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Tiling the Line in Theory and in Practice

Abstract: Covering two- and three-dimensional spaces with repeated tiles has been extensively studied for 2000 years, but tiling in one dimension is much less explored. How can one cover the points of a line with repeated patterns? How does one string beads so that the colors fall in repeating formations? How can a melody be written that contains overlapping canons of other melodies? Continuing from my general presentation of the subject at the Journées d'informatique musicale *(Bourges, June 2001), I want to define* Tiling the Line *more rigorously on a theoretical level, as well as showing some applications in recent musical compositions.*

Résumé : On a étudié les pavages en deux et trois dimensions depuis 2000 ans, mais les pavages d'une seule dimension sont beaucoup moins explorées. Comment couvrir les points d'une ligne avec des formes répétées ? Comment enfiler les perles de manière que chaque couleur ait sa propre forme répétée ? Comment écrire une mélodie qui contient des canons d'autres mélodies qui se chevauchent ? Suivant une présentation générale du sujet au Journées d'informatique musicale *(Bourges, juin 2001) je veux définir* Pavage de la ligne *plus rigoureusement au niveau théorique, et montrer comment je l'applique musicalement dans quelques compositions récents.*

Tiling the Line in Theory

Definitions

A *line* is a sequence of equally spaced points.

A *loop* is a line that cycles back through its point of origin.

A *motif* is a subset of the line, consisting of a particular formation of points.

A *tiling* is a way of covering each point of a line or loop once and only once with repeating motifs.

Motifs with Adjacent Points

The simplest motif is a single point, noted (0), which covers or tiles a line of any length, one motif per point.

A motif consisting of two or more adjacent points (0 1) (0 1 2) (0 1 2 3), etc., tiles lines with lengths that are multiples of the length of the motif.

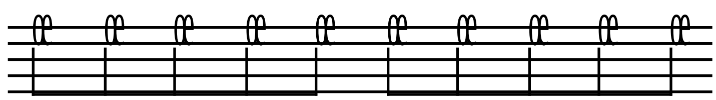
These simplest of tilings can be visualized in many ways:

0	0	0	0	0	0	0	0	0	0

0	0	1	1	2	2	3	3	4	4

1	2	3	4	5	6	7	8	9	10

a	b	c	d	e	**f**	g	h	**i**	**j**



eight points in time

eight motifs

0	1	2	3	4	5	6	7
×							
					×		
	×						
			×				
				×			
		×					
							×
						×	

A line may be tiled with two or more sizes of motifs with adjacent points, such as

aabbbaabbbaabbb

but this remains a rather primitive brick laying process, even when the placement of the various sizes is irregular. Tiling the line only becomes interesting when the points are not always adjacent, and the motifs must overlap.

Motifs with two non-adjacent points:

<u>Even Motifs</u>

The motif (0 2) tiles a line of *4, 8, 12*… points

0	1	2	3
0		2	
	1		3

0	1	2	3	4	5	6	7
0		2					
	1		3				
				4		6	
					5		7

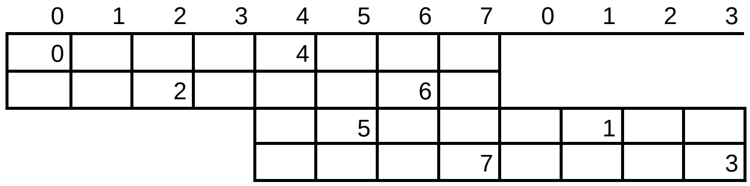
It can also tile *loops* with lengths divisible by *4*. This can be done in countless ways, such as these:

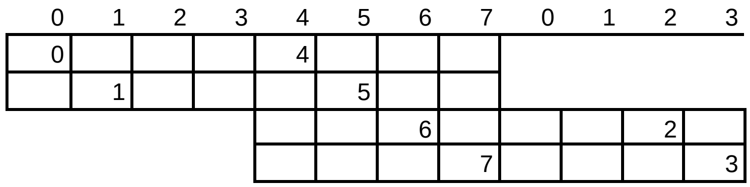
0	1	2	3	4	5	6	7	0	1
0		2							
			3		5				
				4		6			
							7		1

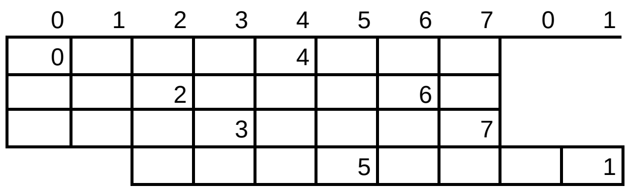
0	1	2	3	4	5	6	7	0	1	2	3
0		2									
				4		6					
					5		7				
									1		3

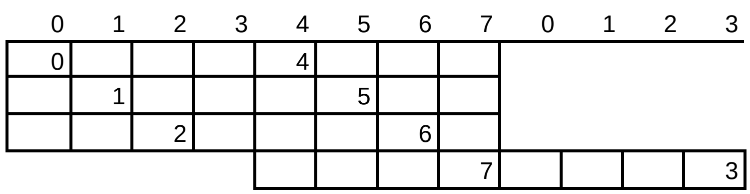
The same is true for other motifs (0 n), where *n* is even. Here the motif (0 6) tiles a line of *12* and the motif (0 4) tiles loops of *8* and *16* in various ways.

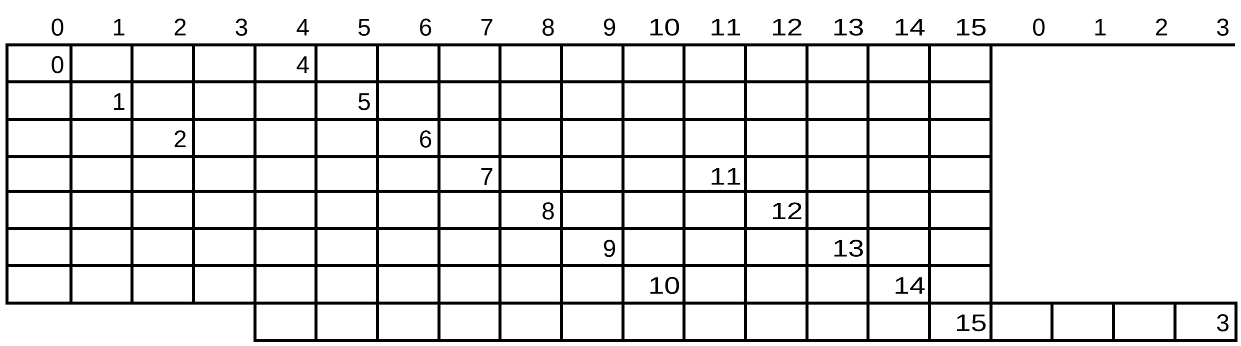
0	1	2	3	4	5	6	7	8	9	10	11
0						6					
	1						7				
		2						8			
			3						9		
				4						10	
					5						11


0 1 2 3 4 5 6 7 0 1 2 3
0 4
2 6
5 1
7 3


0 1 2 3 4 5 6 7 0 1 2 3
0 4
1 5
6 2
7 3


0 1 2 3 4 5 6 7 0 1
0 4
2 6
3 7
5 1


0 1 2 3 4 5 6 7 0 1 2 3
0 4
1 5
2 6
7 3


0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 0 1 2 3
0 4
1 5
2 6
7 11
8 12
9 13
10 14
15 3

Odd Motifs

(0 3) tiles any *line or loop* with a length is divisible by *3.*

0	1	2	3	4	5
0			3		
	1			4	
		2			5

0	1	2	3	4	5	6	7	8	9	10	11
0			3								
	1			4							
		2			5						
						6			9		
							7			10	
								8			11

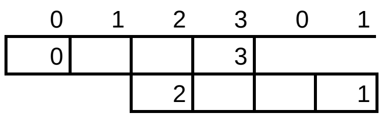
0	1	2	3	4	5	0	1
0			3				
		2			5		
				4			1

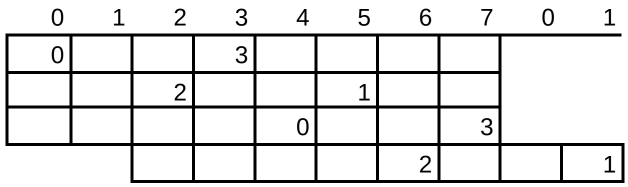
In this example point *1* was omitted at the beginning, but one might also omit the point *2,* or the points *1* and *2,* which is a reversed and inverted version of the same form:

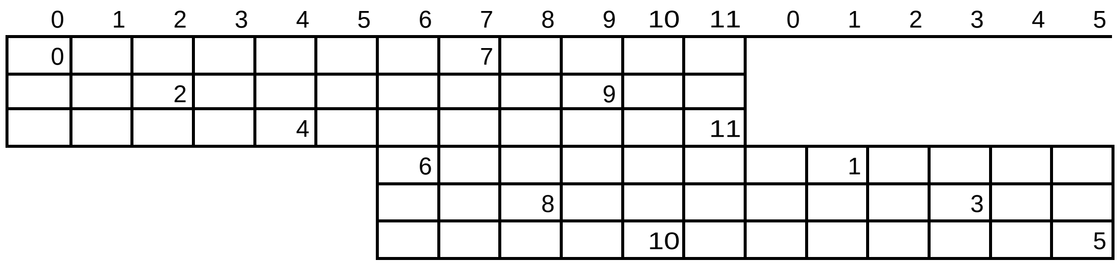
0	1	2	3	4	5	0	1	2
0			3					
	1			4				
					5			2

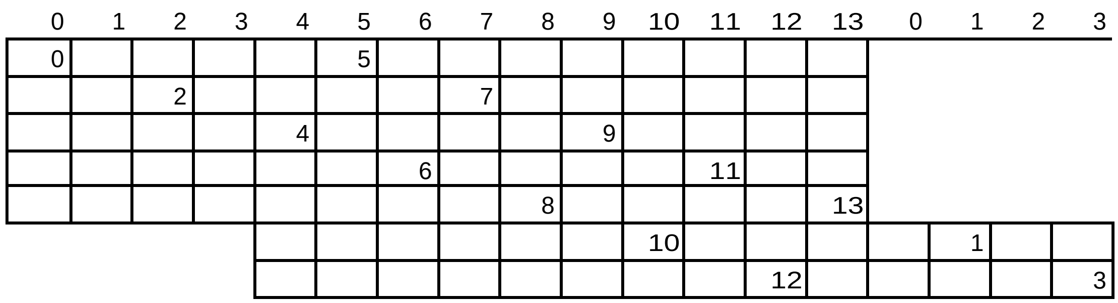
0	1	2	3	4	5	0	1	2
0			3					
				4			1	
					5			2

But unlike the motif (0 2), the motif (0 3) also tiles *loops* that are *not* multiples of *3*. In fact, it tiles any loop longer than itself having an even number of points, and this is the case for all motifs (0 n) where *n* is odd.









Remember as you look at these two-dimensional illustrations that you are looking at one-dimensional phenomena. This last loop could be represented less clearly but more correctly by assigning a digit to each motif and writing the sequence in a strictly linear way:

0_1_20314253640516203142536405162031425 3…

The Rule

Motifs covering two non-adjacent points and having the form (0 n), where *n* is even, tile lines of length *2n* and multiples of that length.

Demonstration: Given a line of length *2n*, points *0* and *n* are covered by the original motif, points *1* and *n + 1* are covered by a second motif, points *2* and *n + 2* are covered by a third, and so forth until points *n - 1* and *2n – 1.* The same procedure may be followed as many times as necessary if the line is a multiple of this length.

These same motifs can also form *loops* of length *2n, 4n, 6n*, etc., in a variety of interlocking forms.

Motifs covering two non-adjacent points and having the form (0 n), where *n* is odd, tile lines of length *2n*, and multiples of *2n* by this same logic. These odd motifs, however, can also form *loops*, not only of length *2n, 4n, 6n* etc., but of any even length *n + 1, n + 3, n + 5*, etc., by simply beginning a motif in each even point.

Motifs with More than Two Points

(0 1 3) can not tile any line or loop, because there is no way to fill the first hole with a second motif.

0	1		3

(0 1 4) can not tile any line or loop, because there is no way to add a third motif.

0	1			4		
		0	1			4

But (0 1 5) tiles any loop whose length is *6* or more, and divisible by *3*, because *0, 1, 5 mod 3* is equivalent to *0, 1, 2*.

With this observation we can pass quickly to a generalization and give a procedure for creating motifs of *n* notes that will tile loops that are multiples of *n*. As an example, suppose we wish to cover a 12-point loop with four 3-point motifs. We begin by writing the 12 digits *0* through *11*, with the digits equivalent to *0 mod 3* in one column, and the digits equivalent to *1 mod 3* and *2 mod 3* in two other columns:

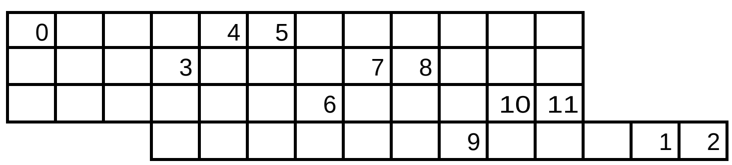
0	*1*	*2*
3	*4*	*5*
6	*7*	*8*
9	*10*	*11*

By selecting three digits, one from each column, we will find 4^3 solutions, but to avoid mere transposing and duplication, we can limit our search to the 16 solutions where *0* is chosen from the first column:

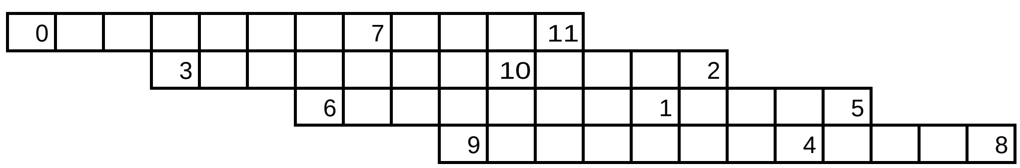
(0 1 2) (0 1 5) (0 1 8) (0 1 11)
(0 4 2) (0 4 5) (0 4 8) (0 4 11)
(0 7 2) (0 7 5) (0 7 8) (0 7 11
0 10 2) (0 10 5) (0 10 8) (0 10 11)

These are the mother motifs, beginning on zero, and the remaining three, necessary to cover all 12 points, are the same thing beginning on points *3, 6,* and *9.* All points are now covered, and two examples will provide visual clarification:

tiling
with
(0 4 5)



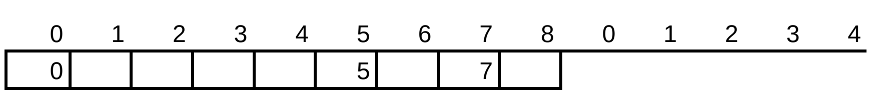
tiling
with
(0 7 11)



Similarly any *n*-point motif can tile any loop with length a multiple of *n* provided that the points of the motif are equivalent to *0, 1, … n-1 mod n.*

But we must consider not only the set of numbers in the motif itself. D.T.Vuza found that there are always two “supplementary sets,” one comprising the points in the motif (Vuza’s supplementary set *R*) and another comprising the starting points of each motif (Vuza’s supplementary set *S*), and that these two sets can always be reversed.

Here is an example with the motif (0 5 7) beginning at points *0 3 6* in order to tile a nine-point loop, and then reversing the sets, with the motif (0 3 6) placed at points *0 5 7*.



			3					8		1			
						6					2		4

0	1	2	3	4	5	6	7	8	0	1	2	3	4
0			3			6							
					5			8			2		
							7			1			4

Vuza found that one of the two sets must be periodic, as is the case with (0 3 6) in this nine-point loop. Other examples of period sets are (0 1 4 5) in the case of an 8-poiint loop or (0 4 8 12 16) in the case of a 20-point loop. Vuza theorized that one could find exceptions, cases where supplementary sets were both non-periodic, and Moreno Andreatta has demonstrated some of these, the shortest of which is a loop 72 points long.

Complementary Motifs

Complementary motifs are motifs that can not tile lines and loops alone, but do so when placed together. One might consider two-point motifs to be complementary in some cases, but it will be more interesting to go directly to the case of complementary 3-point motifs.

It is obvious that any motif, however arbitrary, is complementary with its inversion. If all the points not filled in one motif are filled in another, the line is tiled.

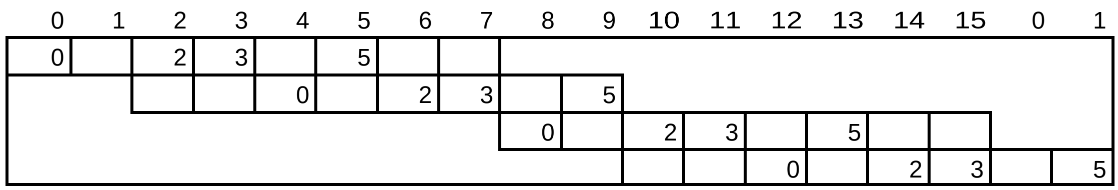
0		2	3
	1		

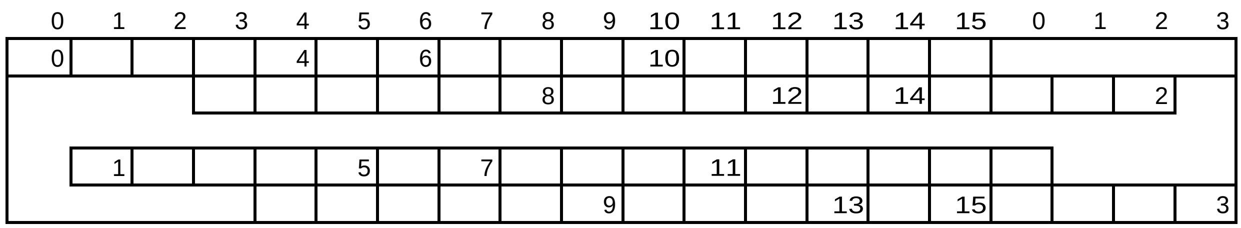
It is somewhat less obvious that one can simply concatenate these two motifs of length four and form a single motif of length 8, which tiles the line by playing a rhythmic canon with itself.

0		2	3		5				
				4		6	7		1

The result is a pattern known to percussionists by its onomatopoetic name "paradiddle." If the first voice corresponds to the left hand and the second voice corresponds to the right hand, we have a classic rudiment, usually defined in terms of left and right as LRLLRLRRLRLLRLRR. It is a valuable technique for drummers, because by playing in this way one can produce a steady barrage of beats quite regularly, quite rapidly, and for a longer time than with other right-and-left alternations.

The paradiddle provides a good example for pointing out how any tiling can be multiplied to twice or three times or *20* times its length, either by simply extending or by stretching and inserting. Here is how the 8-beat paradiddle can be extended to *16* beats, and stretched into *16* beats:





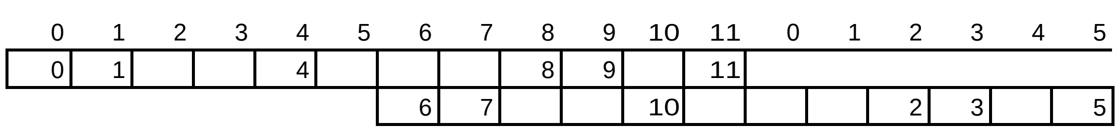
Complementary motifs may also originate quite independently. As an example we can go back to (0 1 3) and (0 1 4), which we observed to be unable to tile a line or a loop alone, but which happen to do so together:

0	1	2	3	4	5
0	1			4	
		0	1		3

Of course, one can multiply or concatenate here too, as in earlier examples. Four of these motifs stretched out to (0 2 6) and (0 2 8) tile a 12-point line

0	1	2	3	4	5	6	7	8	9	10	11
0		2						8			
				4		6				10	
	1		3						9		
					5		7				11

And by concatenating (0 1 3) with (0 1 4) into a single motif with six points, we can produce a rhythmic canon that tiles a loop of 12 points with two voices:



Often motifs are complementary when one is the retrograde of the other. Consider the complementary motifs (0 2 5) and (0 3 5). One can begin with either of the two, and tile a six-point loop with no problem:

0	1	2	3	4	5	0	1	2	3
0		2			5				
				4			1		3

0	1	2	3	4	5	0	1
0			3		5		
		2		4			1

But can one tile a *line*, with a beginning and ending, using only these two motifs? The only way to know if this is possible is to simply follow the chain of possibilities.

Suppose we begin with (0 3 5). The first hole can not be filled by (0 2 5), as both motifs would fall on point *3,* so we have to add another (0 3 5)

0			3		5	
	1			4		6

and here we reach a dead end. The hole at point *2* can not be filled by either motif.

Let's try starting with (0 2 5), followed by (0 3 5)

0		2			5	
	1			4		6

Now we have another dead end. There's no way to fill point *3.* But there is one last possibility, beginning with (0 2 5) twice, and this works better, allowing us to continue by placing (0 3 5) at point *4*.

0	1	2	3	4	5	0	1	2	3
0		2			5				
	1		3			0			
				4			1		3

From here we have two choices. The first possibility is to follow with another (0 2 5), which leaves only the possibility of two more (0 3 5) motifs, leading to a finished line, neatly tiled, with *18* points, which incidentally, is a palindrome, as you can see if you look at the page upside down.

0	1	2	3	4	5	0	1	2	3	4	5	0	1	2	3	4	5
0		2			5												
	1		3			0											
				4			1		3								
								2		4			1				
											5			2		4	
												0			3		5

The second possibility is to continue with another (0 3 5), which obliges us to continue with another (0 2 5) and this returns to the situation we just had.

0	1	2	3	4	5	0	1	2	3						
0		2			5										
	1		3			0									
				4			1		3						
								2			5		1		
										4		0			3

We may keep looping around by adding (0 3 5) followed by (0 2 5) as many times as we want, but the only way to end will be to choose the route shown above, and form a *cadence*, as a musician would say.

Thinking more musically for a moment, suppose that the three notes of the (0 2 5) motif are A, C, D, and the three notes of the (0 3 5) motif are D, C, A, and suppose that we decide to permit unisons. Now things like this become possible:

0	1	2	3	4	5	0	1	2	3	4	5	0	1	2
A		C			D									
	A		C			D								
				D			C		A					
					D			C		A				
									A		C			D

and the total number of routes possible with these two motifs is so great that I gave up counting them. I am almost certain, however, that the length of any solution will be a multiple of 6, that the two motifs will always be used the same number of times, that any solution will be a palindrome, and that all solutions must begin (0 2 5) (0 2 5) (0 3 5) and end (0 2 5) (0 3 5) (0 3 5).

Polyrhythmic Motifs

“Polyrhythmic” motifs are a particularly fruitful category of complementary motifs, in which a single motif is permitted to occur in different tempos. This technique of “augmentation” or “diminution” is standard counterpoint, and it is not difficult to hear the similarity between the variants, even when the difference in tempo is as great as 4 : 1. Furthermore, the number of possible tilings is often far greater than with other types of complementary motifs, though it is difficult to calculate how great. As part of the June 2001 presentation of *Tiling the Line* at the *Journées d’informatique musicale*, I suggested the following problem, hoping to stimulate a response: How many different tilings are possible using only the motif (0 1 4), but permitting also half time and quarter time variants, that is, (0 2 8) and (0 4 16) as well as (0 1 4) ... A short time later, a mathematician who had heard the lecture, Andranik Tangian, sent me some pages of computer output with everything calculated:

1 solution of length 15
6 solutions of length 30
20 solutions of length 45
97 solutions of length 60

This was already rather astounding for me, and when I noticed that all of these *120* solutions began with (0 1 4), I asked Tangian to try the same program beginning with the other tempos. He found *11* additional solutions beginning with (0 2 8), and *107* additional solutions beginning with (0 4 16). That made *238* solutions already, and we were only considering complete lines, not loops, and never looking at lines with more than *60* points. At least in one case, the solutions of length *75* were alone more numerous than all these shorter ones put together.

Thomas Noll did another computer search that indicates the immense number of possibilities for forming polyrhythmic canons. Taking the simple motif (0 1 2) he counted, for example, how many ways one can construct a loop *15* points long, with five different voices, such that each voice turns at a different tempo and each voice falls on its own distinct points. He found that the tempos of the five voices can be in ratios of *1 : 2 : 4 : 7 : 8,* for example, or *1 : 2 : 7 : 8 : 11* or *1 : 4 : 7 : 11 : 13*, or *1 : 7 : 8 :*

13 : 14, or *2 : 4 : 7 : 11 : 14*, and that *51* additional “signatures” are possible. And for each signature, the individual voices may be arranged in several different ways.

Many questions remain, of course:

Would the problem become completely unmanageable if we investigated tiling possibilities with three or more complementary modules?

Why is it that all the solutions Tangian found to the (0 1 4) problem have lengths that are multiples of 15?

Is there a procedure for determining, for example, whether (0 3 8) and (0 4 7) are complementary without just trying to fit the pieces together?

How can one begin to write music that tiles lines and loops in these ways?

I will leave the other questions for now, and try to respond only to the last one.