Supplementary Sets and Regular Complementary Unending Canons (Part One)

Dan Tudor Vuza

0. Introduction

One of the most interesting situations encountered in the study of pitch-class sets is represented by the existence of partitions of the set of all twelve pitch classes into subsets which belong to the same transpositional class. For instance, there exists:

(*a*) a partition into four augmented trichords;

(*b*) a partition into three diminished-seventh tetrachords;

(*c*) a partition into three tetrachords {B, C, F, G♭}, {C♯, D, G, A♭}, {D♯, E, A, B♭};

(*d*) a partition into three minor tetrachords {C, D, E♭, F}, {E, F♯, G, A}, {G♯, A♯, B, C♯};

(*e*) a partition into four trichords {B, C♯, D♯}, {C, D, E}, {F, G, A}, {G♭, A♭, B♭}.

I have examined the theory of such partitions in my papers 1983a and 1983b. A transpositional class (or type) with the property that there is a partition of the set of all twelve pitch classes into subsets belonging to that class was called there a *partitioning class*. A glance at examples (*a*)–(*e*) might leave the impression that there is not an apparent relation between partitioning classes and transpositional symmetry: examples (*a*)–(*c*) involve classes with transpositional symmetry, while the classes in examples (*d*)–(*e*) do not possess such a symmetry. In Vuza 1983a and 1983b I have demonstrated that there is however a close connection between the phenomenon of partitioning classes and the phenomenon of transpositional symmetry. To this end I have introduced the notions of supplementary sets and of supplementary classes. Two pitch-class sets M and N have been called *supplementary* if the product of their numbers of elements equals 12 and the intersection between the set of intervals spanned by the elements in M and the set of intervals spanned by the elements in N is reduced to the null interval.[1] Two transpositional classes have been called *supplementary* if they are respectively the classes of two supplementary sets. From the point of view of the theory of partitioning classes, it makes no difference whether we work with sets of pitch classes or with sets of residue classes (that is, subsets of the group $\mathbf{Z}_{12}$), as the concepts involved are invariant under transposition and hence do not depend on the particular labelling (in the sense of Lewin 1987) of pitch classes by elements of $\mathbf{Z}_{12}$. In particular, the concepts of supplementary sets and of supplementary classes may refer to sets of pitch classes as well as to sets of residue classes.

The notion of supplementary classes is related to that of a partitioning class by the result asserting that a class M is partitioning iff (= if and only if) there is a class N so that M and N are supplementary. For instance, the diagram below exhibits the pairs of supplementary classes in examples (*a*)–(*e*).

$$\begin{tikzcd} d \arrow[r, dash] & a \arrow[r, dash] \arrow[d, dash] & b \arrow[d, dash] \\ & c \arrow[r, dash] & e \end{tikzcd}$$

The connection with transpositional symmetry is now apparent due to the following theorem stated without proof in Vuza 1983a and proved in Vuza 1983b:

THEOREM 0.1. *Given any pair of supplementary sets, at least one set in the pair has transpositional symmetry.*

This theorem has of course both theoretical and practical consequences. The strong restrictions it imposes on the possibilities of finding partitioning classes allowed me to work out the complete list of pairs of supplementary classes in a relatively small number of steps (see Vuza 1983a, 1983b).

It is by now a well-founded principle that the mathematics of sets of residue classes may serve to model phenomena in the universe of pitch-class sets as well as rhythmic phenomena characterized by periodicity: "The possibility of perceiving any twelve-integer set as a succession of pitch classes or as a succession of time-points and the possibility of applying the same systematic operations in the rhythmic domain as have been applied previously in the pitch domain has been presented and explained by Milton Babbitt" [in Babbitt 1962]. (Quoted from Johnson 1984). In accord with this general principle, I try to answer in the present study the following questions:

QUESTION 1. What is the rhythmic analog of a partition into subsets belonging to the same transpositional class? What is the rhythmic analog of supplementary classes?

QUESTION 2. What is the rhythmic interpretation of the fact that one of the classes in a supplementary pair has transpositional symmetry?

QUESTION 3. Does Theorem 0.1 remain true when the group $\mathbf{Z}_{12}$ is replaced by a group $\mathbf{Z}_n$ for some arbitrary integer n?

In Babbitt's model one considers a set T of regular pulses (successive pulses separated by the same time interval which is taken as time unit) and one sets a correspondence between the subsets of $\mathbf{Z}_{12}$ and those periodic subrhythms of T whose periods count 1, 2, 3, 4, 6, or 12 time units. By applying this procedure to a partition of $\mathbf{Z}_{12}$ into subsets belonging to the same transpositional class one obtains a partition of the total rhythm T into periodic subrhythms $R_1, \ldots, R_l$ so that for any couple i, j the rhythm R_i is obtained from R_j via a temporal translation. We analyze separately the musical significance of the properties of the rhythmic partition $R_1, \ldots, R_l$. In the course of this analysis we suppose that each R_i represents the set of time points associated with the rhythmic pattern delivered by a voice V_i.

The fact that the R_i's can be obtained each from the other via a temporal translation means that the voices $V_1, \ldots, V_l$ all together are singing a rhythmic canon in strict style; the fact that each R_i is periodic means that the canon is unending.

The fact that the R_i's form a partition of the total rhythm means on the one hand that there are no beats (attacks) in common among different voices (so that the voices are "complementary") and on the other hand that the resultant rhythm obtained by adding up the beats from all voices equals the total rhythm. In other words, every pulse in the total rhythm corresponds to a beat from one and only one voice. Rhythmic canons of the type described above are called in the present paper *regular complementary unending canons.*

To find the rhythmic analog of supplementary classes, consider first the situation when the voices $V_1, \ldots, V_l$ are singing an arbitrary unending rhythmic canon in strict style and let R_i be the set of time points associated with the voice V_i. We express the fact that each R_i represents the same rhythmic pattern except for a temporal translation by saying that all the sets R_i belong to the same rhythmic class R, which we call the *ground class* of the canon in question. Suppose now that the beginning of each period of R_i is marked by a metric accent. If one adds together the metric accents from all voices one obtains another periodic rhythm, whose rhythmic class S is referred to as the *metric class* of the canon. In this study I define the notion of supplementary rhythmic classes and I show that an unending rhythmic canon is regular and complementary iff its ground class and its metric class are supplementary.

To answer Question 2, observe that the metric class controls the relative distances in time between the voices in a canon. The ratio

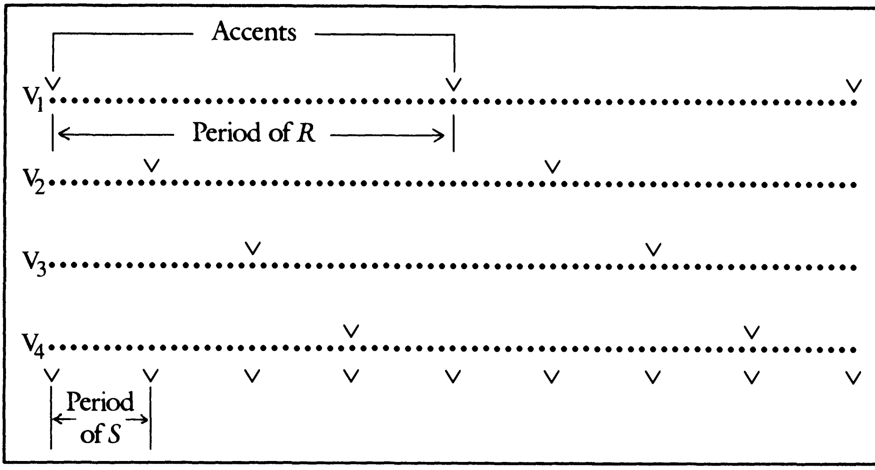
$$\frac{\text{Period of } R}{\text{Period of } S}$$

is an integer which divides the number l of voices, and hence can take values only in the range from 1 to l. (The period of a rhythmic class is defined as the period of any periodic rhythm belonging to that class.) If that ratio equals l, the relative distances divide the period of R into l equal parts (see Example 0.1).

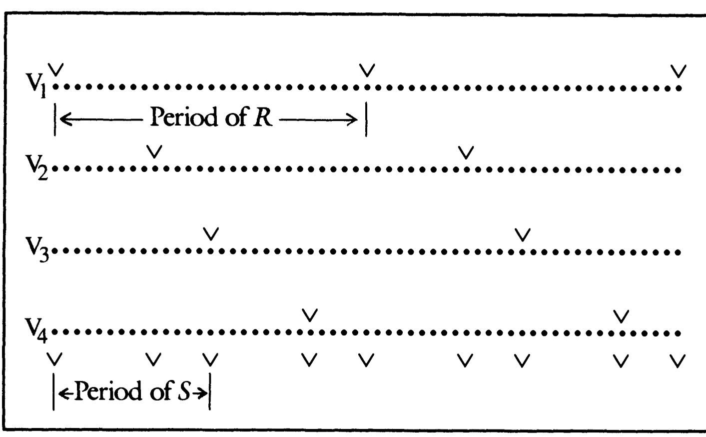
If the ratio equals $l/2$, we have a grouping of the relative distances which is repeated every half a period of R (see Example 0.2).

Similar groupings can be found if the ratio in question is not 1. If however the ratio equals 1, there is no grouping of the relative distances which is regularly repeated within a period of R (see Example 0.3).

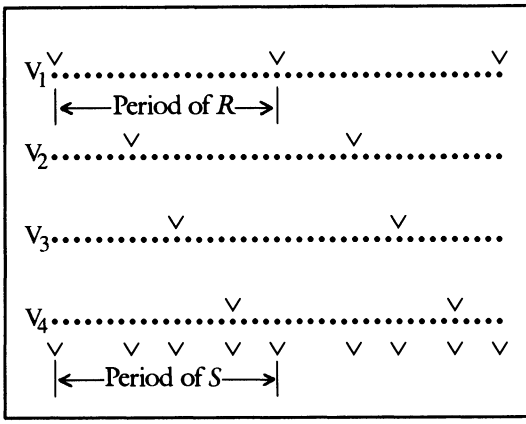
Canons of the latter type will be referred to in the following as *canons of maximal category.*[2] The conclusion of the above discussion is that the statement that one transpositional class in a certain pair of supplementary classes has transpositional symmetry corresponds, in the rhythmic domain, to the statement that a certain regular complementary unending canon is not of maximal category.



EXAMPLE 0.1



EXAMPLE 0.2



EXAMPLE 0.3

We come now to Question 3. If we confined ourselves to canons obtained via rhythmic interpretation of subsets of $\mathbf{Z}_{12}$, then according to Theorem 0.1, we would never obtain regular complementary unending canons of maximal category. A higher level of generality in the study of regular complementary canons is attained by employing the model for periodic rhythm proposed in Vuza 1985, which may be regarded as an extension and a further elaboration of Babbitt's ideas. In this model, various rhythmic classes correspond to translation classes of different groups $\mathbf{Z}_n$, the integer n ranging from 1 to infinity (a translation class is the $\mathbf{Z}_n$-analog of what we used to call, for $n = 12$, a transpositional class). More explicitly, if the rhythmic classes R_1 and R_2, correspond to a translation class of $\mathbf{Z}_{n_1}$ and $\mathbf{Z}_{n_2}$, respectively, then certain set-theoretic operations applied to periodic rhythms belonging to R_1 and R_2 may lead to rhythms whose classes correspond to translation classes of $\mathbf{Z}_{n_3}$, the integer n_3 taking arbitrary large values even if n_1 and n_2 do not vary.

Regular complementary unending canons are still related, from the viewpoint of the extended rhythmic model, to supplementary rhythmic classes, while the latter are closely related to supplementary subsets of $\mathbf{Z}_n$ (their definition is formally the same as previously with the only difference being that 12 is replaced by n). The conclusion is that the study of canons of the mentioned type motivates the study of supplementary subsets of $\mathbf{Z}_n$ for n an arbitrary positive integer. For this reason it is important to answer Question 3. The answer is negative and it is in fact possible to describe explicitly the set of those integers n with the property that Theorem 0.1 is true for all pairs of supplementary subsets of $\mathbf{Z}_n$. It is however worthwhile to note that Theorem 0.1 continues to hold for $n = 24$. In other words, considering again the pitch-class setting, Theorem 0.1 is true both for the twelve-tone system and the twenty-four-tone (or "quarter-tone") system.

From another viewpoint, we may remark that the number of elements in at least one set in a pair of supplementary subsets of $\mathbf{Z}_{12}$ has the form p^k with p a prime (2 or 3 for $\mathbf{Z}_{12}$). We may therefore ask whether it is true that at least one set in a pair of supplementary subsets of $\mathbf{Z}_n$ has "transpositional" symmetry every time the number of elements in at least one set in that pair has the form p^k with p a prime. It turns out that this is indeed true for all values ≥ 2 of n.

The theorems about supplementary subsets to be presented in a separate section of this article are then interpreted within the context of the theory of canons. For instance, the discovery of the fact that Theorem 0.1 is not true for certain values of n has the consequence that *there exist regular complementary unending canons of maximal category*. The theorems about canons proved by means of the theory of supplementary sets have at least two significations. On the one hand, they impose strong restrictions on the construction of regular complementary unending canons. As an illustration, no specialist in rhythmic counterpoint could construct a regular complementary unending canon of maximal category on 2, 3, 4, 5, 7, 8, 9, or p^k voices (p a prime, k an integer ≥ 1) for the reason that this is a mathematical impossibility! On the other hand, those theorems may be regarded in connection with the general principle according to which the simpler the arithmetical structure of a number measuring a certain musical phenomenon, the clearer is the structure of that phenomenon. In our specific situation, it is true that whenever the arithmetical structure of the integers measuring the complexity of a regular complementary canon is simple (in the sense that there are not too many primes dividing them), then the canon itself has a clear and neat (at least from the mathematical viewpoint) recursive structure, as it can be obtained by successive applications of a very neat procedure described in the following sections under the name of "elementary derivation."

Besides elementary derivations, some other transformational procedures such as inversions and multiplicative transformations are discussed. These can be applied to regular complementary canons or even more generally, to complementary canons which are not regular.

Regular complementary unending canons of simple structure often occur in the rhythmic organization of musical works from the preclassical period. Canons of the type mentioned and of more complicated structure seem not to have been used so far, as their construction looks quite difficult without the aid of a mathematical theory. It is the purpose of the present study to lay the foundations of such a theory and to demonstrate that even in connection with this polyphonic form subjected to such strong mathematical restrictions, the composer has quite a large choice of constructive and transformational procedures which may be combined in a creative way.

From the mathematical viewpoint, the theory of supplementary sets represents an instance of application of Fourier analysis of finite groups to music theory, thus continuing a line of research inaugurated by Professor David Lewin in 1959. In fact, the entire theory of supplementary sets can be subsumed to a problem area indicated by Lewin in his remarkable book *Generalized Musical Intervals and Transformations* (see Section 9 of this paper for precise formulations). The problems in this area arising from musical considerations are hard, so that technical difficulties are often unavoidable when trying to solve them. Music is an extremely complex phenomenon; we therefore expect the mathematics used in studying it to support a similar complexity. With this principle in mind, I have decided to present to the readers of *Perspectives of New Music* a detailed account of my own research. In undertaking this task I have been stimulated by the aim given to the journal by its editorial board, particularly by the enthusiastic Editor, Professor John Rahn, to promote works "which may be useful to or inspirational for musical thinkers and doers of the future."

1. Mathematical Preliminaries

Here are some notations to be used throughout the paper.

$\mathbf{Z}$: the ring of all integer numbers.

$\mathbf{Z}_n$: the ring of integers modulo n. Here n ranges over all integers ≥ 1. We agree that $\mathbf{Z}_1 = \{0\}$ (a group reduced to the neutral element).

$\mathbf{Q}$: the field of all rational numbers.

$\mathbf{Q}_+$: the set of all strictly positive rational numbers.

$|a|$: the absolute value of $a \in \mathbf{Q}$ (equal to a if $a \geq 0$, to $-a$ if $a < 0$).

$[a, b)$: the set of all $c \in \mathbf{Q}$ satisfying $a \leq c < b$.

$[a, b]$: the set of all $c \in \mathbf{Q}$ satisfying $a \leq c \leq b$.

$\#\mathrm{M}$: the number of elements in the finite set M.

f^{-1}: the inverse of the bijective map f.

If $\mathrm{f}: \mathrm{A} \rightarrow \mathrm{B}$ is a map and $\mathrm{M} \subset \mathrm{A}$, $\mathrm{N} \subset \mathrm{B}$ are subsets, then we denote by $\mathrm{f}(\mathrm{M})$ the set $\{\mathrm{f}(x) \mid x \in \mathrm{M}\}$ and by $\mathrm{f}^{-1}(\mathrm{N})$ the set $\{x \mid x \in \mathrm{A}, \mathrm{f}(x) \in \mathrm{N}\}$.

Given $a,b \in \mathbf{Q}_+$, we say that *a divides b* and we write $a|b$ if $b/a \in \mathbf{Z}$. The divisibility relation endows $\mathbf{Q}_+$ with a structure of a lattice-ordered set.[3] The greatest lower bound and the least upper bound (with respect to the divisibility relation) of a and b in $\mathbf{Q}_+$ will be denoted by $a \wedge b$ and $a \vee b$, respectively. When a and b are integers, $a \wedge b$ and $a \vee b$ coincide with the usual greatest common divisor and least common multiple, respectively known from arithmetic; it is therefore natural to continue to use the same terminology even in the case when a and b are not integers. The relations

$$c(a \wedge b) = ca \wedge cb, \; c(a \vee b) = ca \vee cb$$

allow one to reduce the computation of $a \wedge b$ and $a \vee b$ for nonintegral values of a and b to the case of integral values. We shall use the symbol a non $|$ b for "a does not divide b."

By a *prime* we shall mean any prime number ≥ 2.

Two integers $m,n \neq 0$ are called *relatively prime* if $|m| \wedge |n| = 1$.

For $a \in \mathbf{Q}_+$ and $\mathrm{M} \subset \mathbf{Q}$, $a\mathrm{M}$ will be the set $\{ax \mid x \in \mathrm{M}\}$.

All groups to be considered in this paper will be commutative; hence by a group we shall mean a commutative group.

Let G be a group. For any two subsets M,N of G we let M + N be the subset of all sums $x + y$ with $x \in \mathrm{M}$ and $y \in \mathrm{N}$. If M contains only one element x, we write $x + \mathrm{N}$ instead of $\{x\} + \mathrm{N}$. If $k \in \mathbf{Z}$ we denote by $k\mathrm{M}$ the set $\{kx| \, x \in \mathrm{M}\}$.[4] In particular, we may define M − N as M + (−N).

The relation "there is $x \in \mathrm{G}$ such that $\mathrm{M} = \mathrm{x} + \mathrm{N}$" is an equivalence relation between subsets of G. The equivalence classes with respect to this relation are referred to as *translation classes* of G; we denote by $T(\mathrm{G})$ the set of all such classes. Capital italics will be used to indicate translation classes. The notation [M] will signify the translation class of the subset M of G.

A structure of a commutative semigroup with unit element is defined on $T(\mathrm{G})$ as follows: if $M,N \in T(\mathrm{G})$ are given, choose sets $\mathrm{M} \in M$ and $\mathrm{N} \in N$ and define the composition $M + N$ as equal to [M + N]. It is easy to see that the definition does not depend on the particular choices of the representants M and N.[5] The unit element of that semigroup is $[\{0\}]$.

For every $M \in T(\mathrm{G})$, the set M − M does not depend on the choice of M in M; we shall denote it by Int M.[6]

Given $k \in \mathbf{Z}$ and $M \in T(\mathrm{G})$, we denote by kM the translation class of $k\mathrm{M}$, where $\mathrm{M} \in M$. In the case when G is finite, we define Nr M as equal to #M for $\mathrm{M} \in M$. (Once again these definitions do not depend on the choices of M.)

Let H be a subgroup of G. The subsets of G in the translation class of H are called the *cosets of* G *modulo* H. Two cosets $x + \mathrm{H}$ and $y + \mathrm{H}$ either coincide (in which case $x - y \in \mathrm{H}$) or are disjoint (in which case $x - y \notin \mathrm{H}$). The unique coset modulo H which contains an element x is also referred to as the *class of x modulo* H. By a *set of representants of* G *modulo* H is meant any subset of G which meets every coset modulo H at one and only one element.

A subset M of G is called H-*periodic* if $\mathrm{H} + \mathrm{M} = \mathrm{M}$. A subset of G is H-periodic iff it is a union of cosets of G modulo H. A subset M of G is called *periodic* if it is H-periodic for some subgroup $\mathrm{H} \neq \{0\}$. The set of all $x \in \mathrm{G}$ satisfying $x + \mathrm{M} = \mathrm{M}$ is a subgroup of G, called the *stability subgroup* of M. The stability subgroups of two subsets in the same translation class coincide; we may therefore speak about the stability subgroup of a translation class $\mathcal{M}$, equal by definition to the stability subgroup of any $\mathrm{M} \in \mathcal{M}$.

If $\mathrm{H}_1,\mathrm{H}_2$ are two subgroups of G, then so are $\mathrm{H}_1 + \mathrm{H}_2$ and $\mathrm{H}_1 \cap \mathrm{H}_2$. The representation of every element in $\mathrm{H}_1 + \mathrm{H}_2$ as $\mathrm{x}_1 + \mathrm{x}_2$ with $\mathrm{x}_i \in \mathrm{H}_i$ $(i = 1,2)$ is unique if $\mathrm{H}_1 \cap \mathrm{H}_2 = \{0\}$; we indicate this situation by saying that the sum $\mathrm{H}_1 + \mathrm{H}_2$ is *direct*.

Given two groups $\mathrm{G}_1,\mathrm{G}_2$ and a group homomorphism $\varphi:\mathrm{G}_1 \rightarrow \mathrm{G}_2$, the set of those $x \in \mathrm{G}_1$ such that $\varphi(x) = 0$ is a subgroup of G_1 called the *kernel* of φ and denoted by Ker φ.

Finally we recall some facts about the groups $\mathbf{Z}_n$. (The ring structure of $\mathbf{Z}_n$ will not be important for our purposes.) The canonical homomorphism from $\mathbf{Z}$ onto $\mathbf{Z}_n$, which maps an integer onto its residue class modulo n, is denoted by φ_n. This notation will be employed only in theoretical considerations. In concrete situations, when there is no confusion on n, we need not resort to such complications; we may simply write $\{0, 4, 7\}$ instead of $\{\varphi_{12}(0), \varphi_{12}(4), \varphi_{12}(7)\}$ if we know that we are working within $\mathbf{Z}_{12}$.

We shall also make use of other canonical homomorphisms, which act between the groups $\mathbf{Z}_n$. For any divisor d of n, we let $\varphi_{n,d}:\mathbf{Z}_n \rightarrow \mathbf{Z}_d$ be the (ring) homomorphism defined by $\varphi_{n,d}(\varphi_n(k)) = \varphi_d(k)$ for every $k \in \mathbf{Z}$. The homomorphism $\varphi_{n,d}$ is onto; its kernel equals $d\mathbf{Z}_n$.

The automorphisms of the group $\mathbf{Z}_n$ are precisely the maps of the form $x \mapsto kx$, where k is any integer $\neq 0$ relatively prime to n.

There is a one-to-one correspondence between the subgroups of $\mathbf{Z}_n$ and the positive divisors of n. Namely, to each positive integer d which divides n there corresponds the subgroup $(n/d)\mathbf{Z}_n$ of $\mathbf{Z}_n$, which is the unique subgroup with d elements of $\mathbf{Z}_n$. If $\mathrm{M} \subset \mathbf{Z}_n$, we use the shorter locution "M is d-periodic" instead of "M is $(n/d)\mathbf{Z}_n$-periodic."

We record the following formulas (for d_1, d_2 divisors of n):

$$d_1\mathbf{Z}_n + d_2\mathbf{Z}_n = (d_1 \wedge d_2)\mathbf{Z}_n\,,$$
$$d_1\mathbf{Z}_n \cap d_2\mathbf{Z}_n = (d_1 \vee d_2)\mathbf{Z}_n\,.$$

In particular, the sum $d_1\mathbf{Z}_n + d_2\mathbf{Z}_n$ is direct iff $d_1 \vee d_2 = n$.

2. Three Theorems about Supplementary Sets

DEFINITION 2.1. *Two subsets* M,N *of a group* G *are called supplementary if every* $x \in$ G *can be written in a unique manner as* $y + z$ *with* $y \in$ M *and* $z \in$ N.
We also say that M is supplementary to N.

In finite groups, supplementary subsets can be given another characterization:

PROPOSITION 2.1. *Let* G *be a finite group and let* M,N *be subsets of* G. *Then "*M *and* N *are supplementary" is equivalent to the conjunction of any two of the following conditions:*

(i) M + N = G;

(ii) (M − M) ∩ (N − N) = {0};

(iii) (#M) (#N) = #G.

Proof. Consider the map f:M × N→G defined by f(y,z) = $y + z$. "M and N are supplementary" is equivalent to "f is bijective"; condition (i) is equivalent to "f is onto" while condition (ii) is equivalent to "f is one-to-one." Hence the proof follows taking into account the above remarks and the fact that for a map f between two finite sets with equal numbers of elements, the conditions "f is onto," "f is one-to-one," and "f is bijective" are all equivalent.

If M is supplementary to N, then x + M is supplementary to y + N for every $x,y \in$ G. Hence we may speak about supplementary translation classes in the following sense: two translation classes M,N are called *supplementary* if each M $\in M$ is supplementary to each N $\in N$.

As explained in the Introduction, the notions of supplementary sets and of supplementary translation classes were introduced in Vuza 1983a and 1983b for the case G = $\mathbf{Z}_{12}$. We have also seen that Theorem 0.1 is true for G = $\mathbf{Z}_{12}$ and we have pointed out that this result is no longer true in the case of a group $\mathbf{Z}_n$ with n arbitrary. The next two theorems aim to give a picture of the situation, as n varies over the set of all positive integers. Before stating them, we introduce some special sets of integers:

$$\mathbf{N}_0 = \{p^k \mid p \text{ prime}, k \geq 0\};$$
$$\mathbf{N}_1 = \{p^k q \mid p,q \text{ distinct primes}, k \geq 1\};$$
$$\mathbf{N}_2 = \{p^2 q^2 \mid p,q \text{ distinct primes}\};$$
$$\mathbf{N}_3 = \{p^k qr \mid p,q,r \text{ distinct primes}, k \in \{1, 2\}\};$$
$$\mathbf{N}_4 = \{pqrs \mid p,q,r,s \text{ distinct primes}\};$$
$$\mathbf{N} = \bigcup_{i=0}^{4} \mathbf{N}_i.$$

THEOREM 2.1. *For every integer $m \geq 1$ the following conditions are equivalent:*

(i) $m \in \mathbf{N}_0$;

(ii) *For every integer $n \geq 2$ and every pair* M,N *of supplementary subsets of* $\mathbf{Z}_n$ *with* #M $= m$, *it is true that at least one of the subsets* M,N *is periodic.*

THEOREM 2.2. *For every integer $n \geq 2$ the following conditions are equivalent:*

(i) $n \in \mathbf{N}$;

(ii) *In every pair of supplementary subsets of* $\mathbf{Z}_n$, *at least one of the subsets is periodic.*

Theorem 0.1 represents a special case of Theorem 2.1 as well as of Theorem 2.2.

An obvious way to produce supplementary subsets is to take any subgroup of G as M and any set of representants of G modulo M as N. However, this procedure is far from yielding all pairs of supplementary sets; even in the simple case of $\mathbf{Z}_{12}$, one finds supplementary sets such that none of them is a coset modulo some subgroup (for instance $\{0, 1, 6, 7\}$ and $\{0, 2, 4\}$). The preceding theorems show that in the general case the situation is even more complicated.

We record here for later use the following property of the set $\mathbf{N}$: *if $n \in \mathbf{N}$ and d is an integer ≥ 1 which divides n, then $d \in \mathbf{N}$.*

Obviously, if M and N are supplementary translation classes in $\mathbf{Z}_n$, so are kM and kN for every integer $k \neq 0$ relatively prime to n (as $x \rightarrow kx$ is an automorphism of $\mathbf{Z}_n$). Less obvious is the fact that if we multiply by k only one of the classes M,N, we obtain again a pair of supplementary classes:

THEOREM 2.3. *Let* M,N *be supplementary subsets of* $\mathbf{Z}_n$. *Then* kM *and* N *are supplementary subsets for every* $k \neq 0$ *relatively prime to* n.

In view of the remark preceding it, Theorem 2.3 is significative only in the situation when $kM \neq M$ and $kN \neq N$. This situation does not occur for $n = 12$, but it does occur for larger values of n (see Section 8).

In this section only the proofs of the implications "(ii) $\Rightarrow$ (i)" in Theorems 2.1 and 2.2 will be presented. The proofs of the converse implications as well as the proof of Theorem 2.3 are postponed until Section 9. There are two reasons for doing so. Firstly, the proofs of the implications "(ii) $\Rightarrow$ (i)" can be cast into the language of the relatively elementary algebraic notions introduced in Section 1, while the proofs of the other facts require more elaborate devices. Secondly, the proofs of the implications mentioned are based on a method of constructing nonperiodic supplementary sets; the reader may wish to apply that method for constructing regular complementary canons of maximal category, as will be explained in Section 7.

The proofs of the implications "(ii) $\Rightarrow$ (i)" in Theorems 2.1 and 2.2 are consequences of the following proposition:

PROPOSITION 2.2. *Suppose that* $n = p_1 p_2 n_1 n_2 n_3$ *with* p_1, p_2 *primes,* $n_i \geq 2$ *for* $1 \leq i \leq 3$ *and* $p_1 n_1$ *relatively prime to* $p_2 n_2$. *Then there are nonperiodic supplementary subsets* M,N *of* $\mathbf{Z}_n$ *such that* $\#\mathrm{M} = n_1 n_2$ *and* $\#\mathrm{N} = p_1 p_2 n_3$.

Indeed, suppose Proposition 2.2 has been proved. Then any integer $m \geq 1$ not in $\mathbf{N}_0$ can be written as $n_1 n_2$ with $n_1 \geq 2$, $n_2 \geq 2$, and $n_1 \wedge n_2 = 1$. By choosing two primes p_1, p_2 such that $p_1 n_1 \wedge p_2 n_2 = 1$ and by applying Proposition 2.2 to $n = 2 p_1 p_2 n_1 n_2$ we see that condition (ii) in Theorem 2.1 is not verified by such an m. On the other hand, it is easy to see that the integers n which can be decomposed as in the hypothesis of Proposition 2.2 are precisely those integers $n \geq 2$ which do not belong to $\mathbf{N}$; hence the implication (ii) $\Rightarrow$ (i) in Theorem 2.2 is also a consequence of the mentioned proposition.

Before proving Proposition 2.2 we state a lemma which will also be used in a later section. The integer n in this lemma is related in no way to the integer n in the statement of Proposition 2.2.

LEMMA 2.1. *For any prime divisor* p *of* n *there is a nonperiodic set of representants of* $\mathbf{Z}_n$ *modulo* $(n/p)\mathbf{Z}_n$.

Proof of Lemma 2.1. If $n = p^k$ for some $k \geq 1$, then any set of representants modulo $(n/p)\mathbf{Z}_n$ will do, as for such an n every periodic subset of $\mathbf{Z}_n$ must be p-periodic.

Suppose therefore that there are at least two distinct primes which divide n. Let $p_1 = p, p_2, \ldots, p_r$ be all the distinct primes which divide n. Set $l = p_1 p_2 \ldots p_r$, $m = p_2 \ldots p_r$. Let $x_1, \ldots, x_s$ be a set of representants of $\mathbf{Z}_n$ modulo $(n/l)\mathbf{Z}_n$. For every $i \in \{2, \ldots, s\}$ let R_i be any set of representants of the group $(n/l)\mathbf{Z}_n$ modulo its subgroup $(n/p)\mathbf{Z}_n$. Choose $y \in (n/p)\mathbf{Z}_n \setminus (n/m)\mathbf{Z}_n$ (which is possible as $(n/p)\mathbf{Z}_n \cap (n/m)\mathbf{Z}_n = \{0\}$) and set $R_1 = ((n/m)\mathbf{Z}_n \setminus \{0\}) \cup \{y\}$. Then R_1 is a set of representants of $(n/l)\mathbf{Z}_n$ modulo $(n/p)\mathbf{Z}_n$ and

$$R = \bigcup_{i=1}^{s} (x_i + R_i)$$

is a set of representants of $\mathbf{Z}_n$ modulo $(n/p)\mathbf{Z}_n$. R is not periodic: for if the stability subgroup of R were not reduced to $\{0\}$, it would contain $(n/p_j)\mathbf{Z}_n$ for some $j \in \{1, \ldots, r\}$. That is, R would be p_j-periodic, and this would imply that R_1 is p_j-periodic, as $(n/p_j)\,\mathbf{Z}_n \subset (n/l)\,\mathbf{Z}_n$. However, R_1 is p_j-periodic for no $j \in \{1, \ldots, r\}$. The proof of the lemma is thus complete.

Proof of Proposition 2.2. Let n satisfy the hypothesis of the proposition in question. Let M_i $(i = 1,2)$ be a nonperiodic set of representants of the group $(n/p_i n_i)\mathbf{Z}_n$ modulo its subgroup $(n/p_i)\mathbf{Z}_n$. The existence of such sets follows in the general case from Lemma 2.1 applied to the group $(n/p_i n_i)\mathbf{Z}_n$ (isomorphic with $\mathbf{Z}_{p_i n_i}$); of course, in concrete situations, many other choices are available. Take the set $M_1 + M_2$ as M. To define N, choose first x_i in $(n/p_i n_i)\mathbf{Z}_n \setminus (n/p_i)\mathbf{Z}_n$ $(i = 1,2)$ and set

$$N_1 = (n/p_1)\mathbf{Z}_n + (((n/p_2)\mathbf{Z}_n \setminus \{0\}) \cup \{x_1\}),$$

$$N_2 = (n/p_2)\mathbf{Z}_n + (((n/p_1)\mathbf{Z}_n \setminus \{0\}) \cup \{x_2\}).$$

Choose then any set R of representants of $\mathbf{Z}_n$ modulo $n_3\mathbf{Z}_n$ and set $S = R \setminus n_3\mathbf{Z}_n$. Finally, take the set $N_1 \cup (N_2 + S)$ as N. Note that by construction we have

$$M \cup N_1 \cup N_2 \subset n_3\mathbf{Z}_n. \tag{1}$$

We count the numbers of elements in M and in N. As $M_i \subset (n/p_i n_i)\mathbf{Z}_n$ and the sum $(n/p_1 n_1)\mathbf{Z}_n + (n/p_2 n_2)\mathbf{Z}_n$ is direct, it follows that

$$\#M = (\#M_1)(\#M_2) = n_1 n_2.$$

To count #N, first remark that the classes modulo $(n/p_1)\mathbf{Z}_n$ of any two distinct elements y_1, y_2 in $((n/p_2)\mathbf{Z}_n \setminus \{0\}) \cup \{x_1\}$ are distinct. This is obvious if $y_1, y_2 \in (n/p_2)\mathbf{Z}_n$ because $(n/p_1)\mathbf{Z}_n \cap (n/p_2)\mathbf{Z}_n = \{0\}$. If $y_1 = x_1$ and $y_2 \in (n/p_2)\mathbf{Z}_n$, we also cannot have $y_1 - y_2 \in (n/p_1)\mathbf{Z}_n$, because this would imply $x_1 \in (n/p_1)\mathbf{Z}_n + (n/p_2)\mathbf{Z}_n = (n/p_1p_2)\mathbf{Z}_n$; on the other hand, $x_1 \in (n/p_1n_1)\mathbf{Z}_n$ by the hypothesis on x_1 and hence $x_1 \in (n/p_1p_2)\mathbf{Z}_n \cap (n/p_1n_1)\mathbf{Z}_n = (n/p_1)\mathbf{Z}_n$ which contradicts the choice of x_1 (observe that $p_1p_2 \wedge p_1n_1 = p_1$ as $p_2 \wedge n_1 = 1$).

It follows from the above that $\#\mathrm{N}_1 = p_1p_2$. In a similar way we obtain $\#\mathrm{N}_2 = p_1p_2$. Hence, taking into account (1), we have

$$\#\mathrm{N} = \#\mathrm{N}_1 + (\#\mathrm{N}_2)(\#\mathrm{S}) = p_1p_2 + p_1p_2(n_3 - 1) = p_1p_2n_3.$$

We prove now that $(\mathrm{M}-\mathrm{M}) \cap (\mathrm{N}-\mathrm{N}) = \{0\}$. Let $t,u \in \mathrm{M}$ and $v,w \in \mathrm{N}$ be such that $t-u=v-w$. Because of (1) and of the fact that the classes modulo $n_3\mathbf{Z}_n$ of the elements in S are distinct and the class of 0 is not represented in S, it follows that the only possibilities left for v and w are either $v,w \in \mathrm{N}_1$ or $v,w \in s+\mathrm{N}_2$ for some $s \in \mathrm{S}$. We consider only the first case, the discussion of the second case being similar to the first. We have thus

$$\begin{aligned} v-w \in \mathrm{N}_1 - \mathrm{N}_1 \subset\ & (n/p_1)\mathbf{Z}_n + ((n/p_2)\mathbf{Z}_n \\ & \cup (x_1 + ((n/p_2)\mathbf{Z}_n \cup \{0\})) \cup (-x_1 + ((n/p_2)\mathbf{Z}_n \setminus \{0\})). \end{aligned}$$

Hence, interchanging if necessary v with w, we have to consider the cases

$$v-w \in (n/p_1)\mathbf{Z}_n + (n/p_2)\mathbf{Z}_n \tag{2}$$

on the one hand and

$$v-w \in x_1 + (n/p_1)\mathbf{Z}_n + ((n/p_2)\mathbf{Z}_n \setminus \{0\}) \tag{3}$$

on the other. Both situations imply that

$$v-w \in (n/p_1n_1)\mathbf{Z}_n + (n/p_2)\mathbf{Z}_n. \tag{4}$$

On the other hand,

$$\begin{aligned} t-u &= t_1 + t_2 - u_1 - u_2 \\ &= (t_1 - u_1) + (t_2 - u_2) \in (n/p_1n_1)\mathbf{Z}_n + (n/p_2n_2)\mathbf{Z}_n, \end{aligned} \tag{5}$$

the sum in the rightmost side of (5) being direct. Comparison of the expressions (4) and (5) for $v-w=t-u$ yields $t_2 - u_2 \in (n/p_2)\mathbf{Z}_n$. The classes modulo $(n/p_2)\mathbf{Z}_n$ of the elements in M_2 being distinct, it follows that $t_2 = u_2$; hence $t-u=t_1-u_1$ and $v-w \in (n/p_1n_1)\mathbf{Z}_n$. This shows that case (3) is not possible, while in case (2) we must have $v-w \in (n/p_1)\mathbf{Z}_n$; as the elements in M_1 belong to distinct classes modulo $(n/p_1)\mathbf{Z}_n$, we obtain $t_1 = u_1$ and finally $t-u=v-w=0$.

We have thus proved that M and N satisfy conditions (ii) and (iii) in Proposition 2.1; they are therefore supplementary subsets. It remains to prove that they are not periodic.

Consider first $y \in \mathbf{Z}_n$ such that $y + \mathrm{M} = \mathrm{M}$. By (1) it follows that $y \in n_3\mathbf{Z}_n = (n/p_1n_1)\mathbf{Z}_n + (n/p_2n_2)\mathbf{Z}_n$, the sum being direct. Hence $y = y_1 + y_2$ with $y_i \in (n/p_in_i)\mathbf{Z}_n$ and the equality $y + \mathrm{M} = \mathrm{M}$ implies $y_i + \mathrm{M}_i = \mathrm{M}_i$ for $i = 1,2$. But M_1 and M_2 are not periodic; consequently $y_1 = y_2 = 0$.

Consider now the set N. We show first that the stability subgroup of N_i equals $(n/p_i)\mathbf{Z}_n$ for $i = 1,2$. The fact that N_i is p_i-periodic is obvious from the definition of N_i. Conversely, suppose for instance that $y + \mathrm{N}_1 = \mathrm{N}_1$; by applying the homomorphism $\varphi_{n,n/p_1}$ to this equality one obtains

$$\varphi_{n,n/p_1}(y) + \varphi_{n,n/p_1}(\mathrm{N}_1) = \varphi_{n,n/p_1}(\mathrm{N}_1).$$

But

$$\varphi_{n,n/p_1}(\mathrm{N}_1) = ((n/p_2)\mathbf{Z}_{n/p_1} \setminus \{0\}) \cup \{\varphi_{n,n/p_1}(x_1)\}$$

and

$$\varphi_{n,n/p_1}(x_1) \notin (n/p_2)\mathbf{Z}_{n/p_1}$$

(otherwise we would have

$$x_1 \in (n/p_1p_2)\mathbf{Z}_n \;\cap\; (n/p_1n_1)\mathbf{Z}_n = (n/p_1)\mathbf{Z}_n,$$

contradicting the choice of x_1). It follows then by a reasoning similar to the one employed in the proof of Lemma 2.1 that $\varphi_{n,n/p_1}(\mathrm{N}_1)$ is a nonperiodic subset of $\mathbf{Z}_{n/p_1}$; hence $\varphi_{n,n/p_1}(y) = 0$, that is, $y \in (n/p_1)\mathbf{Z}_n$.

Let $y \in \mathbf{Z}_n$ be such that $y + \mathrm{N} = \mathrm{N}$. If $y \notin n_3\mathbf{Z}_n$, there is $s \in \mathrm{S}$ such that $y + s \in n_3\mathbf{Z}_n$. From $y + \mathrm{N} = \mathrm{N}$ and from (1) we infer that

$$y + s + \mathrm{N}_2 \subset \mathrm{N} \cap n_3\mathbf{Z}_n = \mathrm{N}_1;$$

as $\#\mathrm{N}_1 = \#\mathrm{N}_2$ the above inclusion becomes $y + s + \mathrm{N}_2 = \mathrm{N}_1$. But this is not possible as the stability subgroups of N_1 and N_2 are distinct. Hence $y \in n_3\mathbf{Z}_n$ and the equality $y + \mathrm{N} = \mathrm{N}$ yields in this case $y + \mathrm{N}_i = \mathrm{N}_i$ for $i = 1,2$. Consequently, $y \in (n/p_1)\mathbf{Z}_n \cap (n/p_2)\mathbf{Z}_n = \{0\}$. Proposition 2.2 is completely proved.

Illustrations of the construction method exposed during the above proof will be presented in Section 7.

3. The Rhythmic Model

The model of periodic rhythm proposed by the author of the present article was presented in the study "Sur le rythme périodique" published in the journal *Revue Roumaine de Linguistique—Cahiers de Linguistique Théorique et Appliquée* (Vuza 1985). In response to the invitation of the Editor, Professor M.G. Boroda from the State Conservatoire of Tbilisi, the study was reprinted in the first volume of the series *Musikometrika*, a new subseries of *Quantitative Linguistics*, which has inscribed among its main goals "to promote the quantitative-systemic approach to musical composition and musical language" (M.G. Boroda, Foreword). I therefore expect that my study about periodic rhythm is by now available to quite a large number of music theorists so that I can confine myself to a brief description of only those parts of the rhythmic model which are relevant for the theory forming the object of the present paper. The definitions and results presented below are all reproduced from Vuza 1985, with only minor changes in notations. The reader is referred to the referenced paper for the detailed proofs; I hope that the material to be found in the following sections will provide enough musical illustrations of the concepts presented here in a quite formal manner.[7]

Definition 3.1. *A periodic rhythm is a (possibly empty) subset* R *of* $\mathbf{Q}$ *satisfying* (R1) *and* (R2) *below*:

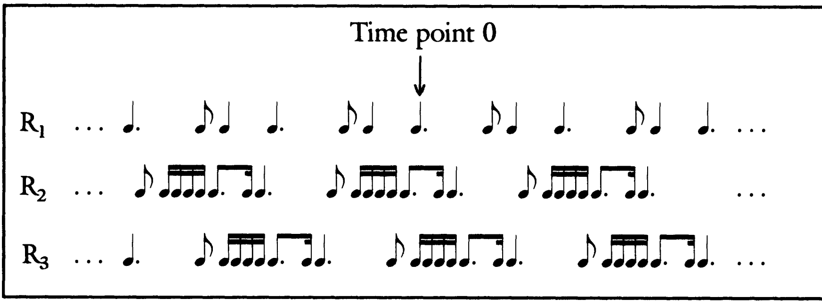
(R1) $t + \mathrm{R} = \mathrm{R}$ for some $t \in \mathbf{Q}_+$ (in other words, R is a periodic subset of the additive group $\mathbf{Q}$);

(R2) For every $a, b \in \mathbf{Q}$ with $a < b$, the set $\mathrm{R} \cap [a,b)$ is finite.

The elements in R should be viewed as marking the transition moments ("beats") from one musical event to another during the discourse delivered by a single voice. (Hence the beginning of a pause is also marked by an element in R.)

As we shall deal throughout the paper only with periodic rhythms in the sense of Definition 3.1, we will by convention omit the adjective "periodic" and speak simply about "rhythms."

The fact that the time axis in my rhythmic model is $\mathbf{Q}$ corresponds to the reality that in European music all durations are denoted by rational numbers. Besides, this fact provides a lot of formal advantages; for the moment we mention only the property that *any finite union and any finite intersection of rhythms is still a rhythm.*



EXAMPLE 3.1

If the time point 0 is chosen as shown (Example 3.1) while the whole note is chosen as time unit, then there correspond to the rhythms R_1, R_2, R_3 the sets

$$R_1 = \{0, 3/8, 1/2\} + (3/4)\mathbf{Z},$$
$$R_2 = \{1/8, 3/16, 9/16, 11/16, 3/4, 13/16, 7/8, 15/16\} + \mathbf{Z},$$
$$R_3 = \{0, 1/16, 1/8, 3/16, 1/4, 7/16, 1/2, 7/8\} + \mathbf{Z}.$$

We have for instance

$$R_1 \cap R_2 = \{-9/8, -1/4, 3/4, 9/8\} + 3\mathbf{Z},$$
$$R_2 \cap R_3 = \{1/8, 3/16, 7/8\} + \mathbf{Z}.$$

DEFINITION 3.2. *A rhythmic class is the translation class (with respect to the additive group* $\mathbf{Q}$*) of a rhythm.*

That is, two rhythms R_1, R_2 belong to the same rhythmic class if there is $t \in \mathbf{Q}$ such that $R_1 = t + R_2$.

To distinguish between rhythms and rhythmic classes we shall use capital Roman letters for the former and capital italics for the latter. The rhythmic class of the rhythm R will be denoted by [R]. The notation *Rhyt* will stand for the set of all rhythmic classes.

The following numerical entities are attached to a rhythm R:

(*a*) the *period* Per R of R, defined as the least $t \in \mathbf{Q}_+$ satisfying $t + R = R$;

(*b*) the *minimal division* Div R of R, defined as the greatest $d \in \mathbf{Q}_+$ which divides every element in $R - R$;

(*c*) the ***number of attacks per period*** Nrp R of R, defined as $\#R \cap [a, a + \text{Per } R)$ (this number does not depend on the particular choice of $a \in \mathbf{Q}$).

If R_1 and R_2 are the rhythms introduced in Example 3.1, then		
Per $R_1 = 3/4$,	Div $R_1 = 1/8$,	Nrp $R_1 = 3$,
Per $R_2 = 1$,	Div $R_2 = 1/16$,	Nrp $R_2 = 8$

EXAMPLE 3.2

Clearly enough, if $[R_1] = [R_2]$ then Per R_1 = Per R_2, Div R_1 = Div R_2, and Nrp R_1 = Nrp R_2. We may therefore define (by use of representants) the corresponding numerical entities Per *R*, Div *R*, and Nrp *R* for a rhythmic class *R*. It is clear from the definitions that Div $R \mid$ Per R for every $R \in \mathit{Rhyt}$.

The fact that periodic rhythms are modelled by infinite sets is imposed by the formal necessities of a theory aiming to study structural relations between rhythms whose periods are ***arbitrary elements*** in $\mathbf{Q}_+$. In concrete situations, when studying a finite collection of rhythms it suffices to consider only the intersections of each of the rhythms in question with a sufficiently large interval of $\mathbf{Q}$. For instance, for the determination of the union or the intersection of the rhythms $R_1, \ldots, R_r$ it suffices to consider the finite sets $R_1 \cap [0,a), \ldots, R_r \cap [0,a)$ where a denotes the least common multiple of the periods of $R_1, \ldots, R_r$ (see Example 3.1).

DEFINITION 3.3. *A **regular rhythm** is a rhythm of the form $a + t\mathbf{Z}$ with $a \in \mathbf{Q}$, $t \in \mathbf{Q}_+$.* A ***regular class*** *is the rhythmic class of a regular rhythm.*

Hence regular classes are the rhythmic classes of the form $[t\mathbf{Z}]$ with $t \in \mathbf{Q}_+$. In order to simplify notation we make the convention of writing $[t]$ instead of $[t\mathbf{Z}]$. Clearly Per $[t]$ = Div $[t] = t$; a rhythmic class R satisfies Nrp $R = 1$ iff it is regular.

DEFINITION 3.4. *Two rhythmic classes R,S are called **intervallically disjoint** if* Int $R \cap$ Int $S \subset (\text{Per } R \vee \text{Per } S)\mathbf{Z}$.

That is, as both R and S repeat over and over indefinitely, the only temporal intervals one can form using time points both from R and also from S, are exactly those temporal intervals that are common multiples of Per R and Per S.

We use the symbol $R \perp S$ for the situation described by the above definition.

The semigroup structure (without a unit element) of *Rhyt* represents the main algebraic structure on that set which proved itself to be a useful device in studying the construction of canons. The set *Rhyt* is endowed with a semigroup structure by means of a (commutative and associative) composition law whose definition is as follows: if R and S are rhythmic classes, then their composition $R+S$ equals $[\mathrm{R}+\mathrm{S}]$, where R and S, respectively, is any rhythm in R and S, respectively. (It is an easy matter to verify that the definition of $R+S$ does not depend on the choices of the representants R and S. It remains to verify the fact that R+ S is a rhythm whenever R and S are rhythms; here again we make use of the fact that the time axis in the rhythmic model is $\mathbf{Q}$.)

DEFINITION 3.5. *Let $R, S \in$ Rhyt. We say that R is a condensation of S, or that S is an extension of R if there is $t \in \mathbf{Q}_+$ such that $S+[t]=R$.*

In symbols, we write $S \rightarrow R$ for "R is a condensation of S."[8]

Another algebraic structure of musical interest is represented by the multiplication of a rhythmic class R by a rational number $t \neq 0$. The result is the rhythmic class tR equal by definition to $[t\mathrm{R}]$, where R is any rhythm in R. Clearly Per $tR = |t|$Per R, Div $tR = |t|$Div R and Nrp $tR =$ Nrp R. As multiplication modifies the period, it is more convenient in some situations to use the condensed multiplication, that is, a multiplication followed by a condensation. The latter procedure is of interest especially in the case when t is an integer relatively prime to Per R/Div R (see Proposition 3.7 below).

Definitions 3.1–3.5 and the algebraic structures exposed above (all of them reproduced from Vuza 1985) provide the formal basis for the study of rhythmic unending canons undertaken in the following sections. In presenting musical examples it is important to have a practical method of labelling rhythmic classes. This is achieved by the use of intervallic structures. Let π_m be the cyclic permutation of the set $\{1, \ldots, m\}$ defined by $\pi_m(i) = i+1$ for $1 \leq i \leq m-1$, $\pi_m(m) = 1$, and let Π_m be the subgroup generated by π_m in the (noncommutative) group of all permutations of $\{1, \ldots, m\}$ (hence $\Pi_m = \{\pi_m^k \mid 1 \leq k \leq m\}$). An *intervallic structure with m elements* is a sequence $s_1, \ldots, s_m$ of elements in $\mathbf{Q}_+$ with the property that the identity map is the only element π in Π_m which satisfies $s_{\pi(i)} = s_i$ for $1 \leq i \leq m$. We say that an intervallic structure $s_1, \ldots, s_m$ *corresponds* to a rhythmic class R, or that R *corresponds* to $s_1, \ldots, s_m$ if there are $\mathrm{R} \in R$ and $t \in \mathrm{R}$ so that $s_i = t_{i+1} - t_i$ for $1 \leq i \leq m$, where $t = t_1 < t_2 \ldots < t_{m+1} = t + \mathrm{Per\ R}$ are the elements in $\mathrm{R} \cap [t, t + \mathrm{Per\ R}]$. In other words, an intervallic structure records the intervals between those successive attacks in a rhythm R which lie inside an interval of length Per R spanned by two attacks in R.

(Our present definition of an intervallic structure corresponds to the notion of a nonperiodic rhythmic structure from Vuza 1985, §8.)

To every intervallic structure $s_1, \ldots, s_m$ there corresponds (in the sense explained above) a unique rhythmic class, which we shall denote by $[s_1, \ldots, s_m]$. (This agrees with our notation introduced earlier for regular classes.) We have the relations

$$\text{Per } [s_1, \ldots, s_m] = \sum_{i=1}^{m} s_i,$$

$$\text{Nrp } [s_1, \ldots, s_m] = m$$

while Div $[s_1, \ldots, s_m]$ equals the greatest common divisor of the numbers $s_1, \ldots, s_m$.

The converse direction of the above correspondence brings in some ambiguity: to one rhythmic class there correspond several intervallic structures. Two intervallic structures $s_1, \ldots, s_m$ and $t_1, \ldots, t_n$ correspond to the same rhythmic class iff $m = n$ and there is $\pi \in \Pi_m$ so that $t_i = s_{\pi(i)}$ for $1 \leq i \leq m$. Hence, if one calls equivalent two intervallic structures related in the above-indicated manner, one may assert that *there is a one-to-one correspondence between rhythmic classes and classes of equivalent intervallic structures.*[9]

Instead of rational numbers, we may use traditional musical notation for writing down an intervallic structure in the situation when this procedure is not too cumbersome.

If R_2 and R_3 are the rhythms introduced in Example 3.1, then $R_3 = 5/16 + R_2$ so that

$[R_2] = [R_3] =$ []

$= 1/16\ [1,1,1,1,3,1,6,2]$

EXAMPLE 3.3

As outlined in the Introduction, there are bijective correspondences between the sets of translation classes $T(\mathbf{Z}_n)$ and certain subsets of *Rhyt* (n ranging over all integers ≥ 1). In the present study, these correspondences will especially be of theoretical importance, as they provide the basis for the mathematical analysis of regular complementary unending canons. Nevertheless, their importance is also practical, as in certain complicated rhythmic situations computations on translation classes are easier than direct computations on rhythmic classes.

We recall the formal construction of these correspondences (see Vuza 1985 §6 for details). For every pair $(a,b) \in \mathbf{Q}_+ \times \mathbf{Q}_+$ such that $a \mid b$, let us denote by $Rhyt_{a,b}$ the set of all $R \in Rhyt$ satisfying $a|\text{Div } R$ and $\text{Per } R|b$. The collection of sets $Rhyt_{a,b}$, as (a,b) ranges over the set of all pairs of the indicated form, is an upwards directed collection of sets[10] whose union equals $Rhyt$. The bijection $H_{a,b}$: $Rhyt_{a,b} \to T(\mathbf{Z}_n)$, where $n=b/a$, is defined as follows. By definition, $R \in Rhyt_{a,\mathrm{b}}$ iff R is the class of a rhythm R verifying the relations $a^{-1}\mathrm{R} \subset \mathbf{Z}$ and $b+\mathrm{R}=\mathrm{R}$. Given $R \in Rhyt_{a,b}$, choose $\mathrm{R} \in R$ satisfying the indicated relations and set $H_{a,b}(R)=[\varphi_n(a^{-1}\mathrm{R})]$. Conversely, if $M \in T(\mathbf{Z}_n)$, then $H^{-1}_{a,b}(M)$ equals $[a\varphi^{-1}_n(\mathrm{M})]$ where M is any set in M.

In concrete situations it is useful to know a practical method of relating a rhythmic class labelled with the aid of an intervallic structure to a translation class via the bijections $H_{a,b}$. To describe such a method, we first introduce the notion of *k-times repetition* of an intervallic structure. Given any integer $k \geq 1$ and any intervallic structure $s_1, \ldots, s_m$, the k-times repetition of the latter is defined as the sequence $t_1, \ldots, t_{km}$ where $t_{im+j}=s_j$ for $0 \leq i \leq k-1$ and $1 \leq j \leq m$. Every finite sequence of elements in $\mathbf{Q}_+$ is the k-times repetition, for some uniquely determined $k \geq 1$, of a uniquely determined intervallic structure, which we call the intervallic structure associated to the given sequence.

Let $R=[s_1, \ldots, s_m]$ be a rhythmic class in $Rhyt_{a,b}$. To compute $H_{a,b}(R)$ take first the k-times repetition of the intervallic structure $a^{-1}s_1, \ldots, a^{-1}s_m$, where $k=b/\text{Per } R$. We obtain thus the sequence of rational numbers $t_1, \ldots, t_{km}$ which are in fact integers because $a|s_i$ for $1 \leq i \leq m$ by the hypothesis on R. Then $H_{a,b}(R)$ is the translation class in $\mathbf{Z}_n$ ($n=b/a$) of the set of partial sums

$$\{\sum_{j=1}^{i} \varphi_n(t_j) \,|\, 1 \leq i \leq km\}.$$

Conversely, given a subset M of $\mathbf{Z}_n$, write its elements in a sequence $\varphi_n(t_1), \ldots, \varphi_n(t_q)$ so that $t_1 < \ldots < t_q$ are integers in $[0,n)$. Let $r_1, \ldots, r_q$ be the sequence defined by $r_i = t_{i+1} - t_i$ for $1 \leq i \leq q-1$, $r_q = n + t_1 - t_q$. The rhythmic class $H^{-1}_{a,b}([\mathrm{M}])$ equals $[s_1, \ldots, s_m]$, where $s_1, \ldots, s_m$ is the intervallic structure associated with the sequence $ar_1, \ldots, ar_q$.

If $R=$ [♩. ♪ ♩] then
$H_{1/8,3/4}(R)=[\{0,3,4\}] \in T(\mathbf{Z}_6)$,
$H_{1/16,3/4}(R)=[\{0,6,8\}] \in T(\mathbf{Z}_{12})$,
$H_{1/16,3/2}(R)=[\{0,6,8,12,18,20\}] \in T(\mathbf{Z}_{24})$.

EXAMPLE 3.4

We close the theoretical part of this section by reproducing from Vuza 1985 a series of results which will be needed in the following sections.

PROPOSITION 3.1. *If* R *is a rhythm and* $t \in \mathbf{Q}\setminus\{0\}$ *satisfies* $t+\mathrm{R}=\mathrm{R}$, *then* R *is a finite union of cosets of* $\mathbf{Q}$ *modulo* $t\mathbf{Z}$.

PROPOSITION 3.2. *For every* $R,S \in$ *Rhyt we have* Per $(R+S)|$Per $R \wedge$ Per S *and* Div $(R+S) =$ Div $R \wedge$ Div S. *For every* $r,s \in \mathbf{Q}_+$ *we have* $[r]+[s]=[r \wedge s]$.

PROPOSITION 3.3. *The condensation relation "*$S\rightarrow R$*" endows the set Rhyt with a structure of a partially ordered set. The relation* $S\rightarrow R$ *is equivalent to* $R=S+$ [Per R] *and implies the relation* [Per S]$\rightarrow$[Per R]. *If* $S_i\rightarrow R_i$ *for* $i=1,2$, then $S_1+S_2\rightarrow R_1+R_2$.

Note the obvious relation $R+[t]=R$ for any $R \in$ *Rhyt* and any $t \in \mathbf{Q}_+$ such that Per $R|t$.

PROPOSITION 3.4. *For every* $R \in Rhyt_{a,b}$ *we have* Per $R=b/\#$G *and* Nrp $R=$ Nr $H_{a,b}$ $(R)/\#$G, *where* G *denotes the stability subgroup of* $H_{a,b}$ (R).

PROPOSITION 3.5. *For every* R_1, $R_2 \in Rhyt_{a,b}$, *the relation* $R_1 \perp R_2$ *is equivalent to* Int $H_{a,b}(R_1) \cap$ Int $H_{a,b}(R_2) \subset \mathrm{G}_1 \cap \mathrm{G}_2$, *where* G_i *denotes the stability subgroup of* $H_{a,b}(R_i)$ $(i=1,2)$.

PROPOSITION 3.6. $Rhyt_{a,b}$ *is a subsemigroup of the semigroup Rhyt and* $H_{a,b}$ *is a semigroup isomorphism of* $Rhyt_{a,b}$ *onto* $T(\mathbf{Z}_n)$ $(n=b/a$; the semigroup structure of the latter set was defined in Section 1).

PROPOSITION 3.7. *Let* $R \in Rhyt_{a,b}$ *and let* $k\neq 0$ *be an integer relatively prime to* b/a. *Then* $H_{a,b}$ $(kR+[b])=kH_{a,b}(R)$, Per $(kR+[b])=$ Per R, Div $(kR+[b])=$ Div R *and* Nrp $(kR+[b])=$ Nrp R.

In the section devoted to the construction of complementary canons, a significant role will be played by the so-called "disjoint extensions" of a rhythmic class R (Vuza 1985). These are those extensions S of R satisfying $S+[t]=R$ and $S \perp [t]$ for some $t \in \mathbf{Q}_+$. It is therefore important, for both theoretical and practical reasons, to know a procedure for finding all disjoint extensions of a given rhythmic class. The last result in this section indicates such a procedure, based on the correspondences between rhythmic classes and translation classes of the groups $\mathbf{Z}_n$. It also indicates a practical method for computing a condensation.

PROPOSITION 3.8. *Let* $R \in Rhyt$, $S \in Rhyt_{a,b}$ *be given and let* $t \in \mathbf{Q}_+$ *be such that* $a|t$ *and* $t|b$. *Set* $m = t/a$, $n = b/a$ *and choose any set* M *in* $H_{a,b}(S)$. *Consider the following conditions on the above objects:*

(i) $S + [t] = R$;

(ii) $S \perp [t]$ *and* $b = \mathrm{Per}\, S \vee t$;

(iii) $R \in Rhyt_{a,t}$ *and* $H_{a,t}(R) = [\varphi_{n,m}(\mathrm{M})]$;

(iv) *The restriction of* $\varphi_{n,m}$ *to* M *is one to one.*

Then (i) *is equivalent to* (iii) *while the conjunction of* (i) *and* (ii) *is equivalent to the conjunction of* (iii) *and* (iv).

The conjunction of (iii) and (iv) may be rephrased as: there is $\mathrm{N} \in H_{a,t}(R)$ so that M meets at precisely one element each coset of $\mathbf{Z}_n$ modulo $m\mathbf{Z}_n$ which is mapped into N by $\varphi_{n,m}$ and it meets no coset modulo $m\mathbf{Z}_n$ which is not mapped into N by $\varphi_{n,m}$.

Let $S =$ [] . We want to compute $S + [1]$. As Per $S = 3/2$ we may write

$$S + [1] = (S + [3/2]) + [1] = S + ([3/2] + [1]) = S + [1/2].$$

We have

$$H_{1/16,3/2}(S) = [\{0,4,9,13\}] \in T(\mathbf{Z}_{24}),$$

$$\varphi_{24,8}(\{0,4,9,13\}) = \{0,1,4,5\}$$

so that

$S + [1] = H^{-1}_{1/16,1/2}([\{0,1,4,5\}]) =$ []

EXAMPLE 3.5: A CONDENSATION

Let us determine two rhythmic classes S_1, S_2 so that $S_i + [1] =$  and $S_i \perp [1]$ for $i = 1,2$ (see Example 5.3 below). We use the notations from Proposition 3.8. We have $a = 1/16$, $t = 1$, $m = 16$, and $H_{1/16,1}$ () $= [\mathrm{N}] \in T(\mathbf{Z}_{16})$ where

$$\mathrm{N} = \{0,3,4,7,8,11,12,15\}.$$

Take in the first place $b = 3$ so that $n = 48$. The cosets of $\mathbf{Z}_{48}$ modulo $16\mathbf{Z}_{48}$ mapped into N by $\varphi_{48,16}$ are

$$\{0,16,32\}, \{3,19,35\}, \{4,20,36\}, \{7,23,39\},$$
$$\{8,24,40\}, \{11,27,43\}, \{12,28,44\}, \{15,31,47\}.$$

By choosing one element from each of these cosets we arrive at the set

$$\mathrm{M}_1 = \{0,4,15,19,24,28,39,43\}.$$

We obtain thus the first solution

$S_1 = H^{-1}{}_{1/16,3}([\mathrm{M}_1]) =$ 

Remark that the period of S_1 is not 3 but 3/2.

Take in the second place $b = 2$ so that $n = 32$. The cosets of $\mathbf{Z}_{32}$ modulo $16\mathbf{Z}_{32}$ mapped into N by $\varphi_{32,16}$ are

$$\{0,16\}, \{3,19\}, \{4,20\}, \{7,23\},$$
$$\{8,24\}, \{11,27\}, \{12,28\}, \{15,31\}.$$

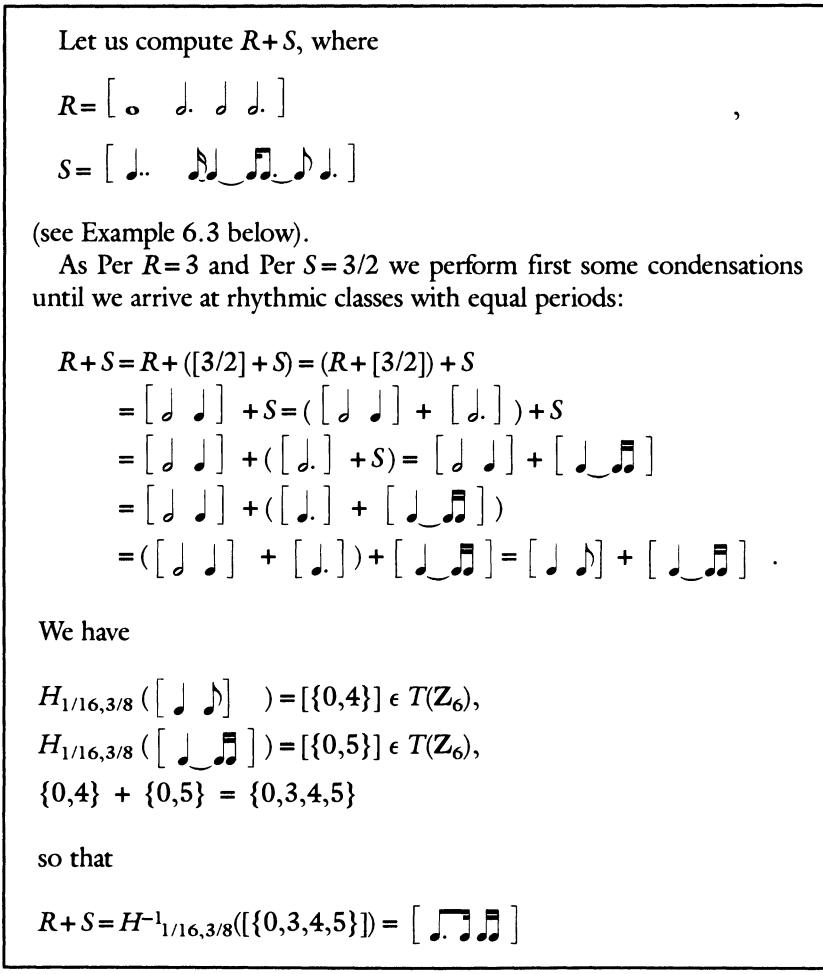
By choosing one element from each of these cosets we arrive at the set

$$\mathrm{M}_2 = \{0,3,8,11,15,20,23,28\}.$$

We obtain thus the second solution

$S_2 = \mathrm{H}^{-1}{}_{1/16,2}([\mathrm{M}_2]) =$ 

EXAMPLE 3.6: DISJOINT EXTENSIONS



EXAMPLE 3.7: A COMPOSITION

This is the first of four installments of this article. Future issues will contain:
Part 2 (Section 4: Generalities about unending rhythmic canons, and Section 5: The inversion of canons),
Part 3 (Section 6: Complementary canons, and Section 7: The structure of regular complementary canons),
and Part 4 (Section 8: Multiplicative transforms of supplementary rhythmic classes, and Section 9: Completion of proofs of Theorems 2.1–2.3. The role of convolution and of Fourier transform in the analysis of supplementary sets.)

Notes

1. Equivalently, the collection of all transpositions of M by the intervals from, say, the pitch-class C to each pitch class in N forms a partition of the set of twelve pitch classes.

2. Examples 0.1–0.3 presented the structure of a canon at a moment *subsequent to the entering of the last voice*; for this reason the relative distance between two voices appears there as being strictly inferior to Per *R*. It is of course possible that the distance between the *entrances* of the two voices may exceed Per *R*; however, this reality is reflected only in the beginning of the canon, up to the entrance of the last voice, and it does not at all affect that intervallic structure of the canon which can be observed (and is periodically repeated) after the entering of all voices.

   In connection with the manner in which a canon may begin, see the introduction of secondary metric classes in Section 4.

3. A lattice-ordered set is a partially ordered set (M, $\leq$) with the property that for every a, $b \in$ M, the set $\{x \mid x \in$ M, $x \geq a$ and $x \geq b)$ has a least element, called the least upper bound of a and b, while the set $(y \mid y \in$ M, $y \leq a$ and $y \leq b\}$ has a greatest element, called the greatest lower bound of a and b.

4. Recall that kx means: $x + \ldots + x$ (k times) if $k > 0$; $(-x) + \ldots + (-x)$ ($|k|$ times) if $k < 0$; 0 if $k = 0$.

5. The applications of the semigroup $T(\mathbf{Z}_{12})$ to modal analysis were first studied by Anatol Vieru (1980).

6. The notation is motivated by the fact that when M is the transpositional class of some pitch set M, then Int M is the set of intervals spanned by all pairs of elements in M.

7. For a comparison between my rhythmic model and Lewin's theory about rhythm (Lewin 1984 and 1987), see Vuza 1988.

8. The condensation as described here should be related to the procedure of "contraction" employed by Johnson (1984).

9. For a rigorous theory of the correspondence between rhythmic classes and intervallic structures see Vuza 1985 and 1986.

10. A collection C of subsets of a certain set is called upwards-directed if for every M_1, $M_2 \in$ C there is N $\in$ C such that $M_1 \cup M_2 \subset$ N.

References

Babbitt, Milton. 1962. "Twelve-Tone Rhythmic Structure and the Electronic Medium." *Perspectives of New Music* 1, no. 1 (Fall-Winter): 49–79.

Johnson, William Marvin Jr. 1984. "Time-Point Sets and Meter." *Perspectives of New Music* 23, no. 1 (Fall-Winter): 278–93.

Lewin, David. 1959. "Intervallic Relations between Two Collections of Notes." *Journal of Music Theory* 3:298–301.

———. 1984. "On Formal Intervals between Time-Spans." *Music Perception* 1, no. 4 (Summer): 414–23.

———. 1987. *Generalized Musical Intervals and Transformations*. New Haven and London: Yale University Press.

Vieru, Anatol. 1980. *Cartea modurilor I* (The Book of Modes I). Bucharest: Editura Musicală.

Vuza, Dan. 1983a. "Classes modales partitionnantes." *Muzica* 5:45–47.

———. 1983b. "Aspects mathématiques dans la théorie modale d'Anatol Vieru." Part 4. *Revue Roumaine de Mathématiques Pures et Appliquées* 28, no. 8:757–73.

———. 1985. "Sur le rythme périodique." *Revue Roumaine de Linguistique—Cahiers de Linguistique Théorique et Appliquée* 22, no. 1:73–103. Reprinted in *Musikometrika* 1, edited by M. G. Boroda, 83–126. Bochum: Studienverlag Dr. N. Brockmeyer, 1988.

———. 1986. "Les structures modales, instrument d'étude des modes et des rythmes." *Revue Roumaine de Linguistique—Cahiers de Linguistique Théorique et Appliquée* 23, no. 1:55–68.

———. 1988. "Some Mathematical Aspects of David Lewin's Book *Generalized Musical Intervals and Transformations*." *Perspectives of New Music* 26, no. 1 (Winter): 258–87.

Notes

[8] **Time-Point Sets and Meter**
William Marvin Johnson, Jr.
Perspectives of New Music, Vol. 23, No. 1. (Autumn - Winter, 1984), pp. 278-293.
Stable URL:
http://links.jstor.org/sici?sici=0031-6016%28198423%2F24%2923%3A1%3C278%3ATSAM%3E2.0.CO%3B2-0

References

Twelve-Tone Rhythmic Structure and the Electronic Medium
Milton Babbitt
Perspectives of New Music, Vol. 1, No. 1. (Autumn, 1962), pp. 49-79.
Stable URL:
http://links.jstor.org/sici?sici=0031-6016%28196223%291%3A1%3C49%3ATRSATE%3E2.0.CO%3B2-8

Time-Point Sets and Meter
William Marvin Johnson, Jr.
Perspectives of New Music, Vol. 23, No. 1. (Autumn - Winter, 1984), pp. 278-293.
Stable URL:
http://links.jstor.org/sici?sici=0031-6016%28198423%2F24%2923%3A1%3C278%3ATSAM%3E2.0.CO%3B2-0

LINKED CITATIONS

Re: Intervallic Relations between Two Collections of Notes
David Lewin
Journal of Music Theory, Vol. 3, No. 2. (Nov., 1959), pp. 298-301.
Stable URL:
http://links.jstor.org/sici?sici=0022-2909%28195911%293%3A2%3C298%3ARIRBTC%3E2.0.CO%3B2-C