Enumeration of Mosaics / Enumeration of Canons

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1 Enumeration of non-isomorphic mosaics

In [8] it was stated that the enumeration of mosaics is an open research problem communicated by Robert Morris from the Eastman School of Music. More information about mosaics can be found in [1]. Here I present some results from [6].

A *partition* π of a set X is a collection of subsets of X such that the empty set is not an element of π and such that for each $x \in X$ there is exactly one $P \in \pi$ with $x \in P$. If π consists of exactly k subsets, then π is called a *partition of size* k. Let $n > 1$ be an integer. A partition of the set $Z_n := \mathbb{Z}/n\mathbb{Z}$ is called a *mosaic*. Let Π_n denote the set of all mosaics of Z_n, and let $\Pi_{n,k}$ be the set of all mosaics of Z_n of size k.

A group action of a group G on the set Z_n induces the following group action of G on Π_n:

$$G \times \Pi_n \to \Pi_n, \qquad (g,\pi) \mapsto g\pi := \{gP \mid P \in \pi\},$$

where $gP := \{gi \mid i \in P\}$. This action can be restricted to an action of G on $\Pi_{n,k}$. Two mosaics are called G-*isomorphic* if they belong to the same G-orbit on Π_n, in other words, $\pi_1, \pi_2 \in \Pi_n$ are isomorphic if $g\pi_1 = \pi_2$ for some $g \in G$. Usually the cyclic group $C_n := \langle (0,1,\ldots,n-1)\rangle$, the dihedral group $D_n := \langle (0,1,\ldots,n-1),(0,n-1)(1,n-2)\ldots\rangle$, or the group of all affine mappings on Z_n are candidates for the group G. If G acts on a set X, then for each $g \in G$ the permutation $x \mapsto gx$ is indicated by $\bar{g}$. It is called the permutation representation of g. A detailed introduction to combinatorics under finite group actions can be found in [9, 10].

It is well known (see [4, 5]) how to enumerate G-isomorphism classes of mosaics (i.e. G-orbits of partitions) by identifying them with $G \times S_{\underline{n}}$-orbits on the set of all functions from Z_n to $\underline{n} := \{1,\ldots,n\}$. (The *symmetric group* of the set $\underline{n}$ is denoted by $S_{\underline{n}}$.) Furthermore, G-mosaics of size k correspond to $G \times S_{\underline{k}}$-orbits on the set of all *surjective* functions from Z_n to $\underline{k}$.

Theorem 1. *Let M_k be the number of $G \times S_{\underline{k}}$-orbits on $\underline{k}^{Z_n}$, then the number of G-isomorphism classes of mosaics of Z_n is given by M_n, and the number of G-isomorphism classes of mosaics of size k is given by $M_k - M_{k-1}$, where $M_0 := 0$.*

Using the Cauchy-Frobenius-Lemma, *we have*

$$M_k = \frac{1}{|G|\,|S_{\underline{k}}|} \sum_{(g,\sigma)\in G\times S_{\underline{k}}} \prod_{i=1}^{n} a_1(\sigma^i)^{a_i(\bar{g})},$$

where $a_i(\bar{g})$ or $a_i(\sigma)$ are the numbers of i-cycles in the cycle decomposition of $\bar{g}$ or σ respectively.

Finally, the number of G-isomorphism classes of mosaics of size k can also be derived by the Cauchy-Frobenius-Lemma *for surjective functions by*

$$\frac{1}{|G|\,|S_{\underline{k}}|} \sum_{(g,\sigma)\in G\times S_{\underline{k}}} \sum_{\ell=1}^{c(\sigma)} (-1)^{c(\sigma)-\ell} \sum_{a} \prod_{i=1}^{k} \binom{a_i(\sigma)}{a_i} \prod_{j=1}^{n} \left(\sum_{d|j} d \cdot a_d \right)^{a_j(\bar{g})},$$

where the inner sum is taken over the sequences $a = (a_1, \ldots, a_k)$ of nonnegative integers a_i such that $\sum_{i=1}^{k} a_i = \ell$, and where $c(\sigma)$ is the number of all cycles in the cycle decomposition of σ.

If $\pi \in \Pi_n$ consists of λ_i blocks of size i for $i \in \underline{n}$, then π is said to be of *block-type* $\lambda = (\lambda_1, \ldots, \lambda_n)$. From the definition it is obvious that $\sum_{i=1}^{n} i\lambda_i = n$, which will be indicated by $\lambda \vDash n$. Furthermore, it is clear that π is a partition of size $\sum_{i=1}^{n} \lambda_i$. The set of mosaics of block-type λ will be indicated as Π_λ. Since the action of G on Π_n can be restricted to an action of G on Π_λ, we want to determine the number of G-isomorphism classes of mosaics of type λ. For doing that, let $\bar{\lambda}$ be any partition of type λ. (For instance, $\bar{\lambda}$ can be defined such that the blocks of $\bar{\lambda}$ of size 1 are given by $\{1\}$, $\{2\}$, …, $\{\lambda_1\}$, the blocks of $\bar{\lambda}$ of size 2 are given by $\{\lambda_1+1, \lambda_1+2\}$, $\{\lambda_1+3, \lambda_1+4\}$, …, $\{\lambda_1+2\lambda_2-1, \lambda_1+2\lambda_2\}$, and so on.) According to [9, 10], the *stabilizer* H_λ of $\bar{\lambda}$ in the symmetric group $S_{\underline{n}}$ is *similar* to the *direct sum*

$$\bigoplus_{i=1}^{n} S_{\underline{\lambda_i}}[S_{\underline{i}}]$$

of *compositions* of symmetric groups, which is a permutation representation of the *direct product*

$$\mathop{\times}_{i=1}^{n} S_{\underline{i}} \wr S_{\underline{\lambda_i}}$$

of *wreath products* of symmetric groups. In other words H_λ is the set of all permutations $\sigma \in S_{\underline{n}}$, which map each block of the partition $\bar{\lambda}$ again onto a block (of the same size) of the partition.

Hence, the G-isomorphism classes of mosaics of type λ can be described as $G \times H_\lambda$-orbits of bijections from Z_n to $\underline{n}$ under the following group action:

$$(G \times H_\lambda) \times \underline{n}_{\text{bij}}^{Z_n} \to \underline{n}_{\text{bij}}^{Z_n}, \qquad ((g,\sigma), f) \mapsto g \circ f \circ \sigma^{-1}.$$

When interpreting the bijections from Z_n to $\underline{n}$ as permutations of the n-set $\underline{n}$, then G-mosaics of type λ correspond to *double cosets* (cf. [9, 10]) of the form

$$G \backslash S_{\underline{n}} / H_\lambda.$$

Theorem 2. *The number of G-isomorphism classes of mosaics of type λ is given by*

$$\frac{1}{|G|\,|H_\lambda|} \sum_{\substack{(g,\sigma)\in G\times H_\lambda \\ z(\bar{g})=z(\sigma)}} \prod_{i=1}^{n} a_i(\sigma)!\, i^{a_i(\bar{g})},$$

where $z(\bar{g})$ and $z(\sigma)$ are the cycle types *of $\bar{g}$ and of σ respectively, given in the form $(a_i(\bar{g}))_{i\in\underline{n}}$ or $(a_i(\sigma))_{i\in\underline{n}}$. In other words we are summing over those pairs (g,σ) such that $\bar{g}$ and σ determine permutations of the same cycle type.*

2 Enumeration of non-isomorphic canons

The present concept of a canon is described by G. Mazzola in [11] and was presented by him to the author in the following way: A *canon* is a subset $K \subseteq Z_n$ together with a covering of K by pairwise different subsets $V_i \neq \emptyset$ for $1 \leq i \leq t$, the voices, where $t \geq 1$ is the number of voices of K, in other words

$$K = \bigcup_{i=1}^{t} V_i,$$

such that for all $i, j \in \{1, \ldots, t\}$

1. the set V_i can be obtained from V_j by a translation of Z_n,
2. there is only the identity translation which maps V_i to V_i,
3. the set of differences in K generates Z_n, i.e. $\langle K - K\rangle := \langle k - l \mid k, l \in K\rangle = Z_n$.

We prefer to write a canon K as a set of its subsets V_i. Two canons $K = \{V_1, \ldots, V_t\}$ and $L = \{W_1, \ldots, W_s\}$ are called *isomorphic* if $s = t$ and if there exists a translation T of Z_n and a permutation π in the symmetric group S_t such that $T(V_i) = W_{\pi(i)}$ for $1 \leq i \leq t$. Then obviously $T(K) = L$.

Here we present some results from [7]. The cyclic group C_n acts on the set of all functions from Z_n to $\{0, 1\}$ by

$$C_n \times \{0,1\}^{Z_n} \to \{0,1\}^{Z_n} \qquad (\sigma, f) \mapsto f \circ \sigma^{-1}.$$

As the *canonical representative* of the *orbit* $C_n(f) = \{f \circ \sigma \mid \sigma \in C_n\}$ we choose the function $f_0 \in C_n(f)$, such that $f_0 \leq h$ for all $h \in C_n(f)$.

A function $f \in \{0,1\}^{Z_n}$ (or the corresponding vector $(f(0), f(1), \ldots, f(n-1))$) is called *acyclic* if $C_n(f)$ consists of n different objects. The canonical representative of the orbit of an acyclic function is called a *Lyndon word*.

As usual, we identify a subset A of Z_n with its *characteristic function* $\chi_A : Z_n \to \{0,1\}$ given by

$$\chi_A(i) = \begin{cases} 1 & \text{if } i \in A \\ 0 & \text{otherwise.} \end{cases}$$

Following the ideas of [7] and the notion of [3], a canon can be described as a pair (L, A), where L is the *inner* and A the *outer rhythm* of the canon. In other words, the rhythm of one voice is described by L and the distribution of the different voices is described by A, i.e. the onsets of the different voices are $a + L$ for $a \in A$. In the present situation, $L \neq 0$ is a Lyndon word of length n over the alphabet $\{0,1\}$, and A is a t-subset of Z_n. But not each pair (L, A) describes a canon. More precisely we have:

Lemma 3. *The pair (L, A) does not describe a canon in Z_n if and only if there exists a divisor $d > 1$ of n such that $L(i) = 1$ implies $i \equiv d - 1 \bmod d$ and $\chi_{A_0}(i) = 1$ implies $i \equiv d - 1 \bmod d$, where χ_{A_0} is the canonical representative of $C_n(\chi_A)$.*

An application of the *principle of inclusion and exclusion* allows to enumerate the number of non isomorphic canons.

Theorem 4. *The number of isomorphism classes of canons in Z_n is*

$$K_n := \sum_{d|n} \mu(d)\lambda(n/d)\alpha(n/d),$$

where μ is the Moebius function, *$\lambda(1) = 1$,*

$$\lambda(r) = \frac{1}{r} \sum_{s|r} \mu(s) 2^{r/s} \text{ for } r > 1,$$

and

$$\alpha(r) = \frac{1}{r} \sum_{s|r} \varphi(s) 2^{r/s} - 1 \text{ for } r \geq 1,$$

where φ is the Euler totient function.

3 Enumeration of rhythmic tiling canons

There exist more complicated definitions of canons. A canon described by the pair (R, A) of inner and outer rhythm defines a *rhythmic tiling canon* in Z_n with voices V_a for $a \in A$ if and only if

1. the voices V_a cover entirely the cyclic group Z_n,

2. the voices V_a are pairwise disjoint.

Rhythmic tiling canons with the additional property

3. both R and A are aperiodic,

are called *regular complementary canons of maximal category.*

Hence, rhythmic tiling canons are canons which are also mosaics. More precisely, if $|A| = t$, then they are mosaics consisting of t blocks of size n/t, whence they are of block-type λ where

$$\lambda_i = \begin{cases} t & \text{if } i = n/t \\ 0 & \text{otherwise.} \end{cases}$$

This block-type will be also indicated as $\lambda = ((n/t)^t)$. So far the author did not find a characterization of those mosaics of block-type λ describing canons, which could be used in order to apply enumeration formulae similar to those for the enumeration of non-isomorphic mosaics. Applying Theorem 2 the numbers of C_n-isomorphism classes of mosaics presented in table 1 were computed.

n	
12	(6^2) : 44 (4^3) : 499 (3^4) : 1306 (2^6) : 902
24	(12^2) : 56450 (8^3) : 65735799 (6^4) : 4008.268588 (4^6) : 187886.308429
	(3^8) : 381736.855102 (2^{12}) : 13176.573910
36	(18^2) : 126047906 (12^3) : 15.670055.601970 (9^4) : 24829.574426.591236
	(6^6) : 103.016116.387908.956698 (4^9) : 10778.751016.666506.604919
	(3^{12}) : 9910.160306.188702.944292 (2^{18}) : 6.156752.656678.674792
40	(20^2) : 1723.097066 (10^4) : 4.901417.574950.588294
	(8^5) : 1595.148844.422078.211829 (5^8) : 11.765613.697294.131102.617360
	(4^{10}) : 88.656304.986604.408738.684375 (2^{20}) : 7995.774669.504366.055054

Table 1: Number of mosaics in Z_n of block-type $((n/t)^t)$

However, the description of the isomorphism classes of canons as pairs $(L, C_n(A))$ consisting of Lyndon words L and C_n-orbits of subsets A of Z_n with some additional properties (c.f. Lemma 3) can also be applied for the determination of complete sets of representatives of non-isomorphic canons in Z_n, as was indicated in the last part of [7].

There exist fast algorithms for computing all Lyndon words of length n over $\{0, 1\}$ and all C_n-orbit representatives of subsets of Z_n. For finding regular tiling canons with t voices (where t is necessarily a divisor of n), we can restrict to Lyndon words L with exactly n/t entries 1 and to representatives A_0 of the C_n-orbits of t-subsets of Z_n. Then each pair (L, A_0) must be tested whether it is a regular tiling canon. In this test we only have to test whether the voices described by (L, A_0) determine a partition on Z_n, because in this case it is obvious that (L, A_0) does not satisfy the assumptions of Lemma 3.

For finding the number of regular tiling canons, we make use of still another result. In [13, 2] it is shown that regular complementary canons of maximal category occur only for certain values of n, actually only for *non-Hajós-groups* Z_n (cf. [12]). The smallest n for which Z_n is not a Hajós-group is $n = 72$ which is still much further than the scope of our computations. Hence, we deduce that for all n such that Z_n is a Hajós-group the following is true:

Lemma 5. *If a pair* (L, A_0) *describes a regular tiling canon in a Hajós-group* Z_n*, then* A_0 *is not aperiodic.*

This reduces dramatically the number of pairs which must be tested.

By applying Theorem 4 we computed K_n, the numbers of non-isomorphic canons of length n, given in the third column of table 2. The construction described above yields a list of all regular tiling canons, which also yields T_n, the number of regular tiling canons of length n, given in the second column of table 2.

The group Z_n is a Hajós group if the decomposition of n is not too complicated. If n is of the form

$$p^k \text{ for } k \geq 0,\ p^k q \text{ for } k \geq 1,\ p^2q^2,\ p^k qr \text{ for } k \in \{1,2\},\ pqrs$$

for distinct primes p, q, r and s, then Z_n is a Hajós group and Vuza proved in [13, 14] that for these n there do not exist regular complementary canons of maximal category. Moreover, he described a method how to construct such canons for all Z_n which are not Hajós groups. If Z_n is not a Hajós group, then n can be expressed in the form $p_1p_2n_1n_2n_3$ with p_1, p_2 primes, $n_i \geq 2$ for $1 \leq i \leq 3$, and $\gcd(n_1p_1, n_2p_2) = 1$. Vuza presents an algorithm for constructing two aperiodic subsets L and A of Z_n, such that $|L| = n_1n_2$, $|A| = p_1p_2n_3$, and $L + A = Z_n$. Hence, L or A can serve as the inner rhythm and the other set as the outer rhythm of such a canon. Moreover, it is important to mention that there is some freedom for constructing these two sets, and each of these two sets can be constructed independently from the other one. He also proves that when L and A satisfy $L + A = Z_n$, then also (kL, A), (kL, kA) have this property for all $k \in Z_n^*$.

So the first step in enumeration of regular complementary canons of maximal category is to determine the number of non-isomorphic canons which can be constructed by this method. (Actually, I wanted to do it this week, but finally there was not enough time to do so.)

4 Some interesting open problems

1. In his papers, Vuza did not prove that each regular complementary canon of maximal category can be constructed with his method. Is it possible to find regular complementary canons of maximal category which cannot be produced by Vuza's approach?

2. Is there a more elegant method for enumerating regular tiling canons?

3. When enumerating isomorphism classes of mosaics in Z_n, we could apply groups different from the cyclic group C_n. How to do this for canons? For the group C_n, a canon was given as a pair (L, A) with certain properties. L was an acyclic vector,

n	T_n	K_n
2	1	1
3	1	5
4	2	13
5	1	41
6	3	110
7	1	341
8	6	1035
9	4	3298
10	6	10550
11	1	34781
12	23	117455
13	1	397529
14	13	1.370798
15	25	4.780715
16	49	16788150
17	1	59451809
18	91	212.178317
19	1	761.456429
20	149	2749.100993
21	121	9973.716835
22	99	36347.760182
23	1	133022.502005
24	794	488685.427750
25	126	1.801445.810166
26	322	6.662133.496934
27	766	24.711213.822232
28	1301	91.910318.016551
29	1	342.723412.096889
30	3952	1281.025524.753966
31	1	4798.840870.353221
32	4641	18014.401038.596400
33	5409	67756.652509.423763
34	3864	255318.257892.932894
35	2713	963748.277489.391403
36	31651	3.643801.587330.857840
37	1	13.798002.875101.582409
38	13807	52.325390.403899.973926
39	40937	198.705759.014912.561995
40	64989	755.578639.350274.265100

Table 2: Number of non-isomorphic canons in Z_n

so probably in all generalizations we must assume that L does not have cyclic symmetries. A was considered to be a subset of Z_n, but actually A describes the onset distribution of the different voices, whence it is actually a subset of the acting group C_n. When considering the group $\mathrm{Aff}_1(Z_n)$, consisting of all affine mappings $\pi_{a,b} : Z_n \to Z_n$, $i \mapsto ai + b$, for $a \in Z_n^*$ and $b \in Z_n$, acting on Z_n, then A must be considered as a subset of this group. If L has just the trivial symmetry, then each $\pi_{a,b}(L)$ describes another voice of the canon. If the stabilizer U of L is non-trivial, but it does not contain symmetries of the form $\pi_{1,b}$ for $b \neq 0$, then A must be considered as a subset of the right-cosets of U in $\mathrm{Aff}_1(Z_n)$. When computing the number of non-isomorphic canons in this setting, we get a much bigger number of different canons, since usually many different voices start at the same onset. So maybe in this situation we should restrict to canons, such that different voices have different onsets in Z_n. But when speaking of onsets of voices we can get some problems with symmetries $\pi_{a,b}$ of L for $a \neq 1$ and $b \neq 0$. So maybe we should not allow any symmetries of L. But then we will not get a complete overview over all canons in Z_n. Still the property that $K - K$ generates Z_n was not considered for these generalizations.

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