

ON SOME PROPERTIES OF PERIODIC SEQUENCES IN ANATOL VIERU’S MODAL THEORY

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ABSTRACT. Algebraic methods have been currently applied to music in the second half of the twentieth-century (see [M. Andreatta: *Group-theoretical Methods applied to Music*, unpublished dissertation, 1997], [M. Chemilier: *Structure et Méthode algébriques en informatique musicale*. Thèse de doctorat, L. I. T. P., Institut Blaise Pascal, 1990] and [G. Mazzola et al.: *The Topos of Music—Geometric Logic of Concepts, Theory and Performance*] for main references). By starting from Anatol Vieru’s compositional technique based on finite difference calculus on periodic modal sequences, as it has been introduced in his book [*Cartea modurilor, 1 (Le livre des modes, 1)*. Ed. Muzicala, Bucarest, 1980. Revised ed. *The book of modes*, 1993], the present essay tries to generalize some properties by means of abstract group theory. Two main classes of periodic sequences are considered: reducible and reproducible sequences, replacing respectively Vieru’s modal and irreducible sequences. It turns out that any periodic sequence can be decomposed in a unique way into a reducible and a reproducible component.

1. Reducible and reproducible sequences

For any sequence f defined on \mathbb{Z} taking values in a finite abelian group G we define the translated sequence Tf and the sequence of differences Df by

$$Tf(x) = f(x + 1), \quad Df(x) = f(x + 1) - f(x).$$

The relationship between the translated sequence and the sequence of differences is expressed by the following equation:

$$D = T - 1.$$

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DEFINITION 1. The sequence f is called m -periodic if $f(x + m) = f(x)$ for any $x \in \mathbb{Z}$.

Affirming that f is m -periodic, it is equivalent to the relation $T^m f = f$. If f is m -periodic, then Tf and Df are also m -periodic.

DEFINITION 2. The sequence f is called *reducible* if an integer $k \geq 0$ does exist such that $D^k f = 0$.

The sequence f is called *reproducible* if an integer $k \geq 0$ does exist such that $D^k f = f$.

By $\text{Red}(G)$ and $\text{Rep}(G)$ we will designate respectively the set of reducible and reproducible sequences taking values in G (also called G -valued sequences).

EXAMPLE 1. Example of a reducible sequence. In Anatol Vieru's book [10] a \mathbb{Z}_{12} -valued sequence of period 72 is considered. It can be represented by means of 6 lines of 12 elements

$$\begin{pmatrix} 4 & 1 & 0 & 8 & 8 & 7 & 0 & 6 & 8 & 1 & 4 & 0 \\ 8 & 11 & 4 & 6 & 0 & 5 & 4 & 4 & 0 & 11 & 8 & 10 \\ 0 & 9 & 8 & 4 & 4 & 3 & 8 & 2 & 4 & 9 & 0 & 8 \\ 4 & 7 & 0 & 2 & 8 & 1 & 0 & 0 & 8 & 7 & 4 & 6 \\ 8 & 5 & 4 & 0 & 0 & 11 & 4 & 10 & 0 & 5 & 8 & 4 \\ 0 & 3 & 8 & 10 & 4 & 9 & 8 & 8 & 4 & 3 & 0 & 2 \end{pmatrix}.$$

Any column is a periodic sequence obtained by adding modulo 12 a constant value to a basis element (i.e., the top of the column). By putting the constant value as an index for the basis element, one has the following compact expression

$$\begin{aligned} & (4_4 \ 1_{10} \ 0_4 \ 8_{10} \ 8_4 \ 7_{10} \ 0_4 \ 6_{10} \ 8_4 \ 1_{10} \ 4_4 \ 0_{10}), \\ & D^1 = (2_6 \ 9_6 \ 11_6 \ 8_6 \ 0_6 \ 11_6 \ 5_6 \ 6_6 \ 2_6 \ 5_6 \ 3_6 \ 8_6), \\ & D^2 = (0 \ 7 \ 2 \ 9 \ 4 \ 11 \ 6 \ 1 \ 8 \ 3 \ 10 \ 5), \\ & D^3 = (7), \\ & D^4 = 0. \end{aligned}$$

EXAMPLE 2. Example of a reproducible sequence

$$\begin{aligned} f &= (8 \ 11 \ 0 \ 1 \ 4 \ 0), \\ D^1 &= (3 \ 1 \ 1 \ 3 \ 8 \ 8), \\ D^2 &= (10 \ 0 \ 2 \ 5 \ 0 \ 7), \\ D^3 &= (2 \ 2 \ 3 \ 7 \ 7 \ 3), \\ D^4 &= f. \end{aligned}$$

THEOREM 1. Let d_i , $0 \leq i \leq N$, be some integers such that at least one of them is relatively prime to the number of elements of G . Then, an integer m does exist such that any sequence verifying

$$\sum_{i=0}^N d_i T^i f = 0$$

is m -periodic.

COROLLARY 2. Reducibles and reproducibles sequences are periodic.

THEOREM 3. All periodic sequence can be decomposed in a unique way

$$f = f_{\text{red}} + f_{\text{rep}}, \quad f_{\text{red}} \in \text{Red}(G), \quad f_{\text{rep}} \in \text{Rep}(G).$$

Proof. Let f be m -periodic. Being the collection of sequences m -periodic finite, two integers $k, l \geq 1$ do exist such that $D^k f = D^{k+l} f$. By induction on r one has $D^k f = D^{k+rl} f$. In the same way it can be shown that two integers $r, s \geq 1$ may exist such that $D^{rl} f = D^{(r+s)l} f$. We put

$$f_{\text{red}} = f - D^{rl} f, \quad f_{\text{rep}} = D^{rl} f.$$

It follows that $D^k(f - D^{rl} f) = 0$, $D^{sl} D^{rl} f = D^{rl} f$, which means that f_{red} and f_{rep} give the needed decomposition.

The unicity comes from the relation $\text{Red}(G) \cap \text{Rep}(G) = \{0\}$. □

2. Decomposition of \mathbb{Z}_n into p -groups

There is a one-to-one relation between the subgroups of \mathbb{Z}_n and the family of integers d that divide n by $1 \leq d \leq n$, i.e., for any such d we may take the unique subgroup of \mathbb{Z}_n with d elements. The latter can be characterized as the set of $z \in \mathbb{Z}_n$ such that $dz = 0$ or, equivalently, as the set $\frac{n}{d}\mathbb{Z}_n$ of elements having the form $\frac{n}{d}z$ where z belongs to \mathbb{Z}_n .

DEFINITION 3. The abelian group G is a *direct sum of a family of subgroups* G_1, \dots, G_m of G if any $x \in G$ may be decomposed in a unique way into a sum $x_1 + \dots + x_m$ with $x_i \in G_i$ for $1 \leq i \leq m$.

We will put $G = \bigoplus_{i=1}^m G_i$.

DEFINITION 4. Let p be a prime number. A finite abelian group is called *p -group* if its cardinality is a power of p .

THEOREM 4. Any group \mathbb{Z}_n is a direct sum of its p -maximals subgroups.

If $n = \prod_{i=1}^m p^{k_i}$ is the decomposition of n into prime factors, the decomposition of \mathbb{Z}_n in maximal p -subgroups can be written as follows

$$\mathbb{Z}_n = \bigoplus_{i=1}^m G_{p^{k_i}},$$

where $G_{p^{k_i}}$ is the subgroup of \mathbb{Z}_n with p^{k_i} elements. The decomposition of any $z \in \mathbb{Z}_n$ defines the elements $\pi_i(z) \in G_{p^{k_i}}$ in a unique way such that $z = \sum_{i=1}^m \pi_i(z)$. The arrows $\pi_i: \mathbb{Z}_n \rightarrow G_{p^{k_i}}$ are group morphisms.

The p_i -component $\pi_i(z)$ of z is the unique element $y \in \mathbb{Z}_n$ satisfying the relations $p^{k_i}y = \frac{n}{p^{k_i}}(z - y) = 0$ in \mathbb{Z}_n . Let q_i be an integer verifying

$$q_i \frac{n}{p^{k_i}} = 1 \pmod{p^{k_i}}.$$

It follows that $p^{k_i} \frac{n}{p^{k_i}} q_i z = \frac{n}{p^{k_i}} (z - \frac{n}{p^{k_i}} q_i z) = 0$ in \mathbb{Z}_n , which means that

$$\pi_i(z) = \frac{n}{p^{k_i}} q_i z.$$

EXAMPLE 3. Subgroups of \mathbb{Z}_{12}

G_1	{0},	
G_2	{0, 6},	
G_3	{0, 4, 8},	maximal 3-group,
G_4	{0, 3, 6, 9},	maximal 2-group,
G_6	{0, 2, 4, 6, 8, 10},	
G_{12}	{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11}.	

EXAMPLE 4. Decomposition of \mathbb{Z}_{12} into p -groups

$$\mathbb{Z}_{12} = G_3 \bigoplus G_4.$$

	0	3	6	9
0	0	3	6	9
4	4	7	10	1
8	8	11	2	5

$$\begin{aligned}
q_3 &= 1, & 1 \cdot \frac{12}{3} &= 1 \pmod{3}, \\
q_4 &= -1, & -1 \cdot \frac{12}{4} &= 1 \pmod{4}, \\
\pi_3(x) &= \frac{12}{3} q_3 x = 4x, & \pi_4(x) &= \frac{12}{4} q_4 x = -3x.
\end{aligned}$$

$$\begin{aligned}
5 &= 8 + 9 = 4 \cdot 5 - 3 \cdot 5 = 4 \cdot 2 - 3 \cdot 1, \\
7 &= 4 + 3 = 4 \cdot 7 - 3 \cdot 7 = 4 \cdot 1 - 3 \cdot 3, \\
11 &= 8 + 3 = 4 \cdot 11 - 3 \cdot 11 = 4 \cdot 2 - 3 \cdot 3.
\end{aligned}$$

Remark 1. Many theoretical works ([2], [8] and [10]) have already shown the musical interest of the decomposition of the cyclic group \mathbb{Z}_{12} into a direct sum of its maximal p -groups from a theoretical, analytical and compositional point of view.

THEOREM 5. Let $\varphi_i: \mathbb{Z}_n \rightarrow \mathbb{Z}_{p^{k_i}}$ be the canonic map. The map $\varphi: \mathbb{Z}_n \rightarrow \prod_{i=1}^m \mathbb{Z}_{p^{k_i}}$ defined by

$$\varphi(z) = (\varphi_1(z), \dots, \varphi_m(z))$$

is a ring homomorphism with inverse given by

$$(z_1, \dots, z_m) \mapsto \sum_{i=1}^m \frac{n}{p^{k_i}} q_i z_i.$$

For any \mathbb{Z}_n -valued sequence f we put $f_i(x) = \pi_i(f(x))$. We will say that $f = \sum_i f_i$ is the decomposition of f corresponding to the decomposition of \mathbb{Z}_n in p -groups.

3. Characterization of reducible sequences

PROPOSITION 6. Let $f = \sum_j f_j$ be the decomposition of f corresponding to the decomposition of \mathbb{Z}_n in p -groups. Then $f \in \text{Red}(\mathbb{Z}_n)$ if and only if $f_j \in \text{Red}(\mathbb{Z}_n)$ for any j .

THEOREM 7. Let f be a \mathbb{Z}_{p^k} -valued sequence. Then f is reducible if and only if it is p^m -periodic for a given $m \geq 0$.

Proof. By induction on k . For $k = 1$ the result comes from the following lemma. □

LEMMA 8. For any \mathbb{Z}_p -valued sequence f , one has

$$(T - 1)^{p^m} f = (T^{p^m} - 1)f.$$

P r o o f. By induction on m one is conducted to the case $m = 1$.

$$(T - 1)^p f = \sum_{i=0}^{p-1} (-1)^{p-i} C_p^i T^i f.$$

Since p divides any C_p^i by $1 \leq i \leq p - 1$ and since $pf = 0$ it follows that the sum of the right member is reduced to $T^p f - f$.

Let now the theorem be true for any $k_1 < k$. We designate f_1 the sequence defined by $f_1(x) = pf(x)$. It is clear that f_1 takes values in the subgroup $G_{p^{k-1}}$ with p^{k-1} elements of \mathbb{Z}_{p^k} , isomorphic to $\mathbb{Z}_{p^{k-1}}$.

Let f be p^m -periodic. Then f_1 is also p^m -periodic, i.e., reducible (induction hypothesis). Therefore, there exists $l \geq 1$ such that $D^l f_1 = 0$. By definition of f_1 , it means that $p(D^l f)(x) = 0$ for any x , i.e., the sequence $D^l f$ takes values into the subgroup G_p with p elements of \mathbb{Z}_{p^k} , isomorphic to \mathbb{Z}_p . Since $D^l f$ is p^m -periodic, the induction hypothesis leads to the existence of $l_1 \geq 1$ such that $D^{l_1} D^l f = 0$, i.e., $D^{l_1+l} f = 0$ and f is reducible.

Let f be reducible. Then $f_1 \in \text{Red}(G_{p^{k-1}})$ by induction hypothesis it follows that there exists $m_1 \geq 0$ such that f_1 is p^{m_1} -periodic. This means that $f(x) - f(x + p^{m_1}) \in G_p$ for any x . We define the sequence f_2 by $f_2(x) = f(x) - f(x + p^{m_1})$. Since $f_2 \in \text{Red}(G_p)$, the induction hypothesis guaranties the existence of $m_2 \geq 0$ such that f_2 is p^{m_2} -periodic. In short, $(1 - T^{p^{m_2}})(1 - T^{p^{m_1}})f = 0$. To conclude we just need the following. \square

LEMMA 9. Let f be a \mathbb{Z}_n -valued sequence such that $(1 - T^k)(1 - T^l)f = 0$. Then f is kln -periodic.

P r o o f. From $(1 - T^l)f = T^k(1 - T^l)f$ one deduces that $(1 - T^l)f = T^{ki}(1 - T^l)f$ for any $i \geq 1$, and in particular that $(1 - T^{kl})(1 - T^l)f = 0$. By the same argument, $(1 - T^{kl})f = T^{li}(1 - T^{kl})f$ for any $i \geq 1$. This means that

$$(1 - T^{kln})f = (1 - (T^{kl})^n)f = \left(\sum_{i=0}^{n-1} T^{kli} \right) (1 - T^{kl})f = n(1 - T^{kl})f = 0.$$

\square

COROLLARY 10. $\text{Red}(\mathbb{Z}_n)$ is a ring.

COROLLARY 11. Let $f \in \text{Red}(\mathbb{Z}_n)$ be such that $f(x)$ is invertible in \mathbb{Z}_n for all x . Then $f^{-1} \in \text{Red}(\mathbb{Z}_n)$.

COROLLARY 12. Let k, l be two integers and let g be defined by $g(x) = f(kx + l)$. If $f \in \text{Red}(\mathbb{Z}_n)$ then $g \in \text{Red}(\mathbb{Z}_n)$.

4. Characterization of reproducible sequences

PROPOSITION 13. *Let $f = \sum_j f_j$ be the decomposition of f corresponding to the decomposition of \mathbb{Z}_n in p -groups. Then $f \in \text{Rep}(\mathbb{Z}_n)$ if and only if $f_j \in \text{Rep}(\mathbb{Z}_n)$ for any j .*

DEFINITION 5. For every m -periodic sequence f and for every integer d dividing m , we define the d -periodised obtained from f as the sequence $\sum_{i=0}^{m/d-1} T^{id} f$.

The equation

$$(1 - T^d) \sum_{i=0}^{m/d-1} (T^d)^i f = (1 - (T^d)^{m/d}) f = (1 - T^m) f = 0$$

shows that the d -periodised sequence is indeed a d -periodic sequence.

THEOREM 14. *Let f be a \mathbb{Z}_{p^k} -valued m -periodic sequence. Then f is reproducible if and only if the p^r -periodised of f is zero, where p^r is the highest power of p that divides m .*

Proof. If f is reproducible, then its p^r -periodised is also reproducible by construction, but also reducible according to Theorem 7, because it is p^r -periodic. Then the p^r -periodised is zero, because $\text{Rep} \cap \text{Red} = \{0\}$.

Conversely, by supposing that the p^r -periodised is zero, we may write the decomposition

$$f = f_{\text{red}} + f_{\text{rep}}, \quad f_{\text{red}} \in \text{Red}, \quad f_{\text{rep}} \in \text{Rep}.$$

According as we have shown, the p^r -periodised of f_{rep} is zero. Then the previous equation shows that the p^r -periodised of f_{red} must be zero too. Due to Theorem 7, f_{red} must be p^s -periodised for a given integer $s \geq 0$. Because of the fact that the m -periodised of f implies that of f_{red} , the period of the latter divides m and p^s , which shows that f_{red} is in fact p^r -periodic. As a consequence its p^r -periodised is equal to $\frac{m}{p^r} f_{\text{red}}$. Since $\frac{m}{p^r} \wedge p^k = 1$, the multiplication by $\frac{m}{p^r}$ is an automorphism of \mathbb{Z}_{p^k} . From $\frac{m}{p^r} f_{\text{red}} = 0$ one may conclude that $f_{\text{red}} = 0$.

□

COROLLARY 15. *Let f be a \mathbb{Z}_{p^k} -valued m -periodic sequence and let p^r be the highest power of p dividing m . Then f is reproducible if and only if the relations*

$$\sum_{i=0}^{m/p^r-1} f(p^r i + x) = 0$$

hold for all x such that $0 \leq x < p^r$.

5. Calculation of reducible and reproducible components of a periodic sequence

PROPOSITION 16. *Let $f = \sum_j f_j$ be the decomposition of the periodic sequence f corresponding to the decomposition of \mathbb{Z}_n in p -groups and let $f_j = f_{j,\text{red}} + f_{j,\text{rep}}$ be the decomposition of f_j in a sum of a reducible and a reproducible sequence. Then the decomposition of f*

$$f = f_{\text{red}} + f_{\text{rep}}, \quad f_{\text{red}} \in \text{Red}, \quad f_{\text{rep}} \in \text{Rep},$$

is given by

$$f_{\text{red}} = \sum_j f_{j,\text{red}}, \quad f_{\text{rep}} = \sum_j f_{j,\text{rep}}.$$

THEOREM 17. *Let f be a \mathbb{Z}_{p^k} -valued m -periodic sequence and let f_{per} be the p^r -periodised of f , where p^r is the highest power of p dividing m . Let $\left(\frac{m}{p^r}\right)^{-1}$ be the inverse of $\frac{m}{p^r} \pmod{p^k}$. Then the decomposition of f*

$$f = f_{\text{red}} + f_{\text{rep}}, \quad f_{\text{red}} \in \text{Red}, \quad f_{\text{rep}} \in \text{Rep},$$

is given by

$$f_{\text{red}} = \left(\frac{m}{p^r}\right)^{-1} f_{\text{per}}, \quad f_{\text{rep}} = f - \left(\frac{m}{p^r}\right)^{-1} f_{\text{per}}.$$

Proof. Since f_{red} is p^r -periodic by construction, it is reducible according to the Theorem 7. The p^r -periodised of f_{rep} is zero:

$$\sum_{i=0}^{m/p^r-1} T^{ip^r} f_{\text{rep}} = \sum_{i=0}^{m/p^r-1} T^{ip^r} f - \left(\frac{m}{p^r}\right) \left(\frac{m}{p^r}\right)^{-1} f_{\text{per}} = f_{\text{per}} - f_{\text{per}} = 0.$$

Due to the Theorem 14, f_{rep} is reproducible. □

COROLLARY 18. *Let m and n be two integers such that $m \wedge n = 1$ and let f be a \mathbb{Z}_n -valued m -periodic sequence. f is reproducible if and only if*

$$\sum_{i=0}^{m-1} f(i) = 0.$$

6. Decomposition algorithm

1. Write the decomposition of n

$$n = \prod_{j=1}^N p_j^{k_j}$$

into prime factors.

2. Find the integers q_j such that

$$q_j \frac{n}{p_j^{k_j}} = 1 \pmod{p_j^{k_j}}.$$

3. For all j build the sequences $f_{j,\text{red}}$ and $f_{j,\text{rep}}$ as follows.
 4. Set $f_j(x) = \varphi_j(f(x))$, where $\varphi_j: \mathbb{Z}_n \rightarrow \mathbb{Z}_{\frac{n}{p_j^{k_j}}}$ is the canonic map.
 5. Let $p_j^{r_j}$ be the highest power of p_j dividing the period m of f . Determine the inverse $\left(\frac{m}{p_j^{r_j}}\right)^{-1}$ of $\frac{m}{p_j^{r_j}} \pmod{p_j^{k_j}}$.
 6. Write down the m elements of the period of f_j as a table with $\frac{m}{p_j^{r_j}}$ lines and $p_j^{r_j}$ columns (if $r_j = 0$, the table will have an unique column). Add a line given by the elements of any column and by multiplying by $\left(\frac{m}{p_j^{r_j}}\right)^{-1}$ module $p_j^{k_j}$.
 7. Let $f_{j,\text{red}}$ be the $p_j^{r_j}$ -periodic sequence defined by the line built at the previous step and build $f_{j,\text{rep}} = f_j - f_{j,\text{red}}$.
 8. By setting

$$f_{\text{red}} = \sum_{i=1}^N q_i \frac{n}{p_i^{k_i}} f_{i,\text{red}}, \quad f_{\text{rep}} = \sum_{i=1}^N q_i \frac{n}{p_i^{k_i}} f_{i,\text{rep}}.$$

we have the decomposition of any periodic sequence in a reducible and reproducible component.

EXAMPLE 5. Decomposition in reducible and reproducible components. Let f be the following \mathbb{Z}_{12} -valued sequence.

$$f = (0 \ 0 \ 7 \ 7 \ 4 \ 4 \ 3 \ 3 \ 4 \ 4 \ 7 \ 7).$$

The sequences corresponding to the decomposition of $\mathbb{Z}_{12} = G_3 \oplus G_4$ are respectively

$$f_1 = (0 \ 0 \ 1 \ 1 \ 1 \ 1 \ 1 \ 0 \ 0 \ 1 \ 1 \ 1 \ 1)$$

$$f_2 = (0 \ 0 \ 3 \ 3 \ 0 \ 0 \ 3 \ 3 \ 0 \ 0 \ 3 \ 3)$$

	0	0	1
	1	1	1
	0	0	1
	1	1	1
	2	2	1
$\div 4 \pmod{3}$	2	2	1
$\times 4 \pmod{12}$	8	8	4

						0	0	3	3									
						0	0	3	3									
						0	0	3	3									
						0	0	1	1									
				$\div 3 \pmod 4$		0	0	3	3									
				$\times -3 \pmod{12}$		0	0	3	3									
						8	8	4	8	8	4							
						0	0	3	3	0	0							
						f_{red}	8	8	7	11	8	4	11	11	4	8	11	7
						f_{rep}	4	4	0	8	8	0	4	4	0	8	8	0

7. The sets of values of reducible and reproducible sequences

THEOREM 19. *Let $\mathbb{Z}_n = \bigoplus_j G_j$ be the decomposition of \mathbb{Z}_n in p -groups. A subset M of \mathbb{Z}_n is the set of values of a reducible sequence if and only if two subsets $M_j \subset G_j$ do exist such that $M = \sum_j M_j$.*

This theorem is a direct consequence of the the following proposition.

PROPOSITION 20. *Let $\mathbb{Z}_n = \bigoplus_j G_j$ be the decomposition of \mathbb{Z}_n in p -groups and let $f = \sum_j f_j$ be the decomposition of the reducible sequence f corresponding to the decomposition of \mathbb{Z}_n . Then $f(\mathbb{Z}) = \sum_j f_j(\mathbb{Z})$.*

Proof. Clearly $f(\mathbb{Z}) \subset \sum_j f_j(\mathbb{Z})$. On the other hand, let $y_j = f_j(x_j) \in f_j(\mathbb{Z})$ and, for all integer p_j dividing n , let p_j^r be the highest power of p_j dividing the period m of f . According to the Theorem 7, f_j is p_j^r -periodic. A well known result of algebra gives the existence of $x \in \mathbb{Z}$ verifying $x = x_j \pmod{p_j^r}$ for all j . It follows that $f_j(x) = f_j(x_j)$ for all j , i.e.,

$$\sum_j y_j = \sum_j f_j(x_j) = \sum_j f_j(x) = f(x) \in f(\mathbb{Z}).$$

□

THEOREM 21. *For any subset $M \subset \mathbb{Z}_n$ it does exist an element $u \in \mathbb{Z}_n$ and a reproducible sequence f such that $f(\mathbb{Z}) = u + M$.*

Proof. It does exist an integer m such that $m \wedge n = 1$ and the m -periodic sequence g such that $g(\mathbb{Z}) = M$. Since $m \wedge n = 1$, the inverse m^{-1} of $m \pmod n$ does exist. We set $u = -m^{-1} \sum_{i=0}^{m-1} g(i)$ and $f(x) = u + g(x)$. It follows that $\sum_{i=0}^{m-1} f(i) = 0$, which gives the reproducibility of f due to the Corollary 18.

□

Remark 2. Generally, it does not exist a reproducible sequence f such that $f(\mathbb{Z}) = M$. For example, take $M = \{3, 9\} \subset \{0, 3, 6, 9\} \subset \mathbb{Z}_{12}$. By supposing that a reproducible f may exist such that $f(\mathbb{Z}) = M$, let $m = 2^r d$ be the period of f , where d is odd. By Theorem 14 one has $\sum_{i=0}^{d-1} f(2^r i) = 0$. Since $f(2^r i) \in \{3, 9\}$, it follows that some integers k_1, k_2 must exist such that $3k_1 + 9k_2 = 0 \pmod{12}$, $k_1 + k_2 = d$. This means that

$$\begin{aligned} 3k_1 + 9(d - k_1) &= 0 \pmod{12}, \\ 6k_1 + 9d &= 0 \pmod{12}, \\ 2 \cdot 9d = -2 \cdot 6k_1 &= 0 \pmod{12}, \\ 3d &= 0 \pmod{2}. \end{aligned}$$

But because of the oddness of d one would conclude that $3 = 0 \pmod{2}$, which is absurd.

EXAMPLE 6. Modal classes of sets of values of \mathbb{Z}_{12} -valued reducible sequences

$$\begin{aligned} \{0\} &+ \{0\} &= \{0\}, \\ \{0\} &+ \{6, 6\} &= \{6, 6\}, \\ \{0\} &+ \{9, 3\} &= \{9, 3\}, \\ \{0\} &+ \{6, 3, 3\} &= \{6, 3, 3\}, \\ \{0\} &+ \{3, 3, 3, 3\} &= \{3, 3, 3, 3\}, \\ \{8, 4\} &+ \{0\} &= \{8, 4\}, \\ \{8, 4\} &+ \{6, 6\} &= \{4, 2, 4, 2\}, \\ \{8, 4\} &+ \{9, 3\} &= \{5, 3, 1, 3\}, \\ \{8, 4\} &+ \{6, 3, 3\} &= \{1, 2, 1, 3, 2, 3\}, \\ \{8, 4\} &+ \{3, 3, 3, 3\} &= \{1, 2, 1, 2, 1, 2, 1, 2\}, \\ \{4, 4, 4\} &+ \{0\} &= \{4, 4, 4\}, \\ \{4, 4, 4\} &+ \{6, 6\} &= \{2, 2, 2, 2, 2, 2\}, \\ \{4, 4, 4\} &+ \{9, 3\} &= \{3, 1, 3, 1, 3, 1, 3, 1\}, \\ \{4, 4, 4\} &+ \{6, 3, 3\} &= \{1, 1, 2, 1, 1, 2, 1, 1, 2\}, \\ \{4, 4, 4\} &+ \{3, 3, 3, 3\} &= \{1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1\}. \end{aligned}$$

8. The reproducibility of rarefied and repeating sequences

Let f be a periodic \mathbb{Z}_n -valued sequence and let d be a positive integer.

DEFINITION 6. The *rarefied sequence* f_{rar} obtained from the sequence f is given by the following relation:

$$f_{\text{rar}}(x) = \begin{cases} f\left(\frac{x}{d}\right) & \text{if } x \equiv 0 \pmod{d}, \\ 0 & \text{if } x \not\equiv 0 \pmod{d}. \end{cases}$$

The integer d is called the *factor of insertion*.

DEFINITION 7. The *repeating sequence* obtained from f is the sequence f_{rpt} defined by

$$f_{\text{rpt}}(x) = f\left(\left[\frac{x}{d}\right]\right)$$

where $[r]$ means the highest integer $\leq r$. The integer d is called the *factor of repetition*.

We will write $f_{\text{rar},d}$ and $f_{\text{rpt},d}$ when we want to make evident the presence of the factor d . It follows that

$$\begin{aligned} f_{\text{rar},d_1d_2} &= (f_{\text{rar},d_1})_{\text{rar},d_2}, \\ f_{\text{rpt},d_1d_2} &= (f_{\text{rpt},d_1})_{\text{rpt},d_2}. \end{aligned} \tag{1}$$

Problem. Find the relationship between the reproducibility of f , f_{rar} and f_{rpt} .

LEMMA 22.

$$f_{\text{rpt}} = \sum_{i=0}^{d-1} T^i f_{\text{rar}}.$$

P r o o f. This result is a trivial consequence of the definition of rarefied and repeating sequences. □

DEFINITION 8. A sequence f is called d -reproducible if $(T^d - 1)^m f = 0$ for an integer $m \geq 1$.

LEMMA 23. *If f is a d -reproducible sequence then f is reproducible.*

P r o o f. Let f be d -reproducible. We write down the decomposition

$$f = f_{\text{red}} + f_{\text{rep}}, \quad f_{\text{red}} \in \text{Red}, \quad f_{\text{rep}} \in \text{Rep}.$$

It follows that $(T^d - 1)^m f = f$ and $(T - 1)^m f_{\text{red}} = D^m f_{\text{red}} = 0$. According to the identity

$$T^d - 1 = P(T)(T - 1), \quad P(T) = \sum_{i=0}^{d-1} T^i, \tag{2}$$

one has that

$$\begin{aligned} f &= (T^d - 1)^m f = P(T)^m (T - 1)^m f \\ &= P(T)^m (T - 1)^m f_{\text{red}} + P(T)^m (T - 1)^m f_{\text{rep}} \\ &= P(T)^m (T - 1)^m f_{\text{rep}} \in \text{Rep}. \end{aligned}$$

□

THEOREM 24. *The reproducibility of f is equivalent to that of f_{rar} .*

Proof. If f is reproducible, the relation

$$(T^d - 1)f_{\text{rar}} = ((T - 1)f)_{\text{rar}}$$

shows that f_{rar} is d -reproducible, i.e., reproducible by the Lemma 23.

Now let suppose that f_{rar} is reproducible. Then $f = \sum_j f_j$ is the decomposition of f corresponding to the decomposition of \mathbb{Z}_n in p -groups. Since the decomposition of f_{rar} is $\sum_j (f_j)_{\text{rar}}$, it is enough to consider the case where $n = p^k$, p prime. According to (1), we do not lose generality by taking d which is prime. By writing down the decomposition

$$f = f_{\text{red}} + f_{\text{rep}}, \quad f_{\text{red}} \in \text{Red}, \quad f_{\text{rep}} \in \text{Rep},$$

$$f_{\text{rar}} = (f_{\text{red}})_{\text{rar}} + (f_{\text{rep}})_{\text{rar}}.$$

one has that $f_{\text{rar}} \in \text{Rep}$ and $(f_{\text{rep}})_{\text{rar}} \in \text{Rep}$ because of i), i.e., $(f_{\text{red}})_{\text{rar}} \in \text{Rep}$ and one is led to show that if $f \in \text{Red}(\mathbb{Z}_{p^k})$ and $f_{\text{rar}} \in \text{Rep}$, then $f = 0$. Now, since f is reducible, it has to be p^m -periodic. Therefore f_{rar} is dp^m -periodic. If $d = p$ then f_{rar} is reducible too, i.e., $f_{\text{rar}} \in \text{Rep} \cap \text{Red} = \{0\}$ which means that $f = 0$. If $d \neq p$, the reproducibility of f_{rar} is expressed by the relations

$$\sum_{i=0}^{d-1} f_{\text{rar}}(ip^m + j) = 0, \quad 0 \leq j < p^m. \quad (3)$$

The only terms different from zero in the left side of (3) are those for which

$$ip^m + j = 0 \pmod{d}. \quad (4)$$

Since $d \wedge p = 1$, the equation (4) in the variable i has an unique solution $i(j)$ for all j . It follows that in the left part of (3), just one term is zero, i.e.,

$$f_{\text{rar}}(i(j)p^m + j) = f\left(\frac{i(j)p^m + j}{d}\right).$$

This means that

$$f\left(\frac{i(j)p^m + j}{d}\right) = 0, \quad 0 \leq j < p^m. \quad (5)$$

The numbers $(i(j)p^m + j)/d$, $0 \leq j < p^m$, form a fundamental system of classes mod p^m . In fact, if

$$\frac{i(j_1)p^m + j_1}{d} = \frac{i(j_2)p^m + j_2}{d} \pmod{p^m},$$

then

$$i(j_1)p^m + j_1 = i(j_2)p^m + j_2 \pmod{p^m},$$

i.e., $j_1 = j_2 \pmod{p^m}$ which means that $j_1 = j_2$. According to (5) and to the p^m -periodicity of f , one obtains $f = 0$. \square

LEMMA 25. *If $d \wedge n = 1$ and if $\sum_{i=0}^{d-1} T^i f$ is reducible then f is reducible.*

P r o o f. Let's write down the decomposition

$$f = f_{\text{red}} + f_{\text{rep}}, \quad f_{\text{red}} \in \text{Red}, \quad f_{\text{rep}} \in \text{Rep},$$

$$P(T)f = P(T)f_{\text{red}} + P(T)f_{\text{rep}},$$

where $P(T)$ is given by (2). Since $P(T)f \in \text{Rep}$ (by hypothesis) and also $P(T)f_{\text{rep}} \in \text{Rep}$, one has $P(T)f_{\text{red}} \in \text{Red} \cap \text{Rep} = \{0\}$. This leads to the relation

$$(T^d - 1)f_{\text{red}} = (T - 1)P(T)f_{\text{red}} = 0,$$

which means that f_{red} is d -periodic. Since f_{red} is reducible, it is also m -periodic where m is an integer that divides n . This means that f is $d \wedge m$ -periodic, which gives the 1-periodicity, i.e., f is a constant. In conclusion

$$f = z + f_{\text{rep}}, \quad z \in \mathbb{Z}_n, \quad f_{\text{rep}} \in \text{Rep},$$

$$P(T)f = dz + P(T)f_{\text{rep}}.$$

By using the same argument $dz \in \text{Red} \cap \text{Rep} = \{0\}$, i.e., $z = 0$ since the multiplication by d is an automorphism of \mathbb{Z}_n . □

THEOREM 26. *The reducibility of f is equivalent to that of f_{rpt} .*

P r o o f. If f is reducible, then (Theorem 24) f_{rar} is reducible, which means that (Lemma 22) f_{rpt} is reducible, too.

Let's now suppose that f_{rpt} is reducible. As in the proof of Theorem 24 one may just consider the case where f takes its values in a p -group and d is prime. If $d = p$ one uses the same argument as in the proof of Theorem 24. If $d \neq p$, then (Lemmas 22 and 25) f_{rar} is reducible, which leads to the conclusion that (Theorem 24) f is reducible too. □

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