Morphologie mathématique et traitement de la spatialité pour l'analyse des images

Séminaire MaMux - IRCAM - 5 avril 2013

Laurent Najman[1]
Principaux contributeurs: Thierry Géraud[2] et Yongchao Xu[1,2]

Labex Bézout
[1]Université Paris-Est LIGM A3SI ESIEE Paris, France
[2]EPITA Research and Development Laboratory (LRDE)

Motivation: going beyond metric spaces

Labex
BÉZOUT

du discret au continu

Motivation: going beyond metric spaces

Labex
BÉZOUT

du continu au discret

Motivation: going beyond metric spaces

Question

Is there a need for metric spaces?

- Discrete Exterior Calculus: algebraic differential calculus
- Discrete (Alexandrov) topology
- Set-valued analysis

What we want

Notions should easily be implementable on a computer

An example of Discrete Exterior Calculus: CCMF

- CCMF: Combinatorial Continuous Maximum Flow
- Incidence matrix of a graph plays the role of a gradient

Continuous MaxFlow

$$\max_{\vec{F}} \quad \vec{F}_{st}$$
$$\text{s.t.} \quad \nabla \cdot \vec{F} = 0,$$
$$\|\vec{F}\| \leq g.$$

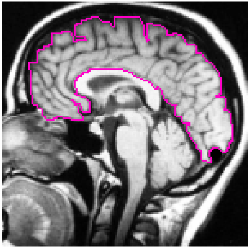
Combinatorial formulation

$$\max_{F} \quad F_{st}$$
$$\text{s.t.} \quad A^T F = 0,$$
$$|A^T| F^2 \leq g^2$$

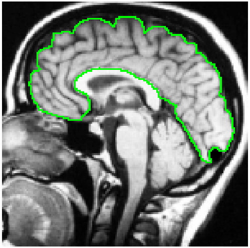
MaxFlow, GraphCuts

$$\max_{F} \quad F_{st}$$
$$\text{s.t.} \quad A^T F = 0,$$
$$|F| \leq g$$

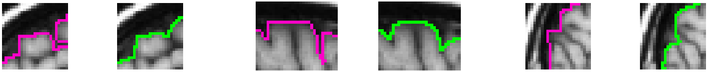
An example of Discrete Exterior Calculus: CCMF



Graph cuts result



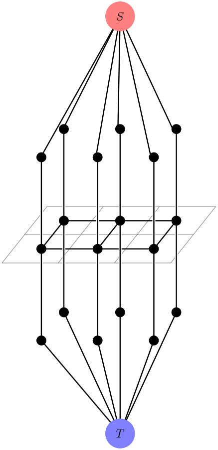
CCMF result

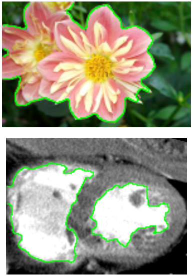


GC CCMF GC CCMF GC CCMF

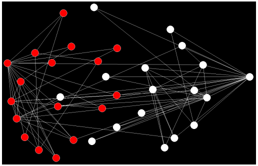
C. Couprie *et al.*: Combinatorial Continuous Max flows. In SIAM journal on imaging sciences, 2011.

An example of Discrete Exterior Calculus: CCMF





Unseeded segmentation



Classification

Continuity is different from differentiablity

Problem

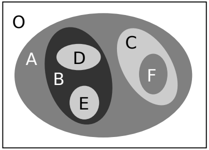
We are going to look at a problem where continuity and topology are keys.

Outline

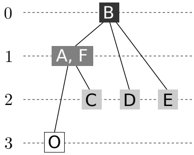
Outline

Trees and shapes: a schematic example

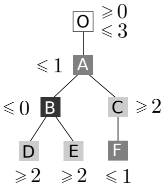
image



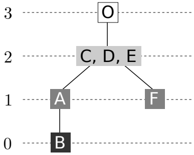
max-tree



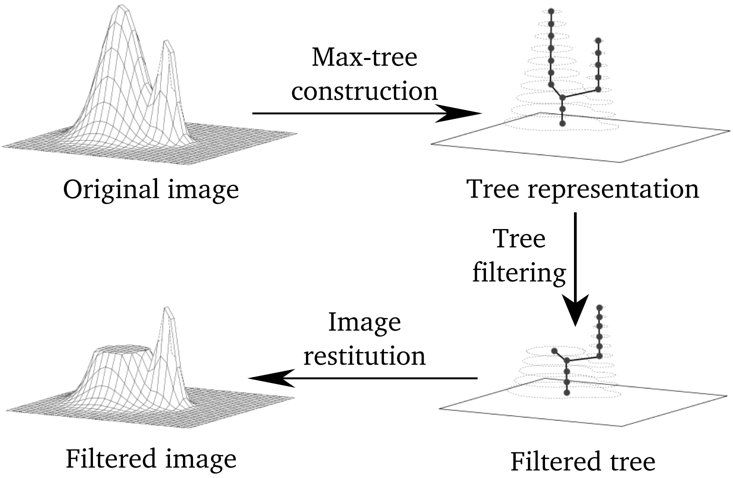
tree of shapes



min-tree



Max-tree filtering [Salembier, ITIP, 2000]



Self-dual connected filters

tree + pruning $\longrightarrow$

- grain filters κ:
  - connected filters (preserve some level lines $\partial[\, u \leq \lambda\,]$)
  - based on a single and self-dual tree, called the *tree of shapes*.

What's stange with the continuous world

Processing either u or $-u$ should give the same result.

Continuous definition:

- connected components of:
  - upper level sets $\{x; u(x) \geq \lambda\}_{\lambda \in \mathbb{R}}$
  - and lower level sets $\{x; u(x) < \lambda\}_{\lambda \in \mathbb{R}}$.

- only a "quasi-self-dual" tree of shapes...
- and some topological inconsistencies...

Some topological inconsistencies

Consider these examples:

1	0
0	1

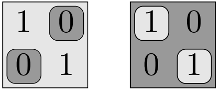
Tree of Shapes?

1	1	1	1	1	1	1	1	1
1	0	0	0	1	2	2	2	1
1	0	1	0	1	2	1	2	1
1	0	0	1	1	1	2	2	1
1	1	1	1	1	1	1	1	1

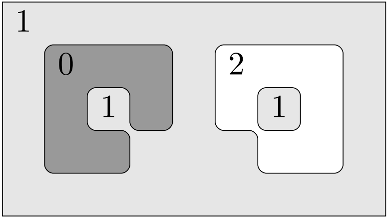
Tree of Shapes?

Some topological inconsistencies

Consider these examples:

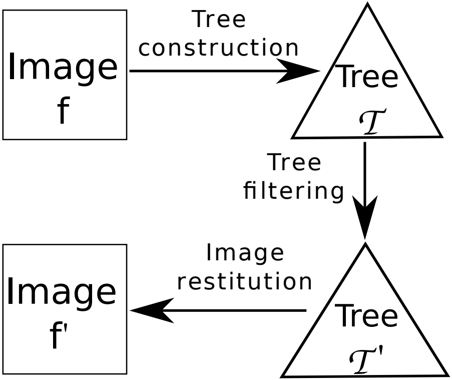


two possible trees!



a non symmetrical tree!

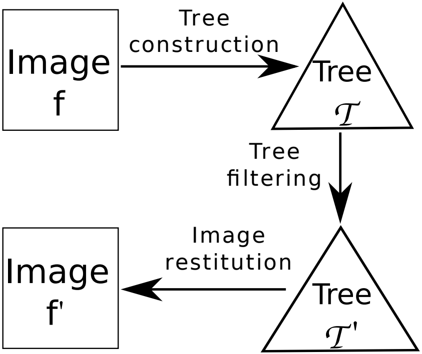
Tree filtering, or why discrete space?



Works well with increasing attributes $X \subseteq Y \Rightarrow \mathcal{A}(X) \leq \mathcal{A}(Y)$

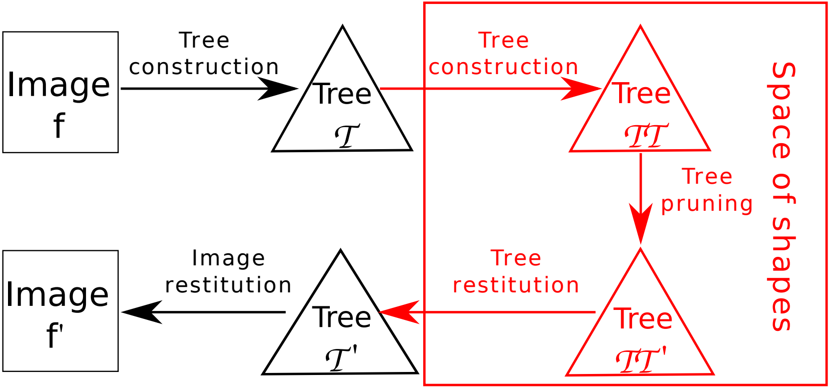
Why discrete spaces: filtering in shape-spaces

[Xu & Géraud & Najman, ICPR, 2012]



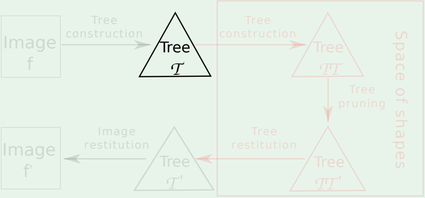
Why discrete spaces: filtering in shape-spaces

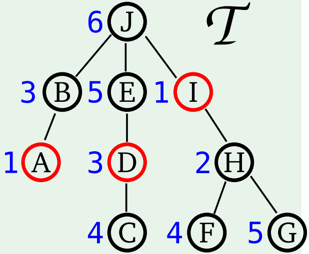
[Xu & Géraud & Najman, ICPR, 2012]



Filtering in shape-spaces: more details

What's $\mathcal{T}$?



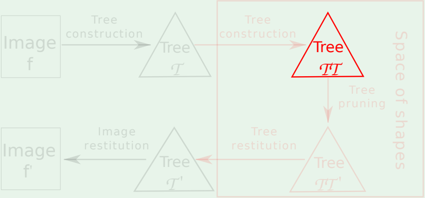


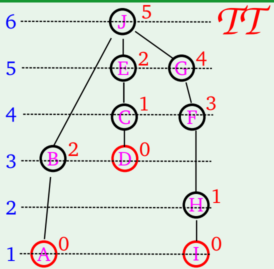
Three local minima.

- Node $\mathcal{N}$: connected component,
- Attribute $\mathcal{A}$: interesting feature,
- Parenthood: inclusion relationship between $\mathcal{N}$.

Filtering in shape-spaces: more details

What's $\mathcal{TT}$?



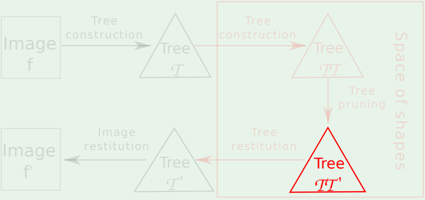


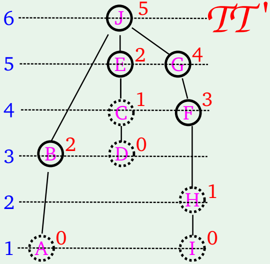
Min-tree of $\mathcal{T}$.

- Node $\mathcal{NN}$: a set of neighboring connected components,
- New attribute $\mathcal{AA}$: new features based on $\mathcal{A}$ or image,
- Parenthood: inclusion relationship between $\mathcal{NN}$.

Filtering in shape-spaces: more details

What's $\mathcal{TT}'$?



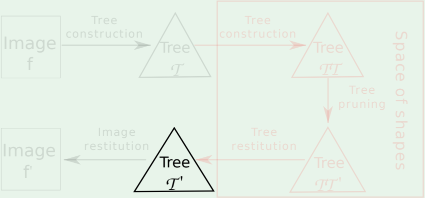


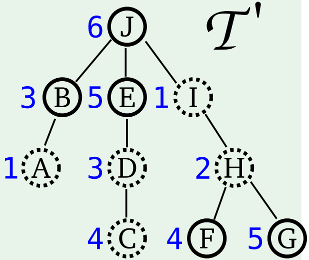
Pruning of $\mathcal{TT}$.

$\mathcal{TT}'$: pruned $\mathcal{TT}$ based on $\mathcal{AA}$.

Filtering in shape-spaces: more details

What's $\mathcal{T}'$?

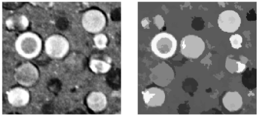




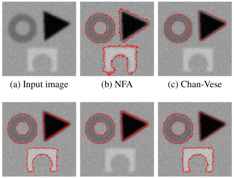
Restituted tree from $\mathcal{TT}'$.

$\mathcal{T}'$: removal of $\mathcal{N}$ represented by pruned $\mathcal{NN}$.

Some illustrations and applications

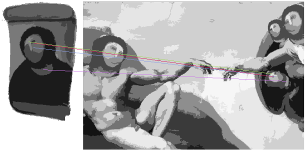


(a) Input image. (b) Shaping 1.



(a) Input image (b) NFA (c) Chan-Vese

(d) Ballester, $\lambda = 2k$ (e) Ballester, $\lambda = 3k$ (f) Our method



Outline

Topological space (Reminder)

A set X is a topological space if

1. The intersection of any number and the union of any finite number of closed sets is a closed set.
2. The whole set X and the empty set $\emptyset$ are closed.

- Closure of $M \subset X$: $cl_X(M) = \bigcap\{C \mid C \subset X, C \text{ closed}, M \subset C\}$
- A set M is degenerate if it contains just one point.

Discrete spaces

Definition

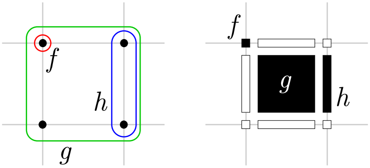
- *A topological space X is said to be a T_0-space if every two distinct degenerate subsets of X have distinct closures in X.*
- *A T_0-space is called a discrete space if the union of an arbitrary number of closed sets of the space is closed.*

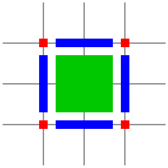
The Khalimsky grid is a discrete space

$$
\begin{aligned}
H_0^1 &= \{\{x\} \mid x \in \mathbb{Z}\} \\
H_1^1 &= \{\{x, x+1\} \mid x \in \mathbb{Z}\} \\
H^1 &= H_0^1 \cup H_1^1 \\
H^n &= \{h_1 \times \ldots \times h_n, \forall i \in [1, n], h_i \in H^1\}
\end{aligned}
$$

The Khalimsky grid is a discrete space

$$
\begin{aligned}
H_0^1 &= \{\{x\} \mid x \in \mathbb{Z}\} \\
H_1^1 &= \{\{x, x+1\} \mid x \in \mathbb{Z}\} \\
H^1 &= H_0^1 \cup H_1^1 \\
H^n &= \{h_1 \times \ldots \times h_n, \forall i \in [1, n], h_i \in H^1\}
\end{aligned}
$$



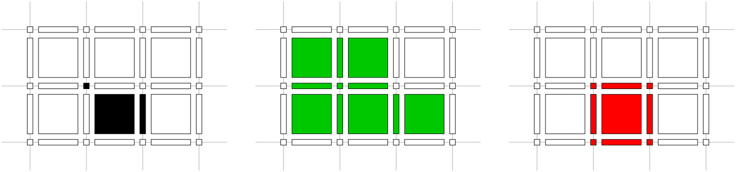


Cubical complex... ...and Khalimsky's grid.

Open sets and stars

- Star of $M \subset X$: $st_X(M) = \bigcap\{O \mid O \subset X, O \text{ open}, M \subset O\}$.
- If $M = \{x\}$ is degenerate, we write $st_X(x) = st_X(\{x\})$.

Topological operators



$E = \{f, g, h\}$

star: $st_X(E)$

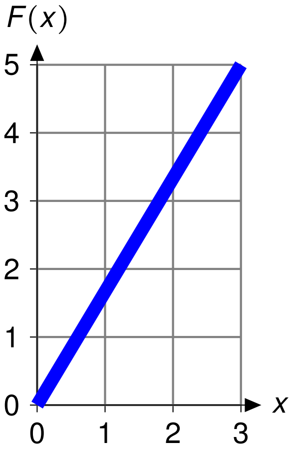
closure: $cl_X(E)$

Connected sets

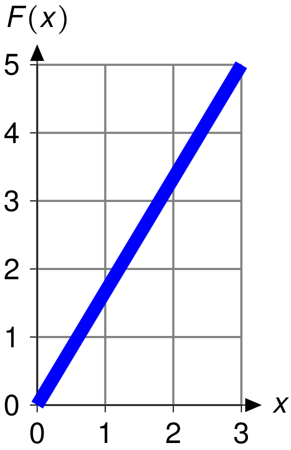
Definition

- *A set is said to be connected if it is not the union of two disjoint nonempty closed sets.*
- *A connected component of X is a connected subset of X that is maximal for the connectivy property.*

Why set-valued analysis?



Why set-valued analysis?

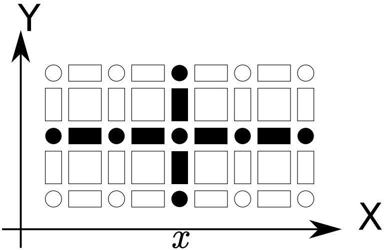


Upper semicontinuity

In the sequel, X and Y denotes two discrete spaces.

Definition

A set-valued map $F : X \rightsquigarrow Y$ is called upper semicontinuous at $x \in Dom(F)$ (USC at x) if and only if for all $y \in st_X(x)$, $F(y) \subseteq st_Y(F(x))$.



Inverse image and core

- Inverse image: $F^{\oplus}(M) = \{x \in X; F(x) \cap M \neq \emptyset\}$.
- Core: $F^{\ominus}(M) = \{x \in X; F(x) \subseteq M\}$.

Property

- *F is* USC *if and only if $F^{\ominus} st_Y = st_X F^{\ominus}$.*
- *F is* USC *if and only if $F^{\oplus} cl_Y = cl_X F^{\oplus}$.*

A single unbroken curve with no “jumps”

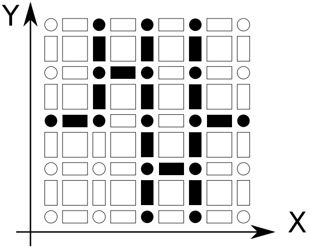
Property

Let F be a USC *set-valued map such that, for all $x \in X$, $F(x)$ is a closed (resp. open) connected set.*
Then, for any connected set M, $F(M)$ is a closed (resp. open) connected set.

Quasi-simple maps

Definition

A USC *set-valued map F is a closed (resp. open) quasi-simple map if for all $x \in X$, $F(x)$ is a closed (resp. open) connected set, and if furthermore, for any $\{x\} = st_X(x) \in X$, $F(x)$ is degenerate.*



Simple maps

Definition

A quasi-simple map F is simple if it is the smallest of all quasi-simple maps with the same data on open points, i.e., a quasi-simple map F_1 is simple if for any quasi-simple map F_2 such that for any $\{x\} = st_X(x) \in X$, $F_1(x) = F_2(x)$, then, for all $x \in X$, $F_1(x) \subseteq F_2(x)$.

Outline

Intervals

- An interval is a connected subset of H^1.
- Open sets of $H^1 \equiv \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$
- Closed sets of $H^1 \equiv \mathbb{Z}$

We thus have

- $\ldots < \{0\} < \{0,1\} < \{1\} < \ldots$,
- $\lambda + \frac{1}{2}$ (where $\lambda \in H^1$),
- max or min on any subset of H^1

Definition

A set-valued map F is an interval-valued map if the images of F are intervals.

Intermediate value theorem

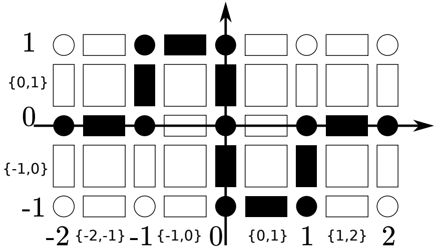
Theorem (Intermediate value theorem)

Let F be an USC *interval-valued map from X to H^1. If x and y are two points of X and λ_0 is a point of H^1 lying between $F(x)$ and $F(y)$ (i.e., $\lambda_0 \in [\min\{\mu; \mu \in F(x) \cup F(y)\}, \max\{\mu; \mu \in F(x) \cup F(y)\}]$), then there exists $z \in X$ such that $\lambda \in F(z)$.*

Plain maps

Definition

A closed-valued, interval-valued simple map F from X to H^1, with $dom(F) = X$ is called a plain map on X.



Strict level sets

In the sequel, F denotes an interval-valued map from X to H^1 and $\lambda \in \mathbb{Z}$.

$$
\begin{aligned}
[F \lhd \lambda] &= F^{\ominus}(]-\infty, \lambda[) = \{ x \in X \mid \forall \mu \in F(x); \mu < \lambda \} \\
[F \rhd \lambda] &= F^{\ominus}(]\lambda, +\infty[)
\end{aligned}
$$

The upper and lower level sets of a plain map are open.

Extrema

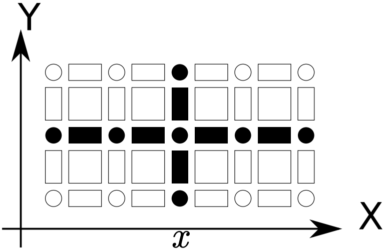
Definition (Extrema)

- *A connected component of* $[F \lhd \lambda]$ *is a* *minimum* *of* F *if it does not contain any other connected component of* $[F \lhd \mu]$ *for any* $\mu < \lambda$.
- *A connected component of* $[F \rhd \lambda]$ *is a* *maximum* *of* F *if it does not contain any other connected component of* $[F \rhd \mu]$ *for any* $\mu > \lambda$.
- *An* *extremum* *of* F *is either a maximum or a minimum of* F.

Flatness

A set M is flat for F if for all $x \in M$, $F(x) = F(M)$.

- $F(M)$ is not always a degenerate set
- An extremum of a set-valued map is not always flat.



The point x is both a minimum and a maximum.
Although this extremum is flat, it is not degenerate.

Plain maps and extrema

Lemma

The extrema of a plain map F are flat open sets.
Furthermore, for any extrema M of F, $F(M)$ is degenerate.

Property

Let F be a plain map on X that is not constant.
Then the extrema of F are separated.

Outline

Definition

Let $M \subseteq X$. We call cavities *of M in X the components of $X \setminus M$.*
Let $p_\infty \in X$ a reference point.
We call saturation *of M with respect to p_∞ the set*

$$sat(M, p_\infty) = X \setminus \mathcal{CC}(X \setminus M, p_\infty).$$

For any $\lambda \in H^1$, we call quasi-shape *of x the set:*

$$S_\lambda(F, x) \;=\; sat(\mathcal{CC}([F \not\equiv \lambda], x), p_\infty)$$

Remark

We have

$$\begin{array}{rrcl} \textit{either} & S_\lambda(F,x) & = & sat(\mathcal{CC}([F \triangleright \lambda], x), p_\infty) \\ \textit{or} & S_\lambda(F,x) & = & sat(\mathcal{CC}([F \triangleleft \lambda], x), p_\infty). \end{array}$$

Tree of shapes ?

$$\mathfrak{S}(F,x) = \{ S_\lambda(F,x) \}_\lambda \setminus \emptyset.$$

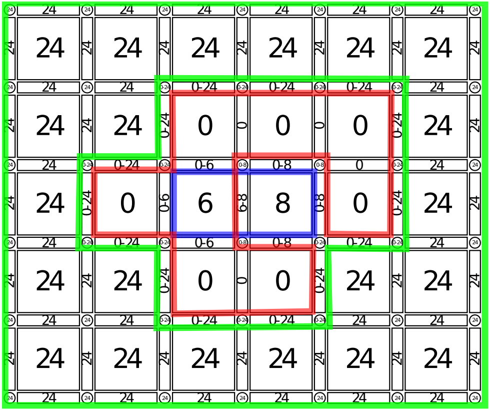
Definition

$$\mathfrak{S}(F) = \cup_{x \in X} \mathfrak{S}(F,x)$$

Question

Is $\mathfrak{S}(F)$ a tree?

A plain map is not enough!



$[F \lhd 7] - [F \rhd 5] - [F \rhd 5]$

A combinatorial definition of surface

Definition (n-surface)

- *A discrete space Y is a 0-surface if Y is made of exactly two points x and y such that $x \notin st_Y(y)$ and $y \notin st_Y(x)$.*
- *A discrete space Y is a n-surface ($n > 0$)*
  - *if Y is connected and*
  - *if, for any $x \in Y$, $cl_Y(x) \cup st_Y(x) \setminus \{x\}$ is a $(n-1)$-surface.*

Well-composedness

Definition (Well-composed set and map)

- *A connected set M is well-composed if $\partial M = cl_X(M) \cap cl_X(X \setminus M)$ is a n-surface.*
- *A set N is well-composed if any connected component of ∂N is a n-surface.*
- *A plain map F is well-composed if if for any $\lambda \in H^1$, both $[F \lhd \lambda]$ and $[F \rhd \lambda]$ are well-composed.*

Tree of quasi-shapes

Theorem

If F is a well-composed plain map of a unicoherent finite discrete space, then any two quasi-shapes are either disjoint or nested. Hence $\mathfrak{S}(F)$ is a tree.

Tree of shapes

Definition

We call shape of x the smallest non-empty quasi-shape of x defined by:

$$S(F,x) = \bigcap_{S \in \mathfrak{S}(F,x)} S$$

We denote by $\mathcal{S}(F)$ the set formed by all the shapes of x for all $x \in X$

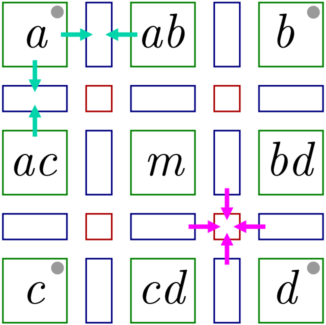
Corollary

If F is a well-composed plain map,
the set $\mathcal{S}(F)$ is a tree (called the tree of shapes).

Outline

Proposed Image Immersion

a	b
c	d



Max/Min interpolation

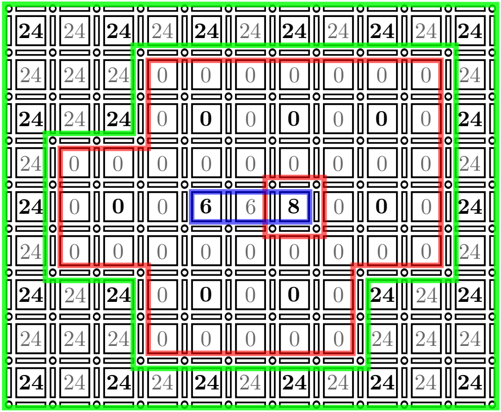
Property

Any plain map obtained thanks to the max- or the min-interpolation, whatever n (the dimension of the space Z^n), is well-composed.

Remark

Such interpolations are not self-dual.

An example of Min-interpolation



$$[F \triangleleft 7] - [F \triangleright 5] - [F \triangleright 5]$$

Median interpolation

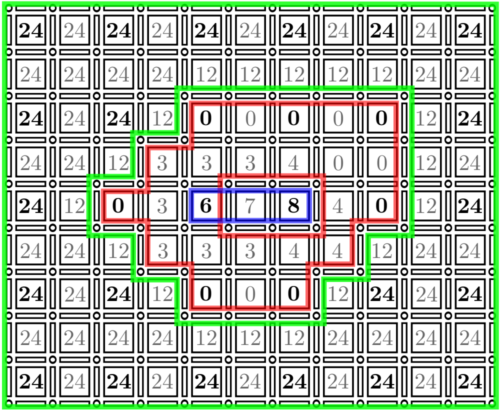
Property

The median interpolation of a function defined on $\mathbb{Z}^2$ leads to a self-dual plain map.

Conjecture

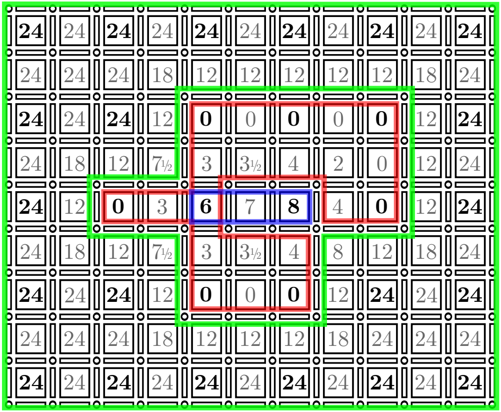
With the suitable space subdivision, the median interpolation leads to a self-dual map whatever the dimension of the space.

An example of Median-interpolation



$$[F \triangleleft 7] - [F \triangleright 5] - [F \triangleright 5]$$

An example of Mean-interpolation: it does not work!



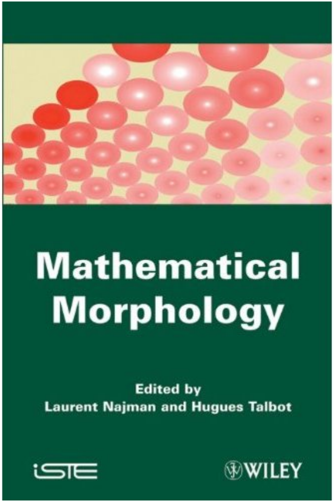
$$[F \lhd 7] - [F \rhd 5] - [F \rhd 5]$$

Outline

Result

- *We retrieve continuity properties in discrete spaces.*
- *From that, we can derive a quasi-linear algorithm for the tree of shapes, whatever the dimension of the space.*
- *We propose a true self-dual interpretation of discrete data.*
- *Many applications follow.*

Thank for your attention !



Pink: http://pinkhq.com
Olena: http://www.lrde.epita.fr/cgi-bin/twiki/view/Olena