EDGE COLORED GRAPHS

Yannis Manoussakis
University Paris XI, ORSAY
France

Joint work

A.Abouelaoualim , R. Agueda-Maté, J. M. Becu, V. Borozan, K. Ch. Das , M. Dah, L. Faria, L. Fernadez de la Vega, M.Karpinski, C. Martinhon , G. Mendy, R. Saad

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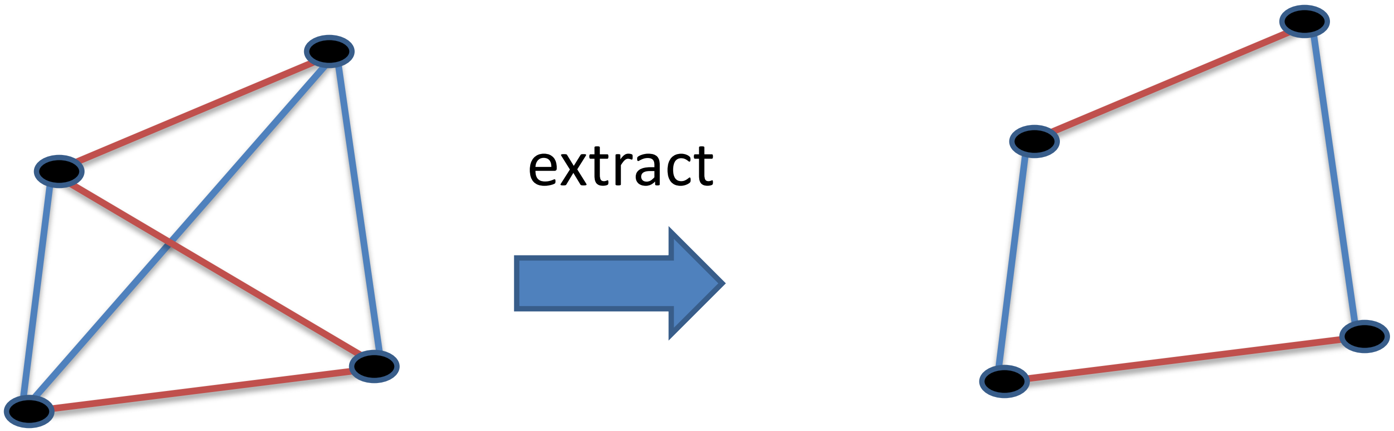
Main Question

Input: Given edge colored graphs
Output: Find sub-graphs (if any) with specified color patterns

Example 1: Proper Cycles

- **Given:** An edge colored graph
- **Find:** Properly edge-colored cycles



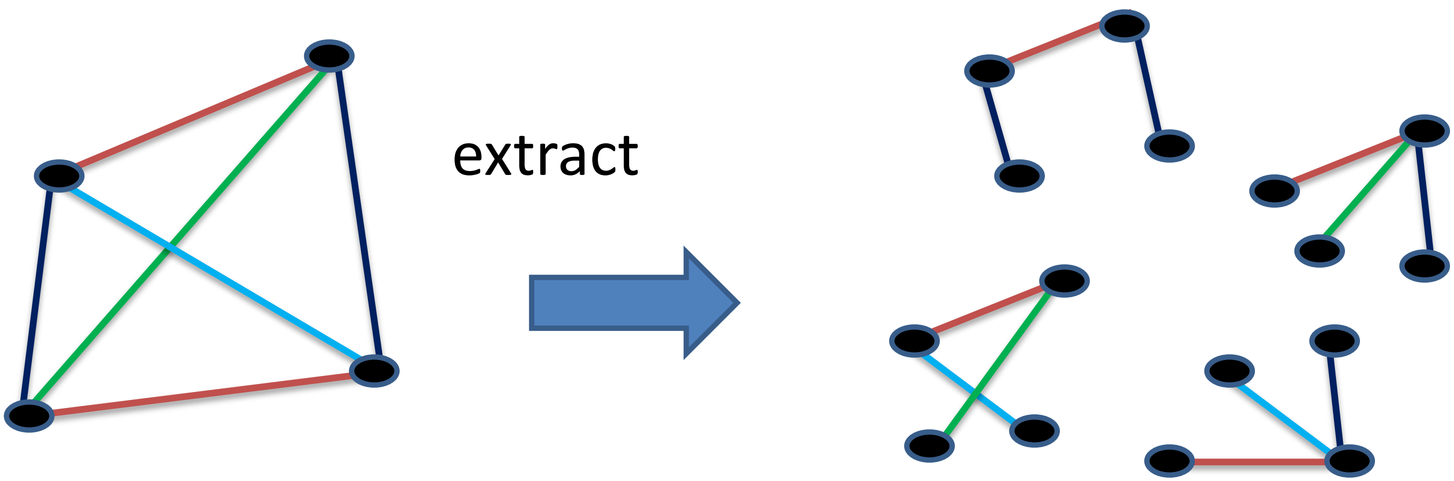
A 2-edge-colored graph

A "proper" cycle

Examples -Continued

Example 2: Proper Trees

- **Given:** An edge colored graph
- **Find:** Properly edge-colored trees



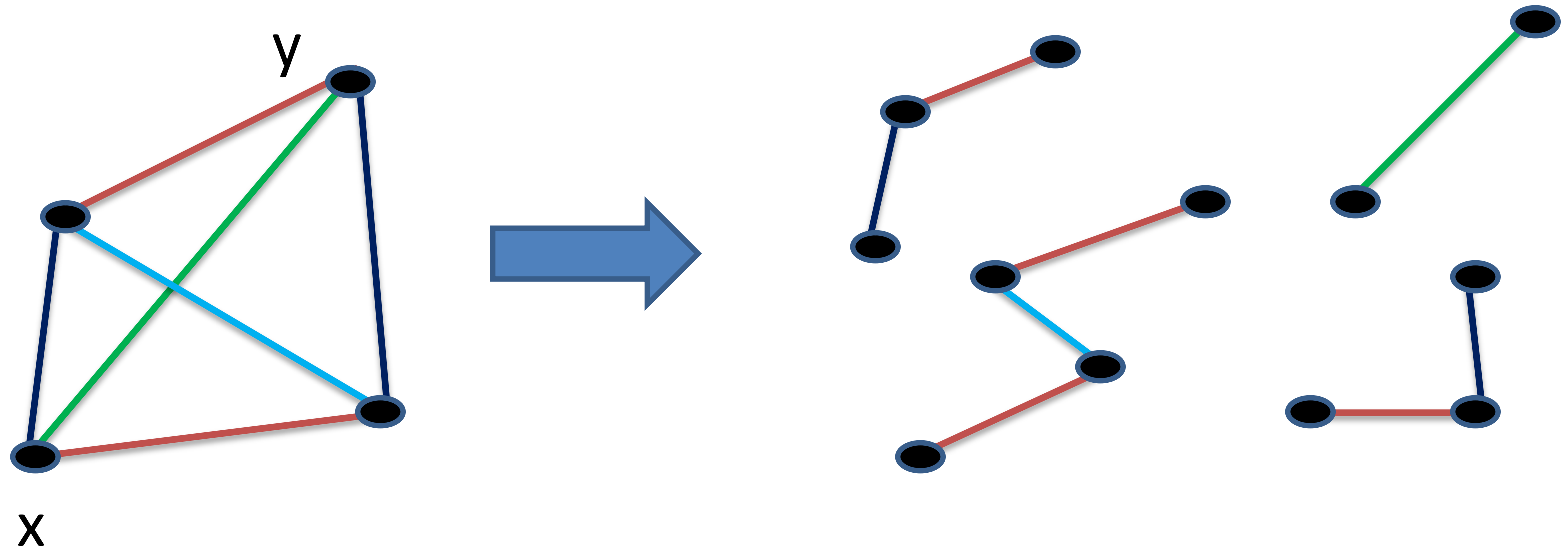
A 4-edge-colored graph

“Proper” spanning trees

Examples -Continued

Example 3: Proper paths

- **Given:** An edge colored graph
- **Find:** Proper paths between pairs of vertices



A 4-edge-colored graph, vertices x and y

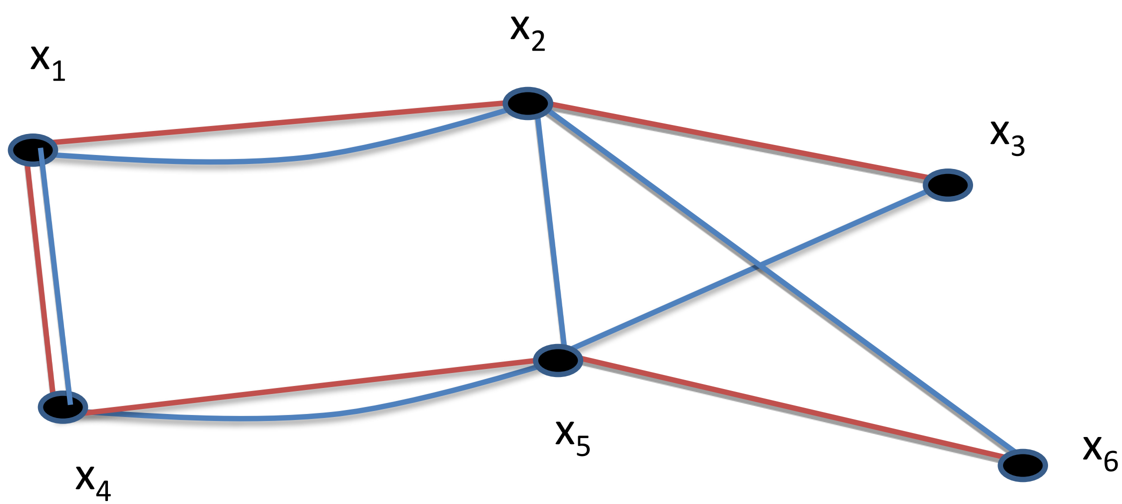
Possible “proper” paths between x and y

Definitions

- Set of given colors, C ={1,2,…, c}.
- G^c (V,E) : Edge-colored graph (simple or multigraph)
- **n**:number of vertices **m**:number of edges
- **xy** denotes an edge, **c(xy)** the color of the edge xy
- The *colored i-degree of x*, denoted by $d^i(x)$ corresponds to the number of vertices of G^c joined to x by edges on color i.
- A subgraph of G^c is said to be *properly edge colored* (shortly PEC), if any two adjacent edges in this subgraph differ in color.
- Path (trail): A set of consecutive edges without vertex repetitions (edge repetitions)
- Factor: Pairwise vertex-disjoint PEC cycles covering the graph
- Color-connectivity: Any two vertices are joined by two PEC paths xx'…yy' and xu…vy such that $c(xx') \neq c(xu)$ and $c(yy') \neq c(vy)$

Examples (for the definitons)

2-edge colored multigraph G^c:

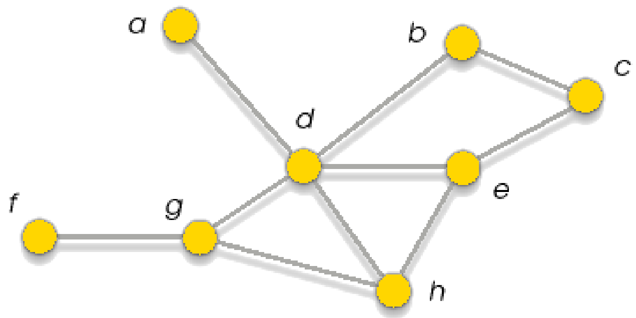


- **Red** degree of x_2 : $d^r(x_2)=2$, **Blue** degree of x_2 : $d^b(x_2)=3$
- **PEC cycle** : $x_1 x_2 x_5 x_4$ **PEC Factor:** $x_1 x_4 x_1$ and $x_2 x_3 x_5 x_6 x_2$
- x3 - x6 **are** color-connected **Trail:** and $x_3 x_5 x_4 x_5 x_6$ **Path:** and $x_3 x_5 x_6$

Connectivity

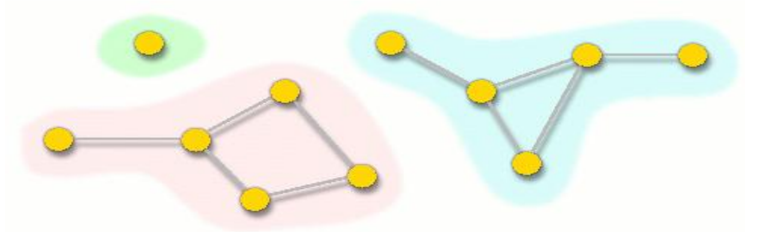
Recall: A graph is connected if there exists a path between any pair of vertices

Example: A Connected graph

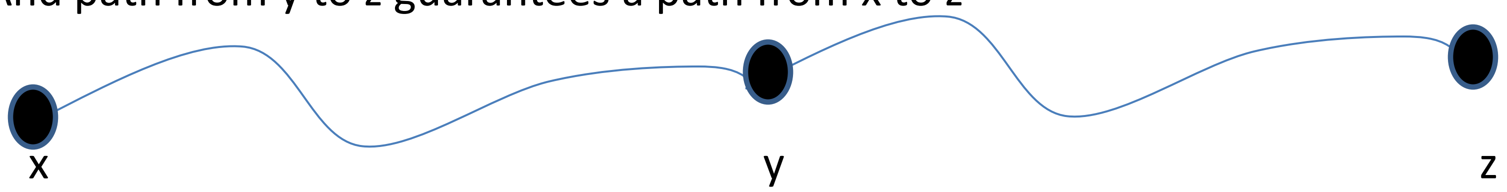


What happens when G is not connected? The graph is a collection of connected graphs, called the components

Example: A graph with 3 components

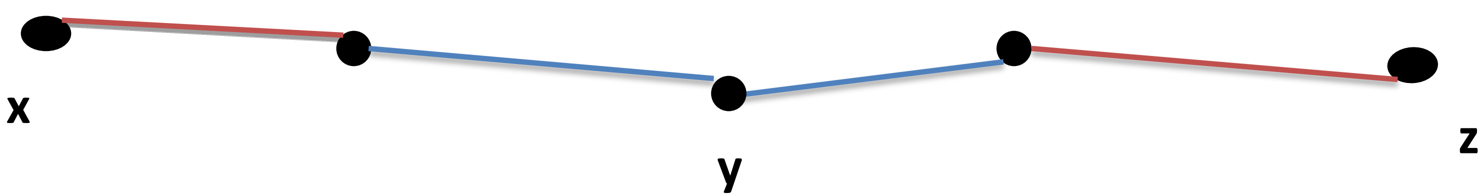


Fundamental observation: Connectivity has the **transitivity property**: path from x to y And path from y to z guarantees a path from x to z



Connectivity of edge colored graphs

Fundamental observation: *Transitivity is not valid for edge-colored graphs.* ***Thus connectivity is not an equivalence relation!***



- a PEC path x-y
- a PEC path y-z
- But not a PEC path x-z

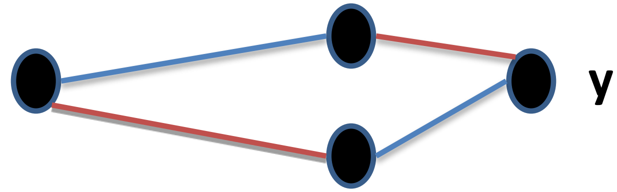
Question: Which **kind of connectivity** we can define here ?

Connectivity issues

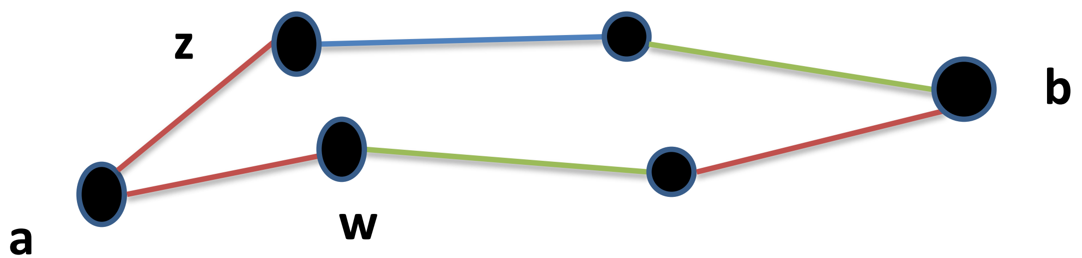
A/ Color-connectivity R. Saad (1991):

Any two vertices are joined by two PEC paths xx'…y'y and xu…vy such that c(xx')≠c(xu) and c(y'y) ≠ c(vy)

x, y are color-connected:

x  y

a, b are not color-connected: **edges az** and **aw** are on a **same** color



Nice fact: Color connectivity my be checked in polynomial time.

Bad Fact: Color connectivity is not always an equivalence relation

Nice Fact: Color connectivity is an equivalence relation for complete graphs

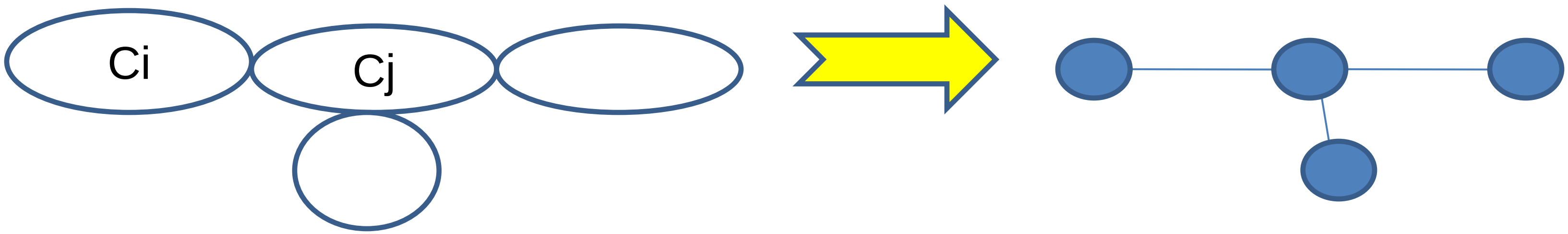
Connectivity issues/general graphs

B/ Cyclic connectivity (Bang-Jensen, Gutin, 1995)

The intersection graph Ω(P):

Set of cycles $P=\{C_1,C_2,\ldots,C_p\}$

Graph Ω(P) has vertex-set *P and* edge-set $\{C_iC_j \mid V(C_i) \quad V(C_j) \cap\}$



Vertices x, y are cyclic connected iff Ω(P) is connected and both x, y belong to same set P.

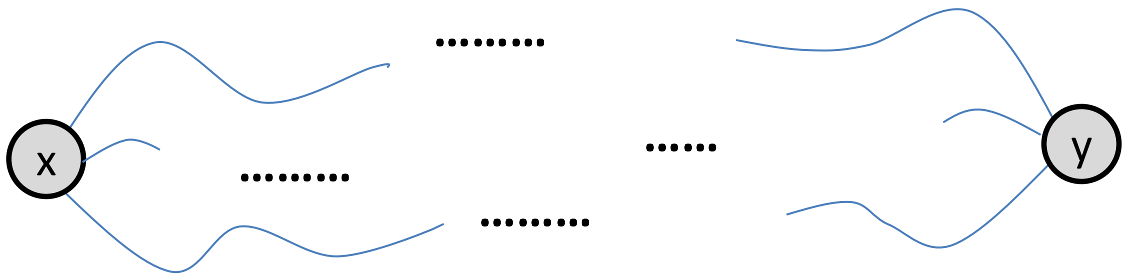
Nice facts:

1. ***Cyclic connectivity is an <u>equivalence</u> relation***
2. ***If x, y are cyclic-connected, then they are color –connected: (Bang-Gutin → Saad)***

Connectivity issues/general graphs

An easiest question:

Problem: *Given two vertices x,y in G^c, can we decide/find efficiently k vertex-disjoint (edge -disjoint) PEC x-y paths (trails), for a given integer k≥ 1?*



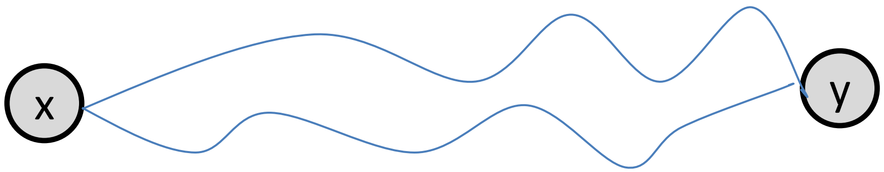
Answer

Yes, in polynomial time **only** for **k=1**

- Path version: by Edmonds, (private communication) using matching techniques
- Trail version (no edge repetition): Manoussakis *et al* (2007) using the path algorithm of Edmonds

Connectivity issues/general graphs

k=2 (Two paths between x and y) : NP-hard**.**



Redaction: From the 2-cyclic problem in directed graphs (Fortune, Hopcroft, Wyllie, 1980)

Thus, surprising result…

Theorem: (Abouelaoualim, Das, Faria, Manoussakis, Martinhon, Saad, 2007**)** *Given two vertices x,y in G^c, decide if there exist 2 pairwise vertex disjoint PEC paths (trails) in G^c is NP-Hard.*

NB: Remains NP-hard even for 2 colors or for $O(n^2)$ colors

Connectivity issues/complete graphs

What about paths in edge-colored **complete** graphs K^c_n?

Theorem (Manoussakis, 1994) *Given two vertices x, y in K^c_n, there exists a $O(n^{2.5})$ algorithm for finding the maximum number of pairwise vertex-disjoint x-y paths in K^c_n.*

Idea: Given x, y, observe that we may find many x-y paths of length **at most 3 in K^c_n.** Then we use matching techniques.

Problem (Manoussakis, 1994) *Given two vertices x, y in K^c_n, does there exist an efficient algorithm for finding the maximum number of edge-disjoint x-y trails in K^c_n?*

Observation: *Probably this problem is NP-complete. But then , if yes, why the complexity of edge-version is much higher than the vertex-version ?*

Connectivity issues/open problems

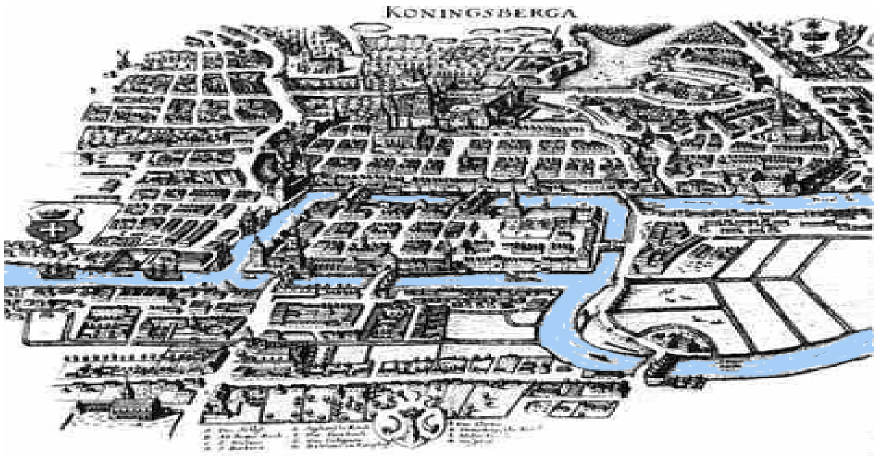
Problem 1. Find an "universal" definition for color-connectivity, preserving transitivity.

Problem 2. *Approximate the k-path problem for edge-colored graphs.*

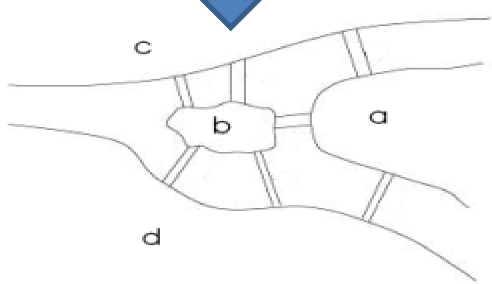
Problem 3. *Approximate the k-trail problem for edge-colored graphs.*

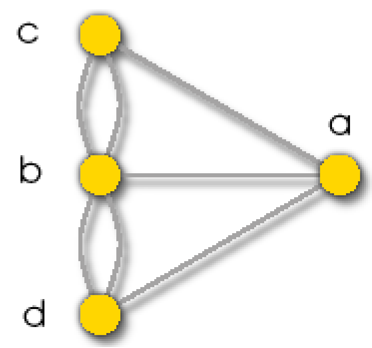
Problem 4 (Manoussakis, 1994) *Given two vertices x, y in* K^c_n*, does there exist an efficient algorithm for finding the maximum number of edge-disjoint x-y trails in* K^c_n*?*

The Euler walks



The question of Euler:
Walk arround the river, going once per bridge and then come back to the initial point.



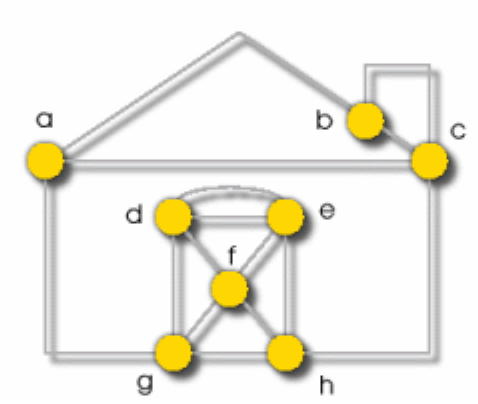


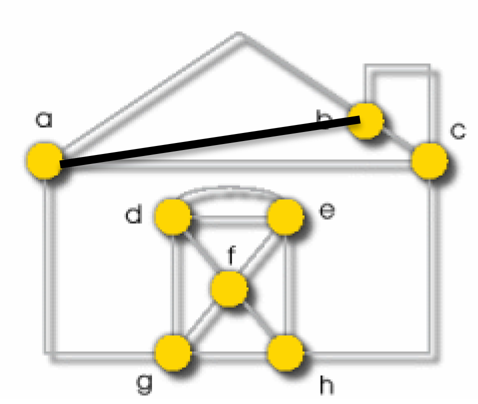
The Euler tour: A cycle passing once per each edge and coming back to the initial point.

Answer to the question of Euler: NO
Reason: Some vertices have odd degrees

The Euler Tour

Theorem. Any graph has an Euler cycle if and only if it is connected and the degree of each vertex is even.





NO

- a and b have degree 3

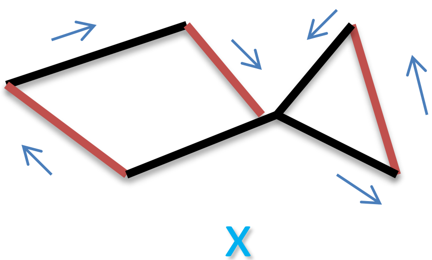
YES

- all vertices have even degree

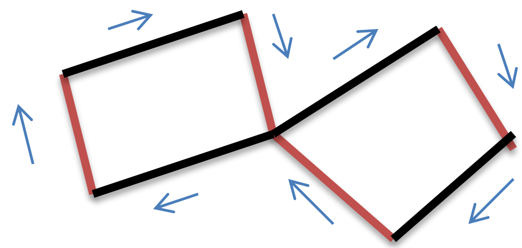
Complexity: Polynomial time for finding such an Eulerian cycle

The Euler Tour/colored graphs

Theorem (2 colors): A 2-edge colored graph has an alternating Euler cycle if and only if the red degree=blue degree for every vertex.



NO



YES

Vertex x has <u>no equal</u> blue-red degrees

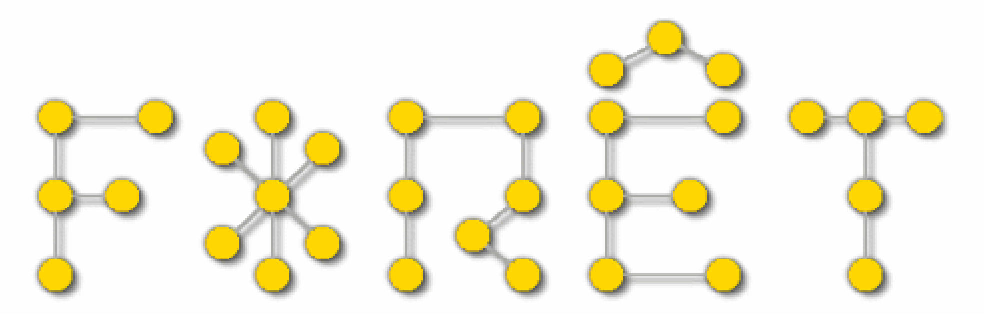
Complexity: Polynomial time to find such an alternating Eulerian cycle

Open problem: Count the number of distinct Euler cycles (applications To DNA structures)

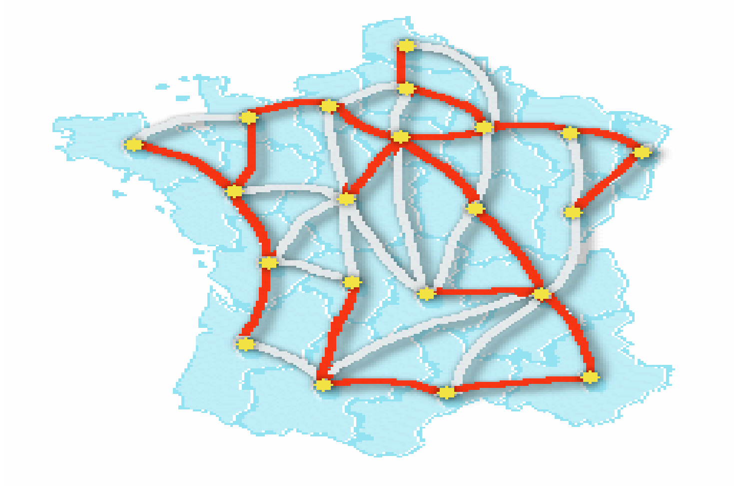
Trees

Tree(definition) : Graph without cycles

Forest: A non connected graph where every component is a tree



Spanning tree: A tree “spanning” all the vertices of the graph

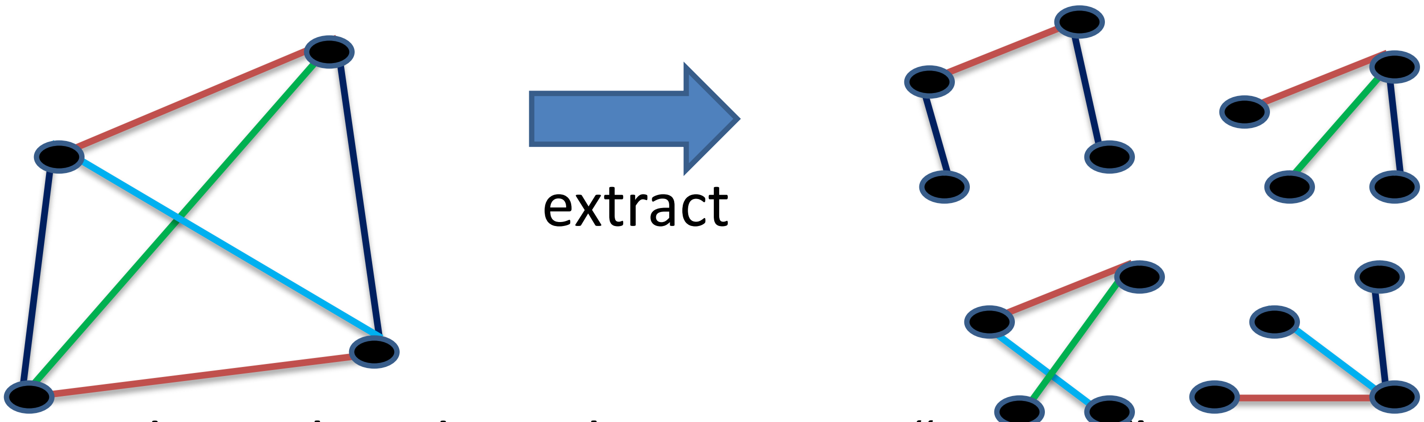


Complexity: Find a spanning tree is polynomial (linear) on the number of edges

Proper trees

Question

- **Given:** An edge colored graph
- **Find:** Properly edge-colored trees



A 4-edge-colored graph

"Proper" spanning trees

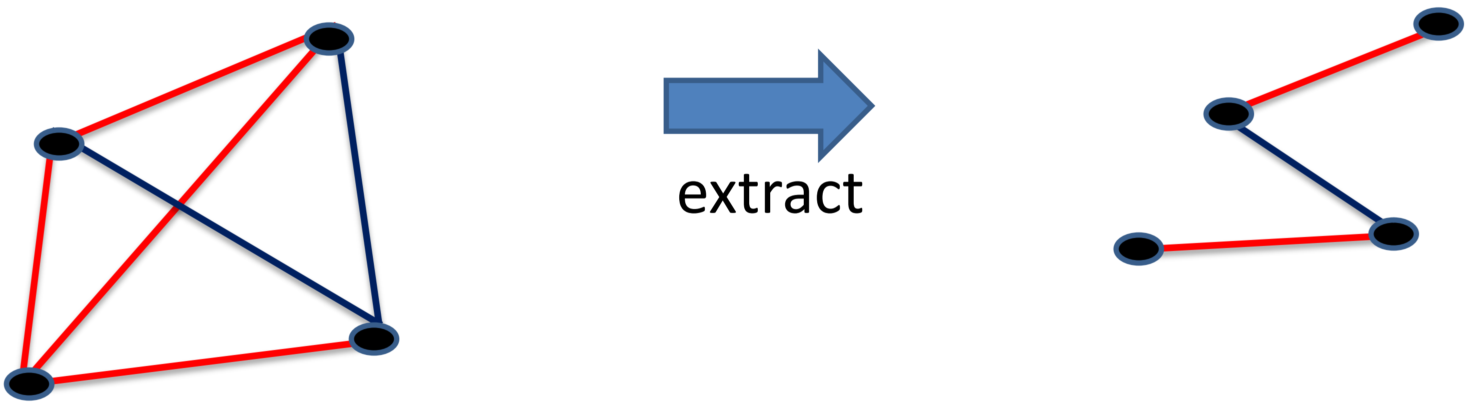
Theorem 1. The spanning proper tree (SPT) problem is NP-hard even for c = (n^2).

Theorem 2. The SPT problem is non-approximable within 55/56 + ε, for $\varepsilon > 0$, unless P = NP.

Proper trees/special cases

Question

- **Given:** An edge colored graph without proper cycles
- **Find:** Spanning proper trees



A 2-edge-colored graph <u>without</u> proper cycles

"Proper" spanning tree

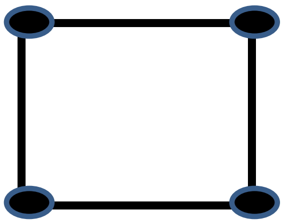
Theorem . The spanning proper tree (SPT) problem is <u>polynomial</u> for acyclic graphs.

Proof: Using matching technics

Trees/Count the spanning trees

Question: How many distinct spanning trees has a connected (non colored) Graph?

Example: Distinct trees of this graph?



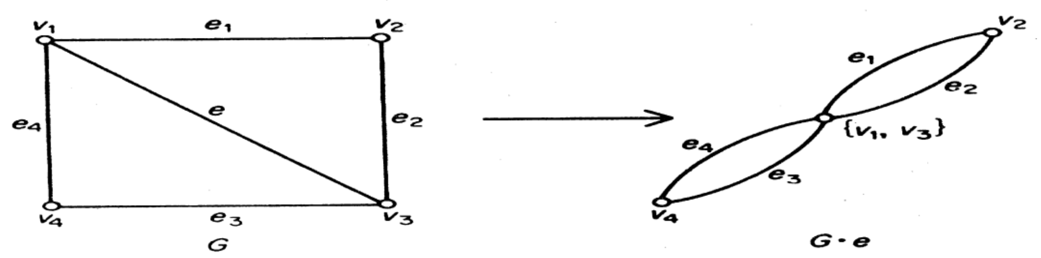
Answer: 4

Trees

Recursive formula of Galey

<u>Concatenation of an edge xy:</u> Replace the extremities x, y of the edge by a new vertex z. The Neighbors of x and y become the neighbors of z.

Example: Concatenation of V1 and V3



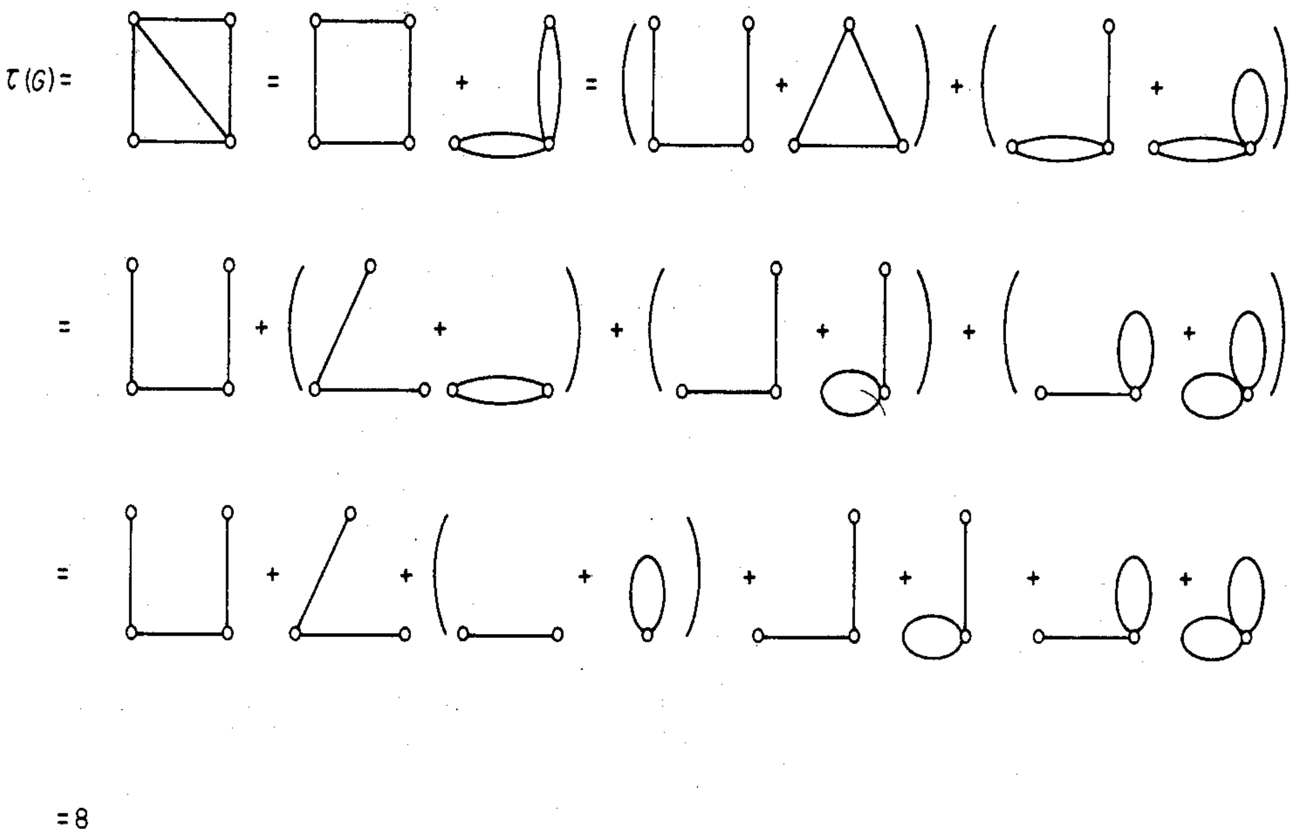
G → G.e

Theorem (Cayley). $\tau(G) = \tau(G-e) + \tau(G.e)$

Where **$\tau(G)$** is the number of spanning trees

Trees
Recursive formula of Galey

Example:

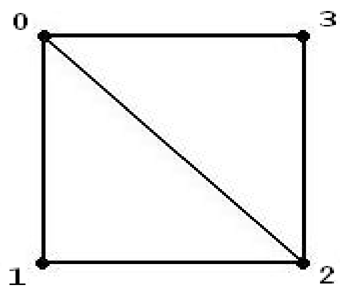


Count trees through the determinant of matrices

Construct a matrix as follows:

- **-If i # j and i is a neighbor of j then set qij=-1, else set qij=0**
- **-f i=j then set qii= degree of vertex i**

Example.



$$Q = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}$$

- **Delete one line & one column from Q**

$$Q^* = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

Det (Q*)=8. Thus this graph has 8 spanning trees.

*****Not known how to do with colored trees****

Proper trees /Open questions

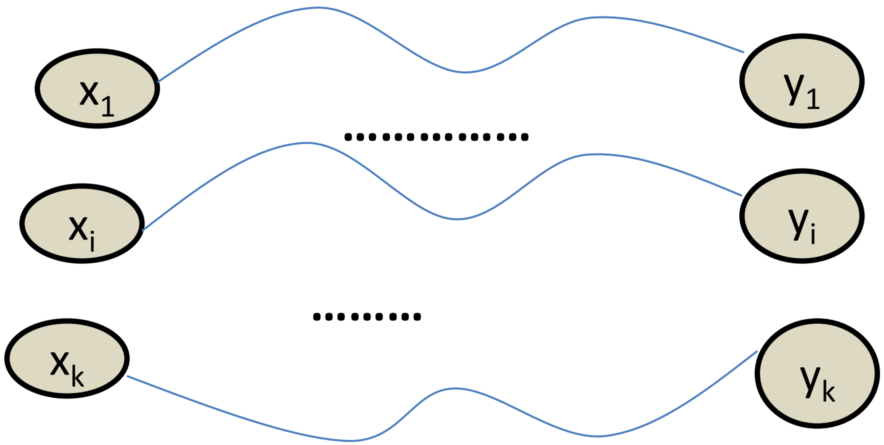
Problem 1. Approximate the maximum (spanning) proper tree problem

Problem 2. Find an efficient method to count the number of distinct proper spanning trees. Or else prove that it is “hard” to count them.

Problem 3 (related to Problem 2). Establish lower bounds on the number of distinct proper spanning trees.

Links in edge-colored graphs

The k-linked problem: *Given k pairs of vertices x_1, y_1, …., x_k, y_k, find k pairwise vertex -disjoint paths, one connecting x_i to y_i , for each i=1, 2, …, k*



Complexity for general non-colored graphs

Non directed graphs: NP-complete for k non fixed (Knuth 1974), but polynomial for k fixed (Robertson, Seymour, 1995)
Directed graphs: NP-complete even **for k=2** (Fortune, Hopcroft, Wyllie, 1980)

What about edge-colored graphs?

Links in edge-colored graphs

Theorem (Manoussakis *et al*, 2007). *The color version of the k-linked problem is NP-complete for c-edge colored graphs. And this happens for* ***any fixed k ≥3*** *and* ***any number of colors c≥2****.*

Known also that it is polynomial for k=1, then some questions arise:

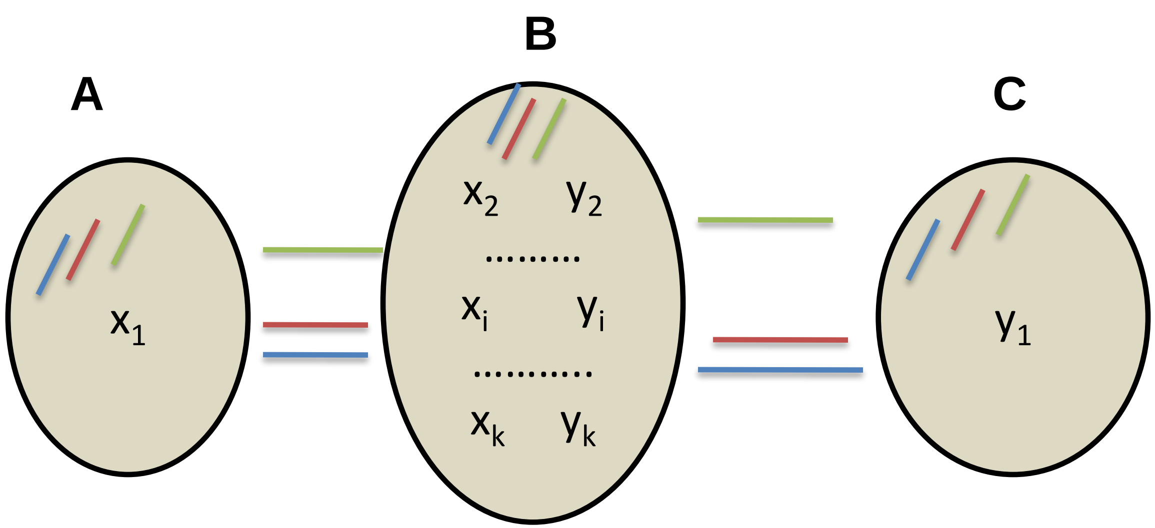
1. *What about k=2 (polynomial or NP-complete)?*
2. *Are there classes of graphs (planar, perfect, etc) for which the problem is polynomial for any k?*
3. *Can we approximate this problem?*
4. *Can we define sufficient conditions on color-degrees or other parameters of the graph guarantying this property?*

Links in edge-colored graphs : Colored-degree conditions

The path version….:

Theorem (Becu,Manoussakis *et al*, 2007)
Let G^c be a c-edge colored multigraph of order $n \geq 242k$, $k \geq 1$. If for every vertex x, $d^c(x) \geq (n/2) + k - 1$, then G^c is k-linked.

Extremal graph:



A=C= (n-2k+2)/2, **B=2k-2** Colored degree= ***[n/2] +k-2***

This graph is **not k-linked,** since any PEC path from x_1 to y_1 goes through the set B

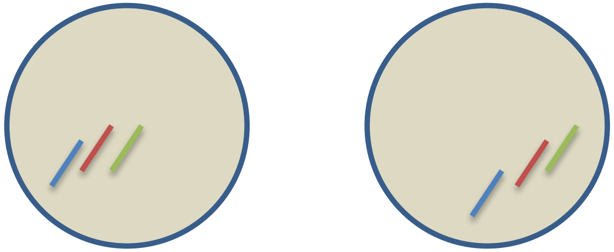
Links in edge-colored graphs : Colored-degree conditions

The trail version…:

Theorem (Becu,Manoussakis *et al*, 2007)

*Let G^c be a c-edge colored multigraph of order n. If for every vertex x, **$d^c(x) \geq n/2$**, then G^c is k-edge-linked.*

Extremal graph:



A, B: Complete multigraphs

A=n/2=B, no edges between A and B

No path **at all** from A to B

Links in edge-colored graphs : Mixed conditions

Conjecture 1. A multigraph G^c on n vertices, k, r integers, $r \geq 2k$, colored degree $\geq r$.

i) if c=2, $n \geq 6r+2k+2$, and $m \geq n^2-n-(r-2k+3)(2n-3r+2k-4)+1$,

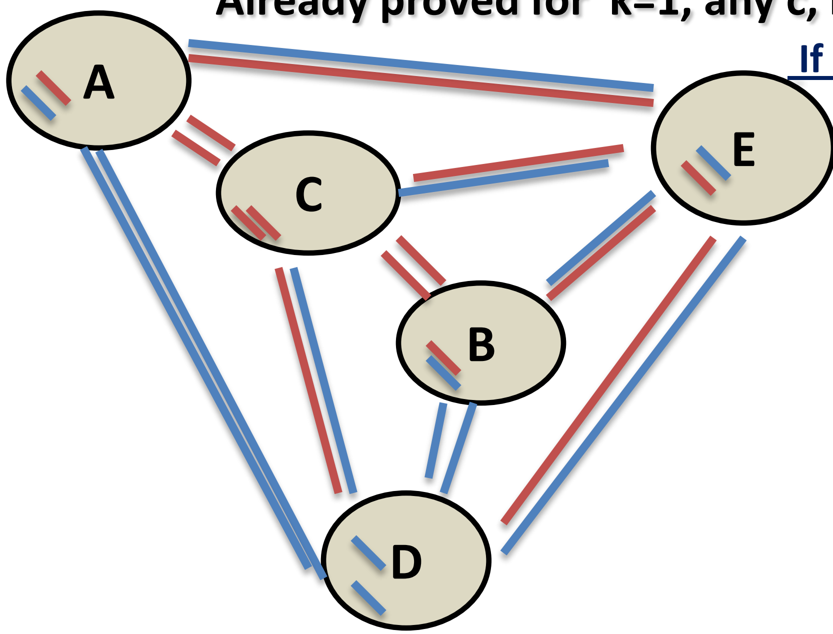
ii) if c=2, $n \leq 6r+2k+2$, and $m \geq [3n^2]/4+n[k-(5/2)]-k(k-3)-1$,

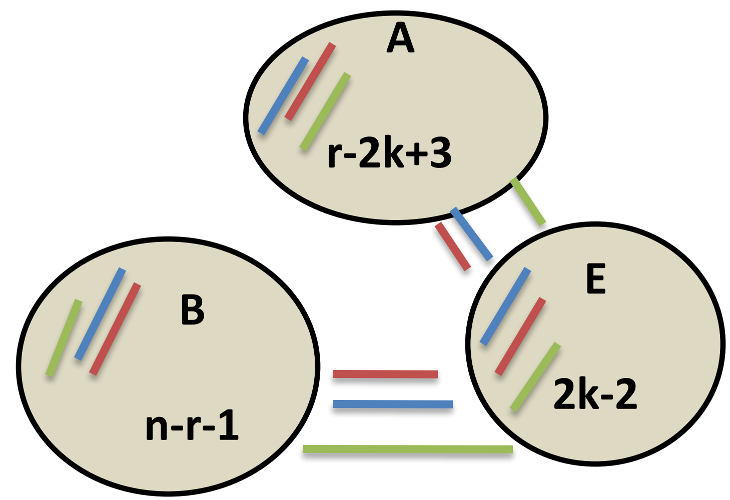
iii) If $c \geq 3$ and $m \geq c[(n^2-n)/2-(r-2k+3)(n-r-1)]+1$,

then G^c is k-linked.

Already proved for k=1, any c, r

If true, best possible:





c=2

c≥3

i) A=1,B=n+2k-2r-3, C=r-2k+2=D, E=2k-2

ii)A=1=B, C=(n/2)-k=D, E=2k-2

iii) any path between A and B goes through E

Links in edge-colored graphs : Mixed conditions

Conjecture 2. An edge-colored multigraph G^c on n vertices, integers $k \geq 0$ *and* $0 \leq r \leq (n/2)-1$, colored degree $\geq r$.

i) If $c=2$, $n \geq 6r+4$ and $m \geq n^2-n(2r+3)+3r^2+5r+3$,

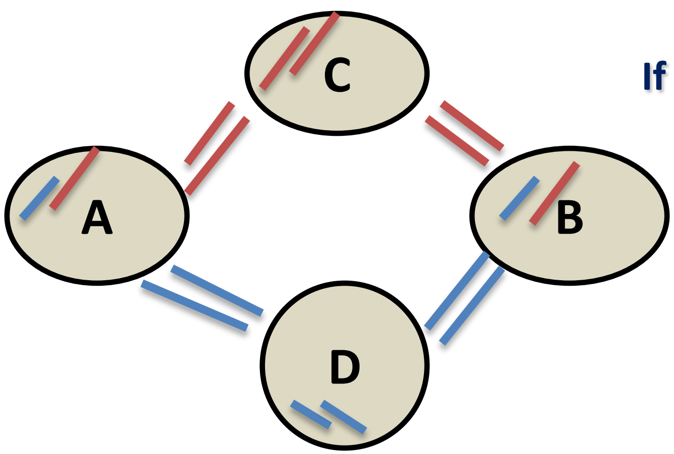
ii) If $c=2$, $n \leq 6r+4$, and $m \geq (3n^2)/4 - 3n/2+1$,

iii) If $c \geq 3$ and $m \geq c[\,(n^2-n(2r+3)+2r(r+2)+2)/2\,] + \min(k, r+1)$,

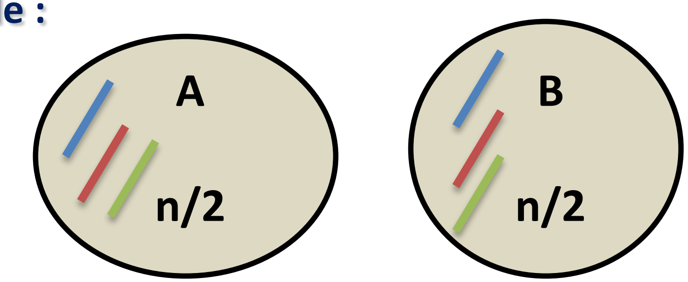
then G^c is k-edge-linked.

Already proved for r=1, any k , c or k=1, any r, c

If true, best possible :



c=2



c≥3

i) A=1, B=n-2r-1 C=r=D,

ii) A=1=B, C= (n/2)-1 =D

iii) (no path between A and B)

PEC cycles/ Two colors

K_n^2 : *a 2-edge colored* ***complete simple*** *graph*
- $V(K_n^2)=\{v_1,v_2,\dots,v_{2p}\}$
- $d^r(v_1) \leq d^r(v_2) \leq \dots \leq d^r(v_{2p})$

Theorem (M. Bankfalvi and Z. Bankfalvi , 1968)

K_n^2 *contains a PEC hamiltonian cycle* <u>*if and only if*</u> *:*

- *It contains a PEC factor*
- *For every* $1<k<n$,
  $d^r(v_1)+d^r(v_2)+\dots+d^r(v_k)+d^b(v_{2p})+d^b(v_{2p-1})+\dots+d^b(v_{2p-k+1})>k^2$.

More intuitive….

Theorem (R. Saad, 1994)

K_n^2 *contains a PEC hamiltonian cycle if and only if :*

- *It contains a PEC factor*
- *It is color-connected.*

Complexity

Theorem (Benkouar, Manoussakis, Paschos, Saad, 1991)
There exists an $O(n^{2.5})$ algorithm for finding a PEC Hamiltonian cycle (path) in K_n^2

Theorem (Saad, 1992)
There exists a $O(n^{2.5})$ algorithm for finding a PEC longest cycle (path) in K_n^2

Complete graphs/PEC cycles/ More than two colors

***$\Delta(K_n^c)$* :** *maximum monochromatic degree of K_n^c*

Conjecture (Bollobas and Erdös , 1976)
*Every K_n^c with **$\Delta(K_n^c) \leq (n/2)-1$** has a PEC Hamiltonian cycle*

Extremal graph : *Consider , for instance, a 2-edge colored complete graph on 2k+1 vertices with regular red/blue degrees k.*

Theorem (Chen and Daykin, 1976)
*Every K_n^c with **$\Delta(K_n^c) \leq n/17$** has a PEC Hamiltonian cycle*

Theorem (Shearer, 1979)
*Every K_n^c with **$\Delta(K_n^c) \leq n/7$** has a PEC Hamiltonian cycle*

Theorem (Alon, Gutin 1997)
*For every $\varepsilon>0$ and n sufficiently large , if **$\Delta(K_n^c) \leq (0.2928\ldots-\varepsilon)n \cong n/3$** , then K_n^c has a PEC Hamiltonian cycle*

General graphs/PEC cycles/More than two colors

Complexity of edge-colored **complete** graphs
on **any number** of colors

- **PEC Hamiltonian path**
  Feng *et al* (2006): Polynomial (through factor techniques)

- **PEC Hamiltonian path with fixed extremities**
  Open problem, even for two colors
  (Manoussakis *et al*, 2007)

- **PEC Hamiltonian Cycle**
  Open problem for $c \geq 3$
  (Benkouar, Manoussakis, Paschos, Saad, 91)

General graphs/PEC cycles/ any number of colors ?

Complexity for edge-colored **general graphs** G^c
on **any number** of colors

- PEC Factor

Saad (1991), Manoussakis *et al* (2007): Polynomial (through matching techniques)

- Find an arbitrary PEC cycle (ie, check acyclicity).

Grossman & Haggkvist (1983), 2 colors : Polynomial

Yeo(1997), any number of colors : Polynomial

- PEC Hamiltonian cycle/path

NP-complete (trivial)

- PEC cycle/closed trail through two fixed vertices

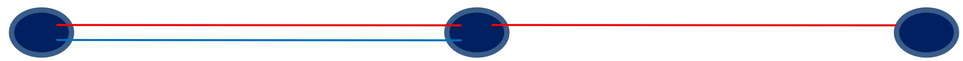
Cycle : NP complete (trivial)

Closed trail: NP-complete (Manoussakis *et al (2007)*)

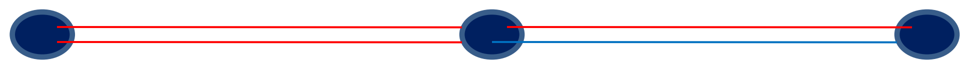
PEC cycles/degree conditions/multigraphs

Edge-colored Multigraphs : *graphs with parallel edges, but any two such edges are not monochromatic*

Multigraph:



<u>Not</u> allowed:



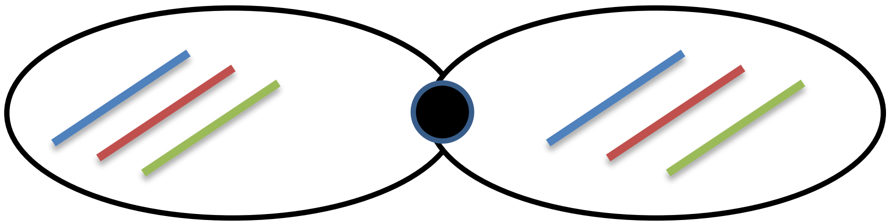
Theorem (Manoussakis *et al, 2007***)**
*Let G^c be a c-edge colored multigraph, $c \geq 2$, colored degree at least d. Then G^c has either a PEC **path** of length at least **min(n-1,2d)** or else a PEC **cycle** of length at least **min(n,d).***

Observation: ***Not** the best possible…*

PEC cycles/degree conditions/multigraphs

Theorem (Manoussakis et al, 2007**)**
Let a c-edge-colored multigraph G^c, $c \geq 2$, colored degree $\geq (n/2)+1$
If c= 2, then G^c has a PEC hamiltonian cycle, n even, and a PEC cycle of length n-1, when n is odd.
- If $c \geq 3$, then G^c has a PEC hamiltonian cycle.

«Almost» best possible:
Degree : (*n-1*)/2



Corollary. *Let a c-edge-colored multigraph G^c, $c \geq 3$, colored degree $\geq n/2$.*
Then G^c has a PEC hamiltonian path.

Corollary. *Let x,y be two fixed vertices in G^c, $c \geq 2$. If colored degree $\geq (n/2)+2$, then G^c has a PEC hamiltonian path with endpoints x,y.*

PEC cycles/degree conditions/multigraphs

Color-Pancyclism: *Existence of PEC cycles of all possible lengths*

Corollary (Manoussakis *et al, 2007***)**
Let G^c be a c-edge colored graph, $c \geq 2$, of colored degree at least $(n/2)+1$
- If $c=2$, then G^c is color-even-pancyclic.
- If $c \geq 3$, then G^c is color-pancyclic.

Another result of probabilistic nature (random graphs)….

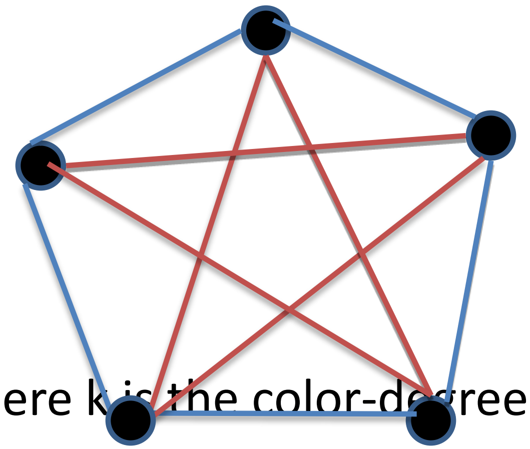
Theorem (Manoussakis *et al, 2007***)**
-k: sufficiently large constant
-G^b, G^r two independent random graphs on the same vertex set $V=\{1,2,...,2n\}$ and with the same edge probability $p=kn^{-1} \log n$.
-The edges of G^b are colored blue, and the edges of G^r are colored red.

Then, the graph $G^b \cup G^r$ has a PEC hamiltonian cycle with high probability.

Problems/Conjectures for cycles & paths

Conjecture . *Any regular c-edge-colored complete simple graph, $c \geq 2$, has a PEC hamiltonian cycle unless if $c=2$, n is odd, in which case it has a cycle of length $n-1$.*

Regular edge-colored complete graph: All its monochromatic sub-graphs are regular and on a same degree

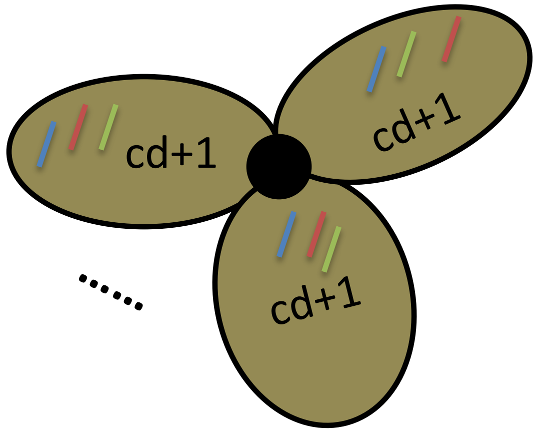


Example on two colors:
Red degree=blue degree=2
Observations:
1- The order of such regular graphs is kc+1 where k is the color-degree and c is the number of used colors.
2- This conjecture is weaker than the conjecture by Bollobas and Erdös (given before) , thus probably easier to prove.
3- Not true for two colors, because a 2-edge colored graph on an odd number of vertices has no PEC hamiltonian cycle (parity constraint).
4- True for some values of c

Problems/Conjectures for cycles & paths

c, d : given integers >0

Edge-colored **simple** extremal graph defined as at least 2 copies of a same graph within a common point :



- **No path** of length **2cd+1**
- **No cycle** of length **cd+2**

Conjecture. *If a simple graph G^c $c\geq 3$ has colored degree $d\geq 1$, then it has a :*

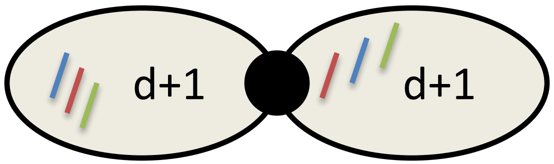
a/ PEC path of length min(n-1, 2cd).

b/ PEC cycle of length min (n, cd+1).

Problems/Conjectures for cycles & paths

given integers ; c≥3, d≥1

Define an c-edge-colored **multigraph** as at least 2 copies of a same **multigraph** with a common point :



- **No path** of length **2d+1**
- **No cycle** of length **d+2**

Conjecture. *A c-edge colored* ***multigraph*** *G^c , c≥3, with colored degree d≥1, has:*
- a PEC path of length min(n-1, 2d).
- a PEC cycle of length min (n, d+1).

Observation: **Not true** for two colors (there exist colored multigraphs with no cycles of length d+1)

Algorithmic problems for cycles & paths

Let us summarize…

Problem 1. *Is there any polynomial algorithm for finding (if any) a PEC hamiltonian cycle in an edge-colored complete graph, $c \geq 3$?*

Problem 2. *Is there any polynomial algorithm for finding (if any) a PEC hamiltonian path between two fixed vertices in an edge-colored complete graph, $c \geq 2$?*

Problem 3. *Is there any polynomial algorithm for finding (if any) a properly edge-colored hamiltonian path starting at fixed vertex in an edge-colored complete graph, $c \geq 2$?*

My favorite problem

Input: A c-edge colored complete graph K^c_n,
$c \geq 3$.

Question: Is there an efficient way to decide/find a properly edge-colored hamiltonian cycle in K^c_n ?

Polynomial or NP-complete **or both**?