1 1	. . . .	. . .
19	1	.	. . . .	. . . .	. . . .	. . . .	. . . .	. . 1 1 0 0 1 .	. . . .	. . .
20	1	.	. . . .	. . . .	. . . .	. . . .	. . . .	. . . . 1 1 0 0	1	. . .
21	4	.	. . . .	. . . .	. . . .	. . . .	. . . .	. . . . . 1 0 0	0 0 0 1 0 1 . .	. . .
22	3	.	. . . .	. . . .	. . . .	. . . .	. . . .	 1	0 0 1 1	. . .
23	4	.	. . . .	. . . .	. . . .	. . . .	. . . .	. . . .	. 1 0 0 0 0 0 1	0 1 .
24	1	.	. . . .	. . . .	. . . .	. . . .	. . . .	. . . .	. . . . 1 1 0 0	1 . .
25	1	.	. . . .	. . . .	. . . .	. . . .	. . . .	. . . .	. . . . . . 1 1	0 0 1
Simult.voices		1	1 1 1 2 2 3 3 3	3 2 2 3 2 3 3 3	2 2 2 2 3 3 3 3	2 3 3 3 2 2 2 2	3 3 2 3 3 3 3 3	2 2 2 2 2 3 3 3	3 3 3 3 3 3 3 3	3 2 1
Pulse train		1	1 0 0 2 1 2 1 1	1 1 1 2 1 2 1 2	1 1 1 1 1 2 1 2	1 2 1 1 1 1 1 2	1 2 1 2 1 1 1 1	1 2 1 2 1 2 1 1	1 1 1 2 1 2 1 2	1 1 1