Self-Similar Structures in my Music: an Inventory

lecture presented in the MaMuX seminar
IRCAM, Paris, Oct. 14, 2006

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lecture presented in the MaMuX seminar, IRCAM, Paris, Oct. 14, 2006 in context with a lecture on self-similarity by mathematician Emmanuel Amiot

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Self-similarity has been a central concern in my music for a long time, and Emmanuel Amiot has done a lot of work with self-similar mathematical structures, so we want to focus our contributions today on that. I have already written a book called *Self-Similar Melodies (1996),* but that was a didactic study, with specially composed musical examples. Ten years later it is useful for me, and hopefully for others, to take inventory and see what self-similarity means in my actual compositions.

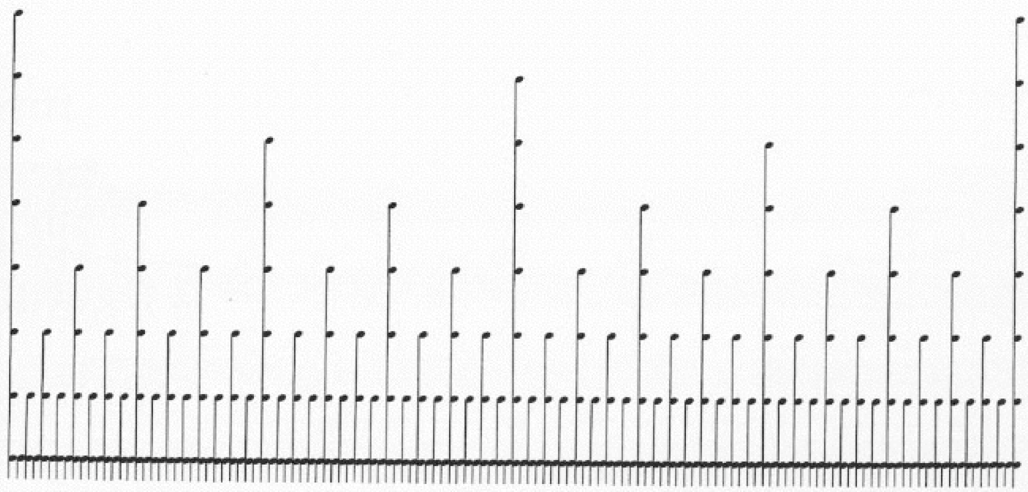
As I was selecting examples for this lecture, I was surprised by two things. First of all I was struck by the variety of techniques involved. Each self-similar piece has a slightly different way of embedding a musical structure within itself, and it is difficult to generalize how this is done. For today I'm going to define self-similar music simply as music that somehow contains itself within itself and does so on at least three different levels of time. We'll look separately at about 20 examples today, in no particular order, and without attempting to generalize or categorize.

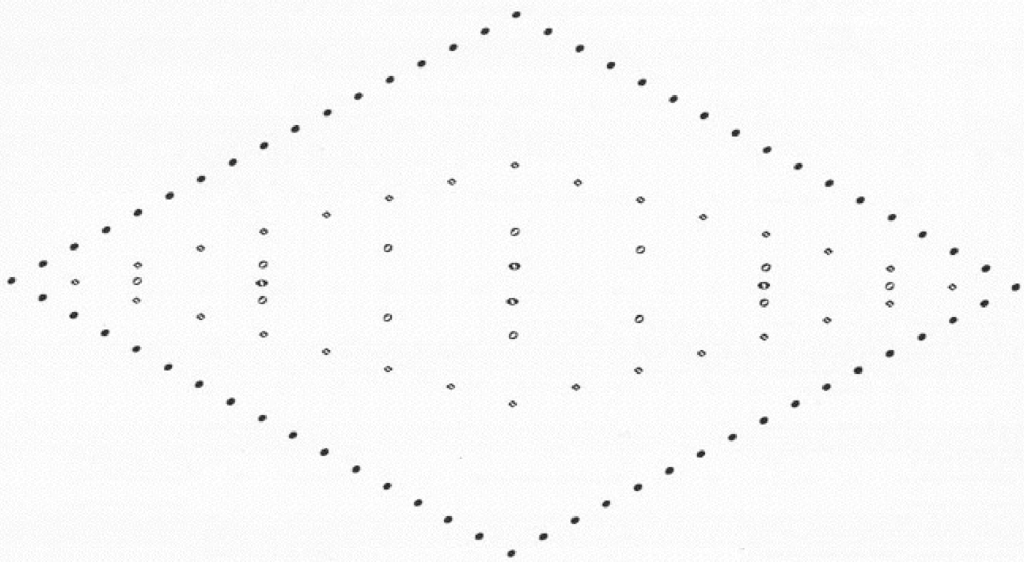
The second surprise for me was to discover that all of this began rather suddenly, around 1979-1980. It generally seems to me that my music evolves in a gradual way, but now it is clear that there was a rather quick shift toward self-similar structure at this time, and I can see several reasons for this change. One was that I had just found some old issues of *Scientific America* with articles of Martin Gardner about "dragons," which is to say the paper-folding formula. More about that later. Another was that in 1979 I met David Feldman, who taught me some group theory and helped me calculate a self-replicating melodic loop. The third, and perhaps the most important reason, was Benoit Mandelbrot's *Fractals.* This book was published in 1977, and I don't remember exactly when I read it, but I'm sure that Mandelbrot's pictures of the Koch curve, the Minkowsky sausage, the Sierpinski sponge, the Peano island and the Cantor triadic bar were for me essentially maps showing the route that I was about to follow.

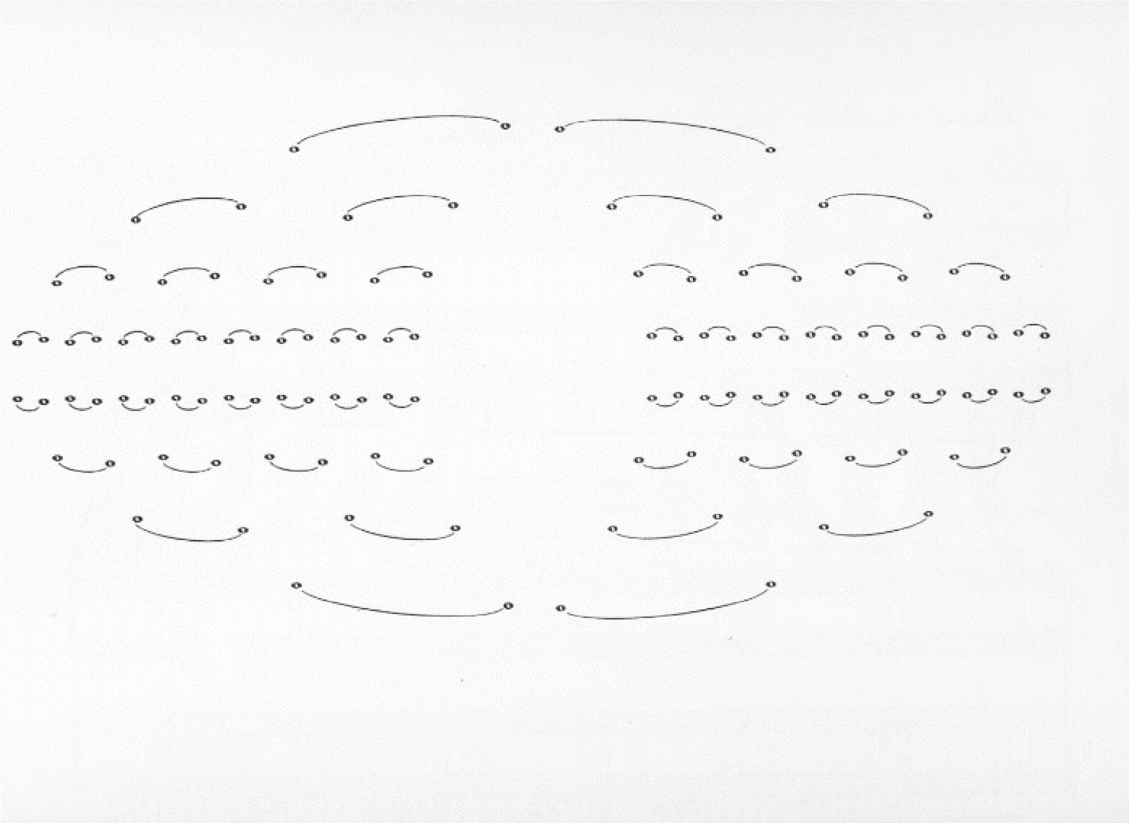
Before we begin with the examples, I should add that, while all of my music is minimal in some way or another, and almost all of it is logical in one way or another, true self-similar structure is relatively rare. Of the 49 *Symmetries*, for example, I found only three that are clearly self-similar. Of the 28 pieces in *Organ and Silence*, only two really fit into this lecture. Of the 21 *Rational Melodies (1983)*, only three or four adhere rigidly to the definition I'm using here. I also work with many other logical procedures such as permutations, combinations, tiling, and block design, and some pieces have no rigorous logical structure at all. The most blatantly non-mathematical score is the *Bonhoeffer Oratorio,* where the emotions, the text, and the political/religious content completely took over.

For the first example I'd like to show you a page from a choral piece called *Trinity*, written in 1978. I don't think I had read Mandelbrot yet, and the technique is simply isorhythm, something already practiced in the 15th century. The three-note melodic loop is the *color,* and the seven-note rhythmic loop, coming from the seven syllables of the text, is the *talea*. The first phrase begins and ends on B-flat, the second on A, and the third on G, and the fourth once again on B-flat, forming the second level of the structure. At the end of this page we once again hear the word "creator" in the first chorus, and a second section begins on A, leading to the third level of this self-similarity. Because isorhythmic structures loop around within themselves, I was writing self-similar music even before I began using the term.

The *Symmetries* began in 1979 with a music typewriter, a curious pre-computer machine that was still used in the US at that time. Three of the 49 drawings have self-similar structure:







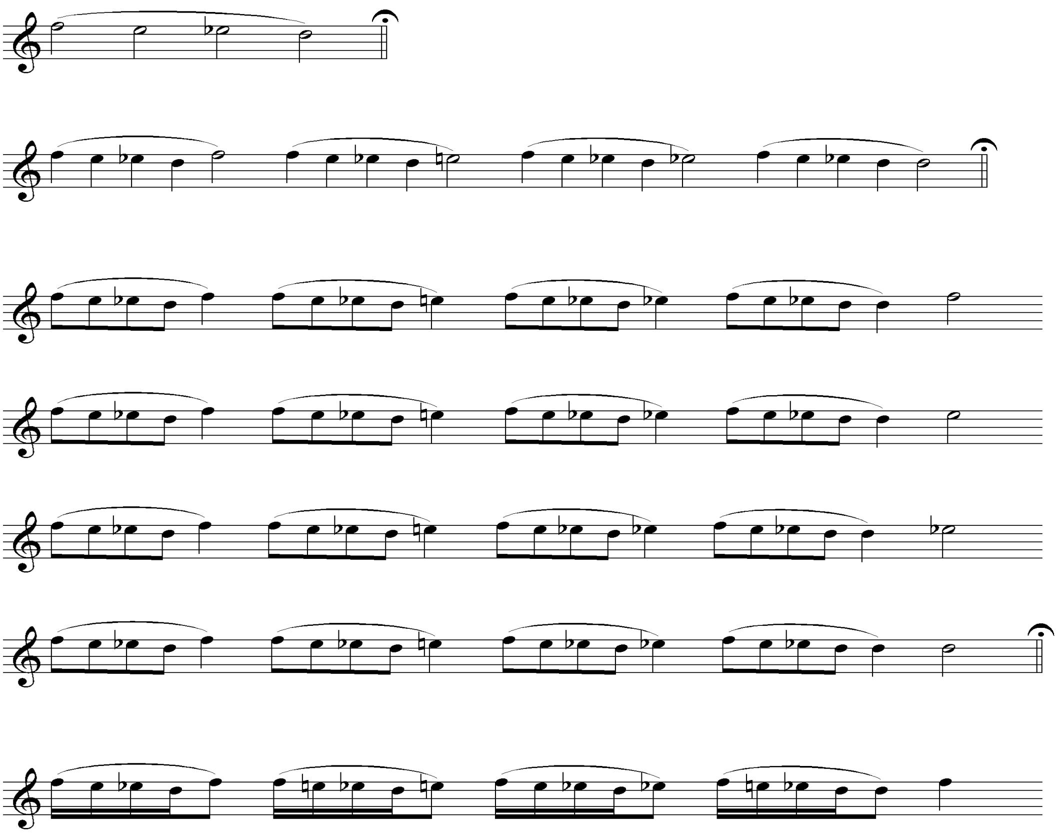
At the time I typed these images I was thinking of a pure visual music, hoping that others would do their own realizations, or would simply imagine the sounds implied by the drawings. A few years later I made my own version for piano four hands, asking people to hear this music as I heard it. This summer, a recording of the whole set was issued by Karnatic Lab recordings, in an interpretation of Dante Oei and Samuel Vriezen (KLR 010).

That same year, 1979, I wrote *Dragons in A*, which came directly from the paper-folding formula I had found in *Scientific America*. In case you don't know the paper-folding formula, it looks like this, beginning with an upward fold in the middle of the piece of paper, then new upward and downward folds dividing the paper into four parts, then four more up and down folds, etc.

```
_______________U_______________
_______U_______U_______D_______
___U___U___D___U___U___D___D___
_U_U_D_U_U_D_D_U_U_U_D_D_U_D_D_
UUDUUDDUUUDDUDDUUUDUUDDDUUDDUDD
```

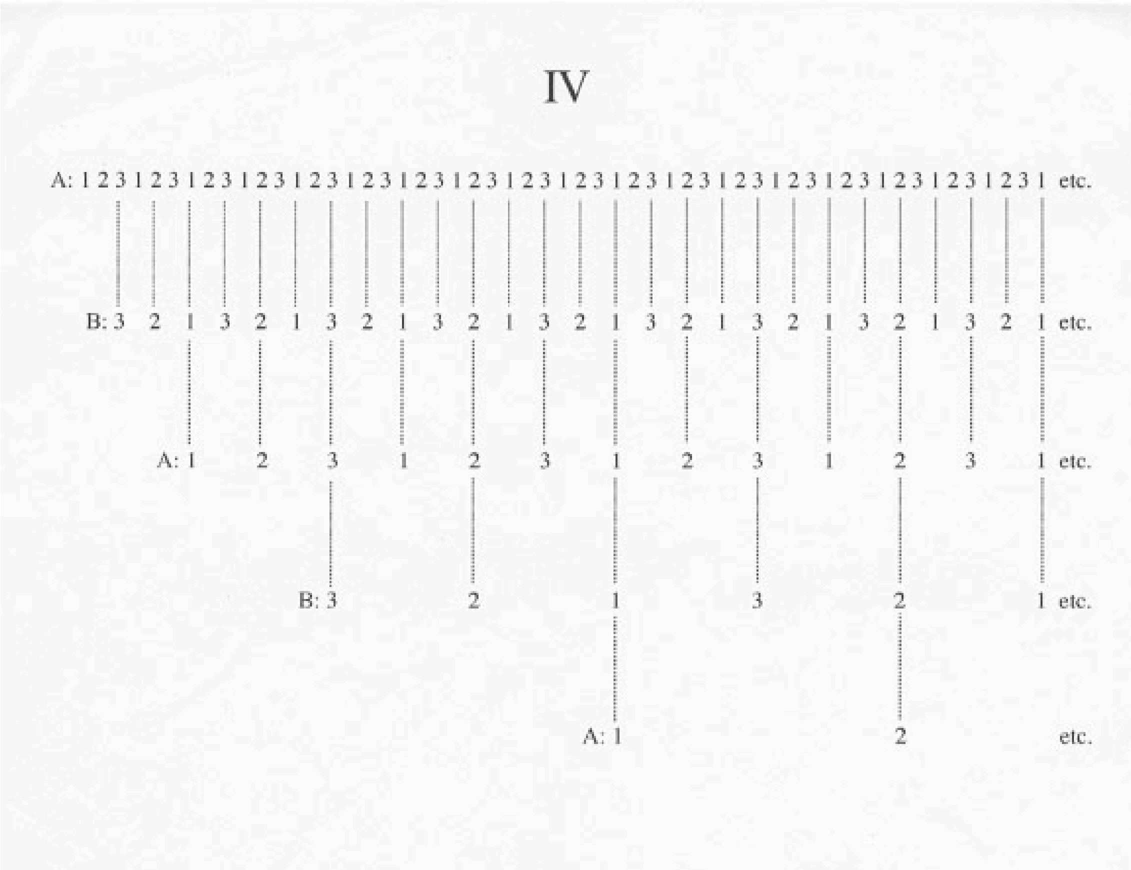
The melody that results, moving up and down the scale in these patterns, does not display its structure within structure in an obvious way, but whenever you repeat a single operation over and over, you inevitably have some sort of self-similar structure. Here is a page from the fifth and last movement, showing the complete melody in the strings, a slower version of it accumulating a major third higher in the woodwinds, and an even slower version building up another major third higher in the brass. The three voices are always in unison.

The *Rational Melodies* were published in 1983, but most of them were written two or three years earlier, and all of them are constructed with rational systems, following rigorous rules. The collection was itself a sort of inventory of the procedures I was working with at this time. *Rational Melody VIII* is constructed with a self-similarity, which you can see clearly here:

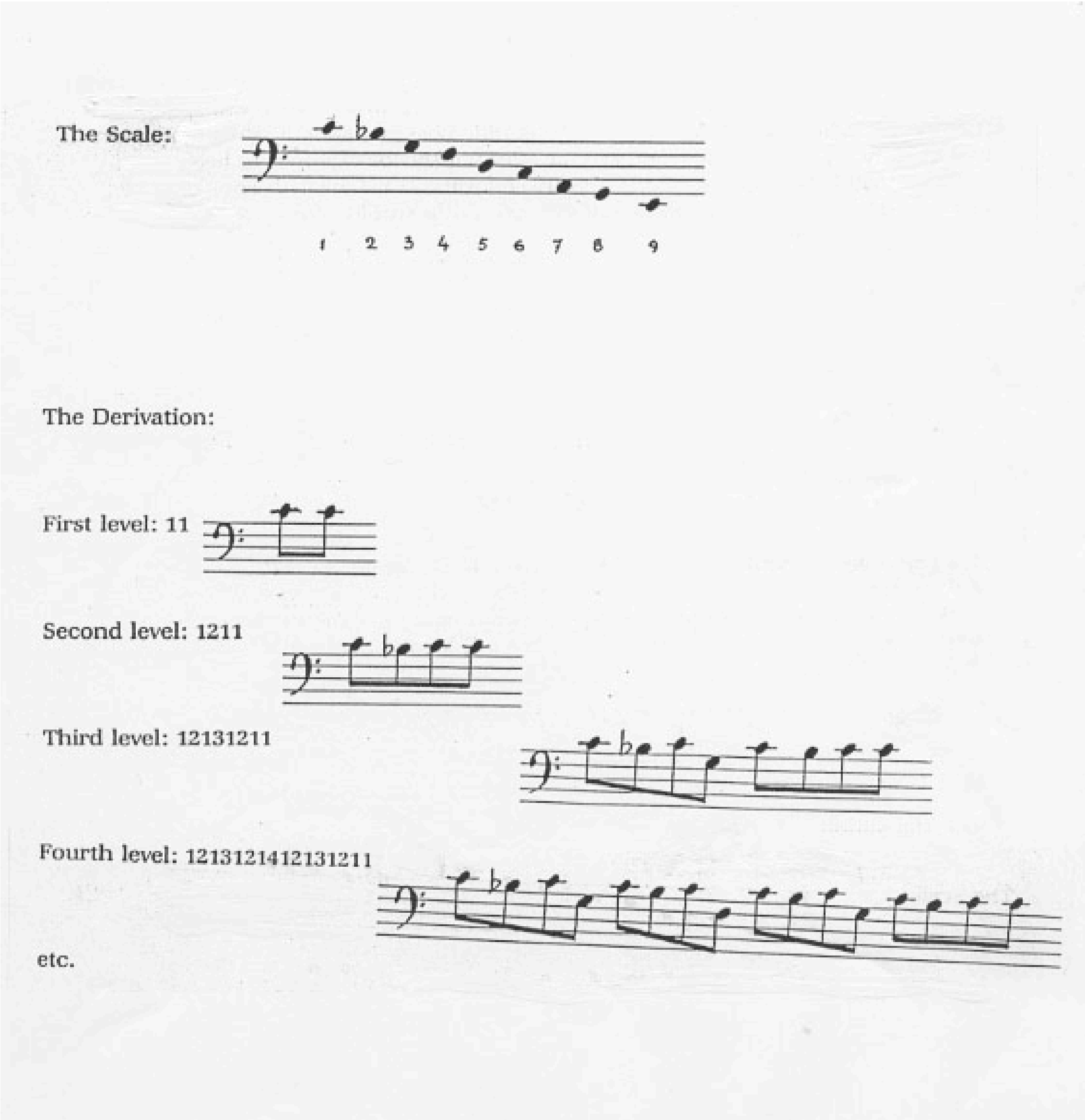


The melody, which can be played on any melodic instrument, begins with the four pitches in half notes, then inserts the quarter-note diminution before each of these notes, then the eighth-note diminution, creating a third level, and finally a 16th-note version, which is as far as I wanted to go. Curiously, this melody takes over 3 _ minutes in Eberhard Blum's Hat Hut recording and less than two minutes in the new Roger Heaton recording on Ants records (AG 12). Since the music is strictly a question of logic, it is really not possible to say that one interpretation is more sensitive than the other. It is strictly a question of taste. I should also add that while the process here looks quite obvious, I have never found a clear way to define it mathematically as an automaton – something that is rather easy to do with the paper-folding formula.

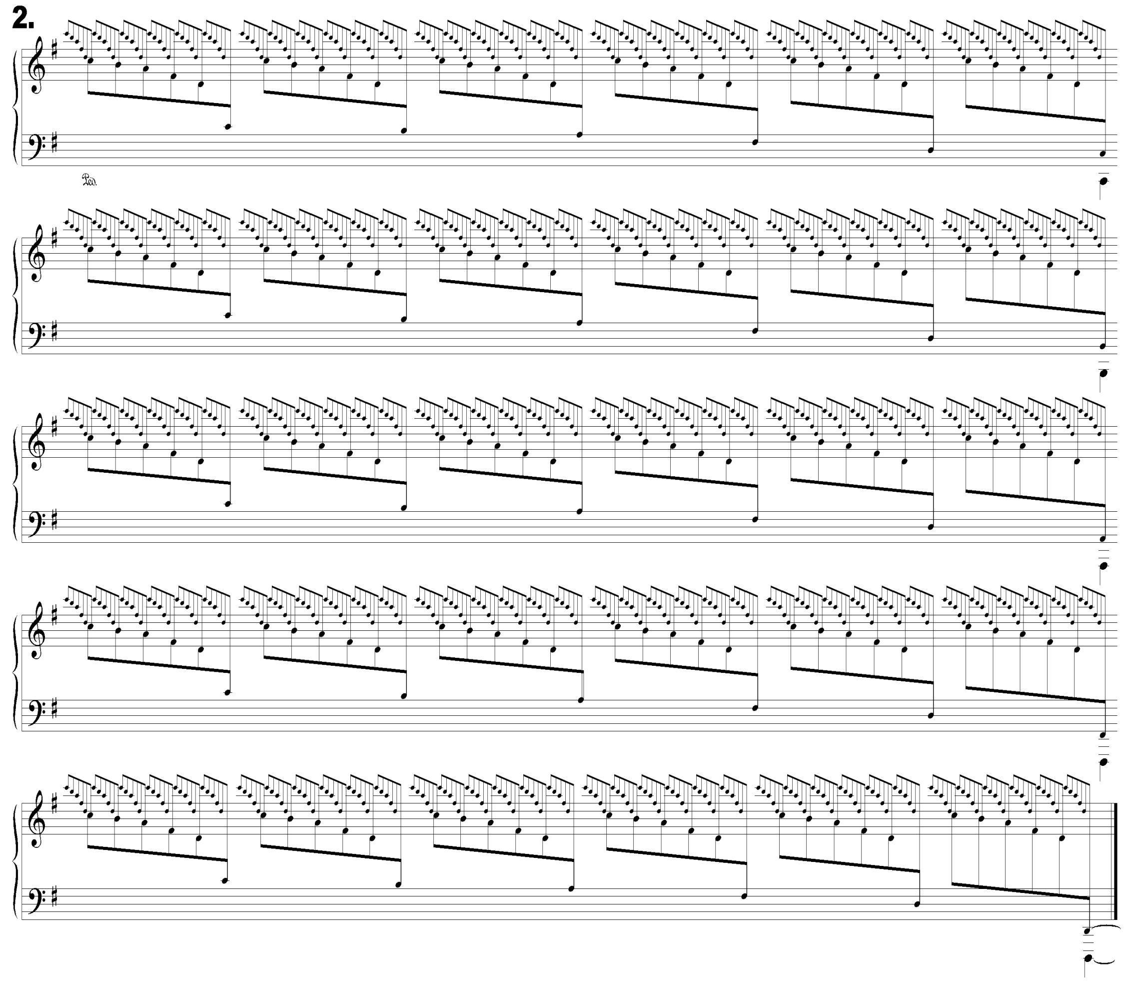
The fourth movement of the *Counting Duets (1982)* begins very fast but then alternates from one speaker to the other, each time at half tempo. Here the self-similarity is obvious, but the theme keeps turning around. In order for the two performers to be always in unison, one performer speaks all the transformations that say 1-2-3 and the other performer speaks the transformations that say 3-2-1. Things get turned around at each step. Here one only actually hears two tempos at once, rather than three, but the other levels continue rather clearly back there in the listener's memory.



Other doubling processes are defined in a score that is titled, logically enough, *Doublings for Double Bass (1980)*. There are six different formulas that the bassists may follow, according to verbal rules, but I think you can see just from the musical notation what the rules are for the first four levels of this one and what you would have to do to construct the fifth and sixth levels. The piece is conceived as an exercise in concentration and memory, which is why I ask the player to construct the remaining elements in his or her head.



The five notes of the second movement of the piano piece, *Counting Keys*, copy themselves exactly at a ratio of 5 : 1, and the structure is pretty obvious when you see it, though it sometimes surprises me how much difficulty people have in perceiving this just from listening. The music turns on a loop of C, B, A, F-sharp, D, which rotates on two levels in the right hand and on two slower levels in the left hand. The score is not necessary, and in fact I played the piece for years without writing it down, and I think a couple of other people did so as well, but it is satisfying to be able to see the structure, and one can do so particularly well in this latest edition, so carefully spaced out by Javier Ruiz.



The organ piece *Six-Note Melody (1987)* is another case where four voices play the same thing in four octaves in four tempos, but here the melody is a chromatic scale, the slower voices play in unison with the faster voices rather than inserting themselves between two notes, and the sound is totally different. The piece states the fast voice alone, then the first and second voices, then the top three voices (shown below) and finally all four voices, the last section requiring the organist to play with both feet.

The fourth movement of *Movements for wind quintet (1980)* follows another doubling procedure, but I'll define it the way mathematicians usually define automata, that is, with a transformation rule:

n → n, n+1
begin with 1, 2:

which gives this development:

first level:	*1, 2*
second:	*1 2, 2 3*
third:	*1 2 2 3, 2 3 3 4*
fourth:	*1 2 2 3 2 3 3 4, 2 3 3 4 3 4 4 5*
etc.	

The progression moves up the chromatic scale, each instrument in another tempo, the bassoon playing the slowest voice and the flute playing the 5th transformation, 32 times faster. Only on the last 16th note of each measure do all five instruments play the same note.

Another doubling procedure occurs *Organ and Silence XX (2003)*. This can be considered a strange sort of fugue, each voice entering in half time and the theme becoming twice as long with each entrance.

While we're looking at *Organ and Silence*, also available on Ants Records (AG 05), let us also consider *Organ and Silence XVIII*, where a theme 12 eighth notes long makes a copy of itself at a tempo five times slower, provided the second voice begins at the right point. The third voice, beginning at the bottom of this first page, is of course 25 times slower than the first voice, admitting much silence, which is the chief idea throughout *Organ and Silence.*

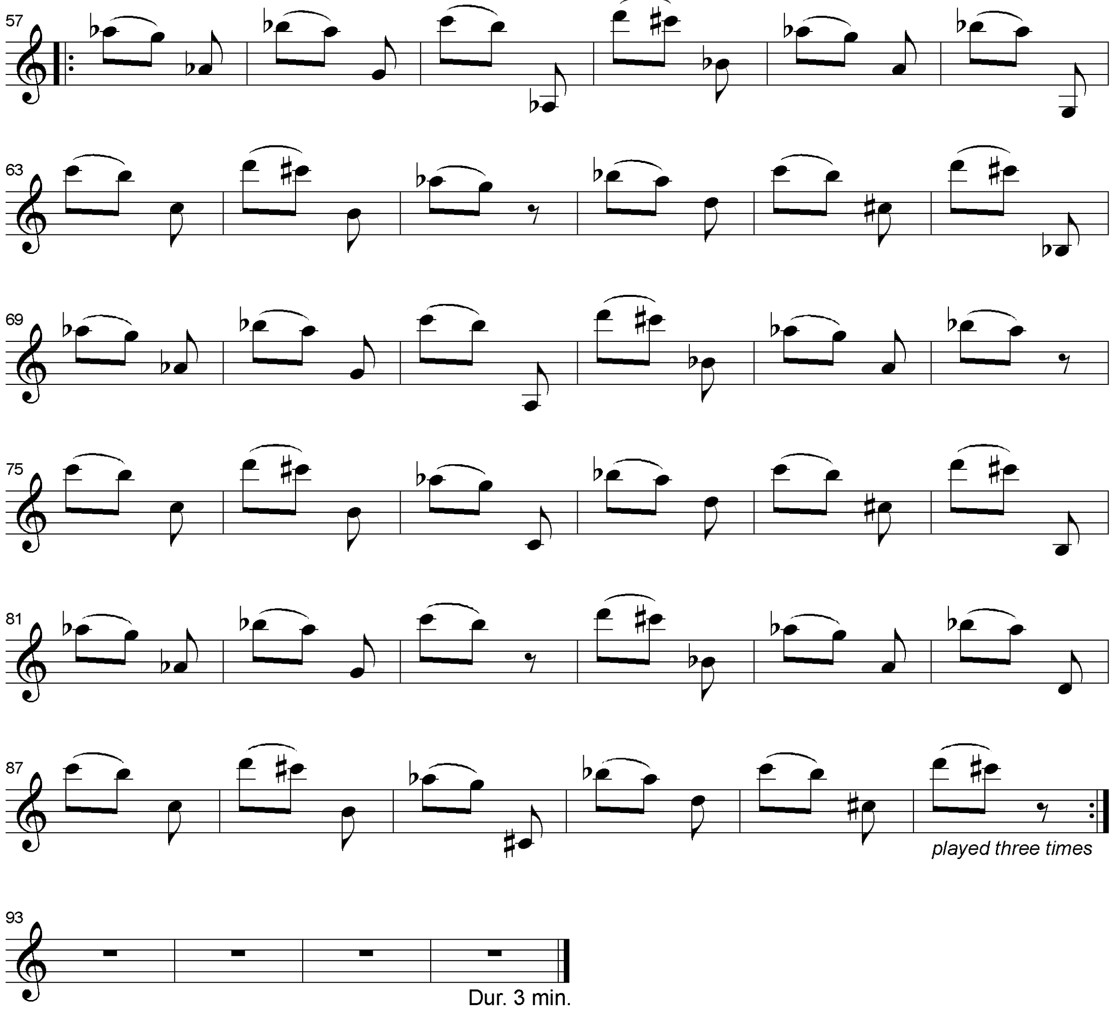
In most of the examples we have been looking at, the different voices occur simultaneously, but sometimes simultaneities are systematically avoided in a texture that we often call “one-dimensional tiling.” In this movement of the non-pitched choral piece *1 2 3 (2002),* for example, a simple 1-2-3 rhythm comes together in 24 beats with 8 voices in proportions: 8 : 8 : 4 : 4 : 4 : 4 : 2 : 2.

Part I, No. 1

In another movement of *1 2 3* each successive voice is four times slower than the voice above it, and there are no simultaneities. The structure appears rather simple, but on the later pages, as the slower and louder voices begin to move, the numbers get quite mixed up. It is not really as simple as 1-2-3.

Part II, No. 2

In general the forms in the *Tilework* series are not self-similar, but *Tilework for Violin (2003)* is an exception. In this excerpt from the first movement one sees easily the three versions of the melody in three different octaves, with tempo ratios of 9 : 3 : 1. Note that the music does not tile every single point of time, but leaves two holes.



In the second and third movements of this piece the ratios between tempos are 4 : 2 : 1 and 5 : 2 : 1, and there are holes in these cases as well. The rhythms are always exactly the same, however. Only the tempos differ.

Rational Melody XVI involves something we haven't seen yet, a technique that in the *Self-Similar Melodies* book I call "sandwiching." We begin with a phrase of three notes, which I'll call 1-2-1, and construct subsequent transformations according to these rules:

- *If there is an interval of two scale degrees between two adjacent notes, sandwich in the note that falls between.*
- *If two adjacent notes fall next to one another on the scale, sandwich in the note that is one degree higher than the highest of the two.*

So beginning with 1-2-1, we must sandwich in two 3s, giving 1-3-2-3-1.

Now we fill in note 2 in places where the melody skips from 1 to 3 or 3 to 1, and we fill in 4 at the other two points, giving 1-2-3-4-2-4-3-2-1.

Since the rules are symmetrical, the result is always a palindrome.

In this case the composition begins with the 7th transformation, a melody of 129 notes, and works back through the shorter forms, so the system is most easily seen if we simply look at the final bars of the piece. Technically this example does not fit my definition, since the original 1-2-1 never appears on three or more levels at the same time, but since it is contained within the five-note transformation, which is contained within the nine-note transformation, and so on, this obviously represents some kind of self-similarity:

The 18 movements of *Automatic Music for six percussionists (1997)* contain all sorts of automata, most of which are not interesting for mathematicians, who like to study sequences that are strictly 2-automatic or 3-automatic or n-automatic, which means that the length of the phrase is multiplied by the same proportion with each successive transformation. The fifth movement of the collection is constructed by this automaton:

1 → 1, 2, 1
2 → 2, 2, 3
3 →___ (The 3, the pause, is to be dropped on the next transformation.)
begin 1, 2

where 1 = dum, 2 = tak, and 3 = a pause, and with this development:

1,2
1,2,1,2,2,3
1,2,1,2,2,3,1,2,1,2,2,3,2,2,3
1,2,1,2,2,3,1,2,1,2,2,3,2,2,3,1,2,1,2,2,3,1,2,1,2,2,3,2,2,3,2,2,3,2,2,3
etc.

In the first two measures of the following fragment you can see the 1,2 in the sixth instrument, the 1,2,1,2,2 in the fifth, and 1,2,1,2,2,1,2,1,2,2,2,2 in the fourth. In the third measure all of this is repeated while the third player introduces the new transformation, playing in eighth notes instead of dotted quarters. This sequence may also not have exactly the same thing on three or more levels. It depends on how you look at it. It is rather curious in another sense, because the total number of notes grows from 2 to 5 to 12 to 28 to 64 to 144. There is a formula for this, but it's kind of a strange sequence all the same. The web site of Sloan's sequences says "I am sorry, but the terms do not match anything in the table."

Another curious automaton, which has a built-in delay, produced *Narayana's Cows*, a score that has been often played by all sorts of large and small ensembles and is interpreted by Daniel Kientzy on a Pogus CD (Pogus21033-2). The sequence was derived from a 14th-century Indian problem in which a mother cow gives birth to a female calf at the beginning of each year and each calf becomes a mother and does the same thing beginning in her fourth year. By the 10th year, the herd numbers 60 mothers and daughters, and while it is obvious that some sort of logic is going on in the music that results from this progression, I don't think people hear much of the self-similarity. But as we saw with the paper-folding formula, repeating one process over and over always results in some kind of self-similarity, and one can see this in the following illustration, which shows the 60-note melody that has evolved by the 10th year. Within this one can also hear an inner layer of the first 19 notes, and within that, a layer of 6 notes. It is rather hard to really perceive the three levels at once here, but the self-similarity is present if you look for it. The first line of numbers below represents the 60 cows of the herd in its 10th year. The second line singles out the 19 mothers, that is, the quarter-notes, of the 10th year, and this happens to be the notes of the 19 cows of the 7th year, and the mothers of the 7th year are equivalent to the six mothers and daughters of the 4th year, shown on the last line.

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122223233233323333423333434423333434434442333343443444344445
1___2_2__2___2___3_2___3_3__2___3_3__3___2___3_3__3___3___4
1____________2_____2________2____________2___3______________
```

The first movement of *Movements for Wind Quintet (1980)* is case of a rather simple self-similar process that produces a rather complicated result. Based on the Fibonacci series, the first voice is literally that: (1,1,2,3,5,8,13…). Since every note of this sequence is the sum of the two preceding notes, the second voice can be formed by substituting (1,1) for the 2 of the first voice, (1,2) for the 3 of the first voice, (2,3) for the 5, and so on. The third voice is derived from the second voice in the same way, and by the time we get to the fifth voice, the melody stutters a lot and takes a long time to say what it has to say. The movement ends with a rather long bassoon solo playing this final transformation of the Fibonacci series.

```
112 3  5    8       d
  1112 2 3  3  5    5    8
     111112 12 2 3  2 3  3  5
          11 111112 1112 12 2 3
                  11   11 111112
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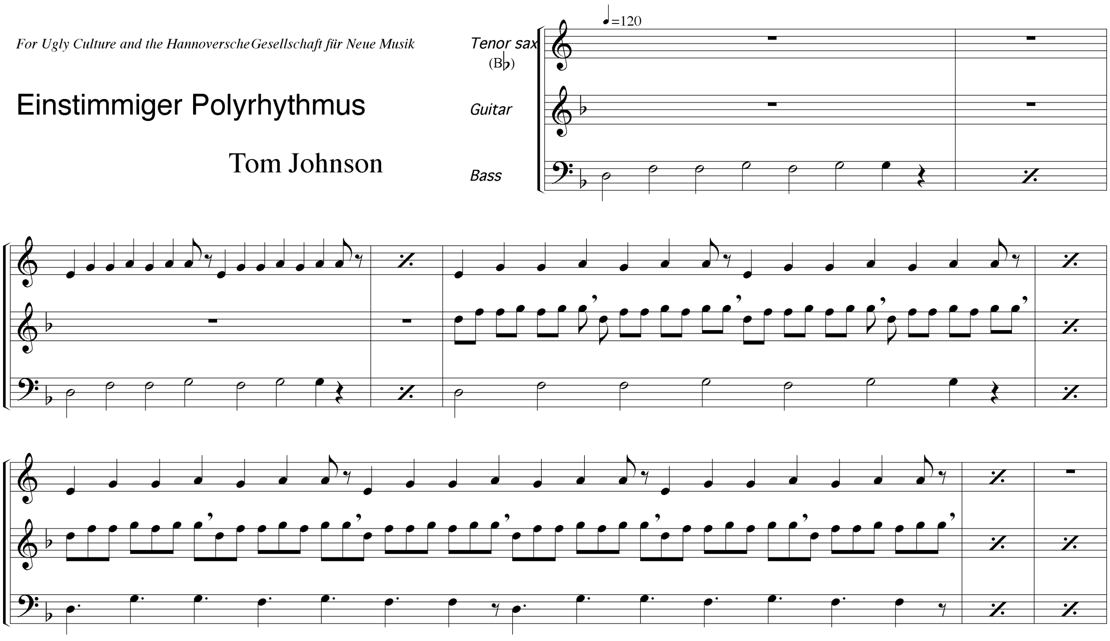
I said I wasn’t going to propose categories here, but sequences that self-replicate are rather special, so I’m going to show several examples of that all together. The first and most obvious example is “La Vie est si court,” a piece for 8 instruments, which self-replicates at 3 : 1 and 7 : 1. You can see the 3 : 1 process clearly already on the cover. The voices are always in unison.

Recycled Ostinato (1996) is a piece I wrote for the trio Ugly Culture, thus for saxophone, electric guitar, and bass. As in *Kientzy Loops*, the eight-note loop is palindromic, so that it self-replicates at both 3 : 1 and 5 : 1. The melody is rather different here than in the saxophone piece though, and as one can see in this excerpt, one can set all three loops in motion from one starting point.

Einstimmmiger Polyrhythmus (1992) is an earlier and longer piece written for Ugly Culture, and here one loop transforms into another as the piece goes on. This excerpt comes from the beginning of the piece, when a seven-note loop makes copies of itself at 2 : 1, beginning with the bass in half notes and continuing with the saxophone in quarter notes and the guitar in eighth notes, all in unison. Then, however, the guitar begins to play the fast line in triplets, thus producing a new loop, a metamorphosis of the original one, which one sees clearly in the dotted quarter note line of the bass.



I will not take time for more examples here, and I promised to leave the generalizations to others, so I will just end with an anecdote that seems relevant. I once asked John Cage what he really learned from Schoenberg, as it seemed to me that the story he often told about learning, or not learning, harmony from Schoenberg was not the important thing. He reflected a moment and then said something that surprised me and may surprise you: “Well, that was a long time ago, and I don’t remember much, but I do recall that he used to talk about how there should be some kind of relationship between the microcosm and the macrocosm.” Since then I have often thought about how the unpredictable leaps in Schoenberg’s 12-tone melodies are reflected in the unpredictable leaps one hears from section to section, and clearly the chance procedures of Cage take place both in the microcosm and in the macrocosm. The Sierpinski gasket, like the structures in my music that we have been examining here, are much more orderly, but is not the essential thing the same? Many different people, in many different ways, all seek the satisfaction of structures where the individual cell has a real connection with the organism as a whole, as is so often the case in nature.

The book Self-Similar Melodies*, and all of the scores are available from Editions 75, 75 rue de la Roquette, 75011 Paris, www.tom.johnson.org*
The CDs are available from Metamkine.com.