A Classification of Dodecaphonic Groups

Franck Jedrzejewski

CEA, Saclay (France)

IRCAM, 15 janvier 2010

Contents

Twelve-Tone Technique

Identify the 12 notes with $\mathbb{Z}_n$ i.e. 0=do, 1= do $\sharp$, etc.

- Choose a 12-tone row.
- Example : Jean Barraqué ...au-delà du hasard
- $P = (0, 8, 7, 1, 4, 2, 10, 3, 11, 5, 6, 9)$
- Apply the 4 basic following transformations
  - *Transpose* up or down : $T_n(x) = x + n \mod 12$
  - *Retrograde* = reverse in time $R(P) = 9{,}6{,}5{,}11$, etc.
  - *Inversion* = reverse in pitch $I_n(x) = -x + n \mod 12$
  - *Retrograde Inverse* = composition RI

  (P, I, R, RI) = Klein group = dihedral group D_2 with presentation

  $$D_2 = \left\langle a, b \mid a^2 = b^2 = (ab)^2 = 1 \right\rangle$$

  with $1 = P$, $a = I$, $b = R$
- Material for composition = at most $4 \times 12 = 48$ rows, called *derivative forms*

Enumeration of Tone Rows

Action of the dihedral group $\mathcal{D}_n$ on the permutation group $\mathfrak{S}_n$

Theorem

In the n-tone equal temperament ($n \geq 3$), under the equivalence of the derived forms (i.e. under the action of the dihedral group), there are

$$\begin{cases} \frac{1}{4}(n-1)! + 2^{(n-4)/2}\frac{1}{n}\left(\frac{n}{2}+1\right)! & \text{if } n \text{ is even} \\ \frac{1}{4}(n-1)! + 2^{(n-5)/2}\left(\frac{n-1}{2}\right)! & \text{if } n \text{ is odd} \end{cases}$$

tone rows of n pitch classes.
In particular, for $n = 12$, $|\mathcal{S}| = 9,985,920$ rows.

The problem of counting the tone rows of $\mathcal{S}$ is equivalent to count the number of orbits of the group $G = D_{12} \times \mathbb{Z}_2$ generated by the transformations T, I and R on the series P. D_{12} denotes the dihedral group (of order 24). Burnside's lemma says that the number of orbits is the average number of fixed points.

Tone Rows with Limited Derivative Forms

Remark

$12!/48 = 479,001,600/48 = 9,979,200$ is not the number of 12-tone rows because some tone rows have less than 48 derivative series.

Example 1

$A = (0,4,8,11,3,7,1,9,5,2,10,6)$ verifies
$T_6(A) = R(A) = (6,10,2,5,9,1,7,3,11,8,4,0)$
Thus $T_n(A) = RT_{n+6}(A)$ and $I_n(A) = RI_{n+6}(A)$.
The series A has only 24 derivative forms.

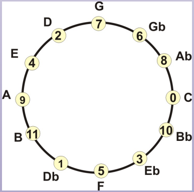
Example 2

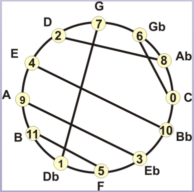
$A = (0,3,11,2,10,1,6,9,5,8,4,7)$ is equal to $RI_7(A)$.

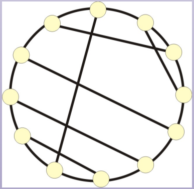
Chord Diagrams

How to build a Chord diagram ?

1. Take a row of twelve pitch classes : 0,8,6,7,2,4,9,11,1,5,3,10
2. Place the numbers on a circle
3. Join the tritone pairs together
4. Just keep the structure







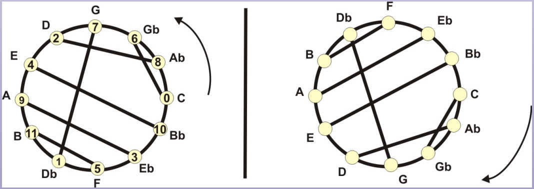
This structure is called a chord diagram.

Proposition

The 48 forms of the series are represented by the same chord diagram. i.e. Chord diagrams are invariant under the action of the dihedral group.

Example :

Retrogradation is obtained by mirror symmetry and rotation.



Theorem (A. Khruzin)

Under the action of the dihedral group, the number of chord diagrams for the equal temperament with 2n degrees is

$$d_n = \frac{1}{2}(c_n + \frac{1}{2}(\kappa_{n-1} + \kappa_n))$$

$$c_n = \frac{1}{2n}\sum_{i|2n}\varphi(i)\nu_n(i), \quad \kappa_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor}\frac{n!}{k!(n-2k)!}$$

$\varphi(i)$ *is the Euler totient function and* ν_n *is given for all divisors of* $2n$ *by*

$$\nu_n(i) = \begin{cases} i^{n/i}(2n/i-1)!! & \text{if } i \text{ is odd} \\ \sum_{k=0}^{\lfloor \frac{n}{i} \rfloor}\binom{2n/i}{2k} i^k (2k-1)!! & \text{if } i \text{ is even} \end{cases}$$

For the first $2n$-equal temperaments,

$2n$	c_n	d_n
6	5	5
8	18	17
10	105	79
12	**902**	**554**
14	9 749	5 283
16	127 072	65 346
18	1 915 951	966 156
20	32 743 182	16 411 700
22	625 002 933	312 702 217
24	13 176 573 910	6 589 356 711

Interest : about 10 millions rows are represented by only 554 diagrams.

Each chord diagram D_n is associated with a permutation P_n
Luigi Nono *Canto Sospeso* D_{358} is associated with (in cyclic notation)

$$(0,1)(2,11)(3,10)(4,9)(5,8)(6,7)$$

The generating power of this row is measured by the order of the *Serial group* defined by

$$G_n = \langle P_n, T, I \rangle$$

T = permutation associated with transposition

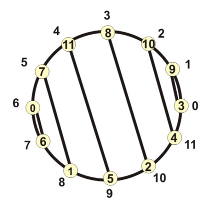
$$T = (0,1,2,3,4,5,6,7,8,9,10,11)$$

I = permutation associted with inversion

$$I = (1,11)(2,10)(3,9)(4,8)(5,7)$$

Serial Groups

Luigi Nono, Canto Sospeso, S=(9,10,8,11,7,0,6,1,5,2,4,3)



The permutation P_n indicates the places where we can exchange the tritones

$$P_n = (0,1)(2,11)(3,10)(4,9)(5,8)(6,7)$$

All combination of exchanges are obtained when we add T and I.

$$G_n = \langle P_n, T, I \rangle$$

Examples of Serial Groups

- A. Webern *Concert for nine instruments*, opus 24.
  $S = 11, 10, 2, 3, 7, 6, 8, 4, 5, 0, 1, 9$

  $$G_{549} = \left\langle \begin{array}{c} a, b, c \mid a^2 = c^2 = (b^{-1}c)^2 = (acb)^2 \\ = (ab^2c)^2 = (ab^{-1})^4 = (abab^{-1})^3 = 1 \end{array} \right\rangle$$

  This group has 192 elements.
- Schoenberg's *Ode to Napoleon,* Kees van Baaren's *Variations in Isometrical Series*

  $$5, 10, 6, 9, 7, 8, 2, 1, 3, 0, 4, 11$$

  is associated with the diagram D_{538}. The associated group

  $$G_{538} = \langle P_{538}, T, I \rangle$$

  has 7680 elements.

- Dallapiccola's *Quaderno Musicale di Annalibera*

$$10, 11, 3, 6, 8, 2, 7, 1, 5, 9, 0, 4$$

is associated with the diagram D_{56}. The group generated by the three operators

$$G_{56} = \langle P_{56}, T, I \rangle$$

has presentation

$$G_{56} = \left\langle \begin{array}{c} a, b, c \mid a^2 = c^2 = (b^{-1}c)^2 = (ac)^4 = acb^{-1}(abc)^2 \\ = (acb^3)^2 = (ab^2ab^{-2})^2 = b^{12} = (acab^{-2})^4 \end{array} \right\rangle$$

It is a group of order 28800.

- Many diagrams have a maximum generating power

$$|\mathfrak{S}_{12}| = 479\ 001\ 600$$

The 26 Serial Groups

Theorem

The 554 chord diagrams generate only 26 serial groups, which form a subset of the 301 transitive permutation groups. Amongst the 26 serial groups, only two are primitive : $PGL(2,11)$ $(r = 218)$ and $\mathfrak{S}_{12}$. 431 chord diagrams have a trivial serial group ($\mathfrak{S}_{12}$).

Computations with GAP (http ://www.gap-system.org/).
A system for Computational Discrete Algebra.

$r =$ index of the group in the table of the 301 transitive groups

$|G| =$ Order of the group (denote now $G_{|G|}$).

Index of the chord diagrams in the table (Mathematical Theory of Music)

Blocks = Block systems Type : B1, ..., B9.

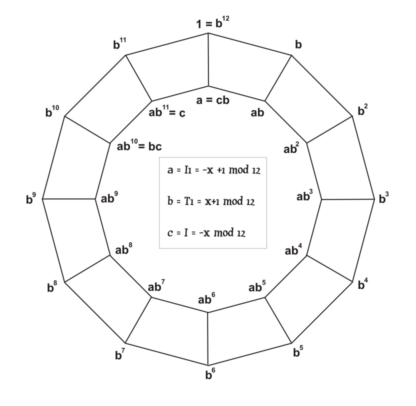
The 26 Serial Groups

r	$\lvert G \rvert$	Chord diagrams	Blocks
12	24	358, 554	B1
28	48	444,509	B1
54	96	186	B2
81	144	1, 381, 553	B3
86	192	491, 549	B3
118	216	50, 359, 474	B4
125	288	487	B3
151	384a	414	B6
152	384b	383, 497, 507, 508	B6
154	384c	136, 536	B2
156	432	248	B4
185	768a	157, 552	B6
186	768b	348, 530	B6
193	768c	490	B2
217	1296	247, 516	B4

The 26 Serial Groups

r	$\lvert G \rvert$	Chord diagrams	Blocks
218	1320	103, 161, 184, 241,395,417, 448, 510	∅
240	2304	43, 351, 440, 537	B5
248	2592	364	B4
260	4608	544	B5
267	5184	11, 282, 303	B8
270	7680	42,61,111,347,355,382,412,441,504,538,551	B9
288	28800	15 knots :24,51,56,150,163,183,252 332,371,376,419,420,439,481,514	B7
293	46080	109,185,346,350,357,488,531,535,546,550	B9
294	82944	170,174,285,289,313,321	B8
299	1036800	31 knots : 3,6,7,26,31,34,36,60,137,142,153 155,168,181,188,190,207,209,281,290,295 305,334,352,362,418,434,482,494,511,532.	B7
301	479001600	The remainder : 431 chord diagrams	∅

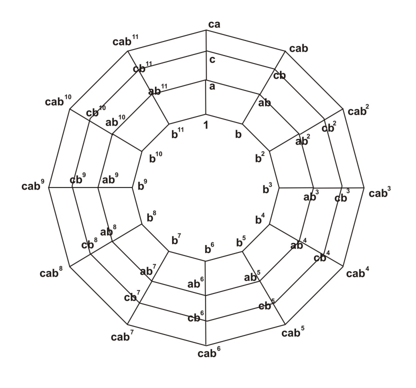
$G_{24} = \langle a, b, c \rangle$ with : a=(0,1)(2,11)(3,10)(4,9)(5,8)(6,7)
b=(0,1,2,3,4,5,6,7,8,9,10,11), c=(1,11)(2,10)(3,9)(4,8)(5,7)



Action of G_{24} : Let S be a 12-tone row of chord diagram D (358 or 554), then $\forall g \in G_{24}, g(S) \in D$, (i.e. always have with the same diagram.)

Example : The Serial Group G_{48} of order 48

$G_{48} = \langle a, b, c \rangle$ with : a=(0,2)(1,7)(3,5)(4,10)(6,8)(9,11)
b=(0,1,2,3,4,5,6,7,8,9,10,11), c=(1,11)(2,10)(3,9)(4,8)(5,7)



Action of G_{48} : Let S be a 12-tone row of chord diagram D (444 or 509), then $\forall g \in G_{48}, g(S) \in D$, (i.e. always have with the same diagram.)

Properties of the Serial Groups

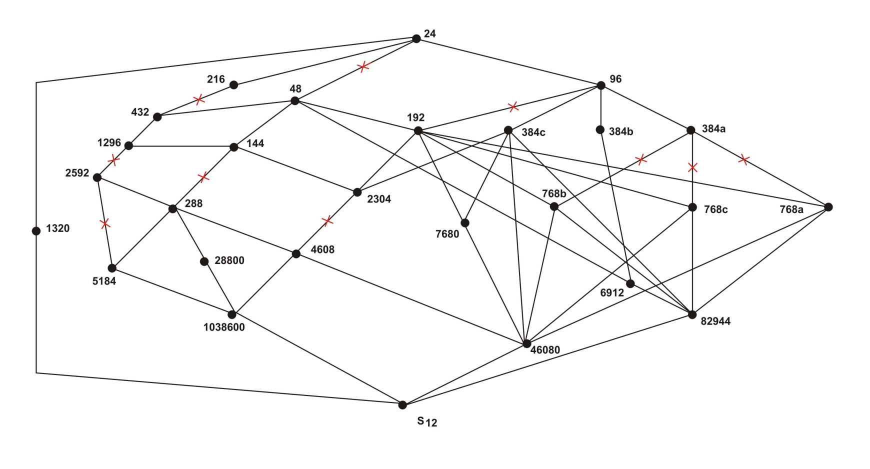
Theorem

Let S be a 12-tone row with chord diagram D, and G the serial group of D. The action of G on S is globally stable in the following sense : for all $g \in G$, the 12-tone row $g(S)$ has a chord diagram associated with G or with one of its subgroups.

Example 1 : The chord diagram of S={0,6,1,7,2,4,11,3,10,8,9,5} is D_{50} of group G_{216}. For all $g \in G_{216}$ $g(S)$ has chord 50 or 358. D_{358} belongs to G_{24} a subgroup of G_{216}.

Example 2 : The chord diagram of S={0,8,6,7,3,11,9,2,4,1,10,5} is D_{474} and its group G_{216}. For all $g \in G_{216}$ $g(S)$ has chord 359 or 474 (The chord diagram 50 is not reached).

Lattice of the 26 Serial Groups



Groups are represented by their order.

Red cross = normal subgroup, $\forall g \in G, gH = Hg$.

Stable Actions

Theorem

Only 9 chord diagrams have a stable action i.e. such that if S is a 12-tone row of chord diagram D with serial group G (depending on D), then for all $g \in G$, $g(S)$ has the same chord diagram D,

$$\forall g \in G,\ g(S) \in D$$

$\|G\|$	CD
24	358, 554
48	444, 509
192	491
288	487
384b	497, 508
768b	530

The 12-tone rows created by the action of G on S have always the same chord diagram.

A permutation group G acting on a set X is **primitive** if G preserves no nontrivial partition of X, otherwise, G is imprimitive. For $X = \{0, 1, ..., 11\}$, there are only 6 primitive groups (R. Carmichael, 1937).

Name	$\|G\|$	GAP index
L(2,11)	660	179
PGL(2,11)	1320	218
M11	7920	272
M12	950540	295
A12	239500800	300
S12	479001600	301

There are only 2 primitive serial groups (G_{1320} and S_{12}).

Block Systems

A subset B of $X = \{0, 1, ..., 11\}$ is a **block of imprimitivity** if

$$\forall g \in G, B \cap g(B) = B \text{ or } \varnothing$$

The trivial blocks $\varnothing, \{x\}$ and X are always blocks of imprimitivity.

If the only blocks of imprimitivity are the trivial ones then the action of G on X is imprimitive.

A partition of X composed of blocks of imprimitivity is a system of imprimitivity.

Symbolic names of the Block Systems for Serial Groups (Table)

B1 = [1, 3, 5, 7, 9, 11], [1, 4, 7, 10], [1, 5, 9], [1, 7]
B2 = [1, 3, 5, 7, 9, 11], [1, 4, 7, 10], [1, 7]
B3 = [1, 3, 5, 7, 9, 11], [1, 5, 9], [1, 7]
B4 = [1, 3, 5, 7, 9, 11], [1, 5, 9]
B5 = [1, 3, 5, 7, 9, 11], [1, 7]
B6 = [1, 4, 7, 10], [1, 7]
B7 = [1, 3, 5, 7, 9, 11]
B8 = [1, 4, 7, 10]
B9 = [1, 7]

Example : Imprimitivity of G_{24}

G_{24} is generated by the permutations
a=(0,1)(2,11)(3,10)(4,9)(5,8)(6,7) b=(0,1,2,3,4,5,6,7,8,9,10,11),
c=(1,11)(2,10)(3,9)(4,8)(5,7)

It has four block systems

$$b1 = \{\{0,2,4,6,8,10\},\{1,3,5,7,9,11\}\}$$
$$b2 = \{\{0,3,6,9\},\{1,4,7,10\},\{2,5,8,11\}\}$$
$$b3 = \{\{0,4,8\},\{1,5,9\},\{2,6,10\},\{3,7,11\}\}$$
$$b4 = \{\{0,6\},\{1,7\},\{2,8\},\{3,9\},\{4,10\},\{5,11\}\}$$

The sub-blocks are limited transposition sets.

Compositional motivation : We move from one block to the other or remain in the same block by the action of G.