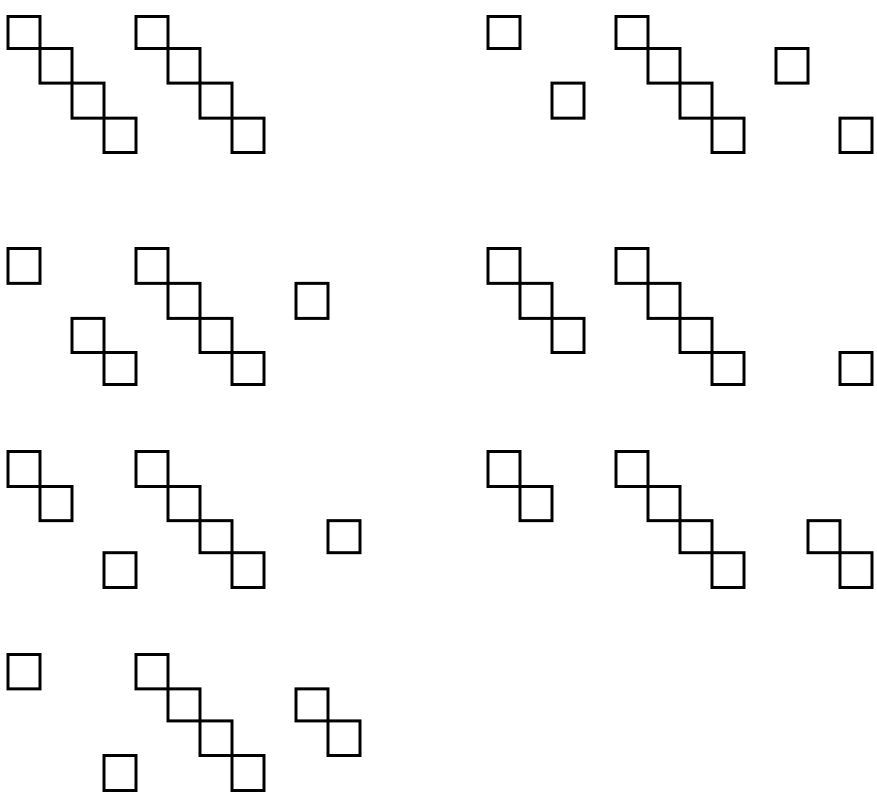
Tiling the Line in Practice

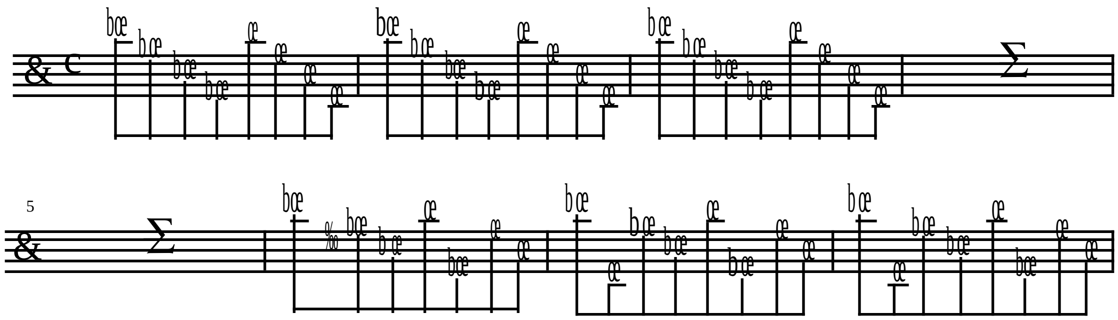
It seems important now to try to explain how I have tiled the line in actual musical compositions. Of course, composers are never the best analysts of their music. They are too close to the ideas to have a clear overview or to make competent judgements and comparisons. So I am not going to really analyze anything, but rather, following the model of Raymond Roussel, simply relate “how I wrote certain of my pieces.” Like Roussel, my procedures are often quite systematic, so I can at least tell you how I calculated the music in some recent pieces.

Tiling with (0 4)

Sometimes a sequence that is mathematically trivial can produce a music that is rather subtle and elegant. One movement from *Tilework for Clarinet* is simply the list of seven ways of tiling an eight-point loop with the rhythmic motif (0 4).



The second, third, and fourth voices are simply delayed four beats, which is obvious visually, yet translated into a melody, the variations are more difficult to perceive. The ear detects the logic and the symmetry, and when the melody ends, one senses that all the possibilities have been heard, yet each phrase is a bit surprising, not quite predictable. Here is the notation for the first two phrases:



Tiling with “1 2 3”

1 2 3 is a work for a large mixed chorus, and one of the movements turns around on a loop 24 beats long, broken up into eight motifs sung/spoken in eight voices. The rhythmic motif is simply three counts in a steady beat, but with tempos in the proportions 1 : 1 : 2 : 2 : 2 : 2 : 4 : 4. It is a normal polyrhythmic tiling, with each beat covered once and only once.

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In some cases, when I want to tile a loop of a certain size in a certain way, I consult the lists that have been prepared by Thomas Noll and Andranik Tangian, but in this case the problem is relatively simple, with many possible solutions, and one can quickly find a way of putting the three tempos together without resorting to computer output. In fact, this movement was written before I even had computer lists to consult.

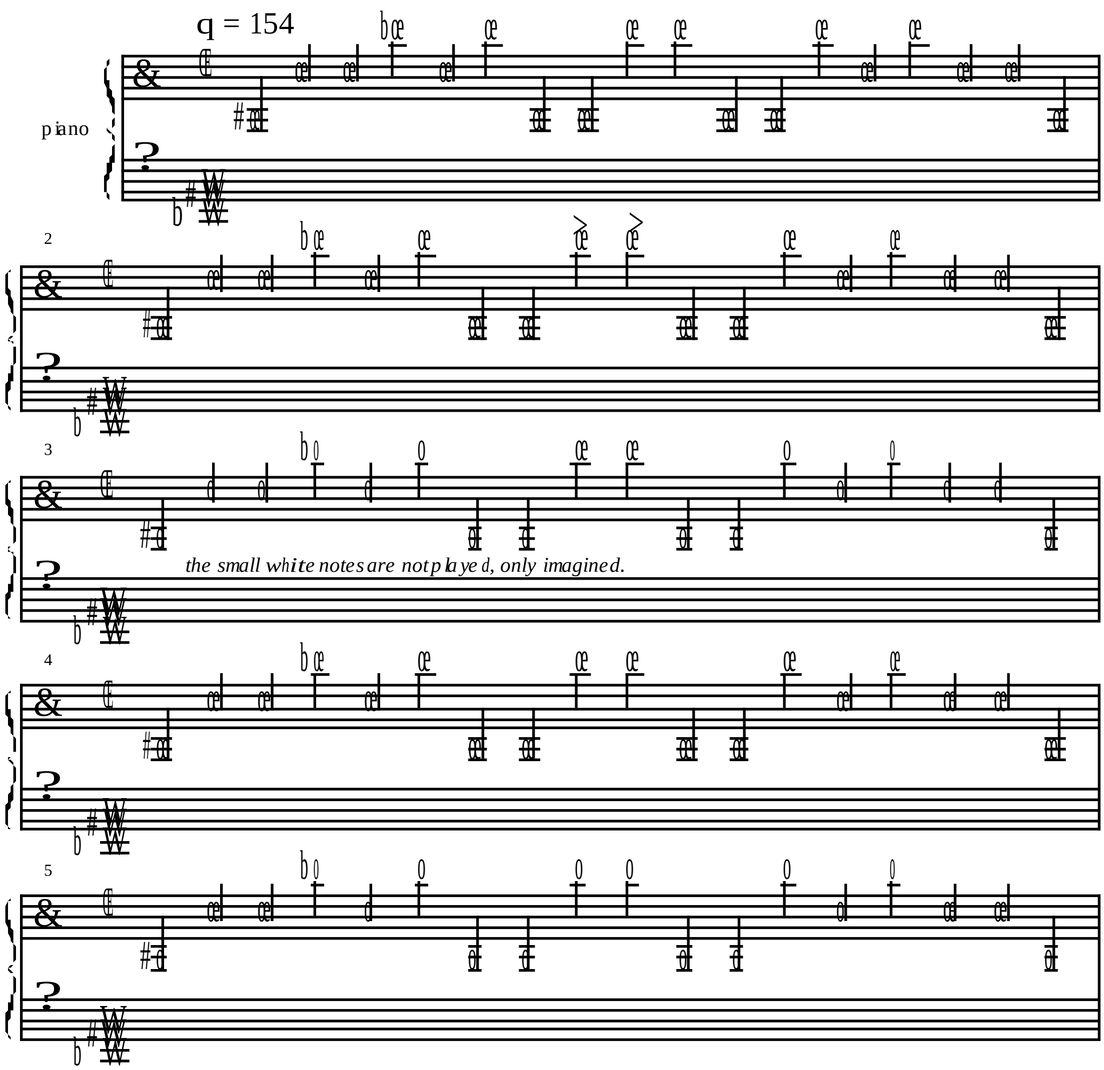
The piece begins as shown in the table, completely contrapuntal, each “1 2 3” sung by a separate group of singers, but gradually the eight parts come together, and after about two minutes everyone is singing/speaking everything, completely in unison. At this point the counterpoint becomes completely garbled, and the numbers make no more sense, but slowly everything moves back into eight separate voices, as at the beginning.

Tiling with All-Distances Subsets

In most cases the fascination in tiling the line is doing it with the same or very similar repeating motifs, so that the whole melody is a unified rhythmic canon of some sort, but sometimes interesting tilings are possible by combining very different motifs. One day I was looking at some tables I have made for the orbits necessary to compose a *19*-point loop that makes a copy of itself at *8 : 1 or 12 : 1*:

(0) (1 8 7 18 11 12) (2 16 14 17 3 5) (4 13 9 15 6 10)

Noticing that there were adjacent pairs in each of these six-point orbits, *7-8* in the first case, *16-17* in the second, and *9-10* in the third, I went on to observe that all the other distances were present in all three orbits as well. These “all-distances subsets,” (not to be confused with all-interval sets!) can perhaps best be seen by going directly to the musical notation, where the singular zero orbit is represented by a recurring chord in the left hand and the three other orbits are represented by F-sharp, A, and B-flat:



In this first page, the complete loop is played twice, and then only the two adjacent B-flats. Later the two instances of adjacent A's are singled out, and then the two cases of adjacent F-sharps. The piece continues by showing the cases where there are distances of *2*, then *3,* and so on.

I later looked at all-distances subsets more carefully and found that it is possible to do this for a cyclic group of *19* with a subset of only five points. For a cyclic group of *12* points, an all-distances subset may be produced with only four points (0 1 3 7). It would be interesting to know why such subsets tend to show up in self-replication orbit structures, and there are no doubt other musical applications, but this is all I can tell you for now.

Tiling with (O 2 5) and (0 3 5)

In the lecture last June I also explored the logic of how to tile a line with (0 2 5) and (0 3 5). This was an arbitrary exploration, and I was not able to analyze the situation completely. Later I worked it all out, and as I explained earlier (p. 12). I found that the shortest solution was an *18*-point line with six motifs. This formation began to seem rather special, and I found a way to make a short piece with it, which became a short movement in the *Tilework for Clarinet,* the full score of which is shown here. I went one day to talk with Steve McAdams, the psycho-acoustician at IRCAM, about when we perceive individual tiles and when they fuse together, and he said that this particular piece is almost a classical demonstration of this phenomenon. Independently heard elements stream together into a single melody when they are close and overlapping, but separate back out into different melodies when they are not.

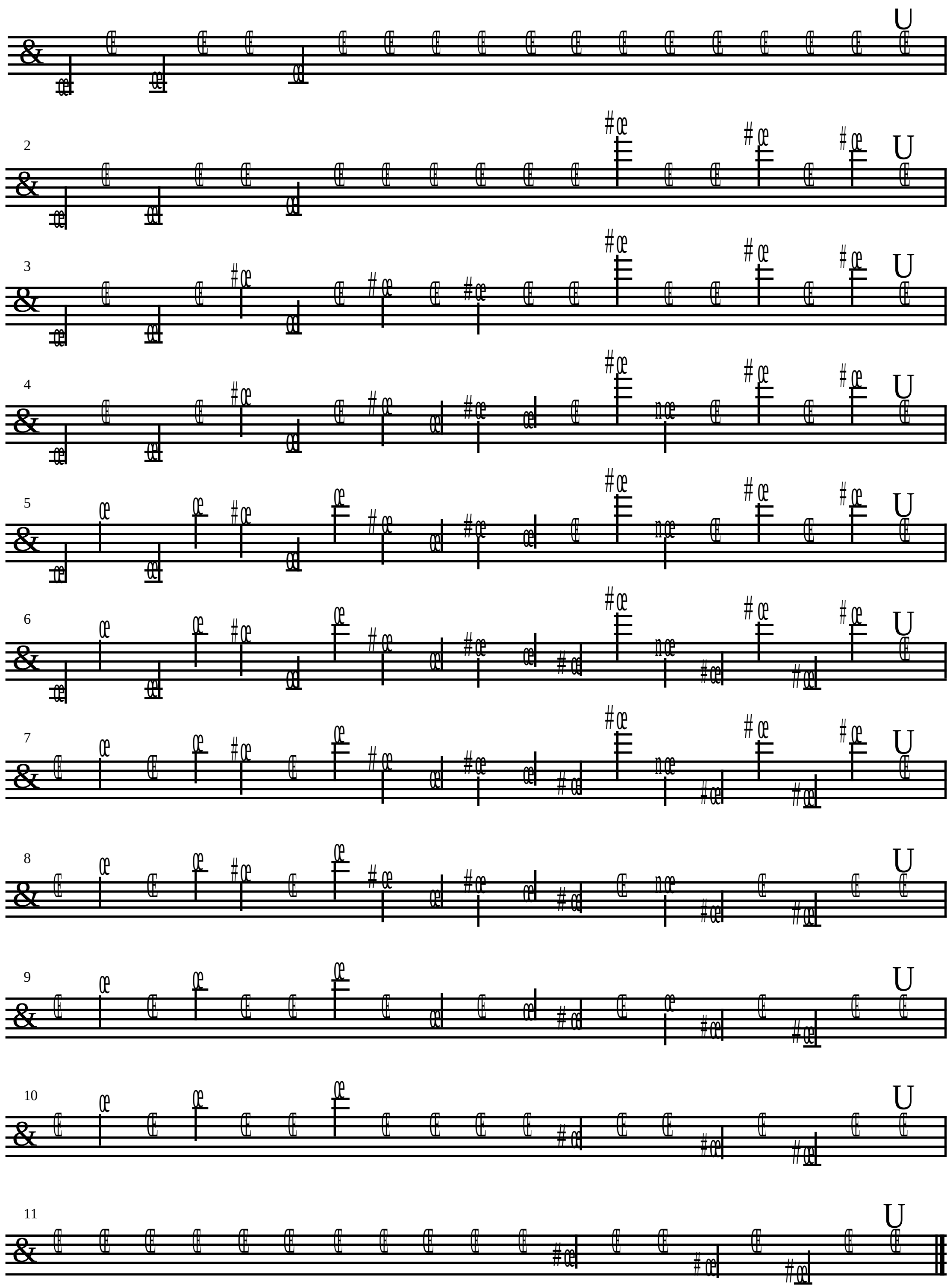
Tiling with a Self-Replicating Melody

I can't take the time today to explain how to construct self-replicating melodies, but this is amply explained in my book *Self-Similar Melodies,* as well as in David Feldman's *Leonardo* article concerning this book. If you are not familiar with any of this, you can still get the general idea simply by following this example. Suppose we want to construct a melodic loop of 15 notes, or as a mathematician would say, a "cyclical group in Z/15Z," in such a way that it will self-replicate at *2 : 1.* In this case the 15 points fall into these orbits:

(0) (1 2 4 8) (3 6 12 9) (5 10) (7 14 13 11)

and in order to have the same thing, whether we play every note, every second note, or every fourth note, the requirement is that all the points within each orbit must be the same pitch. This is exactly the format I used when I composed the first self-replicating melody in 1982, *Rational Melody No. 15*.

Movement from *Tilework for Clarinet*



I always thought that this structure could not possibly serve for a second piece, and yet I came back to it 20 years later in this tiling context, and made a totally different piece out of the same orbit structure. The melody this time is on a different scale, and it is only a four-note scale, because the (0 1 2 4) orbit has been given over to rests. This illustration is all lined up in one octave, in order to make it clear how the tempos fit together, but the actual score is separated out into three different octaves.

Most of the tiling we have been doing today has filled each point once and only once, while here there are lots of holes and overlaps. To me, however, this seems like tiling all the same. Every point on the loop has been calculated into the system in one way or another. The melody occurs in three tempos in the ratio *1 : 2 : 4,* but the polyrhythmic motifs are not inserted around one another. They are written as if they contain one another, which they do anyway in a self-replicating system.

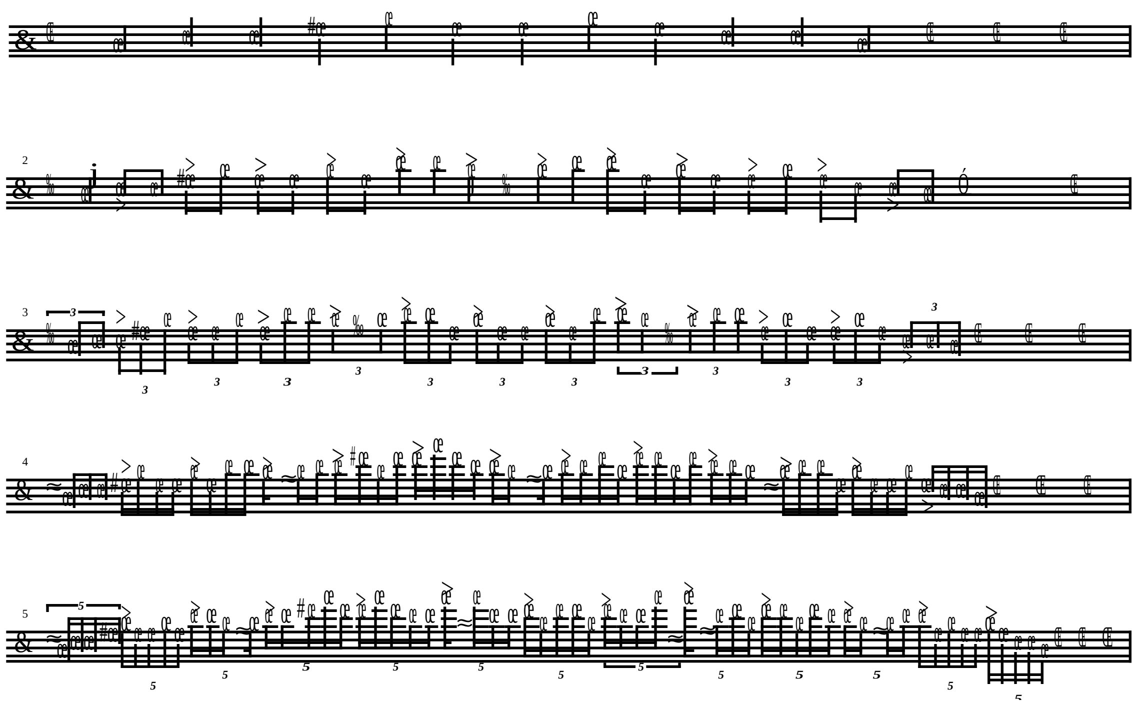
Tiling on Paper

Good mathematics does not always make good music, and I want to show you an example of that. This little piece is too stiff, too rigid, a little didactic, I don't know, for some reason it just doesn't come alive. But it is worth looking at, because it employs a discovery that at first surprised me very much, and which may have potential in some other musical or mathematical context.

This example turns in a loop of *13* notes that self-replicates at *5:1*, provided the notes follow these orbits.

(0) (1 5 8 12) (2 10 13 3) (4 7 9 6)

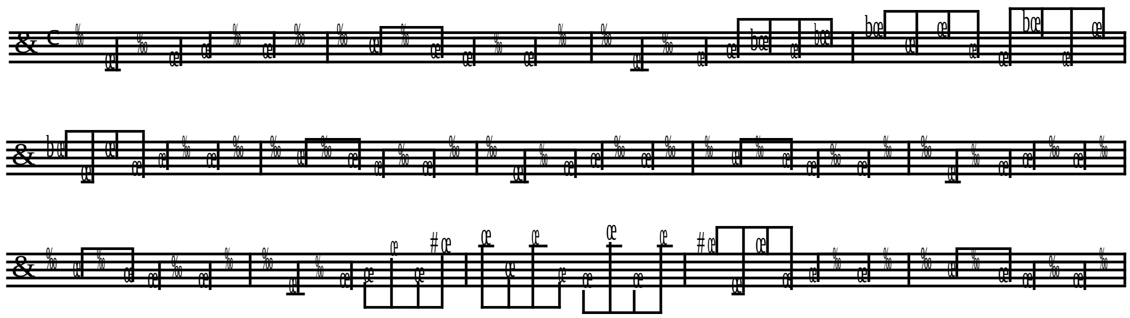
The rest at the beginning of each line represents the zero orbit. Three additional rests are added at the end of each line, just to separate the phrases. The (1 5 8 12) orbit is represented by F, the second orbit by C-sharp, and the third by A. So far this could be just another self-replicating melody, but what is new is that I decided this time to look not only at the solution loop, which self-replicates at *5 : 1,* but also at its related loops, that is, the melodies that result if one plays only every second note or every third note or every fourth note of the solution loop. By arranging the orbits in a particular way, selecting an augmented chord as my three-note scale, and raising the two central F’s to the next octave, I constructed this:



The *2 : 1* melody and the *3 : 1* melody are the same as the original, simply transposed a third higher, the *4 : 1* is also the same, simply transposed a sixth higher, and the *5 : 1* is exactly the same, an octave higher, because the original melody was composed to self-replicate at *5 : 1*, and there you are. A perfect structure. Unplayable, unlistenable, unmusical, but really beautiful to look at and think about. But maybe that’s important also. Music isn’t everything, after all. Ideas and structures are important too, even when they don’t sound very good.

Tiling with (0 2 3 5)

By contrast with this sophistication, I want to end with a more musical example that is about as easy to understand as the (0 4) tilings in that first clarinet piece, and it’s strictly in 4/4 time. As long as we are only hearing the basic four-note motif everything sounds simple, but when it starts intertwining with itself in two other tonalities, as in this excerpt, one can sometimes even lose track of where the downbeat is.



References

I am indebted to several mathematicians, whose insights have helped me greatly: Jean-Paul Allouche (CNRS-Orsay), David Feldman (University of New Hampshire), Andranik Tangian (Fern Universität, Hagen), Thomas Noll (Technische Universität Berlin), and Markus Reineke (Wuppertal).

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