SUPPLEMENTARY SETS AND REGULAR COMPLEMENTARY UNENDING CANONS (PART THREE)

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6. COMPLEMENTARY CANONS

ONE USUALLY SAYS that two voices are complementary if no beat from the first voice coincides with any beat from the second. Within the framework of the rhythmic model in Section 3, the fact that two voices V_1, V_2, delivering the respective periodic rhythms R_1, R_2, are complementary is expressed by the equality $R_1 \cap R_2 = \emptyset$. When applied to canons, these considerations lead to the following definition:

> DEFINITION 6.1. *A canon* $\{R_1, \ldots, R_l\}$ *is called complementary if* $R_i \cap R_j = \emptyset$ *for* $i \neq j$.

Examples 6.1 and 6.2 show excerpts from two three-voiced works of Johann Sebastian Bach in each of which two of the voices are constructed, from the rhythmic (though not the melodic) viewpoint, in the form of an unending complementary canon.

EXAMPLE 6.1: *The Well-Tempered Clavier*, BOOK I, PRELUDE IN G MINOR, MEASURES 9–10

EXAMPLE 6.2: *The Well-Tempered Clavier*, BOOK II, FUGUE IN A MAJOR, MEASURES 17–18

The next proposition shows that there is a close connection between complementarity of canons and intervallic disjointness of rhythmic classes. (In fact, when I introduced the latter notion, in Vuza 1985, it was just such a later application to canons that I had in mind.)

PROPOSITION 6.1. *For any canon $\mathcal{C}$ the following conditions are equivalent:*

(i) *$\mathcal{C}$ is complementary;*

(ii) *Grd $\mathcal{C}$ $\perp$ Met $\mathcal{C}$;*

(iii) *Grd $\mathcal{C}$ is intervallically disjoint from some metric class admitted by $\mathcal{C}$;*

(iv) *Grd $\mathcal{C}$ is intervallically disjoint from all metric classes admitted by $\mathcal{C}$.*

Proof. Before beginning the proof we observe that, according to Proposition 4.2, $\mathcal{C}$ is equivalent to the canon $\{s + \mathrm{R} \mid s \in \mathrm{S}\}$ where $\mathrm{R} \in$ *Grd* $\mathcal{C}$ and S is the resultant of any meter on $\mathcal{C}$. Hence $\mathcal{C}$ is complementary iff for any $s_1, s_2 \in \mathrm{S}$, the relation $(s_1 + \mathrm{R}) \cap (s_2 + \mathrm{R}) \neq \emptyset$ implies $s_1 + \mathrm{R} = s_2 + \mathrm{R}$ (or equivalently, Per R $\mid s_1 - s_2$). Now we begin the proof.

(i) $\rightarrow$(iv) Let S be the resultant of any meter on $\mathcal{C}$. If $r_1, r_2 \in \mathrm{R}$ and $s_1, s_2 \in \mathrm{S}$ are such that $r_1 - r_2 = s_1 - s_2$, then $(s_1 + \mathrm{R}) \cap (s_2 + \mathrm{R}) \neq \emptyset$. As $\mathcal{C}$ is complementary, it follows that Per R $\mid s_1 - s_2$ or equivalently, $s_1 - s_2 \in$ Int [Per R]. By Proposition 4.4, S $\perp$ [Per R]; consequently we must have Per S $\mid s_1 - s_2$ and finally Per R $\vee$ Per S $\mid s_1 - s_2$. We have thus proved that $(\mathrm{R} - \mathrm{R}) \cap (\mathrm{s} - \mathrm{S}) \in (\text{Per R} \vee \text{Per S})\mathbf{Z}$, that is, *Grd* $\mathcal{C} \perp [\mathrm{S}]$.

(iv) $\rightarrow$ (iii) is clear.

(iii)→(i) Let S belong to a metric class admitted by $\mathcal{C}$ which is intervallically disjoint from *Grd* $\mathcal{C}$. If $(s_1 + R) \cap (s_2 + R) \neq \emptyset$ for some $s_1, s_2 \in S$, then $s_1 - s_2 \in$ Int *Grd* $\mathcal{C}$. As *Grd* $\mathcal{C} \perp [S]$ by hypothesis, it follows that Per R | $s_1 - s_2$. In conclusion, $\mathcal{C}$ is complementary.

The implication (i) → (ii) is now a consequence of (i) → (iv), while (ii) → (i) is a consequence of (iii) → (i).

PROPOSITION 6.2. *Let R and S be two intervallically disjoint rhythmic classes and let $\mathcal{C}$ be a canon in Can (R,S). Then $\mathcal{C}$ is a complementary canon which admits S as a metric class of order* $k = (\text{Per } R \vee \text{Per } S)/\text{Per } R$.

Proof. Follows from Propositions 4.5, 6.1, and 4.3, noting that the relation $R \perp S$ implies $[\text{Per } R] \perp S$.

COROLLARY 6.1. *Every complementary canon is invertible. Rhythmically meaningful inversions of complementary canons lead to complementary canons.*

Before proceeding with the theory of complementary canons we present a result which, apart from its theoretical character, has the practical importance of a criterion for intervallic disjointness.

PROPOSITION 6.3. *For any R, S ∈ Rhyt we have*

$$\frac{\text{Per } R \wedge \text{Per } S}{\text{Per } (R+S)} \text{Nrp}(R+S) \leq (\text{Nrp } R)(\text{Nrp } S). \qquad (1)$$

Equality holds in (1) *iff* $R \perp S$.

Proof. Set $a = \text{Div } R \wedge \text{Div } S$, $b = \text{Per } R \vee \text{Per } S$. By Proposition 3.4 we have

$$\text{Nrp } R = \text{Nr } H_{a,b}(R) \text{ Per } R \,/\, b\,,$$

$$\text{Nrp } S = \text{Nr } H_{a,b}(S) \text{ Per } S \,/\, b\,,$$

$$\text{Nrp } (R+S) = \text{Nr } H_{a,b}(R+S) \text{ Per } (R+S) \,/\, b.$$

Choose the sets $M \in H_{a,b}(R)$ and $N \in H_{a,b}(S)$. By Proposition 3.6, $H_{a,b}(R+S) = H_{a,b}(R) + H_{a,b}(S)$ so that $M + N \in H_{a,b}(R+S)$ and hence

$$\text{Nr } H_{a,b}(R) = \#M, \quad \text{Nr } H_{a,b}(S) = \#N,$$

$$\text{Nr } H_{a,b}(R+S) = \#(M+N).$$

Substituting all these into (1) and taking into account the identity $rs = (r \wedge s)(r \vee s)$ true for every $r, s \in \mathbf{Q}_+$, the inequality to prove becomes

$$\#(M + N) \leq (\#M)(\#N). \tag{2}$$

Equality holds in (1) iff equality holds in (2).

Consider the map f: $M \times N \rightarrow M + N$ defined by $f(x,y) = x + y$. As f is onto, the inequality (2) is always true. Now remark that equality holds in (2) iff f is one-to-one. The condition "f is one-to-one" is equivalent to the condition

$$(M - M) \cap (N - N) = \{0\}. \tag{3}$$

Let G, H denote the stability subgroup of M, N, respectively. By Proposition 3.4,

$$G = (\text{Per } R/a)\mathbf{Z}_n, \quad H = (\text{Per } S/a)\mathbf{Z}_n$$

so that

$$G \cap H = ((\text{Per } R \vee Per\ S)/a)\mathbf{Z}_n = n\mathbf{Z}_n = \{0\}.$$

Consequently, condition (3) is equivalent to

$$\text{Int } H_{a,b}(R) \cap \text{Int } H_{a,b}(S) \subset G \cap H$$

which, by virtue of Proposition 3.5, is equivalent to $R \perp S$. The proof is complete.

COROLLARY 6.2. *For every $R \in Rhyt$ and $t \in \mathbf{Q}_+$ we have*

$$\frac{Per\ R \wedge t}{\text{Per } (R + [t])} \text{Nrp } (R + [t]) \leq \text{Nrp } R.$$

Equality holds iff $R \perp [t]$.

PROPOSITION 6.4. *Let $\bar{\mathcal{C}}$ be a canon in the minmax condensation of the class of a complementary canon $\mathcal{C}$. Then $\bar{\mathcal{C}}$ is a complementary canon whose ground number divides the ground number of $\mathcal{C}$ and whose category divides the category of $\mathcal{C}$.*

Proof. Set $R = Grd\ \mathcal{C}$, $S = Met\ \mathcal{C}$ and consider the sequence $(R_n, S_n)_{n \geq 0}$ associated to (R,S) via formulas (5)–(6) from Section 5. It will suffice to prove the relations

$$R_n \perp S_n \tag{4}$$

and

$$\text{Nrp}\, R_{n+1} \mid \text{Nrp}\, S_n, \quad \text{Nrp}\, S_{n+1} \mid \text{Nrp}\, R_n \tag{5}$$

for every $n \geq 0$. Indeed, suppose (4) and (5) have been proved. We know that the class of $\bar{\mathcal{C}}$ equals $Can(R_{n_0}, S_{n_0})$ for some even integer n_0. From (4) we see that $\bar{\mathcal{C}}$ is complementary (by Proposition 6.1). The relations (5) imply that Nrp R_{n+2} | Nrp R_n and Nrp S_{n+2} | Nrp S_n for every $n \geq 0$; consequently Nrp R_{n_0} | Nrp R and Nrp S_{n_0} | Nrp S as n_0 is even.

The proof of (4) is done by induction on n. For $n=0$, (4) is true as $\mathcal{C}$ is complementary. Suppose it true for n and let us prove it for $n + 1$. As $S_n \perp R_n$, the canons in the class Can(S_n, R_n) are complementary (by Proposition 6.2). But $Can(S_n, R_n) = Can(S_n, R_n + [\text{Per}\, S_n])$, the pair in the right side being normal; hence $R_{n+1} = S_n \perp (R_n + [\text{Per}\, S_n]) = S_{n+1}$ by Proposition 6.1.

The first of the relations (5) is obviously true as $R_{n+1} = S_n$. For the second, we have $S_{n+1} = R_n + [\text{Per}\, S_n]$ and $R_n \perp [\text{Per}\, S_n]$ because $R_n \perp S_n$ by (4). We may therefore apply Corollary 6.2 to R_n and Per S_n in order to obtain

$$\frac{\text{Per}\, R_n \wedge \text{Per}\, S_n}{\text{Per}\, S_{n+1}} \text{Nrp}\, S_{n+1} = \text{Nrp}\, R_n.$$

Now observe that Per S_{n+1} | Per R_n ^ *Per* S_n so that (Per R_n ^ *Per* S_n)/Per S_{n+1} is an integer and the above equality tells that Nrp S_{n+1} | Nrp R_n. The proof is complete.

I describe now a procedure of "tilling" a complementary canon based on a simple device I call ***elementary derivation***. By definition, the latter means any of the following transformations applied to a pair (R,S) of intervallically disjoint rhythmic classes:

— replacing R by any $R' \in Rhyt$ satisfying the relations $R' + [\text{Per}\, R \vee \text{Per}\, S] = R$ and $R' \perp [\text{Per}\, R \vee \text{Per}\, S]$;

— replacing S by any $S' \in Rhyt$ satisfying the relations $S' + [\text{Per}\, R \vee \text{Per}\, S] = S$ and $S' \perp [\text{Per}\, R \vee \text{Per}\, S]$.

The reader will have no difficulty in verifying that the classes in a pair obtained by an application of an elementary derivation to a pair of intervallically disjoint rhythmic classes are still intervallically disjoint. Moreover, the composition of the two classes in the pair does not change under elementary derivation. In fact, we have some more precise results as shown by the following propositions.

PROPOSITION 6.5. *Let $R,S \in Rhyt$ be such that $R \perp S$ and let (R',S') be obtained from (R,S) by an elementary derivation. Then the minmax condensations of $Can\ (R,S)$ and of $Can\ (R',S')$ coincide.*

Proof. We have

$$Can\ (R,S) = Can\ (R, S + [\text{Per}\, R]),$$
$$Can\ (R',S') = Can\ (R', S' + [\text{Per}\, R']),$$

the pairs in the right side being normal. Write the minmax condensations of $Can(R,S)$, $Can(R',S')$ as $Can(\bar{R},\bar{S})$, $Can(\bar{R}',\bar{S}')$, respectively, so that $\text{Per}\, \bar{R} = \text{Per}\, \bar{S}$ and $\text{Per}\, \bar{R}' = \text{Per}\, \bar{S}'$. The definition of condensation of canons implies the relations

$$R \to \bar{R},$$
$$S \to S + [\text{Per}\, R] \to \bar{S},$$
$$R' \to \bar{R}',$$
$$S' \to S' + [\text{Per}\, R'] \to \bar{S}'.$$

We consider first the case when R has been extended to R' and S has been left invariant (hence $S' = S$). Since $R' \to R$ we have

$$Can(R',S) \to Can(R,S) \to Can(\bar{R},\bar{S})$$

which implies by Proposition 5.3

$$Can(\bar{R}', \bar{S}') \to Can(\bar{R},\bar{S}). \tag{6}$$

On the other hand, if we let $S'_0 = S + [\text{Per}\, R']$, then $\text{Per}\, S'_0 \mid \text{Per}\, S$ and consequently

$$\begin{aligned} R' + [\text{Per}\, S'_0] &= R' + [\text{Per}\, R \vee \text{Per}\, S] + [\text{Per}\, S'_0] \\ &= R + [\text{Per}\, S'_0]. \end{aligned}$$

From $S'_0 \to \bar{S}'$ we infer that $[\text{Per}\, S'_0] \to [\text{Per}\, \bar{S}']$; hence

$$\begin{aligned} R \to R + [\text{Per}\, S'_0] &= R' + [\text{Per}\, S'_0] \to \bar{R}' + [\text{Per}\, \bar{S}'] \\ &= \bar{R}' + [\text{Per}\, \bar{R}'] = \bar{R}'. \end{aligned}$$

In particular we obtain $[\text{Per}\, R] \to [\text{Per}\, \bar{R}']$ implying that

$$S + [\text{Per}\, R] \to \bar{S}' + [\text{Per}\, \bar{R}'] = \bar{S}' + [\text{Per}\, \bar{S}'] = \bar{S}'.$$

In conclusion, $Can(R,S) \rightarrow Can(\bar{R}', \bar{S}')$ which implies by Proposition 5.3

$$Can(\bar{R},\bar{S}) \rightarrow Can(\bar{R}', \bar{S}'). \tag{7}$$

Comparison of (6) and (7) yields $Can(\bar{R},\bar{S}) = Can(\bar{R}', \bar{S}')$.

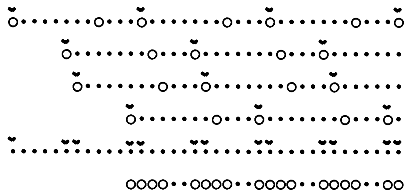
The case when S has been extended and R has been left invariant is reduced to the case considered above by using Corollary 5.1.

PROPOSITION 6.6. *Let $\mathcal{C}$ be a complementary canon and let $Can(\bar{R},\bar{S})$ be the minmax condensation of the class of $\mathcal{C}$, so that* Per $\bar{R}$ = Per $\bar{S}$. *Then the pair (Grd $\mathcal{C}$, Met $\mathcal{C}$) is obtained from the pair $(\bar{R},\bar{S})$ by applying a finite number of successive elementary derivations.*

Proof. Consider the sequence $(R_n, S_n)_{n\geq 0}$, associated to $(Grd\,\mathcal{C}, Met\,\mathcal{C})$ via formulas (5)–(6) from Section 5. We know that there is an even integer n_0 so that $(\bar{R},\bar{S}) = (R_{n_0},S_{n_0})$. If one goes through the sequence in question in the opposite sense (form $n = n_0$ to $n = 0$) taking into considerations only the terms with an even index, we see that $(Grd\,\mathcal{C}, Met\,\mathcal{C})$ is obtained from (R_{n_0}, S_{n_0}) by n_0 successive elementary derivations. This is so because the passage from (R_n,S_n) to (R_{n-2},S_{n-2}) is accomplished by two elementary derivations: the first from (R_n,S_n) to (R_n,S_{n-2}) and the second from (R_n,S_{n-2}) to (R_{n-2},S_{n-2}).

By applying several successive elementary derivations to the pair $(Grd\,\mathcal{C}$, $Met\,\mathcal{C})$ where $\mathcal{C}$ is a complementary canon, the composer has the opportunity to till the canon $\mathcal{C}$ by enlarging both its temporal dimension (the period of its ground class) and its spatial dimension (the number of voices) *without changing the resultant class.* It should be remarked that the process of successive elementary derivations is in some sense the reverse of the process of successive inversions and condensations used in finding the minmax condensation (see the proof of Proposition 6.6); the main difference between them lies in the fact that, while in the latter the passage from the n-th pair to the next one is uniquely determined, in the former there is an infinity of choices for the $(n+1)$-th pair, as there is an infinity of extensions of the rhythmic classes R_n or S_n which may be used in an elementary derivation applied to (R_n,S_n).

Consider the complementary canon $\mathcal{C}_3$:



We have

$$Grd\,\mathcal{C}_3 = [𝅗𝅥 ♩] = R_0, \quad Met\,\mathcal{C}_3 = [♩‿♫] = S_0.$$

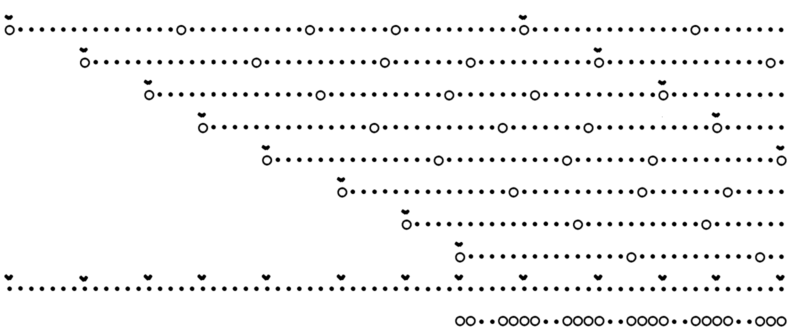
By applying an elementary derivation to the pair (R_0,S_0) we obtain the pair (R_1,S_1) with $R_1 = R_0$ and

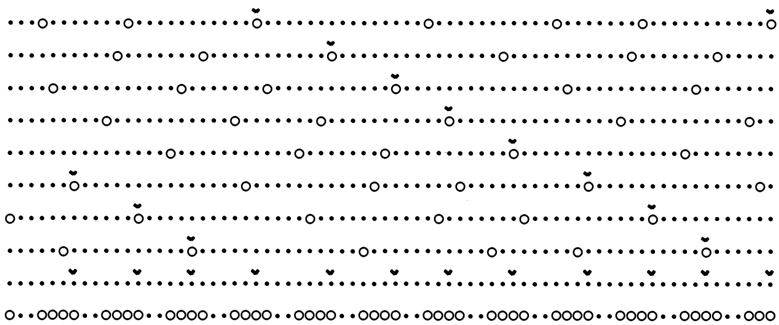
$$S_1 = [♩.. 𝅘𝅥𝅯♩‿♫‿𝅘𝅥𝅮♩.] .$$

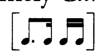
Indeed, S_1 + [𝅗𝅥.] = S_0 and $S_1 \perp$ [𝅗𝅥.] . By applying another elementary derivation to (R_1,S_1), we arrive at the pair (R_2,S_2) with $S_2 = S_1$ and

$$R_2 = [𝅝 \; 𝅗𝅥. \; 𝅗𝅥 \; 𝅗𝅥.] .$$

Indeed, $R_2 + [3/2] = R_1$ and $R_2 \perp [3/2]$. Here is a canon $\mathcal{C}'_3$ from the class $Can(R_2,S_2)$:





The canon classes of $\mathcal{C}_3$ and $\mathcal{C}'_3$ have the same minmax condensation, namely *Can*( , ). In particular, *Res* $\mathcal{C}_3$ = *Res* $\mathcal{C}'_3$ =  .

EXAMPLE 6.3

In order to start a process of successive elementary derivations we need a pair (R,S) of intervallically disjoint rhythmic classes. That is, some preliminary work is necessary for finding such a pair. There are however situations when this preliminary work is not needed at all, the entire work of constructing the complementary canon being therefore concentrated in the process of successive elementary derivations. These situations occur when the pair we start with is an intervallically disjoint pair having the simplest form, namely $(R, [\mathrm{Per}\,R])$ or $([\mathrm{Per}\,R], R)$ for *any* $R \in Rhyt$. These remarks motivate the following definitions.

DEFINITION 6.2. *An elementary pair is any pair of the form* $(R, [\mathrm{Per}\,R]$ *or* $([\mathrm{Per}\,R], R)$ *with* $R \in Rhyt$. *An elementary canon class (or an elementary canon, respectively) is a class of the form Can(R,S) with (R,S) an elementary pair (or a canon whose class is elementary).*

Every elementary canon is a complementary canon of maximal category.

DEFINITION 6.3. *A pair (R,S) of rhythmic classes is said to be constructible by elementary derivations if there is a finite sequence* $(R_0,S_0), \ldots, (R_m,S_m)$ *of pairs of rhythmic classes such that* (R_0,S_0) *is an elementary pair,* $(R_m,S_m) = (R,S)$ *and for every* $i \in \{0, \ldots, m-1\}$, *the pair* (R_{i+1},S_{i+1}) *is obtained from* (R_i,S_i) *by an elementary derivation. A canon is said to be constructible by elementary derivations if its class can be* *represented as Can(R,S), the pair (R,S) being constructible by elementary derivations.*

Every canon constructible by elementary derivations is complementary and hence invertible, its inverses being also constructible by elementary derivations.

The next proposition characterizes constructibility in terms of the elementarity of the minmax condensation.

PROPOSITION 6.7. *The following conditions on a complementary canon $\mathcal{C}$ are equivalent:*

(i) *$\mathcal{C}$ is constructible by elementary derivations;*

(ii) *The minmax condensation of the class of $\mathcal{C}$ is an elementary class;*

(iii) *The pair (Grd $\mathcal{C}$, Met $\mathcal{C}$) is constructible by elementary derivations.*

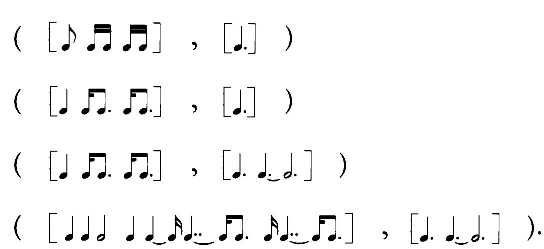
Proof.

(i) → (ii) Follows from Proposition 6.5.
(ii) → (iii) Follows from Proposition 6.6.
(iii) → (i) is obvious.

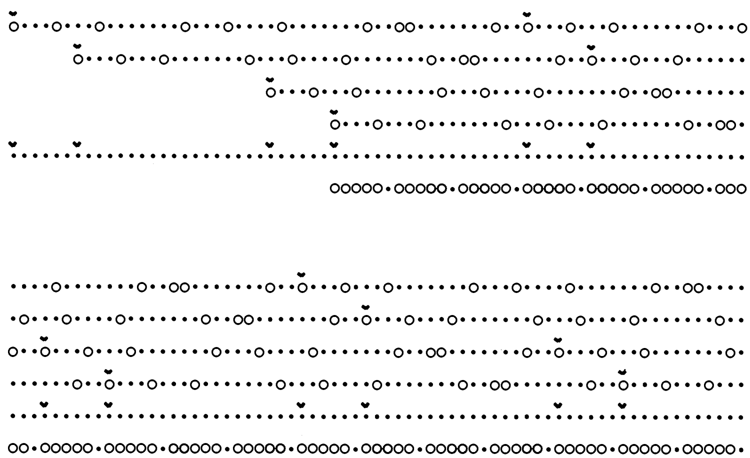
In particular, a complementary canon of maximal category is not constructible by elementary derivations unless it is itself elementary.

COROLLARY 6.3. *Every complementary canon whose ground number or whose category equals 1 is constructible by elementary derivations.*

We construct a complementary canon by successive elementary derivations starting with the elementary pair ([♪ ♬ ♬] , [♩.]).
The succession of pairs is listed below:



Here is a canon in the canon class associated to the last pair in the list.



EXAMPLE 6.4

7. The Structure of Regular Complementary Canons

Definition 7.1. *A regular complementary canon is a complementary canon whose resultant class is regular.*

We have seen in the preceding section that the notion of a complementary canon was related to the notion of intervallically disjoint rhythmic classes. Following the same idea, regular complementary canons are related to the notion of supplementary rhythmic classes to be introduced below.

Definition 7.2. *Two rhythmic classes R, S are called supplementary if $R \perp S$ and $R + S$ is regular.*

Proposition 7.1. *Two rhythmic classes R, S are supplementary iff*

$$\frac{\text{Per}\, R \wedge \text{Per}\, S}{\text{Per}\, (R + S)} = (\text{Nrp}\, R)(\text{Nrp}\, S).$$

Proof. The necessity follows from Proposition 6.3. For the sufficiency, suppose the above relation to hold. Then by the inequality in Proposition 6.3 we have

$$\frac{\text{Per } R \wedge \text{Per } S}{\text{Per } (R + S)} \text{Nrp } (R + S) \leq (\text{Nrp } R)(\text{Nrp } S) = \frac{\text{Per } R \wedge \text{Per } S}{\text{Per } (R + S)}.$$

It follows that Nrp $(R + S) = 1$, that is $R + S$ is regular. In particular, the sign "≤" in the above inequality may be replaced by "="; we infer then from Proposition 6.3 that $R \perp S$.

Thus, to verify that two rhythmic classes R, S are supplementary, we compute first $R + S$. If the latter is not a regular class, R and S cannot be supplementary; if $R + S$ is regular, then we check whether the relation in the statement of Proposition 7.1 is satisfied.

PROPOSITION 7.2. *For any canon $\mathcal{C}$ the following conditions are equivalent:*

(i) *$\mathcal{C}$ is regular and complementary;*

(ii) *Grd $\mathcal{C}$ is supplementary to Met $\mathcal{C}$;*

(iii) *Grd $\mathcal{C}$ is supplementary to some metric class admitted by $\mathcal{C}$;*

(iv) *Grd $\mathcal{C}$ is supplementary to any metric class admitted by $\mathcal{C}$.*

The proof follows from Propositions 4.2 and 6.1.

COROLLARY 7.1. *Every regular complementary canon is invertible. Rhythmically meaningful inversions of regular complementary canons lead to regular complementary canons.*

COROLLARY 7.2. *The modulus of a regular complementary canon equals the product between its ground number and its number of voices.*

Proof. Let $\mathcal{C}$ be a regular complementary canon on l voices. By Proposition 4.3,

$$l = (\text{Nrp } \mathit{Met}\, \mathcal{C})(\text{Per } \mathit{Grd}\, \mathcal{C})/\text{Per } \mathit{Met}\, \mathcal{C}. \tag{1}$$

By Propositions 7.1 and 7.2,

$$\text{Per } \mathit{Met}\, \mathcal{C} / \text{Div } \mathit{Res}\, \mathcal{C} = (\text{Nrp } \mathit{Grd}\, \mathcal{C})(\text{Nrp } \mathit{Met}\, \mathcal{C}); \tag{2}$$

in obtaining (2) we used the facts that $\mathit{Res}\, \mathcal{C} = \mathit{Grd}\, \mathcal{C} + \mathit{Met}\, \mathcal{C}$ and that Div $\mathit{Res}\, \mathcal{C}$ = Per $\mathit{Res}\, \mathcal{C}$ as $\mathit{Res}\, \mathcal{C}$ is regular. By multiplying both sides of (2) by Per $\mathit{Grd}\, \mathcal{C}$/Per $\mathit{Met}\, \mathcal{C}$ and by taking into account (1), we arrive at the equality we look for.

We present some instances of usage of regular complementary canons in the work of Bach in Examples 7.1–7.8.

> *The Well-Tempered Clavier*, Book II,
> Fugue in C Major, measures 42–45.
>
> *The Well-Tempered Clavier*, Book II,
> Fugue in B Major, measures 67–71.

EXAMPLE 7.1: THE CLASS *Can*([♩. ♪] , [♩])

> Three-Part Inventions,
> Invention in A Major, measures 5, 15, and 24.
>
> Twelve Little Preludes,
> Prelude in F Major, measures 4–5, 15, and 23–24.

EXAMPLE 7.2: THE CLASS *Can*([♪. 𝅘𝅥𝅯] , [♪])

> *The Well-Tempered Clavier*, Book, I,
> Prelude in G Minor, measure 3.

EXAMPLE 7.3: THE CLASS *Can*([♪] , [𝅘𝅥𝅯])

Note that the class in Example 7.3 is the inverse of the class in Example 7.2.

> *The Well-Tempered Clavier*, Book I,
> Fugue in F Minor, measures 26–27, 39–40, and 56.
>
> *The Well-Tempered Clavier*, Book I,
> Fugue in G Minor, measures 25–27.

EXAMPLE 7.4. THE CLASS *Can*([♩‿𝅘𝅥𝅯𝅘𝅥𝅯𝅘𝅥𝅯𝅘𝅥𝅯] , [♩])

Two-Part Inventions,
Invention in C Major, measures 15–18.

EXAMPLE 7.5. THE CLASS *Can*()

The Well-Tempered Clavier, Book I,
Prelude in G-sharp Minor, measures 19–21.

The Well-Tempered Clavier, Book I,
Fugue in A Major, measures 17–18.

EXAMPLE 7.6: THE CLASS *Can*()

The Well-Tempered Clavier, Book II,
Fugue in F Major, measures 38–44, 56–60, 61–66, and 72–76.

EXAMPLE 7.7: THE CLASS *Can*()

The Well-Tempered Clavier, Book I,
Fugue in F-sharp Minor, measures 35–36.

EXAMPLE 7.8: THE CLASS *Can*()

We come now to the problem of the construction of regular complementary canons. We know from the preceding section that the resultant class is left unchanged under an elementary derivation. Therefore, an elementary derivation applied to a regular complementary canon yields a regular complementary canon; successive elementary derivations applied to a given regular complementary canon allow us to enlarge it to regular complementary canons of arbitrary large size.

In particular, one can construct regular complementary canons by applying successive elementary derivations to a pair of the form $([t], [t])$ with $t \in \mathbf{Q}_+$. The canons obtained in this manner are precisely those regular complementary canons which are constructible by elementary derivations in the sense of Definition 6.3:

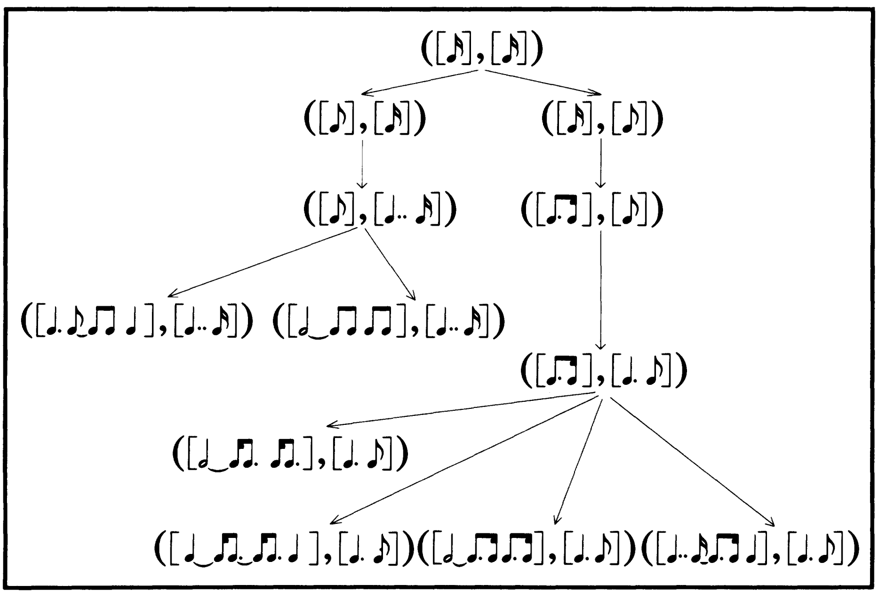
PROPOSITION 7.3. *For any regular complementary canon $\mathcal{C}$ the following conditions are equivalent:*

(i) *$\mathcal{C}$ is constructible by elementary derivations;*

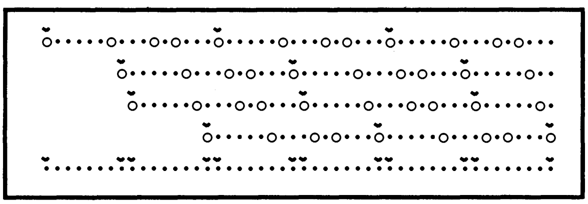
(ii) *The minmax condensation of the class of $\mathcal{C}$ has the form Can$([t], [t])$ for some $t \in \mathbf{Q}_+$;*

(iii) *The pair (Grd $\mathcal{C}$, Met $\mathcal{C}$) is constructible by elementary derivations from a pair of the form $([t], [t])$ for some $t \in \mathbf{Q}_+$.*

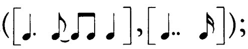
The proof follows from Proposition 6.7, noting that an elementary canon is regular iff its class has the form *Can*$([t],[t])$.



EXAMPLE 7.9: PAIRS OF SUPPLEMENTARY RHYTHMIC CLASSES OBTAINED BY SUCCESSIVE ELEMENTARY DERIVATIONS (THE LATTER ARE INDICATED BY ARROWS)



EXAMPLE 7.10: A REGULAR COMPLEMENTARY CANON $\mathcal{C}$ ON FOUR VOICES CONSTRUCTED BY SUCCESSIVE ELEMENTARY DERIVATIONS

The canon class of $\mathcal{C}$ equals Can(); see Example 7.9.

Despite their arbitrary large size, the regular complementary canons which are constructible by elementary derivations are the regular complementary canons with the simplest conceivable structure. This assertion is supported by the next two theorems, which represent the main results in this study. They show that the method of successive elementary derivations allows us to obtain *all regular complementary canons whose numerical invariants* (introduced in Section 4) *have a not too complicated arithmetical structure*. They also show that the problem of constructibility of regular complementary canons by elementary derivations is closely related to the problem of the maximality of their category. In particular, they assert that *there exist nonelementary regular complementary canons of maximal category* (not an obvious fact, insofar as the regular complementary canons of not too large size, which occur in most common situations, must obey the next two theorems, which strictly forbid their category to be maximal).

Before stating those theorems, we make clear the relation between the study of regular complementary canons and the theory of supplementary sets presented in Section 2.

PROPOSITION 7.4. *For every $R,S \in Rhyt_{a,b}$ the following are true:*

(i) *If R and S are supplementary and $a = \mathrm{Div}\, R \wedge \mathrm{Div}\, S$, $b = \mathrm{Per}\, R \vee \mathrm{Per}\, S$, then $H_{a,b}(R)$ and $H_{a,b}(S)$ are supplementary translation classes of $\mathbf{Z}_n$, where $n = b/a$.*

(ii) *If $H_{a,b}(R)$ and $H_{a,b}(S)$ are supplementary translation classes of $\mathbf{Z}_n$ $(n = b/a)$, then R and S are supplementary.*

Proof. (i) → (ii) As in the proof of Proposition 6.3 we see that because of the equality $b = \mathrm{Per}\, R \vee Per\ S$, the intersection of the stability subgroups of $H_{a,b}(R)$ and $H_{a,b}(S)$ is reduced to $\{0\}$. Consequently, the relation $R \perp S$ implies, by virtue of Proposition 3.5,

$$\mathrm{Int}\, H_{a,b}(R) \cap \mathrm{Int}\, H_{a,b}(S) = \{0\}. \tag{3}$$

By Proposition 3.4,

$$\mathrm{Nrp}\, R = \mathrm{Nr}\, H_{a,b}(R)\ \mathrm{Per}\, R/b, \quad \mathrm{Nrp}\, S = \mathrm{Nr}\, H_{a,b}(S)\ \mathrm{Per}\, S/b.$$

As $R + S$ is regular, we also have

$$\mathrm{Per}\,(R + S) = \mathrm{Div}\,(R + S) = \mathrm{Div}\, R \wedge \mathrm{Div}\, S = a.$$

Substituting all these into the equality

$$\frac{\text{Per } R \wedge \textit{Per } S}{\text{Per } (R + S)} = (\text{Nrp } R)(\text{Nrp } S)$$

given by Proposition 7.1 and taking into account the relation (Per R)(Per S) = (Per R ^ Per S)(Per R v Per S), we finally obtain

$$\text{Nr } H_{a,b}(R) \text{ Nr } H_{a,b}(S) = b/a = n. \tag{4}$$

The relations (3) and (4) tell that any set $M \in H_{a,b}(S)$ together with any set $N \in H_{a,b}(S)$ satisfy conditions (ii) and (iii) in Proposition 2.1, so that they are supplementary. Thus, the classes $H_{a,b}(R)$ and $H_{a,b}(S)$ are supplementary.

(ii) → (i) If $H_{a,b}(R)$ and $H_{a,b}(S)$ are supplementary, then (3) holds, so that $R \perp S$ by Proposition 3.5. Also, as $H_{a,b}(R) + H_{a,b}(S) = [\mathbf{Z}_n] = H_{a,b}([a])$, Proposition 3.6 implies that $R + S = [a]$.

In the following we shall make use of the sets of integers $\mathbf{N}_0$ and $\mathbf{N}$ introduced in Section 2.

THEOREM 7.1. *For every integer $m \geq 1$ the following conditions are equivalent:*

(i) $m \in \mathbf{N}_0$;

(ii) *Every nonelementary regular complementary canon whose ground number or whose category equals m is not a canon of maximal category;*

(iii) *Every regular complementary canon whose ground number or whose category equals m is constructible by elementary derivations.*

THEOREM 7.2. *For every integer $n \geq 1$ the following conditions are equivalent:*

(i) $n \in \mathbf{N}$;

(ii) *Every nonelementary regular complementary canon of modulus n is not a canon of maximal category;*

(iii) *Every regular complementary canon of modulus n is constructible by elementary derivations.*

(Note that whenever $m \geq 2$ or $n \geq 2$, the adjective "nonelementary" in condition (ii) in the above theorems is superfluous.)

Proof of Theorems 7.1 and 7.2. (i) → (ii) *and* (iii) Let $\mathcal{C}$ be a nonelementary regular complementary canon satisfying at least one of the relations

$$\mathrm{Nrp}\ \mathit{Grd}\,\mathcal{C} \in \mathbf{N}_0, \tag{5}$$

$$\mathrm{Nrp}\ \mathit{Met}\,\mathcal{C} \in \mathbf{N}_0, \tag{6}$$

$$n = \mathrm{Per}\ \mathit{Grd}\,\mathcal{C}/\mathrm{Div}\ \mathit{Res}\,\mathcal{C} \in \mathbf{N}. \tag{7}$$

Suppose, if possible, that $\mathcal{C}$ is a canon of maximal category. Set $R = \mathit{Grd}\,\mathcal{C}$, $S = \mathit{Met}\,\mathcal{C}$, $a = \mathrm{Div}\ \mathit{Res}\,\mathcal{C} = \mathrm{Div}\,R \wedge \mathit{Div}\,S$, $b = \mathrm{Per}\,R = \mathrm{Per}\,S$. By Propositions 7.2 and 7.4, $H_{a,b}(R)$ and $H_{a,b}(S)$ are supplementary translation classes of $\mathbf{Z}_n$; by Proposition 3.4, the stability subgroups of $H_{a,b}(R)$ and of $H_{a,b}(S)$ are reduced to $\{0\}$ and we have

$$\mathrm{Nr}\,H_{a,b}(R) = \mathrm{Nrp}\,R, \quad \mathrm{Nr}\,H_{a,b}(S) = \mathrm{Nrp}\,S.$$

We also have $n>1$, as $\mathcal{C}$ is not elementary by hypothesis. We may apply Theorem 2.1 in case that (5) or (6) holds or Theorem 2.2 in case that (7) holds in order to conclude that the stability subgroup of at least one of the classes $H_{a,b}(R)$, $H_{a,b}(S)$ is not reduced to $\{0\}$. The contradiction we have arrived at proves that $\mathcal{C}$ cannot be a canon of maximal category.

Now let $\mathit{Can}(\bar{R},\bar{S})$ (with $\mathrm{Per}\,\bar{R} = \mathrm{Per}\,\bar{S}$) be the minmax condensation of $\mathit{Can}(R,S)$ and let $\bar{\mathcal{C}}$ be a canon in $\mathit{Can}(\bar{R},\bar{S})$. We know from Proposition 5.4 that the modulus of $\bar{\mathcal{C}}$ divides the modulus of $\mathcal{C}$; we also know from Proposition 6.4 that $\mathrm{Nrp}\,\bar{R} \mid \mathrm{Nrp}\,R$ and $\mathrm{Nrp}\,\bar{S} \mid \mathrm{Nrp}\,S$. Besides, every positive integer which divides an integer in $\mathbf{N}$ also belongs to $\mathbf{N}$; the same is obviously true for $\mathbf{N}_0$. These remarks enable us to conclude that whenever $\mathcal{C}$ satisfies at least one of the relations (5)–(7), the same is true for $\bar{\mathcal{C}}$. It follows then by the above part of the proof that $\bar{\mathcal{C}}$, which is a regular complementary canon of maximal category, must be elementary; by Proposition 6.7, this means that $\mathcal{C}$ is constructible by elementary derivations.

(ii) *or* (iii) → (i) If $m \notin \mathbf{N}_0$, there are by Theorem 2.1 an integer $n \geq 2$ and two nonperiodic supplementary subsets M,N of $\mathbf{Z}_n$ such that $\#\mathrm{M} = m$. If $n \notin \mathbf{N}$ there are by Theorem 2.2 two nonperiodic supplementary subsets M,N of $\mathbf{Z}_n$. In both situations, choose any $a \in \mathbf{Q}_+$ and set $R = H^{-1}_{a,na}([\mathrm{M}])$, $S = H^{-1}_{a,na}([\mathrm{N}])$. By Proposition 3.4, $\mathrm{Per}\,R = \mathrm{Per}\,S = na$; by Proposition 7.4(ii), R and S are supplementary rhythmic classes. Hence any canon $\mathcal{C}$ in $\mathit{Can}(R,S)$ is a nonelementary regular complementary canon of maximal category; in particular, $\mathcal{C}$ is not constructible by elementary derivations. In the second situation the modulus of $\mathcal{C}$ equals n, while in the first situation we have (Proposition 3.4) $\mathrm{Nrp}\ \mathit{Grd}\,\mathcal{C} = \mathrm{Nrp}\,R = \#\mathrm{M} = m$. In conclusion, if condition (i) in either Theorem 7.1 or Theorem 7.2 does not hold, then conditions (ii) and (iii) in the respective theorems also do not hold. Theorems 7.1 and 7.2 are thus completely proved.

In connection with conditions (ii) and (iii) in Theorem 7.1, we observe that by inverting a regular complementary canon of maximal category whose ground number equals m we obtain a regular complementary canon of maximal category equal to m.

COROLLARY 7.3. *A regular complementary canon on p^k voices (p prime, $k \geq 1$) is constructible by elementary derivations and it is not a canon of maximal category.*

Proof. By Proposition 4.3, the category of any canon divides its number of voices; if the latter belongs to $\mathbf{N}_0$, the former also belongs to $\mathbf{N}_0$. The corollary appears thus as a consequence of Theorem 7.1.

We have seen that constructing a regular complementary canon of maximal category amounts to constructing a pair of nonperiodic supplementary subsets of some group $\mathbf{Z}_n$. We illustrate the construction of such subsets by the method indicated in the proof of Proposition 2.2. We use the notations introduced during the proof of that proposition.

Take $p_1 = 2, p_2 = 3, n_1 = 2, n_2 = 3, n_3 = 2$ so that $p_1 n_1 \wedge p_2 n_2 = 1$ and $n = p_1 p_2 n_1 n_2 n_3 = 72$. The subgroups of $\mathbf{Z}_{72}$ to be needed in the following are:

$$36\mathbf{Z}_{72} = \{0,36\},$$

$$24\mathbf{Z}_{72} = \{0,24,48\},$$

$$18\mathbf{Z}_{72} = \{0,18,36,54\},$$

$$8\mathbf{Z}_{72} = \{0,8,16,24,32,40,48,56,64\}$$

and $2\mathbf{Z}_{72}$, the subgroup with thirty-six elements.

Choose a nonperiodic set M_1 of representants of $18\mathbf{Z}_{72}$ modulo its subgroup $36\mathbf{Z}_{72}$:

$$M_1 = \{0,18\}.$$

Choose a nonperiodic set M_2 of representants of $8\mathbf{Z}_{72}$ modulo its subgroup $24\mathbf{Z}_{72}$:

$$M_2 = \{0,32,40\}.$$

Form M:

$$M = M_1 + M_2 = \{0,18,32,40,50,58\}.$$

Choose x_1 in $18Z_{72}\backslash 36\mathbf{Z}_{72} : x_1 = 18$.

Choose x_2 in $8\mathbf{Z}_{72}\backslash 24\mathbf{Z}_{72} : x_2 = 8$.

The set S is here a set consisting of a single element y chosen from $\mathbf{Z}_{72}\backslash 2\mathbf{Z}_{72}$; we take $y = 9$.

Form

$$N_1 = \{0,36\} + \{18,24,48\} = \{12,18,24,48,54,60\}$$

and

$$N_2 = \{0,24,48\} + \{8,36\} = \{8,12,32,36,56,60\}.$$

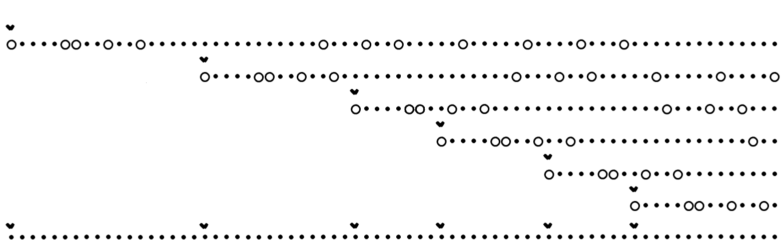
Finally form N:

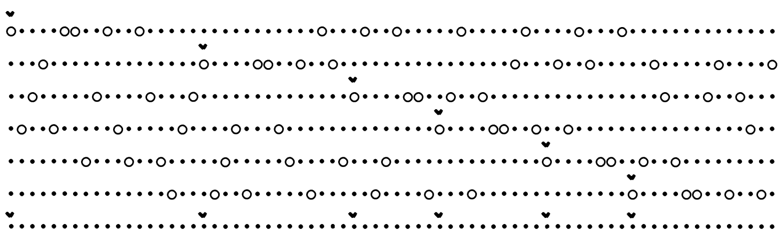
$$N = N_1 \cup (9+N_2) = \{12,17,18,21,24,41,45,48,54,60,65,69\}.$$

The rhythmic classes corresponding to [N] and [M] via $H_{1,72}$ are

$$R = H^{-1}_{1,72}([N]) = [5,1,3,3,17,4,3,6,6,5,4,15],$$
$$S = H^{-1}_{1,72}([M]) = [18,14,8,10,8,14].$$

A regular complementary canon $\mathcal{C}$ of maximal category whose class equals $Can(R,S)$ is presented below.





We needed six voices for the construction of $\mathcal{C}$. By using Corollary 7.3, it is seen that six is the least number of voices needed for a nonelementary regular complementary canon of maximal category.

EXAMPLE 7.11: A REGULAR COMPLEMENTARY CANON OF MAXIMAL CATEGORY ON SIX VOICES

The canon $\mathcal{C}$ constructed in Example 7.11 is primarily invertible; however, the number of voices in the canons whose classes equal the inverse of the class of $\mathcal{C}$ is raised from six to twelve. It is therefore natural to ask: is there a nonelementary regular complementary canon $\mathcal{C}'$ of maximal category with the property that the number of voices in a canon whose class equals the inverse of the class of $\mathcal{C}'$ is the same as the number of voices in $\mathcal{C}'$? Equivalently (by virtue of Proposition 4.3): is there a nonelementary regular complementary canon $\mathcal{C}'$ whose category is maximal and equals the ground number of $\mathcal{C}'$? That the answer is affirmative is shown by the next example.

All we need is a pair (M,N) of nonperiodic supplementary subsets of some group $\mathbf{Z}_n$ with $n>1$ such that #M = #N. The construction of such a pair offers another opportunity to illustrate the method of construction of nonperiodic supplementary subsets described in Section 2.

Take $p_1 = 2, p_2 = 3, n_1 = 4, n_2 = 3, n_3 = 2$, so that $p_1 n_1 \wedge p_2 n_2 = 1$ and $n = p_1 p_2 n_1 n_2 n_3 = 144$. The subgroups of $\mathbf{Z}_{144}$ to be needed in the following are:

$$72\mathbf{Z}_{144} = \{0,72\},$$
$$48\mathbf{Z}_{144} = \{0,48,96\},$$
$$18\mathbf{Z}_{144} = \{0,18,36,54,72,90,108,126\},$$
$$16\mathbf{Z}_{144} = \{0,16,32,48,64,80,96,112,128\}$$

and $2\mathbf{Z}_{144}$, the subgroup with 72 elements.

Choose a nonperiodic set M_1 of representants of $18\mathbf{Z}_{144}$ modulo its subgroup $72Z_{144}$:

$$M_1 = \{0,18,36,126\}.$$

Choose a nonperiodic set M_2 of representants of $16\mathbf{Z}_{144}$ modulo its subgroup $48\mathbf{Z}_{144}$:

$$M_2 = \{0,80,112\}.$$

Form M:

$$M = M_1 + M_2 = \{0,4,18,36,62,80,94,98,112,116,126,130\}.$$

Choose x_1 in $18\mathbf{Z}_{144} \backslash 72\mathbf{Z}_{144}$: $x_1 = 36$.

Choose x_2 in $16\mathbf{Z}_{144} \backslash 48\mathbf{Z}_{144}$: $x_2 = 80$.

The set S is here a set consisting of a single element y chosen from $\mathbf{Z}_{144} \backslash 2\mathbf{Z}_{144}$; we take $y = 31$.

Form

$$N_1 = \{0,72\} + \{36,48,96\} = \{24,36,48,96,108,120\}$$

and

$$N_2 = \{0,48,96\} + \{72,80\} = \{24,32,72,80,120,128\}.$$

Finally form N:

$$N = N_1 \mathrm{U}(31+N_2) = \{7,15,24,36,48,55,63,96,103,108,111,120\}.$$

The rhythmic classes corresponding to [M] and N via $H_{1,144}$ are

$$R = H^{-1}_{1,144}([M]) = [4,14,18,26,18,14,4,14,4,10,4,14],$$

$$S = H^{-1}_{1,144}([N]) = [8,9,12,12,7,8,33,7,5,3,9,31].$$

The ground number and the category of any canon in *Can*(*R*,*S*) equal 12.

EXAMPLE 7.12: A NONELEMENTARY REGULAR COMPLEMENTARY CANON WHOSE CATEGORY IS MAXIMAL AND EQUAL TO THE GROUND NUMBER OF THE CANON

By using Theorem 7.2, it is seen that twelve is the least number of voices needed for a canon with the properties listed at the beginning of Example 7.12.

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