Some Observations on Tiling Problems

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Not being a mathematician, I can not offer any theorems or proofs, but since I am working on tiling problems in my music everyday, I observe many things that help me find the materials I am looking for, and it seems worth sharing some of these. I offer them strictly on what Guerino Mazzola calls the "naïve" level, which at least has the advantage of being relatively comprehensible, and which may help someone else to lead us to a deeper understanding of how one can tile the line.

Observation on ritarding rhythmic canons

In composing *Tilework for Cello*, I wanted to make a six-voice canon with a ritarding rhythm. I found a solution containing eight notes, turning in a loop 48 points long, simply by making sure that my eight notes, modulo 7, were equivalent to 0 – 7. If five additional voices enter at intervals of 8 points, all points are filled and everyone is ritarding.

110100100010000010000010000000100000000000

01_3__6___2_____7_____5______4___________

This basic technique can be extremely useful when one wishes to form a rhythmic canon with a tile of a particular sort. One can imagine structuring not only ritards, but also accelerandos, waltz rhythms, periodic rhythms, or tiles that quote Beethoven.

Observation on lines and loops

In previous discussions we have often distinguished between “lines” and “loops,” and sometimes we have felt that a tiling with a distinct beginning and ending is somehow more correct or more musical than one where the end interlocks with the beginning. I am now convinced that the difference between “lines” and “loops” is not very important, and that the two are essentially a single phenomenon. This becomes particularly clear in Fripertinger’s lists, where both are counted equally, and where one finds that the vast majority of rhythmic canons are in fact loops, not lines. Consider this detail from Fripertinger’s list, where we see all the ways of tiling a 20-point loop with two-note rhythms. At the left are the 2-note rhythms, which Vuza would call the R set, at the right are the points of departure, which Vuza would call the supplementary S set, and the sequences are written from right to left.

[00000000000000000011][01010101010101010101]
Here the 10-voice canon forms 10 separate lines.

[00000000000000000101][00110011001100110011]
Here the 10-voice canon forms 5 separate lines.

[00000000000000001001][01010101010101010101]
This one is a loop.

[00000000000000100001][00000111110000011111]
Here the first 5 voices cover points 0-9 and the second 5 voices cover points 10-19, making two lines.

[00000000000000100001][00010111010001011101]
[00000000000000100001][00100110110010011011]
[00000000000000100001][01010101010101010101]
[00000000000001000001][00110011001100110011]
[00000000000010000001][01010101010101010101]
[00000000001000000001][01010101010101010101]

These last six solutions are all loops.

Loop solutions are even more frequent when 10-note rhythms form two-voice canons. In Fripertinger's list, the first solution forms two separate lines, and the other 50 all form loops:

[00000000001111111111][00000000010000000001]
[00000000101111111101][00000000010000000001]
[00000001001111111011][00000000010000000001]
[00000001101111111001][00000000010000000001]
[00000010001111110111][00000000010000000001]
[00000010101111110101][00000000010000000001]
[00000011001111110011][00000000010000000001]
[00000011101111110001][00000000010000000001]
[00000100001111101111][00000000010000000001]
[00000100101111101101][00000000010000000001]
[00000101001111101011][00000000010000000001]
[00000101101111101001][00000000010000000001]
[00000110001111100111][00000000010000000001]
[00000110101111100101][00000000010000000001]
[00000111001111100011][00000000010000000001]
[00000111101111100001][00000000010000000001]
[00001000101111011101][00000000010000000001]
[00001001001111011011][00000000010000000001]
[00001001101111011001][00000000010000000001]
[00001010001111010111][00000000010000000001]
[00001010101111010101][00000000010000000001]
[00001011001111010011][00000000010000000001]
[00001011101111010001][00000000010000000001]
[00001100001111001111][00000000010000000001]
[00001100101111001101][00000000010000000001]
[00001101001111001011][00000000010000000001]
[00001101101111001001][00000000010000000001]
[00001110001111000111][00000000010000000001]
[00001110101111000101][00000000010000000001]

[00010001001110111011][00000000010000000001]
[00010001101110111001][00000000010000000001]
[00010010001110110111][00000000010000000001]
[00010010101110110101][00000000010000000001]
[00010011001110110011][00000000010000000001]
[00010100101110101101][00000000010000000001]
[00010101001110101011][00000000010000000001]
[00010101101110101001][00000000010000000001]
[00010110001110100111][00000000010000000001]
[00010110101110100101][00000000010000000001]
[00010111001110100011][00000000010000000001]
[00011001001110011011][00000000010000000001]
[00011001101110011001][00000000010000000001]
[00011010001110010111][00000000010000000001]
[00011010101110010101][00000000010000000001]
[00011011001110010011][00000000010000000001]
[00100100101101101101][00000000010000000001]
[00100101001101101011][00000000010000000001]
[00100110101101100101][00000000010000000001]
[00101010101101010101][00000000010000000001]
[00101011001101010011][00000000010000000001]
[00101100101101001101][00000000010000000001]
number of canons found: 51

For those who have not consulted these lists before, it may be useful to point out that Fripertinger eliminates all periodic rhythms (thus 00110011001100110011 does not appear here), and that only one starting point is permitted (the full list here, with all starting points, would be 10 times longer).

Observations on tiling (0,3,6) and (0,5,10), and how to count the solutions

Consider how one may tile loops with three - note canons in tempos 3 : 5, that is to say, by combining motifs (0, 3, 6) and (0, 5,10). A loop of nine can be tiled if one uses (0,3,6) three times or (0,5,10) three times. One can not combine the two. In other words, the two motifs must be supplementary sets. One set defines the rhythm and the other set defines the points of departure.

{{0, 5, 1}, {3, 8, 4}, {6, 2, 7}}

{{0, 3, 6}, {5, 8, 11}, {1, 4, 7}}

There is only one way to tile a loop of 12, and it requires us to use (0, 5, 10) once and (0, 3, 6) three times

{{0, 5, 10}, {1, 4, 7}, {3, 6, 9}, {8, 11, 2}}

One can tile a loop of 15, using (0, 5, 10) twice and (0, 3, 6) three times in six ways.

{{0, 5, 10}, {1, 4, 7}, {3, 8, 13}, {6, 9, 12}, {11, 14, 2}}
{{0, 5, 10}, {1, 4, 7}, {6, 9, 12}, {8, 13, 3}, {11, 14, 2}}
{{0, 5, 10}, {1, 4, 7}, {6, 9, 12}, {11, 14, 2}, {13, 3, 8}}
{{0, 5, 10}, {2, 7, 12}, {3, 6, 9}, {8, 11, 14}, {13, 1, 4}}
{{0, 5, 10}, {3, 6, 9}, {7, 12, 2}, {8, 11, 14}, {13, 1, 4}}
{{0, 5, 10}, {3, 6, 9}, {8, 11, 14}, {12, 2, 7}, {13, 1, 4}}

But we must question how we counted these six solutions. The first three are essentially the same, having only the difference between (3, 8, 13) (8, 13, 3), and (13, 3, 8), just as the second three are essentially the same. And if we had not insisted that the point zero mark the beginning of a (0, 5, 10) rhythm, we would have found many more solutions.

Finally both, or all 6, or all 12, or all 3000 solutions (depending on how one counts) can be read in the following table, which is not really very complicated. The two top lines represent the fives and the three bottom lines represent the threes:

```
1000010000100001000010000100001000010000100...
0010000100001000010000100001000010000100001...
0100100000000100100100000000100100100000000...
0001001001000000001001001000000001001001000...
0000000010010010000000010010010000000010010...
```

In short, the number of solutions you find is mostly a matter of how many zero points you wish to accept. And if you wish to consider (0, 3, 12) or (0, 5, 25) or (-5, 0, 5) to be unique motifs, the number of solutions can become infinite.

This case dramatically confirms Guerino Mazzola’s concern for finding deeper understanding of musical structure. We need to go beyond the “naive approach,” beyond simple numerical lists and “encyclopedic” ordering, beyond the “denotators.” We need to identify the “forms” and the “coordinators” and the “categories” involved. As Mazzola puts it, we do not want simply to identify the addresses where things are. We want to know the “variable addresses” where these things really are.

Observation on rhythmic canons in two tempos

The previous rhythmic canon in tempos 3 : 5 is not difficult to construct. Many others may be formed from two “supplementary sets” (Vuza). For example, to have a rhythmic canon with voices in tempos 3 : 4, one can construct a rhythmic canon where R, the rhythm, = (0, 4, 8) and S, the entrance points, = (0, 3, 6):

x				x				x						
			x				x				x			
						x				x				x

One can do the same thing with the supplementary sets reversed, where R = (0, 3, 6) and S = (0, 4, 8):

x			x			x									
				x				x				X			
							x				x				x

The accumulative rhythm is the same in both cases:

x			x	x		x	x	x			x	x			x

And this 9-point rhythm tiles a loop of 18,

x			x	x		x	x	x		x	x			x																					
									x			x	x		x	x	x		x	x			x												

																		x			x	x		x	x	x		x	x			x	x			x

which we can state as a six-voice canon in tempos 3 : 4, with the threes on the top three lines and the fours on the lower three lines:

x			x			x																	
				x			x			x													
								x			x			x									
									x				x				x						
												x				x				x			
															x				x				x

It is not necessary to have the same number of points in the two supplementary sets. As examples we can consider the five-voice tiling of the supplementary sets (0 2 4) and (0 5)

x		x		x											
					x		x		x						
						x					x				
								x					x		
										x					x

Or this seven-voice tiling derived from the supplementary sets (0, 2, 4, 6, 8) and (0, 3):

x		x		x		x		x															
			x		x		x		x		x												
										x			x										
												x			x								
														x			x						
																x			x				
																		x			x		

It is surprising how many canons in two tempos may be constructed in this way, though this is not possible in cases where the accumulative rhythm does not tile the line with itself. This is true, for example, of the supplementary sets (0,4,8) and (0,5,10).

Observation on (0, 1, 4)

We were all surprised a year ago when Andranik Tangian found so many ways of tiling lines (not loops) of length 15, 30 and other multiples of 15 using only the rhythm (0, 1, 4) and two augmentations (0, 2, 8) and (0, 4, 16). Later it was proved by Emmanuel Amiot and Harald Fripertinger that all solutions to this problem must have a length divisible by 15. It seemed that the subject was closed, but as I gradually became more adept at Mathematica, I also became more convinced that I should consider both lines and loops. Looking for both, I ran the problem through my own computer and found many additional loop solutions of length 15, as well as solutions for lengths that are not multiples of 15. Since the number of solutions generally increases as loops become longer, one is tempted to assume that a loop of any length that is a multiple of three can be tiled by these motifs. However, there is at least one length (12) for which no solution seems to exist.

Here is one solution for tiling a loop of nine with (0, 1, 4) and its augmentations :

{0, 4, 16},(1, 5, 17},(2, 3, 6}

Practically the same program, shown here, found no solution at all for length 12.

```
motif[1] = {0, 1, 4};
motif[2] = {0, 2, 8};
motif[3] = {0, 4, 16};
Do[
  Do[
    Do[
      Do[
        AttemptedSolution =
```

```
      Union[motif[w], motif[x] + a, motif[y] + b, motif[z] + c];
     fixed = Sort[Mod[AttemptedSolution, 12]];
     If[fixed == {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11},
      Print[motif[w], motif[x] + a, motif[y] + b, motif[z] + c]],
     {a, 0, b}],
    {b, 0, c}],
   {c, 0, 13}],
  {w, 3},
  {x, 3},
  {y, 3},
  {z, 3}]
```

Very many solutions for five voices (loops of 15) were found. There is much duplication here since (0, 1, 4) and (0, 4, 16) are equivalent in Z/15Z. the two single line solutions found by Andranik Tangian are in bold face.

{0, 1, 4}, {2, 3, 6}, {7, 8, 11}, {9, 10, 13}, {12, 14, 20}

{0, 1, 4}, {5, 6, 9}, {7, 8, 11}, {10, 12, 18}, {13, 14, 17}

{0, 1, 4}, {5, 6, 9}, {7, 8, 11}, {10, 12, 18}, {13, 17, 29}

{0, 1, 4}, {2, 3, 6}, {7, 8, 11}, {9, 13, 25}, {12, 14, 20}

{0, 1, 4}, {2, 3, 6}, {5, 7, 13}, {8, 9, 12}, {10, 11, 14}

{0, 1, 4}, {2, 3, 6}, {5, 7, 13}, {8, 9, 12}, {10, 14, 26}

{0, 1, 4}, {2, 3, 6}, {5, 7, 13}, {8, 12, 24}, {10, 11, 14}

{0, 1, 4}, {2, 3, 6}, {5, 7, 13}, {8, 12, 24}, {10, 14, 26}

{0, 1, 4}, {2, 3, 6}, {7, 11, 23}, {9, 10, 13}, {12, 14, 20}

{0, 1, 4}, {5, 6, 9}, {7, 11, 23}, {10, 12, 18}, {13, 14, 17}

{0, 1, 4}, {5, 6, 9}, {7, 11, 23}, {10, 12, 18}, {13, 17, 29}

{0, 1, 4}, {2, 3, 6}, {7, 11, 23}, {9, 13, 25}, {12, 14, 20}

{0, 1, 4}, {3, 5, 11}, {6, 7, 10}, {8, 9, 12}, {13, 14, 17}

{0, 1, 4}, {3, 5, 11}, {6, 7, 10}, {8, 9, 12}, {13, 17, 29}

{0, 1, 4}, {3, 5, 11}, {6, 7, 10}, {8, 12, 24}, {13, 14, 17}

{0, 1, 4}, {3, 5, 11}, {6, 7, 10}, {8, 12, 24}, {13, 17, 29}

{0, 1, 4}, {5, 7, 13}, {6, 8, 14}, {9, 11, 17}, {10, 12, 18}

{0, 1, 4}, {3, 5, 11}, {6, 10, 22}, {8, 9, 12}, {13, 14, 17}

{0, 1, 4}, {3, 5, 11}, {6, 10, 22}, {8, 9, 12}, {13, 17, 29}

{0, 1, 4}, {3, 5, 11}, {6, 10, 22}, {8, 12, 24}, {13, 14, 17}

{0, 1, 4}, {3, 5, 11}, {6, 10, 22}, {8, 12, 24}, {13, 17, 29}

{0, 1, 4}, {2, 6, 18}, {7, 8, 11}, {9, 10, 13}, {12, 14, 20}

{0, 1, 4}, {5, 9, 21}, {7, 8, 11}, {10, 12, 18}, {13, 14, 17}

{0, 1, 4}, {5, 9, 21}, {7, 8, 11}, {10, 12, 18}, {13, 17, 29}

{0, 1, 4}, {2, 6, 18}, {7, 8, 11}, {9, 13, 25}, {12, 14, 20}

{0, 1, 4}, {2, 6, 18}, {5, 7, 13}, {8, 9, 12}, {10, 11, 14}

{0, 1, 4}, {2, 6, 18}, {5, 7, 13}, {8, 9, 12}, {10, 14, 26}

{0, 1, 4}, {2, 6, 18}, {5, 7, 13}, {8, 12, 24}, {10, 11, 14}

{0, 1, 4}, {2, 6, 18}, {5, 7, 13}, {8, 12, 24}, {10, 14, 26}

{0, 1, 4}, {2, 6, 18}, {7, 11, 23}, {9, 10, 13}, {12, 14, 20}

{0, 1, 4}, {5, 9, 21}, {7, 11, 23}, {10, 12, 18}, {13, 14, 17}

{0, 1, 4}, {5, 9, 21}, {7, 11, 23}, {10, 12, 18}, {13, 17, 29}

{0, 1, 4}, {2, 6, 18}, {7, 11, 23}, {9, 13, 25}, {12, 14, 20}

{0, 2, 8}, {3, 4, 7}, {5, 6, 9}, {10, 11, 14}, {12, 13, 16}

{0, 2, 8}, {3, 4, 7}, {5, 6, 9}, {10, 11, 14}, {12, 16, 28}

{0, 2, 8}, {3, 4, 7}, {5, 6, 9}, {10, 14, 26}, {12, 13, 16}

{0, 2, 8}, {3, 4, 7}, {5, 6, 9}, {10, 14, 26}, {12, 16, 28}

{0, 2, 8}, {3, 4, 7}, {5, 9, 21}, {10, 11, 14}, {12, 13, 16}

{0, 2, 8}, {3, 4, 7}, {5, 9, 21}, {10, 11, 14}, {12, 16, 28}

{0, 2, 8}, {3, 4, 7}, {5, 9, 21}, {10, 14, 26}, {12, 13, 16}

{0, 2, 8}, {3, 4, 7}, {5, 9, 21}, {10, 14, 26}, {12, 16, 28}

{0, 2, 8}, {1, 3, 9}, {6, 7, 10}, {11, 13, 19}, {12, 14, 20}

{0, 2, 8}, {1, 3, 9}, {4, 6, 12}, {5, 7, 13}, {10, 11, 14}

{0, 2, 8}, {1, 3, 9}, {4, 6, 12}, {5, 7, 13}, {10, 14, 26}

{0, 2, 8}, {1, 3, 9}, {6, 10, 22}, {11, 13, 19}, {12, 14, 20}

{0, 2, 8}, {3, 7, 19}, {5, 6, 9}, {10, 11, 14}, {12, 13, 16}

{0, 2, 8}, {3, 7, 19}, {5, 6, 9}, {10, 11, 14}, {12, 16, 28}

{0, 2, 8}, {3, 7, 19}, {5, 6, 9}, {10, 14, 26}, {12, 13, 16}

{0, 2, 8}, {3, 7, 19}, {5, 6, 9}, {10, 14, 26}, {12, 16, 28}

{0, 2, 8}, {3, 7, 19}, {5, 9, 21}, {10, 11, 14}, {12, 13, 16}

{0, 2, 8}, {3, 7, 19}, {5, 9, 21}, {10, 11, 14}, {12, 16, 28}

{0, 2, 8}, {3, 7, 19}, {5, 9, 21}, {10, 14, 26}, {12, 13, 16}

{0, 2, 8}, {3, 7, 19}, {5, 9, 21}, {10, 14, 26}, {12, 16, 28}

{0, 4, 16}, {2, 3, 6}, {7, 8, 11}, {9, 10, 13}, {12, 14, 20}

{0, 4, 16}, {5, 6, 9}, {7, 8, 11}, {10, 12, 18}, {13, 14, 17}

{0, 4, 16}, {5, 6, 9}, {7, 8, 11}, {10, 12, 18}, {13, 17, 29}

{0, 4, 16}, {2, 3, 6}, {7, 8, 11}, {9, 13, 25}, {12, 14, 20}

{0, 4, 16}, {2, 3, 6}, {5, 7, 13}, {8, 9, 12}, {10, 11, 14}

{0, 4, 16}, {2, 3, 6}, {5, 7, 13}, {8, 9, 12}, {10, 14, 26}

{0, 4, 16}, {2, 3, 6}, {5, 7, 13}, {8, 12, 24}, {10, 11, 14}

{0, 4, 16}, {2, 3, 6}, {5, 7, 13}, {8, 12, 24}, {10, 14, 26}

{0, 4, 16}, {2, 3, 6}, {7, 11, 23}, {9, 10, 13}, {12, 14, 20}

{0, 4, 16}, {5, 6, 9}, {7, 11, 23}, {10, 12, 18}, {13, 14, 17}

{0, 4, 16}, {5, 6, 9}, {7, 11, 23}, {10, 12, 18}, {13, 17, 29}

{0, 4, 16}, {2, 3, 6}, {7, 11, 23}, {9, 13, 25}, {12, 14, 20}

{0, 4, 16}, {3, 5, 11}, {6, 7, 10}, {8, 9, 12}, {13, 14, 17}

{0, 4, 16}, {3, 5, 11}, {6, 7, 10}, {8, 9, 12}, {13, 17, 29}

{0, 4, 16}, {3, 5, 11}, {6, 7, 10}, {8, 12, 24}, {13, 14, 17}

{0, 4, 16}, {3, 5, 11}, {6, 7, 10}, {8, 12, 24}, {13, 17, 29}

{0, 4, 16}, {5, 7, 13}, {6, 8,14}, {9, 11, 17}, {10, 12, 18}

{0, 4, 16}, {3, 5, 11}, {6, 10, 22}, {8, 9, 12}, {13, 14, 17}

{0, 4, 16}, {3, 5, 11}, {6, 10,22}, {8, 9, 12}, {13, 17, 29}

{0, 4, 16}, {3, 5, 11}, {6, 10, 22}, {8, 12, 24}, {13, 14, 17}

{0, 4, 16}, {3, 5, 11}, {6, 1022}, {8, 12, 24}, {13, 17, 29}

{0, 4, 16}, {2, 6, 18}, {7, 8, 11}, {9, 10, 13}, {12, 14, 20}

{0, 4, 16}, {5, 9, 21}, {7, 8, 11}, {10, 12, 18}, {13, 14, 17}

{0, 4, 16}, {5, 9, 21}, {7, 8, 11}, {10, 12, 18}, {13, 17, 29}

{0, 4, 16}, {2, 6, 18}, {7, 8,11}, {9, 13, 25}, {12, 14, 20}

{0, 4, 16}, {2, 6, 18}, {5, 7, 13}, {8, 9, 12}, {10, 11, 14}

{0, 4, 16}, {2, 6, 18}, {5, 7, 13}, {8, 9, 12}, {10, 14, 26}

{0, 4, 16}, {2, 6, 18}, {5, 7, 13}, {8, 12, 24}, {10, 11, 14}

{0, 4, 16}, {2, 6, 18}, {5, 7, 13}, {8, 12, 24}, {10, 14, 26}

{0, 4, 16}, {2, 6, 18}, {7, 11, 23}, {9, 10, 13}, {12, 14, 20}

{0, 4, 16}, {5, 9, 21}, {7, 1123}, {10, 12, 18}, {13, 14, 17}

{0, 4, 16}, {5, 9, 21}, {7, 11, 23}, {10, 12, 18}, {13, 17, 29}

{0, 4, 16}, {2, 6, 18}, {7, 11, 23}, {9, 13, 25}, {12, 14, 20}

Observation on magic squares as tilings

A magic square using all the numbers 1 to *x* tiles a line of *x* points.

6	7	2
1	5	9
8	3	4

In the case of this 3 x 3 square, one can tile a line of nine points with the three columns:

1					6		8	
		3		5		7		
	2		4					9

or with the three lines:

	2				6	7		
1				5				9
		3	4				8	

This magic square four by four tiles a line of 16 in two ways.

1	12	13	8
15	6	3	10

4	9	16	5
14	7	2	11

1			4										14	15	
					6	7		9			12				
	2	3										13			16
				5			8		10	11					

1							8				12	13			
		3			6				10					15	
			4	5				9							16
	2					7				11			14		

Since in a magic square all lines and columns have the same total, there can be no numbers less than zero or greater than the total number of points, so the results always tile lines rather than loops. No magic square can be a rhythmic canon, as the intervals always fall differently in the different lines and columns. Magic square rhythms often come in mirrored pairs, however, and they have a strong symmetrical logic of another kind.

What can we learn from magic square mathematics?

If we calculate how many magic squares can be constructed with the digits 1-25, will this help us tile lines 25 points long?

Observations on one - dimensional polyominos

There are five ways of placing three notes on five points. These five rhythms are defined here as (m1, m2, m3, m4, m5).I eliminated (0,1,2), which is already a line in itself, and which can not interlock with the others. We may consider these modules as parallel in one dimension to polyomino packing problem in two dimensions. How many ways are there to tile a loop of 15 points with these five rhythms, if no rhythm can occur more than once? The answer is two. One joins m2 to m5 as a line of six and one joins m1, m3, and m4 together as a line of nine, and one then brings these two lines together as a solution of 6 plus 9 or 9 plus 6. The program here recognizes all possible starting points on the loop and so gives 15 solutions instead of only two.

```
m1 := {0, 1, 3}
m2 := {0, 1, 4}
m3 := {0, 2, 3}
m4 := {0, 2, 4}
m5 := {0, 3, 4}
Do[
  Do[
    Do[
      Do[
        Do[
          tiling = Union[Mod[{m1 + a, m2 + b, m3 + c, m4 + d, m5 + e}, 15]] ;
          newtiling = Sort[Flatten[tiling]];
          If [newtiling == {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13,
                14},
            Print[Sort[{m1 + a, m2 + b, m3 + c, m4 + d, m5 + e}]],],
          {a, 0, 14}],
        {b, 0, 14}],
      {c, 0, 14}],
    {d, 0, 14}],
  {e, 0, 14}]
```

{{0, 3, 4}, {1, 2, 5}, {6, 7, 9}, {8, 10, 12}, {11, 13, 14}}

{{1, 4, 5}, {2, 3, 6}, {7, 8, 10}, {9, 11, 13}, {12, 14, 15}}

{{2, 5, 6}, {3, 4, 7}, {8, 9, 11}, {10, 12, 14}, {13, 15, 16}}

{{3, 6, 7}, {4, 5, 8}, {9, 10, 12}, {11, 13, 15}, {14, 16, 17}}

{{0, 2, 3}, {4, 7, 8}, {5, 6, 9}, {10, 11, 13}, {12, 14, 16}}

{{1, 3, 4}, {5, 8, 9}, {6, 7, 10}, {11, 12, 14}, {13, 15, 17}}

{{2, 4, 5}, {6, 9, 10}, {7, 8, 11}, {12, 13, 15}, {14, 16, 18}}

{{0, 2, 4}, {3, 5, 6}, {7, 10, 11}, {8, 9, 12}, {13, 14, 16}}

{{1, 3, 5}, {4, 6, 7}, {8, 11, 12}, {9, 10, 13}, {14, 15, 17}}

{{0, 1, 3}, {2, 4, 6}, {5, 7, 8}, {9, 12, 13}, {10, 11, 14}}

{{1, 2, 4}, {3, 5, 7}, {6, 8, 9}, {10, 13, 14}, {11, 12, 15}}

{{2, 3, 5}, {4, 6, 8}, {7, 9, 10}, {11, 14, 15}, {12, 13, 16}}

{{3, 4, 6}, {5, 7, 9}, {8, 10, 11}, {12, 15, 16}, {13, 14, 17}}

{{4, 5, 7}, {6, 8, 10}, {9, 11, 12}, {13, 16, 17}, {14, 15, 18}}

{{0, 1, 4}, {5, 6, 8}, {7, 9, 11}, {10, 12, 13}, {14, 17, 18}}

Here is another experiment I ran, based on the kinds of polyominos that have been explored in two dimensions. In this case, I calculated the 10 three-point rhythms possible on a line of six points, again emliminating (0,1,2) to make only nine:

(0,1,3)
(0,1,4)
(0,1,5)
(0,2,3)
(0,2,4)
(0,2,5)
(0,3,4)
(0,3,5)
(0,4,5)

Considering only lines, no loops, and not permitting the same motif to be used twice, I found that six points may be tiled in six ways.

{2, 3, 5},{0, 1, 4}
{0, 1, 3},{2, 4, 5)
{1, 3, 4},{0, 2, 5}
{1, 2, 5},{0, 3, 4)
{0, 2, 3},(1, 4, 5}
{1, 2, 4},{0, 3, 5}

Three of these motifs form lines of nine in 48 ways, of which I will only show a few:

{0, 4, 5} ,{3, 7, 8),{1, 2, 6)
{0, 2, 5),(3, 7, 8},{1, 4, 6}
{0, 2, 4},{3, 7, 8},{1, 5, 6}
{0, 3, 5},{4, 7,8},{1, 2, 6}
{0, 2, 5),(4, 7, 8),{1, 3, 6}…

The number of solutions will probably be increasingly greater with lines of 12, 15, 18, 21, and 24. Lines of 27, requiring each motif to be used once and only once, are probably rather few, but even with an efficient program, the computer would have to consider 9!=362,880 possibilities to find them .