

# Master I.C.A.

 <u>Master I.C.A.</u>		<b>A</b> rt <b>S</b> ciences <b>T</b> echnologies
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Traitement interactif de l'image et du son

## Méthodes mathématiques pour la création musicale - II canons rythmiques mosaïques

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# Le modèle des canons rythmiques mosaïques



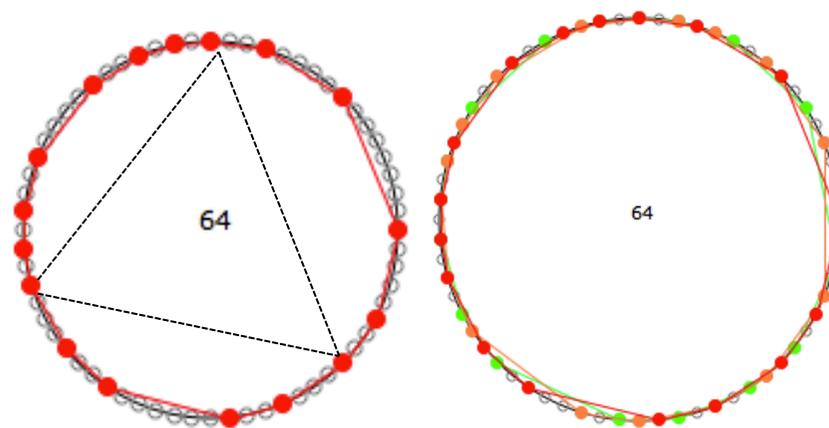
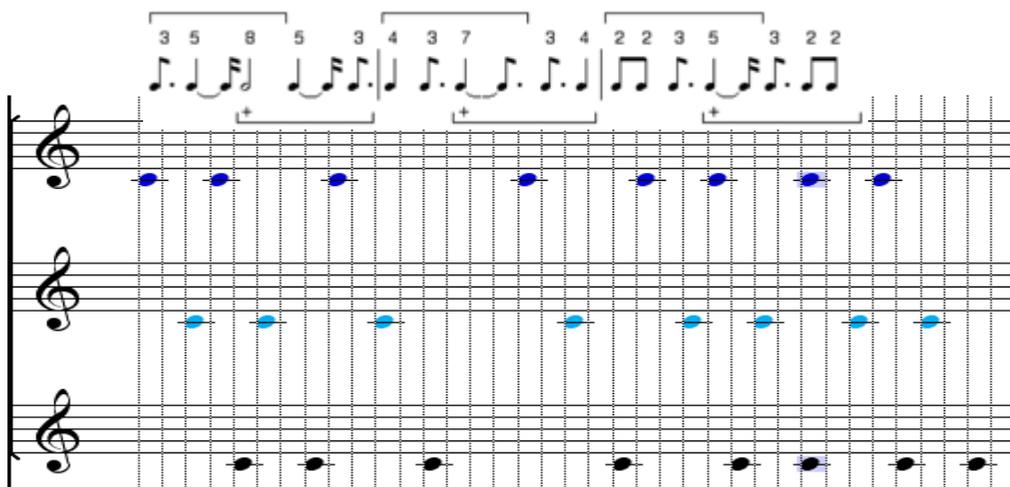
 *Harawi* (1945)



*Visions de l'Amen* (1943)



O. Messiaen



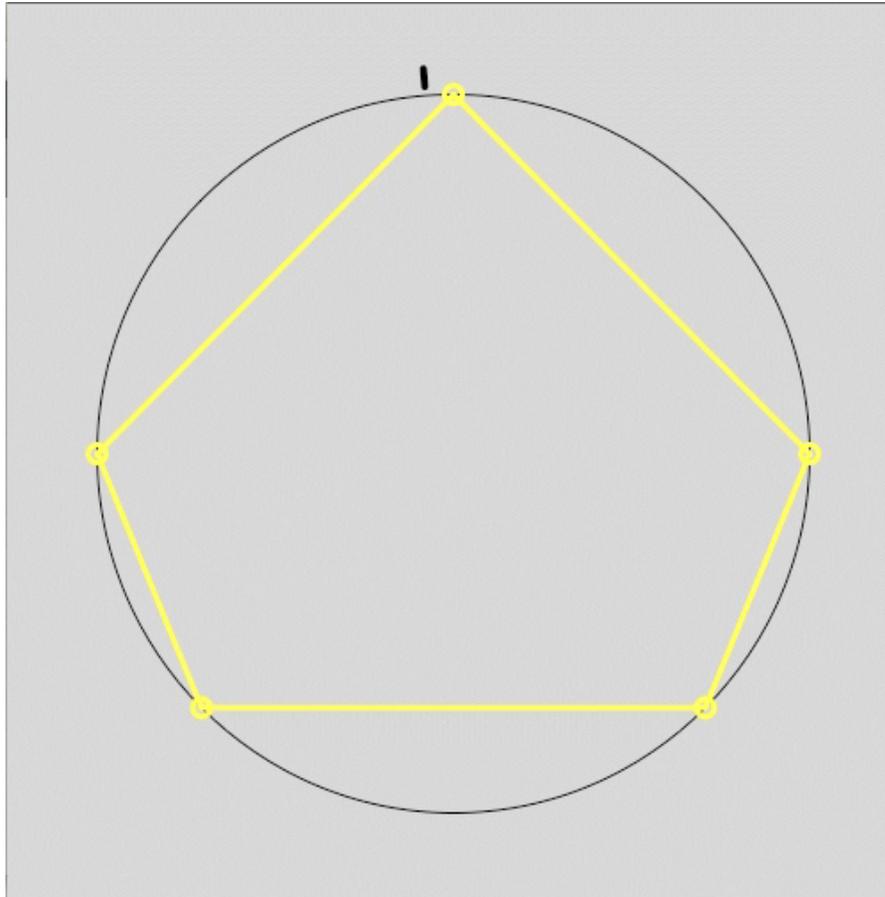
« ...il résulte de tout cela que les différentes sonorités se mélangent ou s'opposent de manières très diverses, jamais au même moment ni au même endroit [...]. C'est du désordre organisé »

O. Messiaen : *Traité de Rythme, de Couleur et d'Ornithologie*, tome 2, Alphonse Leduc, 1992.

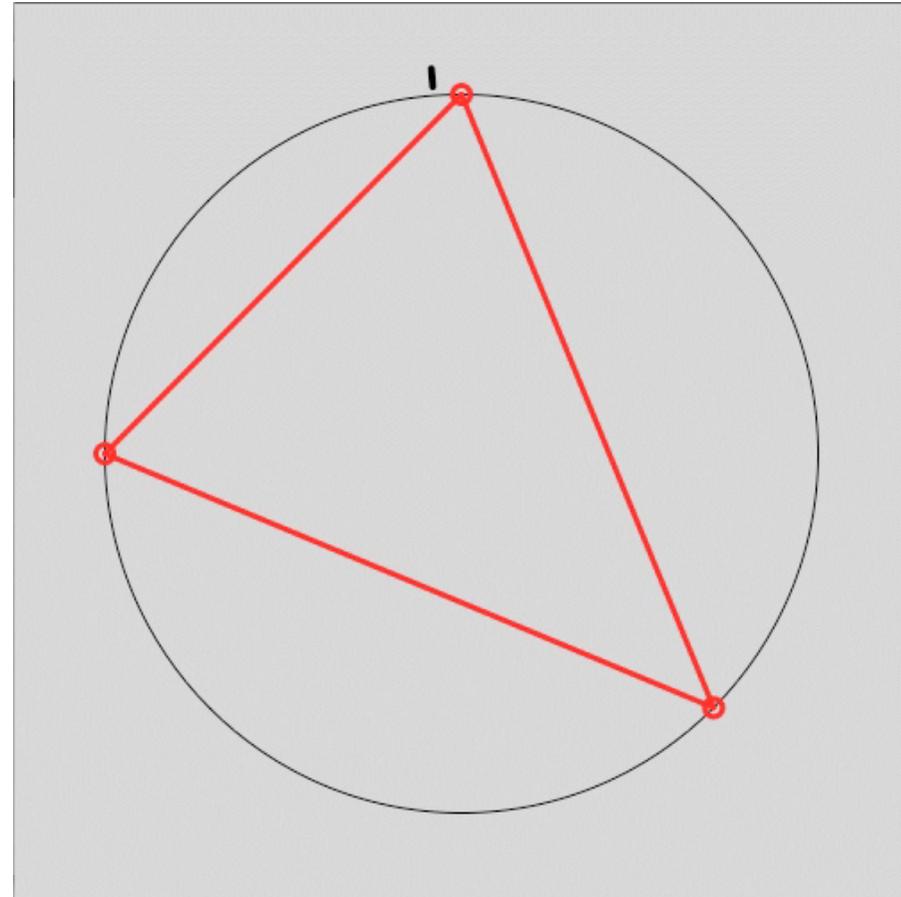
# Deux rythmes traditionnels qui ne paivent pas

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## El cinquillo

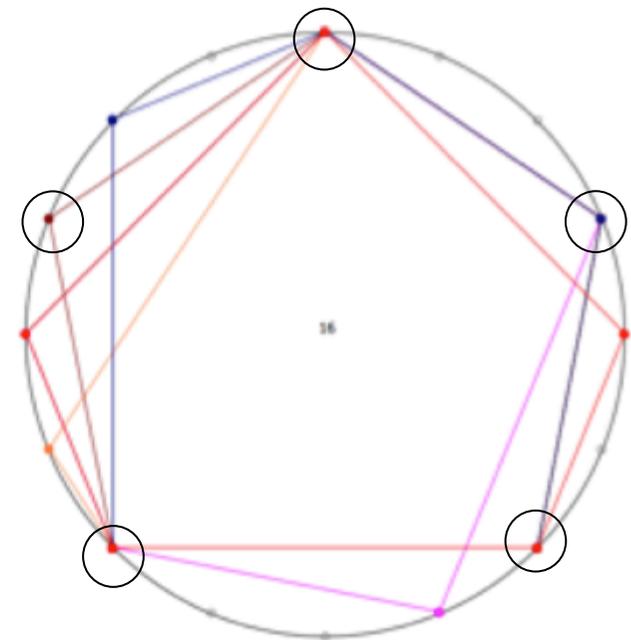
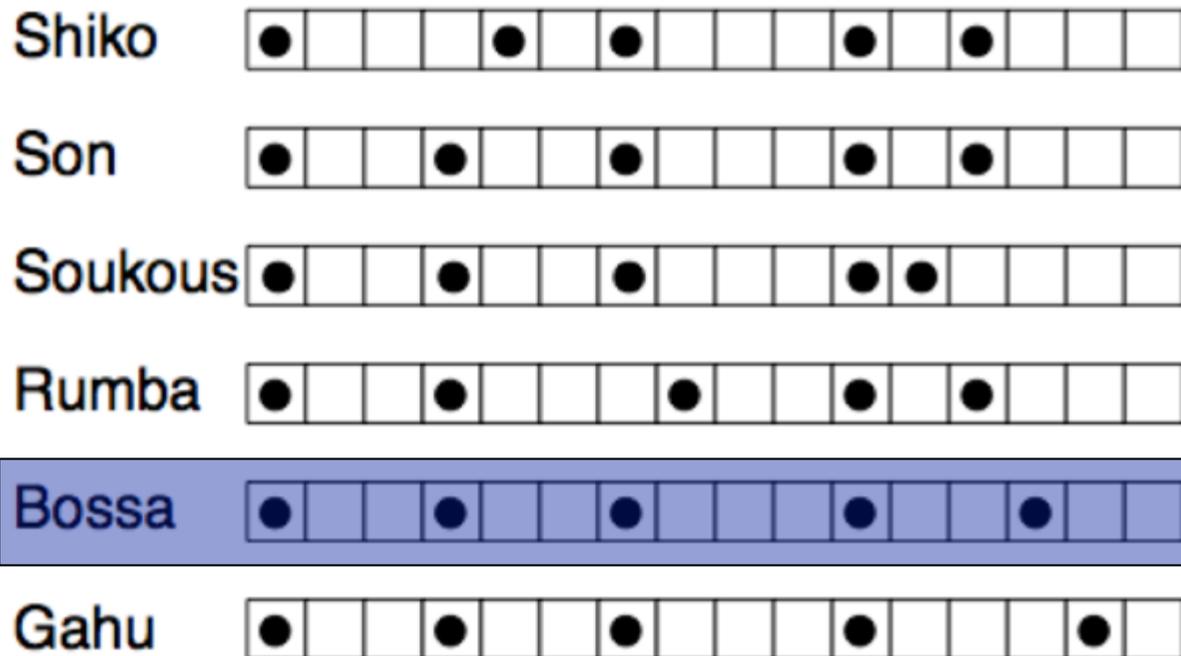
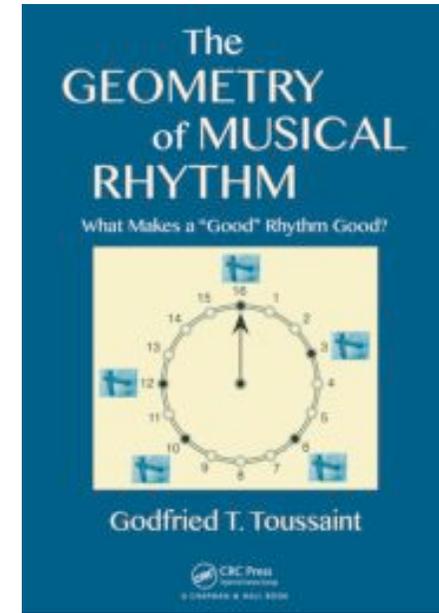


## El trecillo

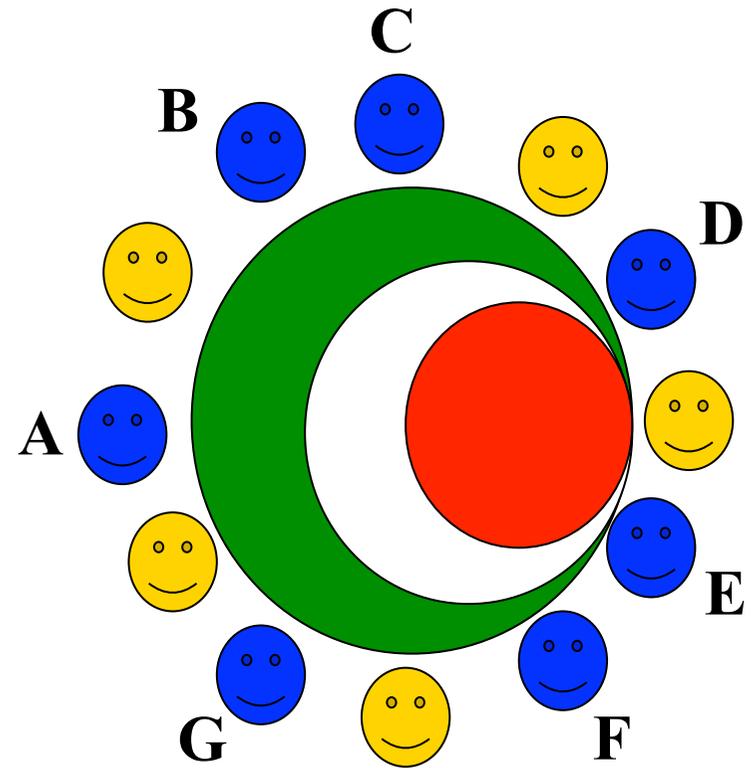
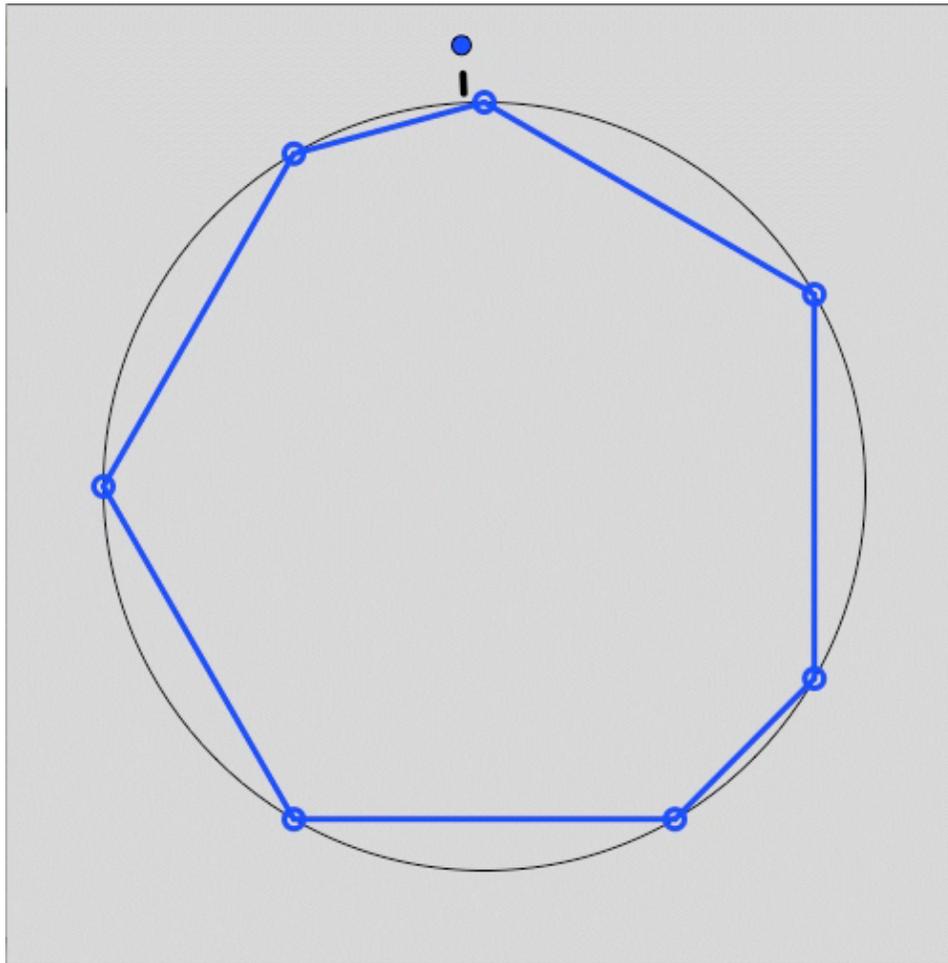


# Autres exemples de rythmes qui ne pavent pas

Godfried T. Toussaint, The Geometry of Musical Rhythm: What Makes a "Good" Rhythm Good?, Chapman and Hall/CRC, 2013

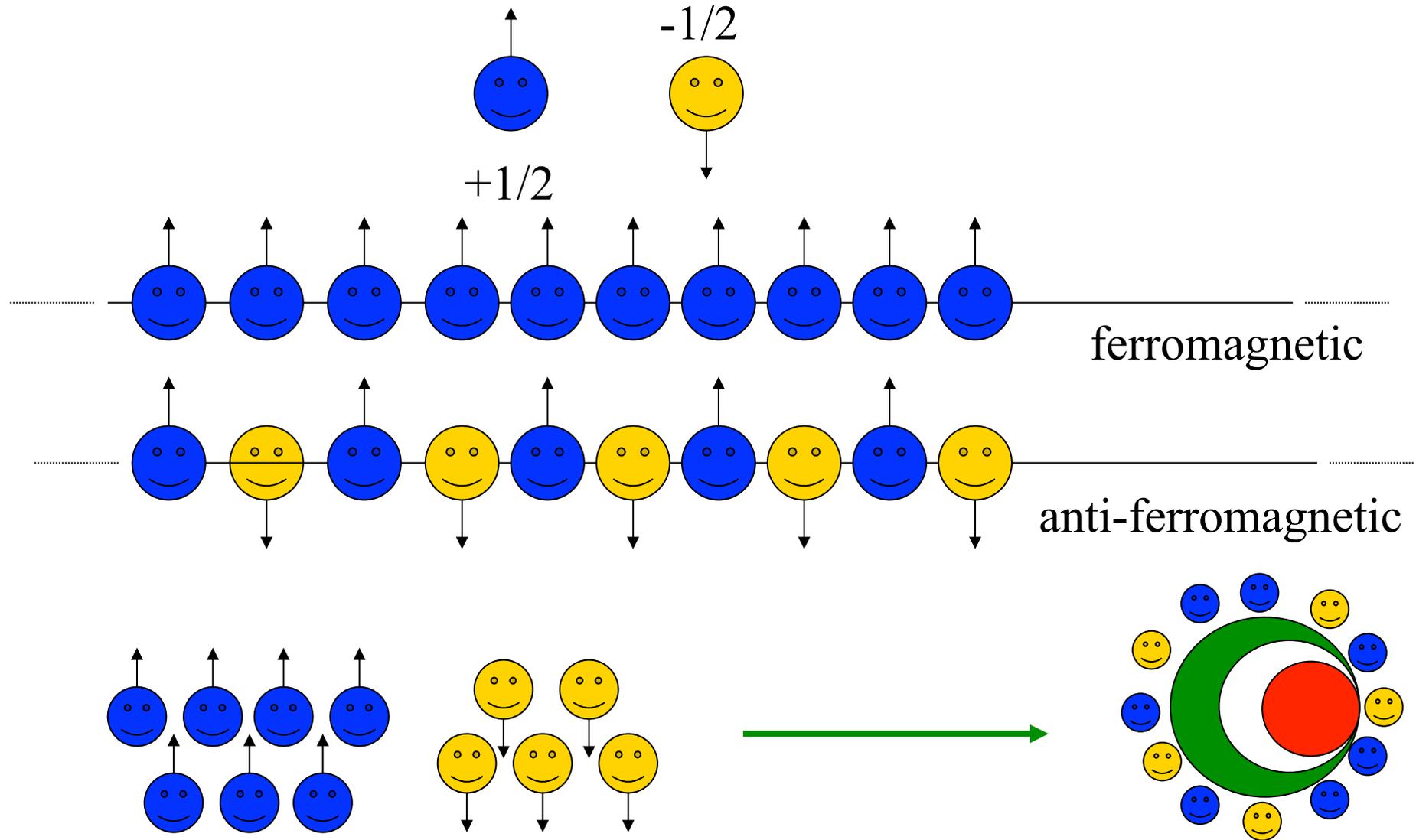


# Abadja/Bembé ou rythme « diatonique »



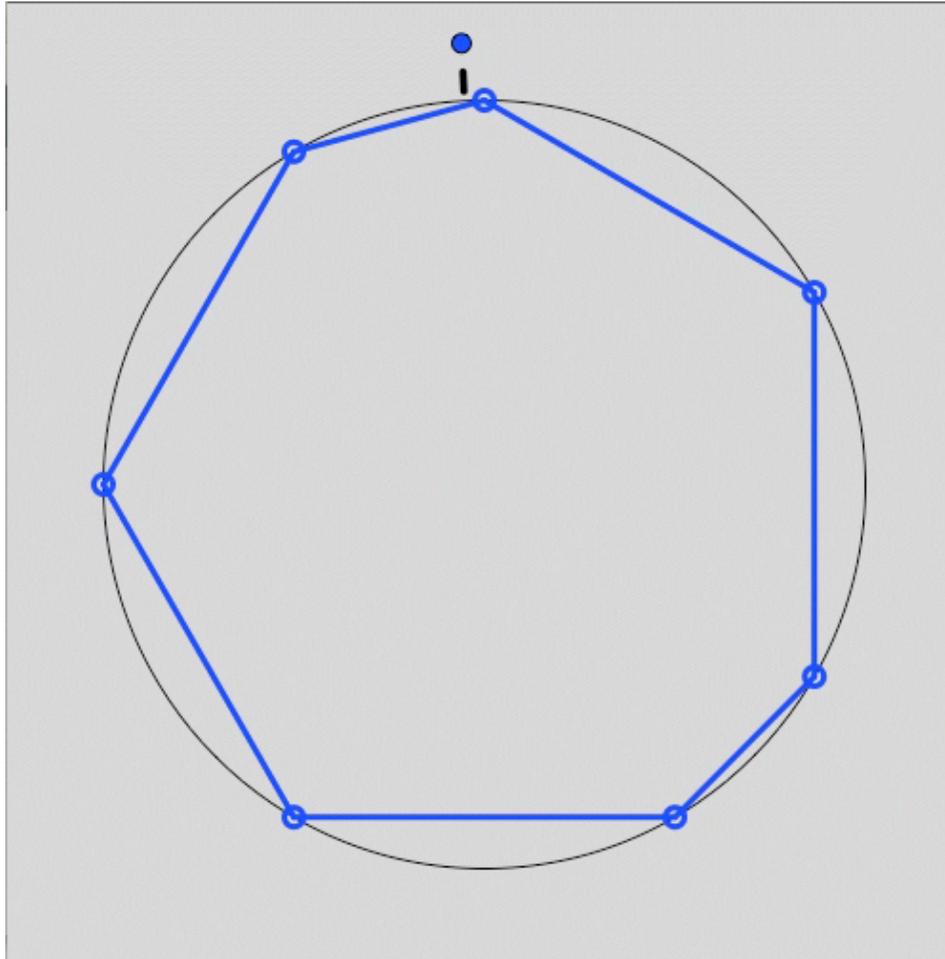
# The one-dimensional antiferromagnetic spin-1/2 Ising Model

Jack Douthett & Richard Krantz, "Energy extremes and spin configurations for the one-dimensional antiferromagnetic Ising model with arbitrary-range interaction", *J. Math. Phys.* 37 (7), July 1996



# ME-sets avec cardinalité première avec $n$

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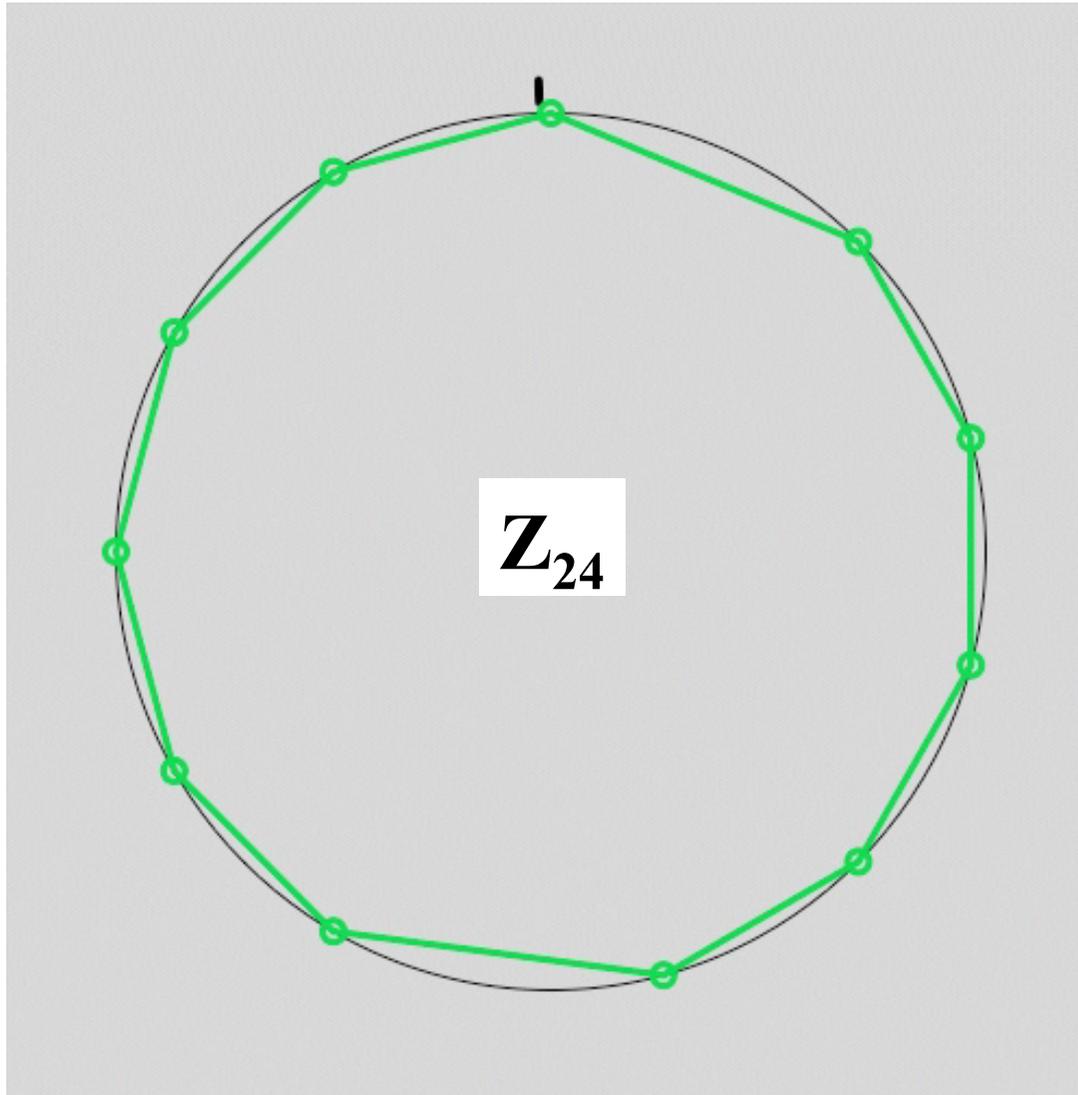


Calcul explicite d'un ME-set  $A$  de cardinalité  $d$  dans  $\mathbb{Z}/n\mathbb{Z}$  avec  
 $\text{PGCD}(d, n) = 1$   
 $\rightarrow A = d^{-1}\{0, 1, \dots, d-1\}$

Exercice : retrouver la gamme diatonique



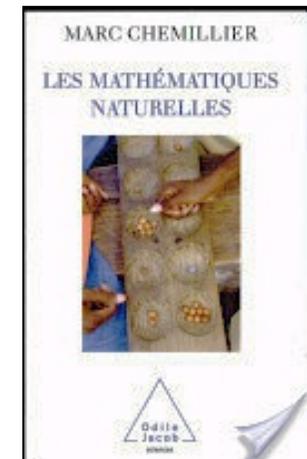
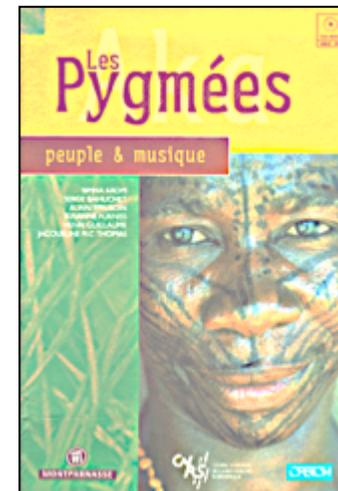
# Imparité rythmique et musiques de tradition orale



Simha Arom



Marc Chemillier



**0-(3 2 2 2 2 3 2 2 2 2 2)**

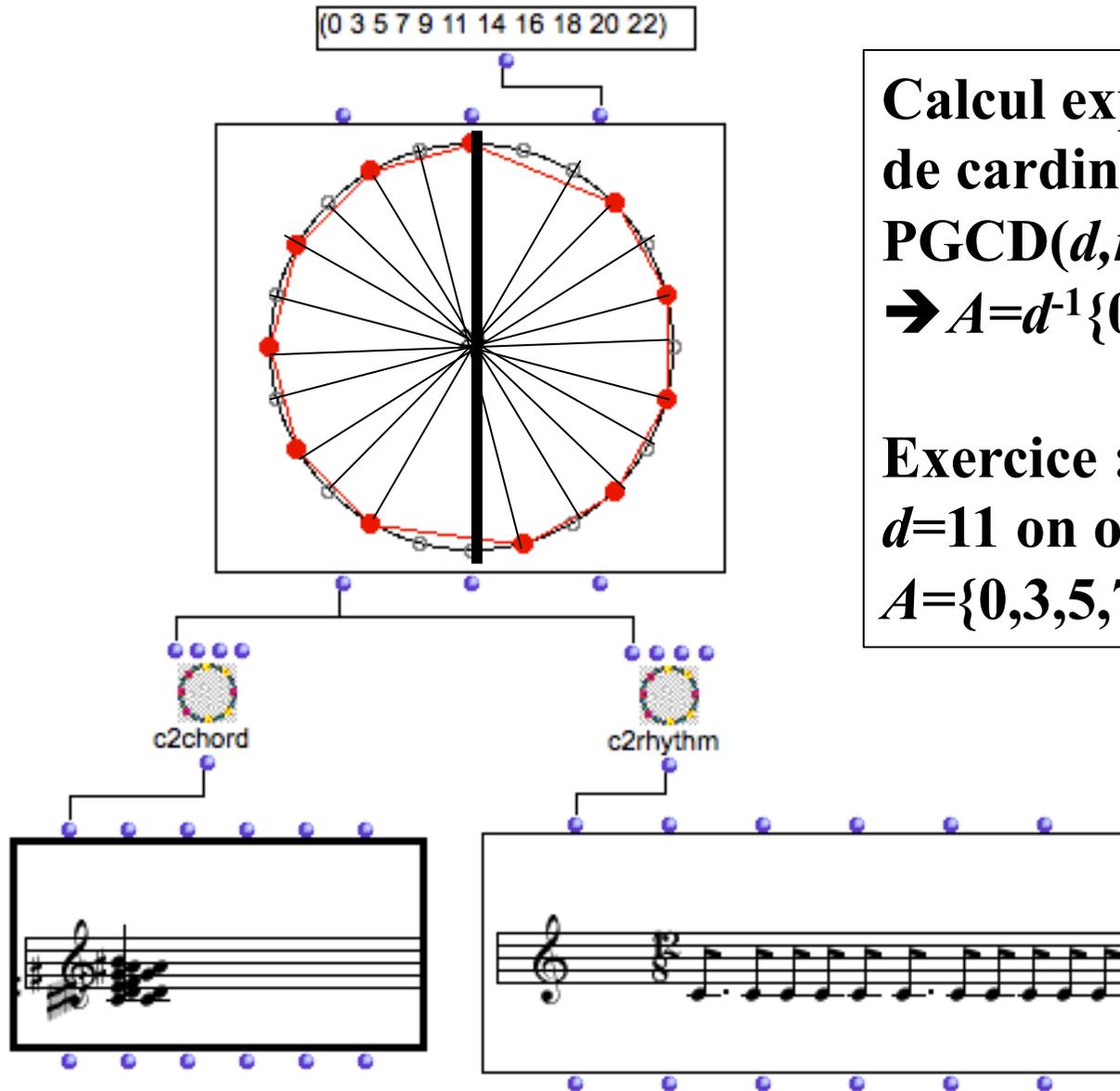
**musimédiane**

publiée avec le concours de la SFAM

revue audiovisuelle et multimédia d'analyse musicale

# Rythmes asymétriques et ME-sets

(Simha Arom & Marc Chemillier)



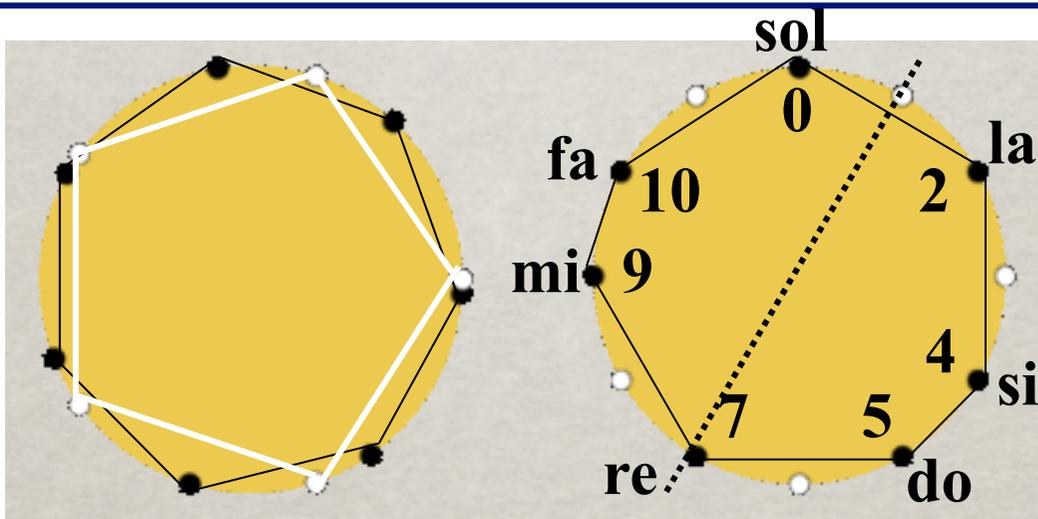
Calcul explicite d'un ME-set  $A$   
de cardinalité  $d$  dans  $\mathbb{Z}/n\mathbb{Z}$  avec  
 $\text{PGCD}(d,n)=1$

$$\rightarrow A = d^{-1}\{0, 1, \dots, d-1\}$$

Exercice : vérifier que pour  
 $d=11$  on obtient :

$$A = \{0, 3, 5, 7, 9, 11, 14, 16, 18, 20, 22\}$$

# Ensembles bien repartis (ME sets)



Gamme diatonique:

{0, 2, 4, 5, 7, 9, 10}

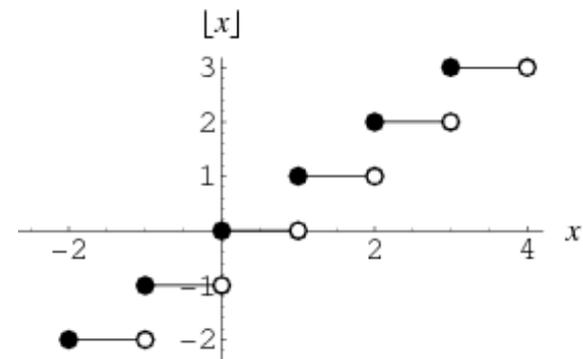
Gamme pentatonique:

{1, 3, 6, 8, 11}

**Definition** (Clough-Myerson-Douthett) A set  $A$  with cardinality  $d$  in a given equal tempered space  $\mathbf{Z}_c$  is maximally even if  $A = \{a_k\}$

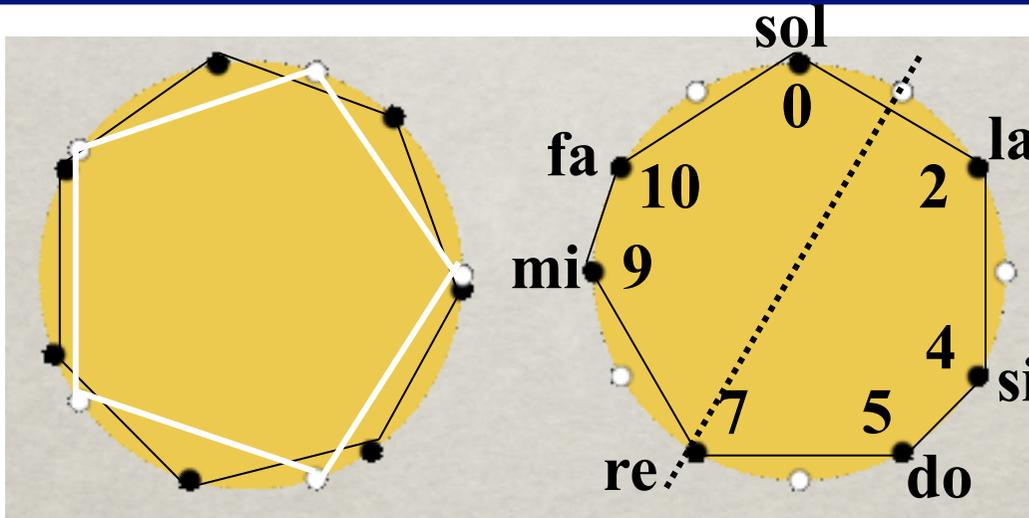
$$a_k = J_{c,d}^\alpha(k) = \left\lfloor \frac{kc + \alpha}{d} \right\rfloor$$

where  $\alpha \in \mathbf{R}$   
 $\lfloor x \rfloor$  is the integer part of  $x$



$$J_{12,7}^5 = \left\{ \left\lfloor \frac{12k + 5}{7} \right\rfloor \right\}_{k=0}^6 = \{0, 2, 4, 5, 7, 9, 11\}$$

# Ensembles bien repartis (ME sets)



Gamme diatonique:

$\{0, 2, 4, 5, 7, 9, 10\}$

Gamme pentatonique:

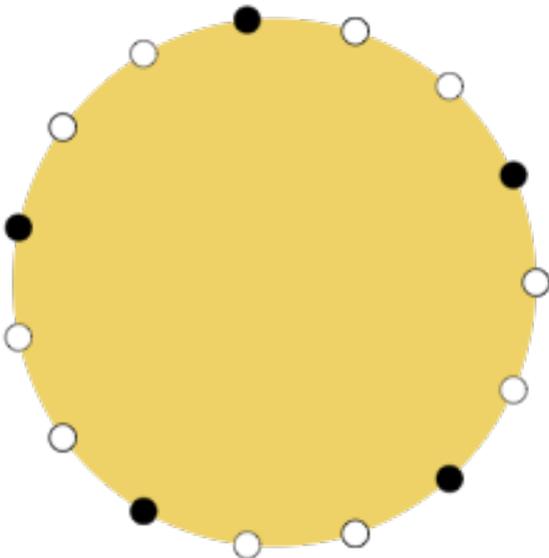
$\{1, 3, 6, 8, 11\}$

**Definition** (Amiot, 2005) A set  $A$  with cardinality  $d$  given equal tempered space  $\mathbf{Z}_c$  is maximally even if  $|F_A(d)| \geq |F_B(d)|$  for all subsets  $B$  of cardinality  $d$  in  $\mathbf{Z}_c$ .

$$\text{where } F_{\text{set}}(t) := \sum_{k \in \text{set}} e^{2i\pi kt/12}$$

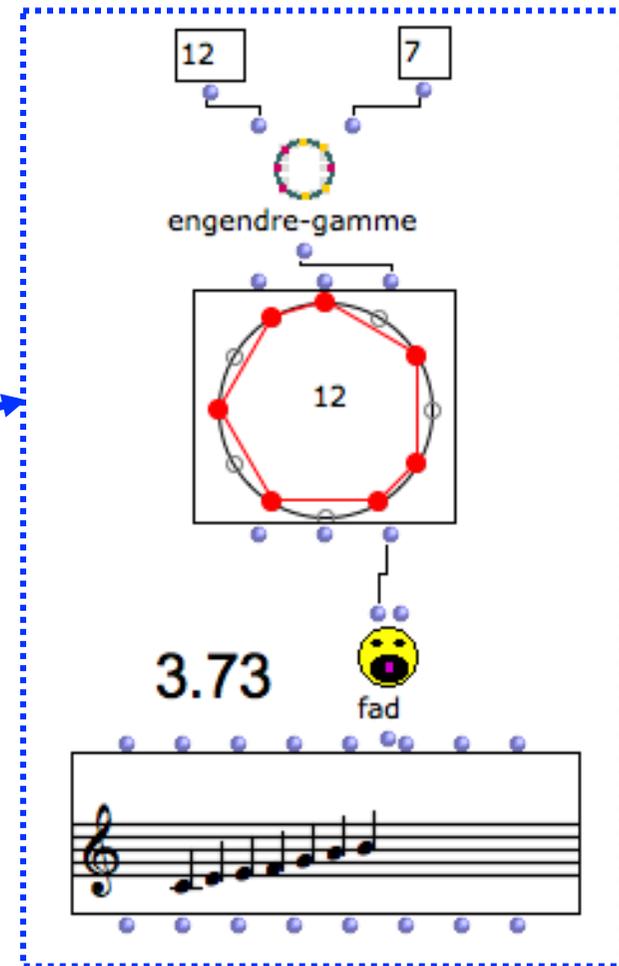
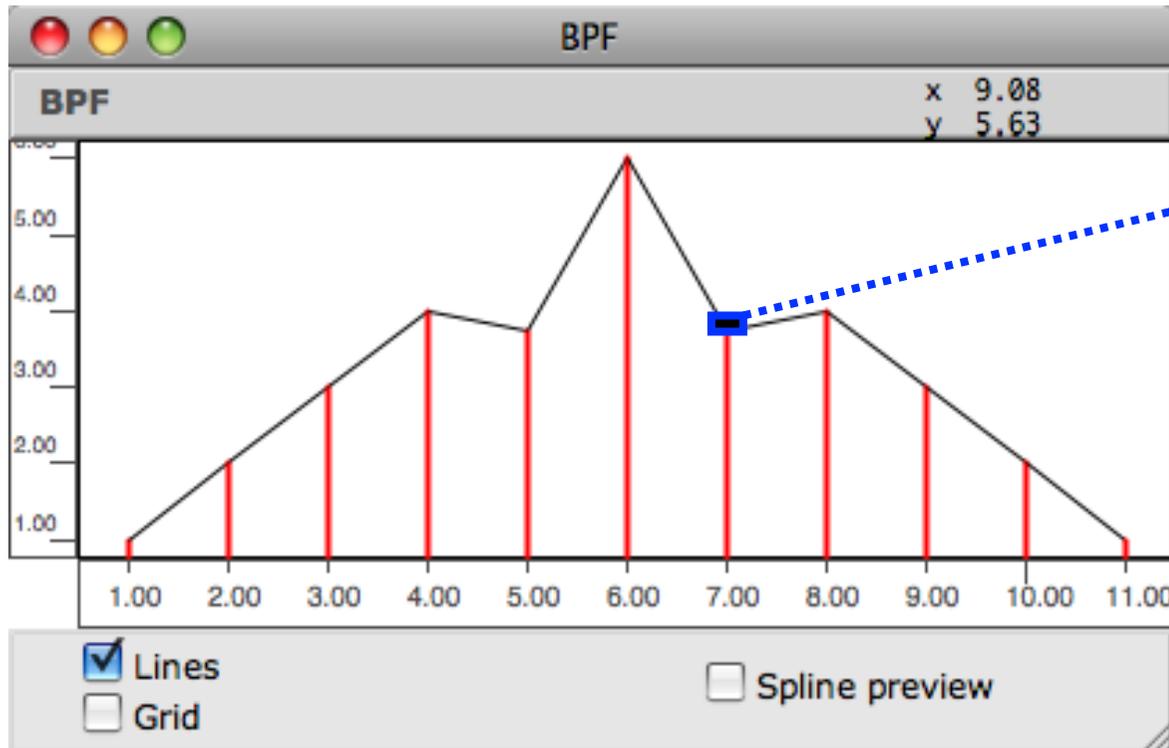
$$|F_A(5)| = 1 + 1 + 1 + 1 + 1 = 5$$

En général,  $|F_A(t)| \leq \#A$

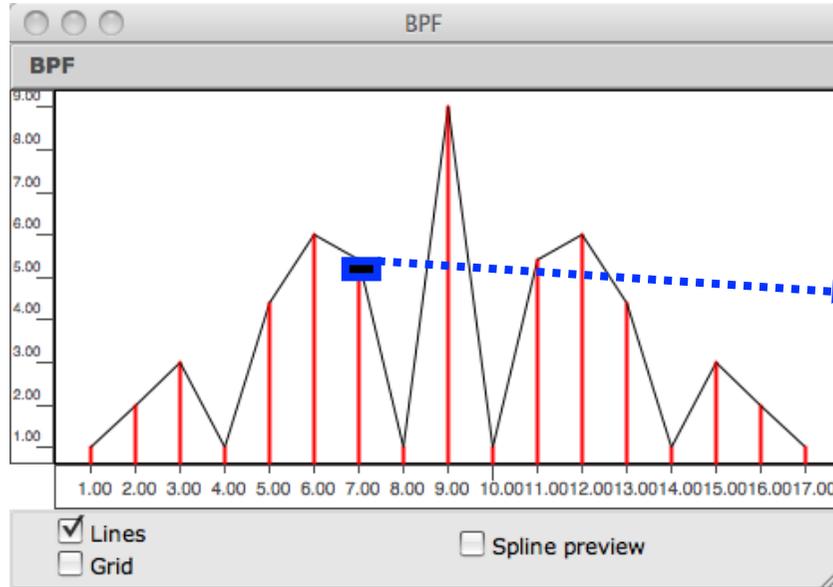


# Nouvelle classification des structures musicales à l'aide de la DFT

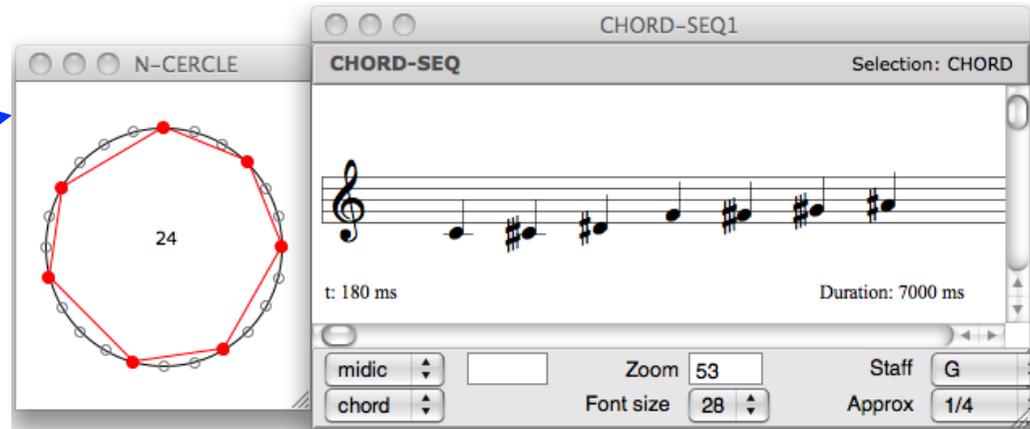
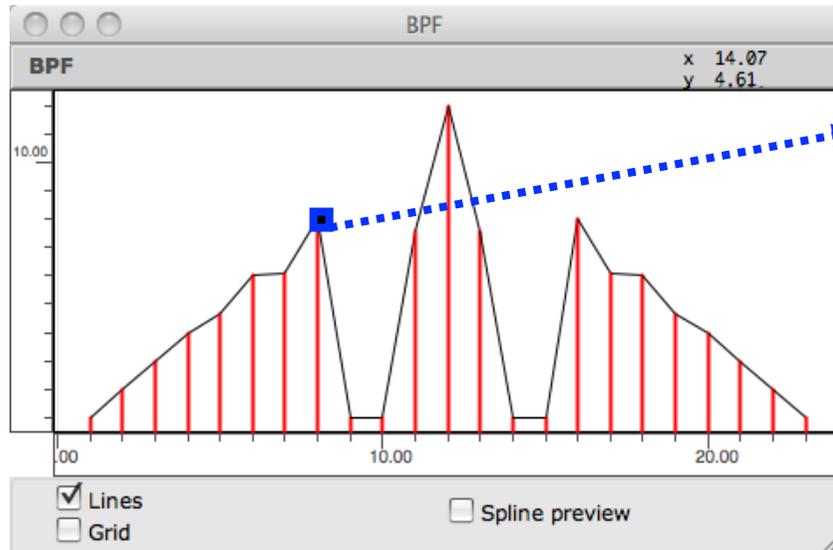
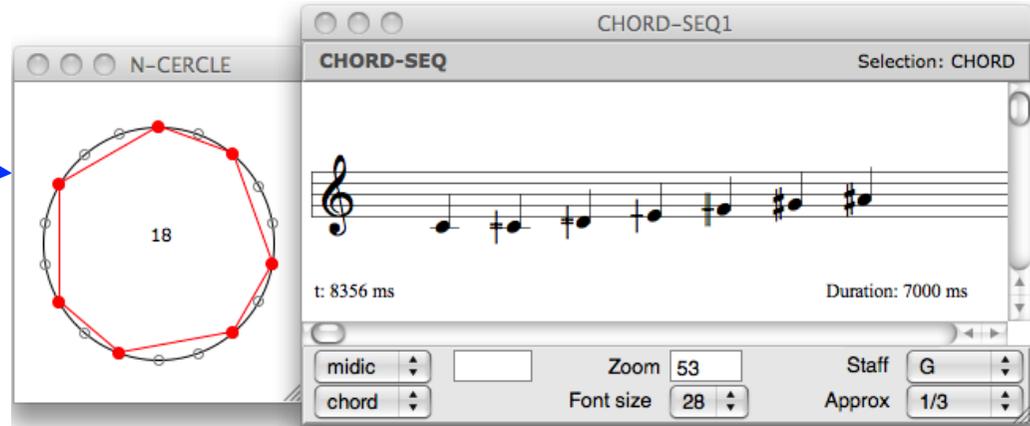
$$\mathcal{F}_A : t \mapsto \sum_{k \in A} e^{-2i\pi kt/c}$$



# Classification des structures microtonales à l'aide de la DFT



*Systeme en tiers de ton*



*Systeme en quarts de ton*

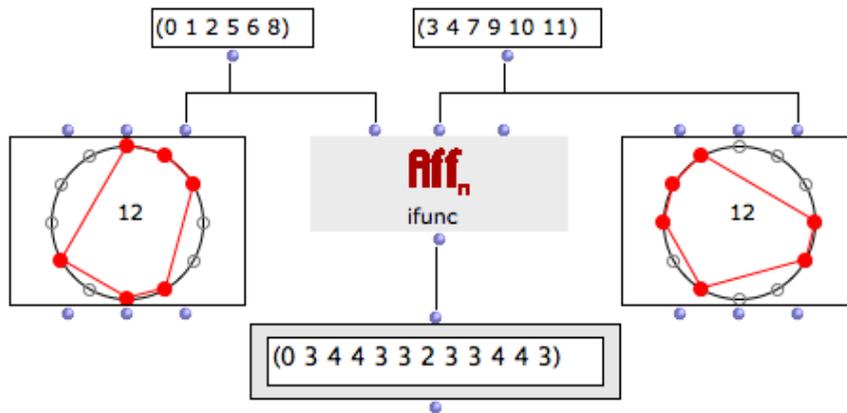
# Relation Z, DFT et théorie de l'homométrie

- Le contenu intervallique est équivalent à un produit de convolution de fonctions caractéristiques (Lewin, 1958)

$$IC_A(k) = (1_A \star 1_{-A})(k)$$

$$1_A \star 1_{-B}(k) = \sum_{x \in \mathbb{Z}_c} 1_A(x) \cdot 1_{-B}(k-x) = \sum_{x \in \mathbb{Z}_c} 1_A(x) \cdot 1_B(x-k) = \sum_{\substack{(x,y) \in A \times B \\ y=x-k}} 1$$

$$\text{IFUNC}(A, B)(k) = \#\{(a, b) \in A \times B \mid a = b + k\}$$



$$\text{IFUNC}(A, A) = IC_A$$

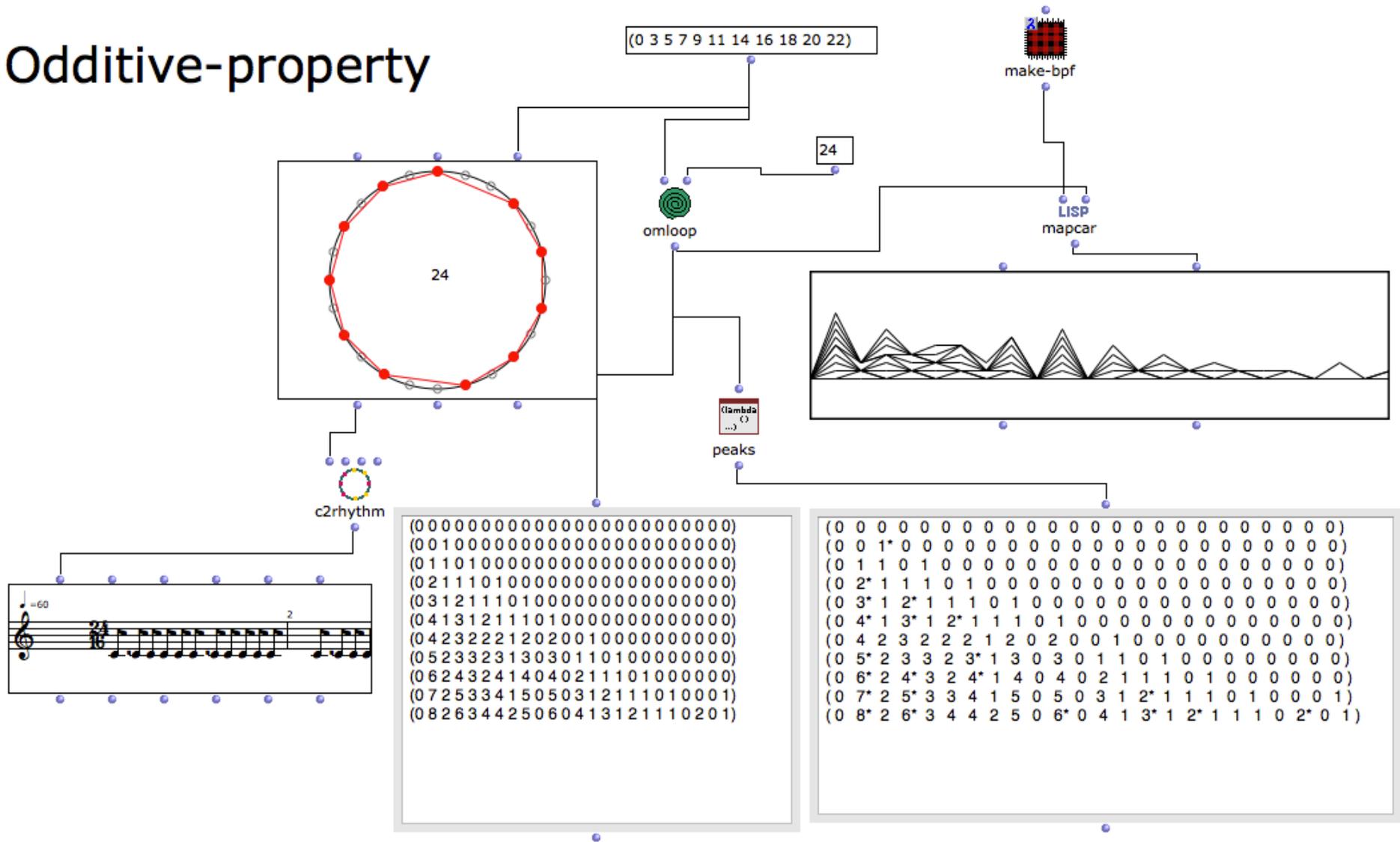
$$\text{IFUNC}(A, B)(3) = \#\{(0, 9), (1, 10), (2, 11), (6, 3)\}$$





# Exemple de modélisation informatique : l'imparité rythmique

## Odditive-property

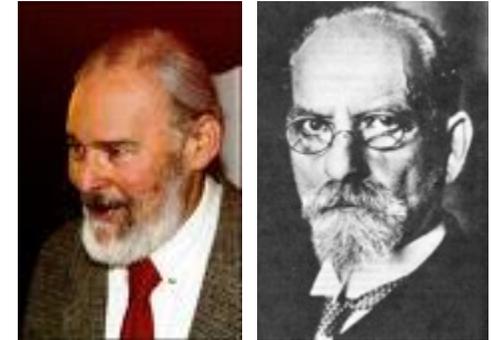


Unfolding Rhythmic Interval Vector



# Interprétation rythmique de la fonction IFUNC

- David Lewin, “Some Investigations into Foreground Rhythmic and Metric Patterning,” *Music Theory: Special Topics*, ed. Richmond Browne (New York: Academic Press, 1981), 101–136.
- David Lewin (1986), « Music Theory, Phenomenology, and Modes of Perception », *Music Perception*, 3, 327-382.



« L'article [Lewin 1981] développe un modèle numérique qui compte, à chaque position-comme-maintenant  $t$  [*“now”-time  $t$* ] le nombre de laps de temps [*time-spans*] que je retiens d'un passé récent pertinent ayant (eu) durée égale à  $d$ . On construit ainsi une fonction  $W(d,t)$  qui donne un vecteur progressif d'intervalles de durées [*“unfolding durational-interval vector”*] au fur et à mesure que le curseur-présent  $t$  avance. Le concept à la base de cette construction utilise un modèle husserlien bidimensionnel de la perception du temps [*Husserlian two-dimensional model of perceptual time*], un modèle qui exprime aussi bien les « impressions primaires » chez Husserl, impressions qui suivent le curseur-présent  $t$ , mais aussi les « retentions » chez Husserl, projections d'instant temporels passés [*projections of remembered past times*] (ainsi que durées passées) dans ma conscience présente [*into my present consciousness*]. Ensuite, dans le même article, j'envisage en quelque sorte les « protentions » chez Husserl, des projections d'attentes futures dans la conscience du présent » (Lewin, 1986/2006) »

# Relation Z, DFT et théorie de l'homométrie

- Le contenu intervallique est équivalent à un produit de convolution de fonctions caractéristiques (Lewin, 1958)

$$IC_A(k) = (1_A \star 1_{-A})(k)$$

$$\mathbb{1}_A \star \mathbb{1}_{-B}(k) = \sum_{x \in \mathbb{Z}_c} \mathbb{1}_A(x) \cdot \mathbb{1}_{-B}(k - x) = \sum_{x \in \mathbb{Z}_c} \mathbb{1}_A(x) \cdot \mathbb{1}_B(x - k) = \sum_{\substack{(x,y) \in A \times B \\ y=x-k}} 1$$

Expression polynomiale :

$$D(x) = 1 + x + x^2 + x^5$$

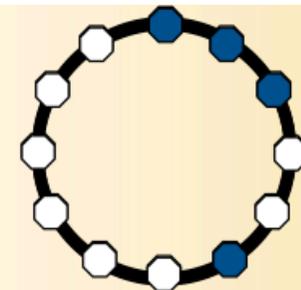
Fonction de Patterson (ou d'autocorrelation):

$$D(x)D(x^{-1}) =$$

$$= (1 + x + x^2 + x^5)(1 + x^{11} + x^{10} + x^7)$$

$$= 1 + x + x^2 + x^5 + x^7 + x^8 + x^9 + x^{10} + 2x^{11} + 3x^{12} + x^{13} + x^{15} + x^{16}$$

$$= 4 + 2x + x^2 + x^3 + x^4 + x^5 + x^7 + x^8 + x^9 + x^{10} + 2x^{11}$$



# Z-relation, homométrie et reconstruction de la phase

- Contenu intervallaire et Fonction de Patterson

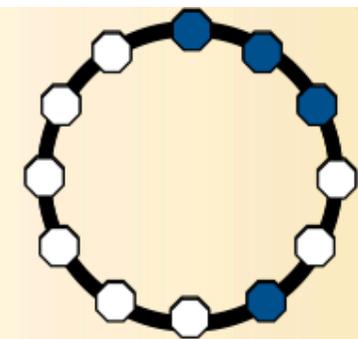
$$IC_A(k) = (1_A \star 1_{-A})(k)$$

Expression polynomiale :

$$D(x) = 1 + x + x^2 + x^5$$

Fonction de Patterson (ou d'autocorrelation):

$$\begin{aligned} D(x)D(x^{-1}) &= \\ &= (1 + x + x^2 + x^5)(1 + x^{11} + x^{10} + x^7) \\ &= 1 + x + x^2 + x^5 + x^7 + x^8 + x^9 + x^{10} + 2x^{11} + 3x^{12} + x^{13} + x^{15} + x^{16} \\ &= 4 + 2x + x^2 + x^3 + x^4 + x^5 + x^7 + x^8 + x^9 + x^{10} + 2x^{11} \end{aligned}$$



$$IC_D = [4, 2, 1, 1, 1, 1, 1, 1, 1, 2]$$

# Relation Z, DFT et théorie de l'homométrie

- Le contenu intervallique est équivalent à un produit de convolution de fonctions caractéristiques (Lewin, 1958)

$$\mathcal{F}_A : t \mapsto \sum_{k \in A} e^{-2i\pi kt/c}$$

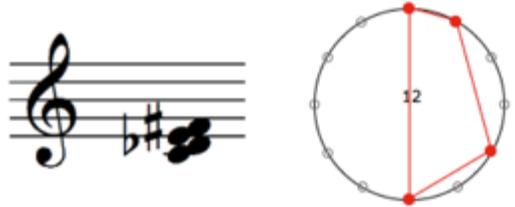
$$IC_A(k) = (\mathbb{1}_A \star \mathbb{1}_{-A})(k)$$

$$\mathbb{1}_A \star \mathbb{1}_{-B}(k) = \sum_{x \in \mathbb{Z}_c} \mathbb{1}_A(x) \cdot \mathbb{1}_{-B}(k-x) = \sum_{x \in \mathbb{Z}_c} \mathbb{1}_A(x) \cdot \mathbb{1}_B(x-k) = \sum_{\substack{(x,y) \in A \times B \\ y=x-k}} 1$$

$$\mathcal{F}(IC_A) = \mathcal{F}_A \times \mathcal{F}_{-A} = |\mathcal{F}_A|^2$$

➔ Relation Z

$\{0, 1, 4, 6\}_{12}$



$iv = [4, 1, 1, 1, 1, 1, 2, 1, 1, 1, 1, 1]$

$\{0, 1, 3, 7\}_{12}$



- D. Ghisi, "From Z-relation to homometry: an introduction and some musical examples", Séminaire MaMuX, 10 décembre 2010
- P. Beauguitte, *Transformée de Fourier discrète et structures musicales*, Master ATIAM, 2011

# Ensembles homométriques

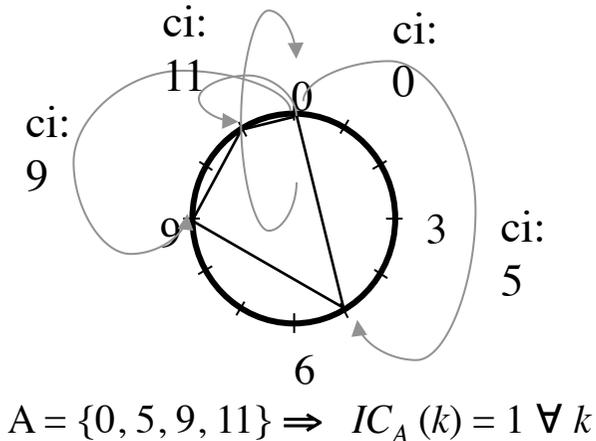
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$$\mathcal{F}_A : t \mapsto \sum_{k \in A} e^{-2i\pi kt/c}$$

$$IC_A(k) = (1_A \star 1_{-A})(k)$$

$$1_A \star \tilde{1}_B(k) = \sum_i 1_A(i) \times 1_B(i - k) = \sum_{\substack{i \in A \\ i - k \in B}} 1$$

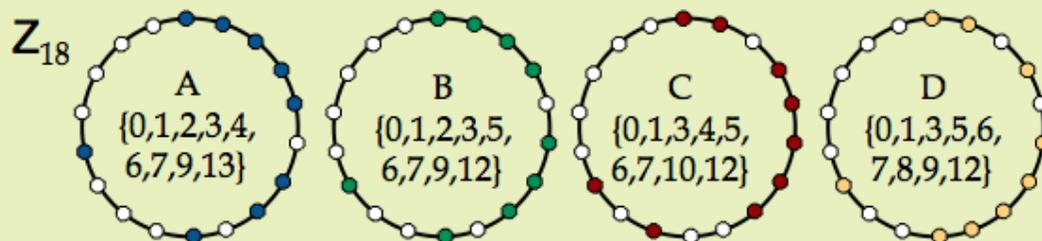
$$\mathcal{F}(1_A \star \tilde{1}_B) = \mathcal{F}(1_A) \times \mathcal{F}(\tilde{1}_B)$$



$$\mathcal{F}(IC_A) = \mathcal{F}_A \times \mathcal{F}_{-A} = |\mathcal{F}_A|^2$$

➔ **Relation Z**

Not only do we have Z-couples, but we also have Z-related t-uples for all t's.



- D. Ghisi, "From Z-relation to homometry: an introduction and some musical examples", Séminaire MaMuX, 10 décembre 2010
- P. Beauguitte, *Transformée de Fourier discrète et structures musicales*, Master ATIAM, 2011

# Relation Z en musique et théorie de l'homométrie

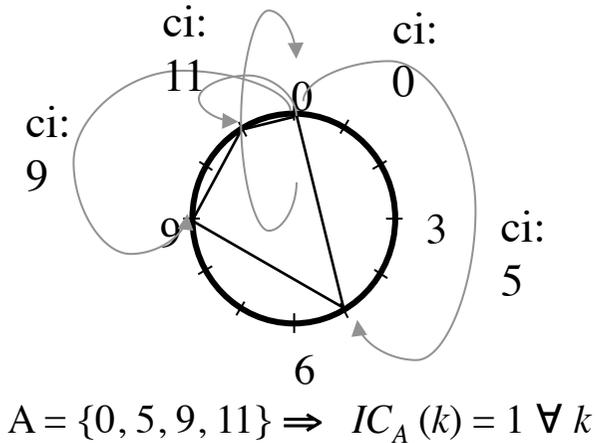
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$$\mathcal{F}(1_A \star \tilde{1}_B) = \mathcal{F}(1_A) \times \mathcal{F}(\tilde{1}_B)$$

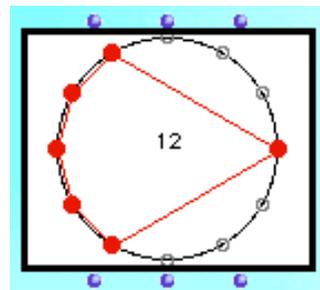
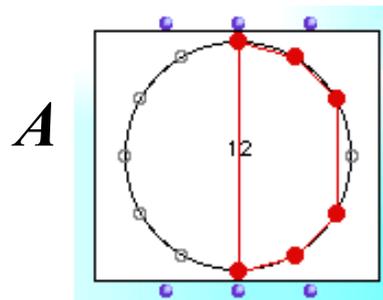


$$\mathcal{F}(IC_A) = \mathcal{F}_A \times \mathcal{F}_{-A} = |\mathcal{F}_A|^2$$

➔ Relation Z

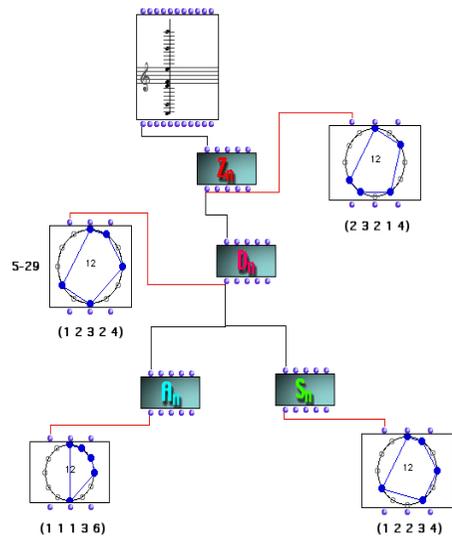
$$\forall k \mathcal{F}(IC_{\mathbb{Z}_c \setminus A})(k) = \mathcal{F}(IC_A)(k)$$

➔ Théorème de l'hexacorde

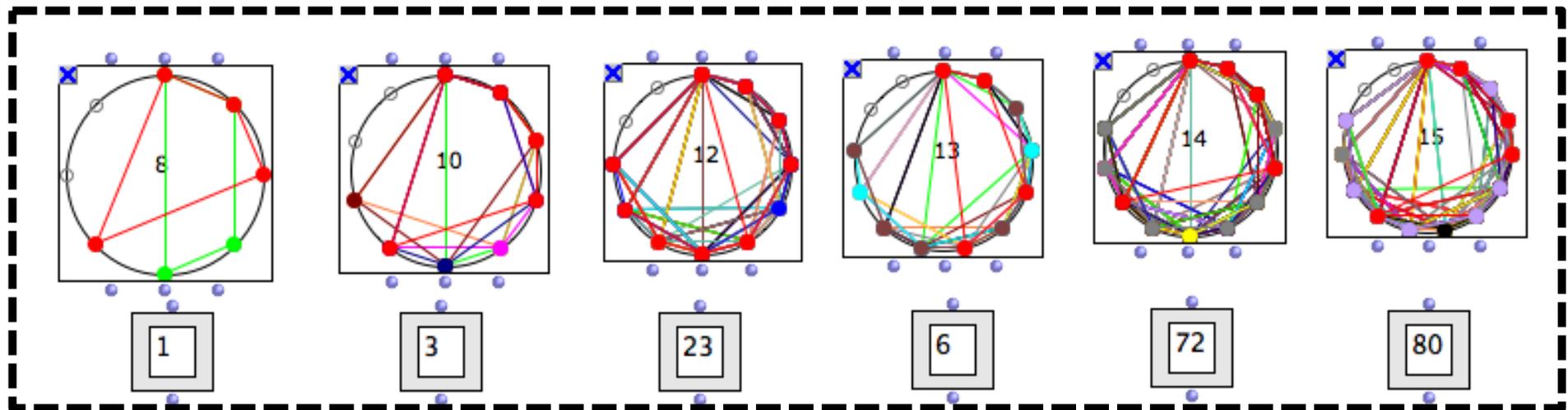


➔  $IC_A = IC_{A'}$

# Classification des ensembles en Z-relations



- *Is there a (non-trivial) group action whose orbits are the equivalence classes of homometric sets?*
- *Is there an enumeration formula for homometric sets?*

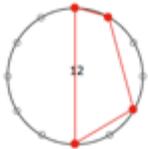


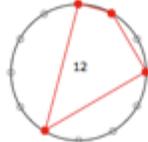
• John Mandereau, *Étude des ensembles homométriques et leur application en théorie mathématique de la musique et en composition assistée par ordinateur*, Master Thesis, ATIAM, Ircam/Université Paris 6, juin 2009



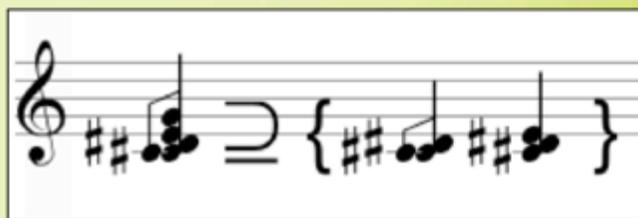
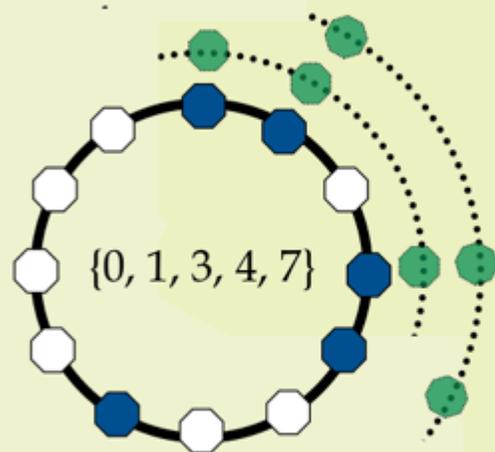
11-catalogues-z-relation

# Relation Z d'ordre supérieur (vecteur $mv^k$ de Lewin)

$\{0, 1, 4, 6\}_{12}$     $iv = [4, 1, 1, 1, 1, 1, 2, 1, 1, 1, 1, 1]$  → 2-homométrie

$\{0, 1, 3, 7\}_{12}$   

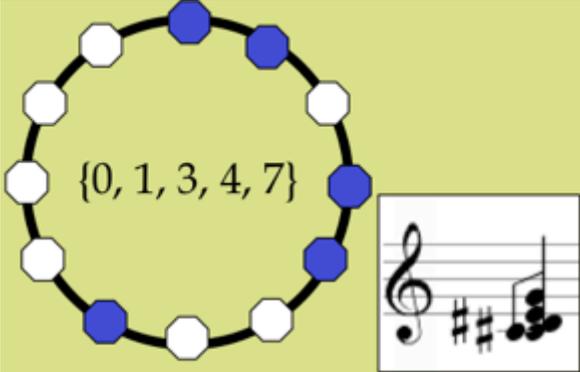
Le vecteur  $mv^k$  (d'après Lewin) nous dit combien de copies d'un certain ensemble de cardinalité  $k$  sont contenues dans un ensemble



→ k-homométrie  
(k-deck ou k-Deck)

- Daniele Ghisi, "From Z-relation to homometry: an introduction and some musical examples", Séminaire MaMuX, 10 décembre 2010 [<http://repmus.ircam.fr/mamux/saisons/saison10-2010-2011/2010-12-10>]

# High-order 'interval' content: Lewin's $mv^k$ vector

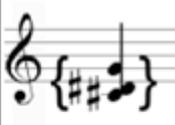


$\{0, 1, 3, 4, 7\}$

3-set (*prime forms*):

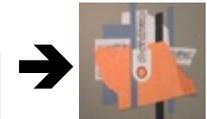
- $\{0, 1, 2\} \rightarrow 0$  copies
- $\{0, 1, 3\} \rightarrow 2$  copies
- $\{0, 1, 4\} \rightarrow 3$  copies
- $\{0, 1, 5\} \rightarrow 0$  copies
- $\{0, 1, 6\} \rightarrow 1$  copie
- $\{0, 2, 4\} \rightarrow 0$  copies
- $\{0, 2, 5\} \rightarrow 0$  copies
- $\{0, 2, 6\} \rightarrow 1$  copie
- $\{0, 2, 7\} \rightarrow 0$  copies
- $\{0, 3, 6\} \rightarrow 1$  copie
- $\{0, 3, 7\} \rightarrow 2$  copies
- $\{0, 4, 8\} \rightarrow 0$  copies

3-Subsets:

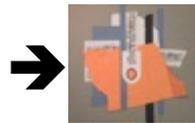
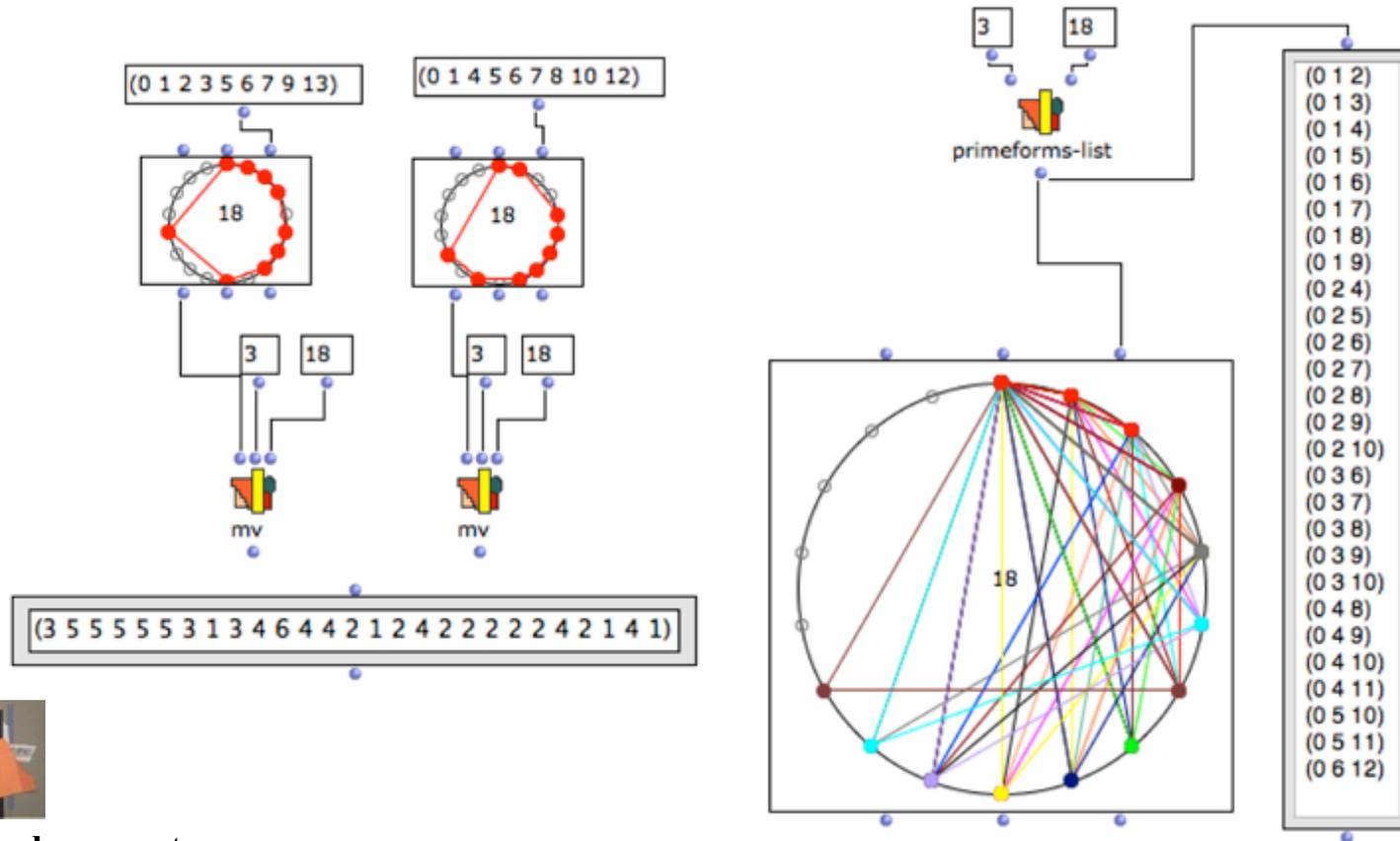
- $2 / \{0, 1, 3\} /$  
- $3 / \{0, 1, 4\} /$  
- $1 / \{0, 1, 7\} /$  
- $1 / \{0, 2, 6\} /$  
- $1 / \{0, 3, 6\} /$  
- $2 / \{0, 3, 7\} /$  

$mv^3(\{0,1,3,4,7\})=[0, 2, 3, 0, 1, 0, 0, 1, 0, 1, 2, 0]$

• Daniele Ghisi, "From Z-relation to homometry: an introduction and some musical examples", Séminaire MaMuX, 10 décembre 2010 [<http://repmus.ircam.fr/mamux/saisons/saison10-2010-2011/2010-12-10>]



# High-order Z-relation and $k$ -homometric nesting

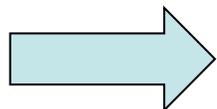
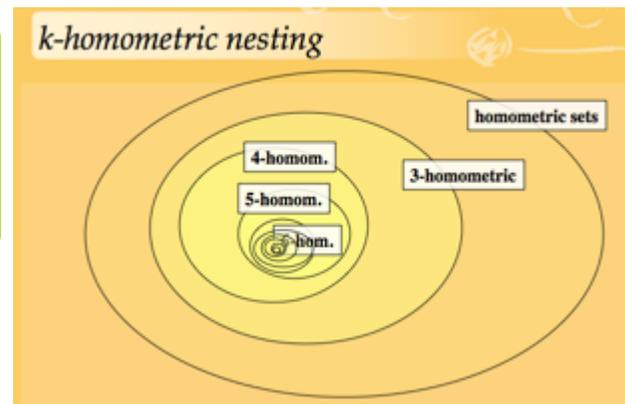


19-mv3-and-zm-partners

$$\mathbf{Z}_{18} \quad A = \{0, 1, 2, 3, 5, 6, 7, 9, 13\}$$

$$\quad \quad B = \{0, 1, 4, 5, 6, 7, 8, 10, 12\}$$

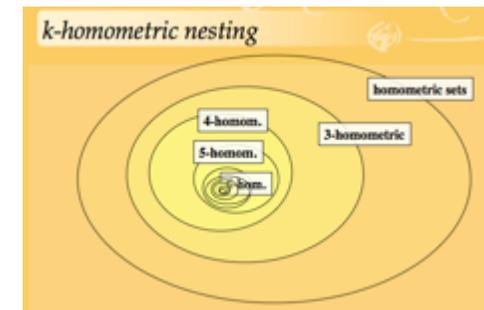
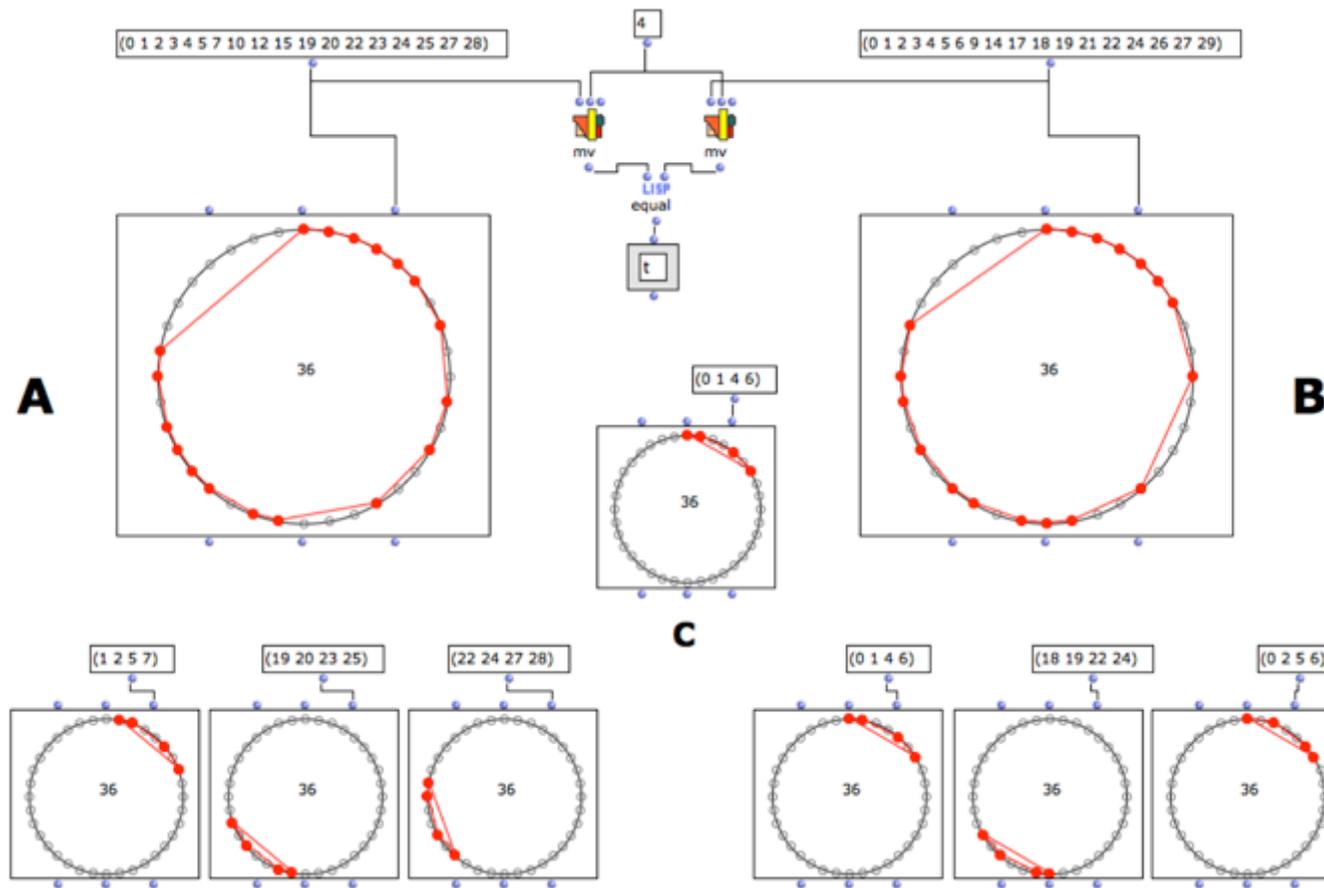
$$mv^3(A) = mv^3(B) \Rightarrow 3\text{-homometry} (\Rightarrow 2\text{-homometry})$$



Where does this nesting stop?

# Premiers résultats sur la $k$ -homométrie

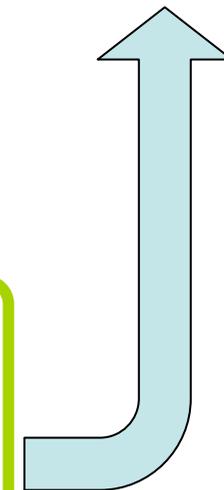
- Mandereau J., D. Ghisi, E. Amiot, M. Andreatta, C. Agon, (2011a), « Z-relation and homometry in musical distributions », *JMM*, vol. 5, n° 2, p. 83-98.
- Mandereau J., D. Ghisi, E. Amiot, M. Andreatta, C. Agon (2011b), « Discrete phase retrieval in musical structures », *JMM*, vol. 5, n° 2, p. 99-116.



$$\mathbf{Z}_{36} \quad A = \{0, 1, 2, 3, 4, 5, 7, 10, 12, 15, 19, 20, 22, 23, 24, 25, 27, 28\}$$

$$\mathbf{Z}_{36} \quad B = \{0, 1, 2, 3, 4, 5, 6, 9, 14, 17, 18, 19, 21, 22, 24, 26, 27, 29\}$$

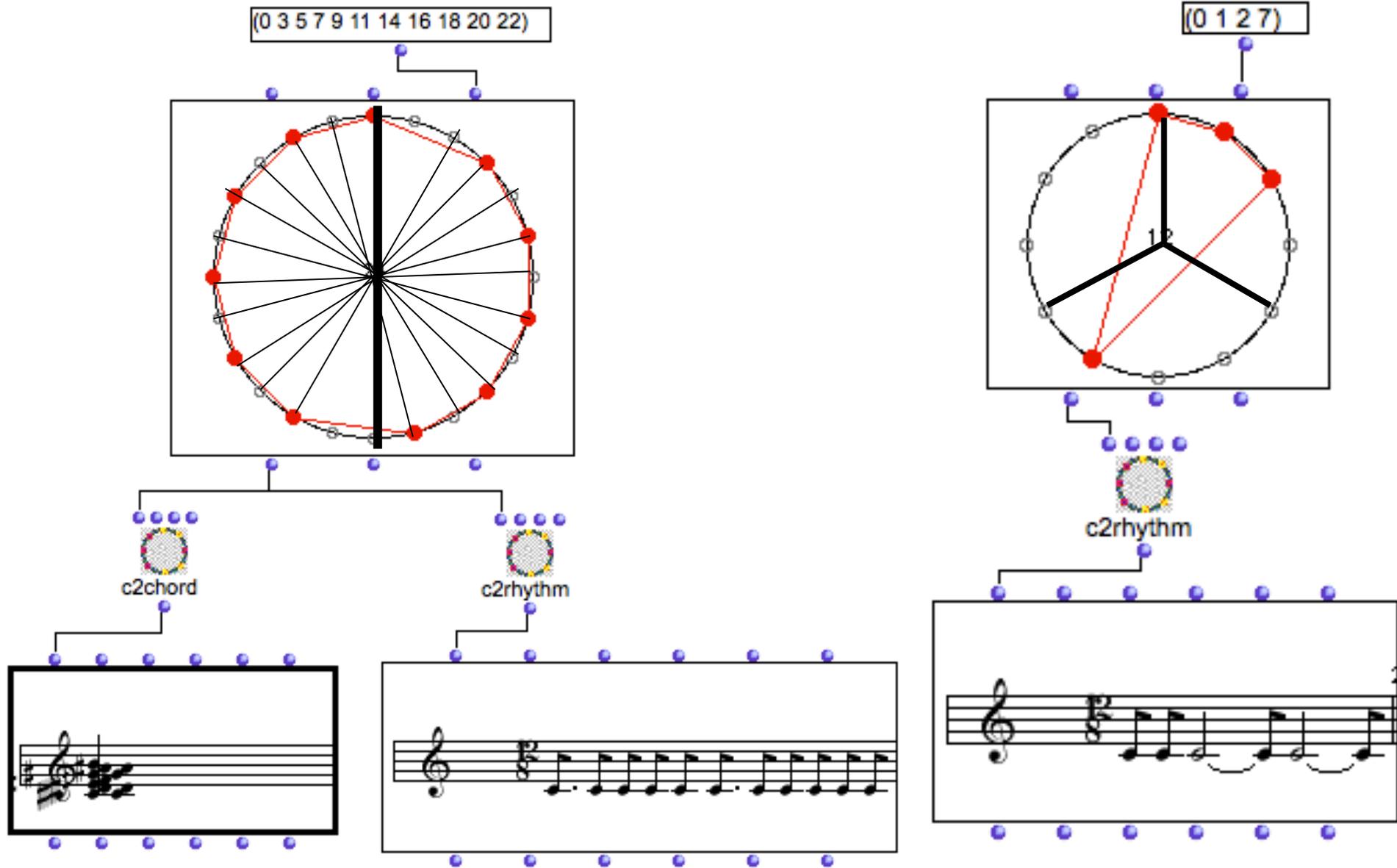
$$mv^4(A) = mv^4(B) \rightarrow \mathbf{4\text{-homométrie}}$$



# Rythmes $k$ -asymétriques

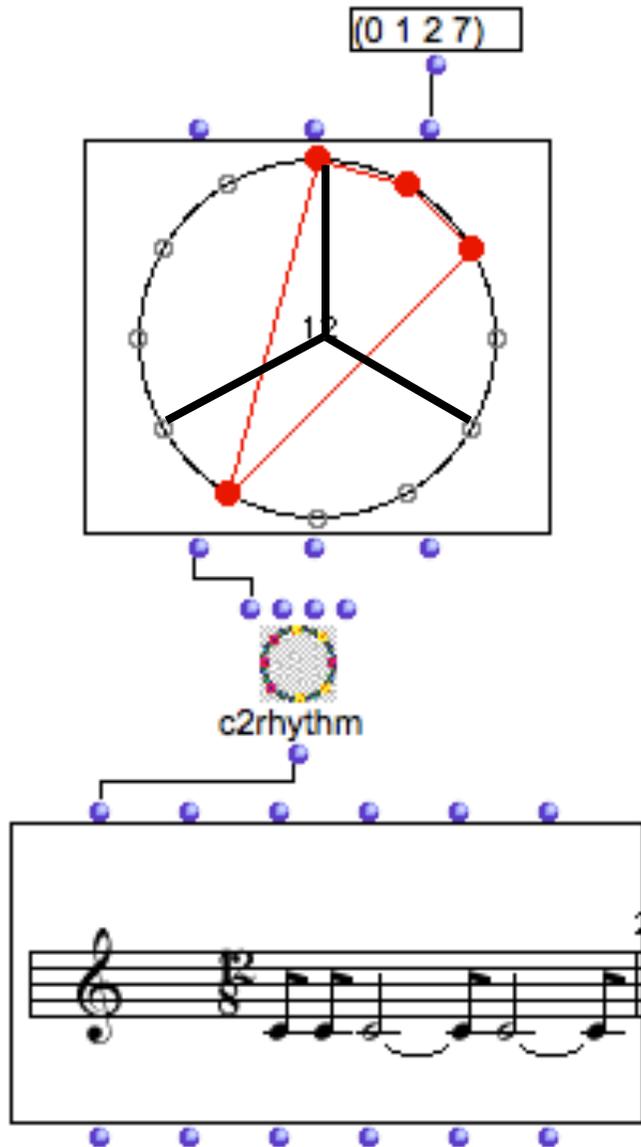
(Simha Arom & Marc Chemillier)

(Rachel W. Hall & P. Klingsberg)



# Rythmes 3-asymétriques

( Rachel W. Hall & P. Klingsberg)



Un rythme périodique  $R$  de période  $kh$  est  $k$ -asymétrique s'il est tel que si une attaque de  $R$  occupe la position  $x$  alors toutes les autres positions  $y$  telles que

$$y \sim x \pmod{h}$$

ne correspondent pas à des attaques du rythme  $R$ .

# Rythmes 3-asymétriques

( Rachel W. Hall & P. Klingsberg)

The diagram illustrates 3-asymmetric rhythms using a circle of 12 and musical notation. It is organized into two rows of three columns each.

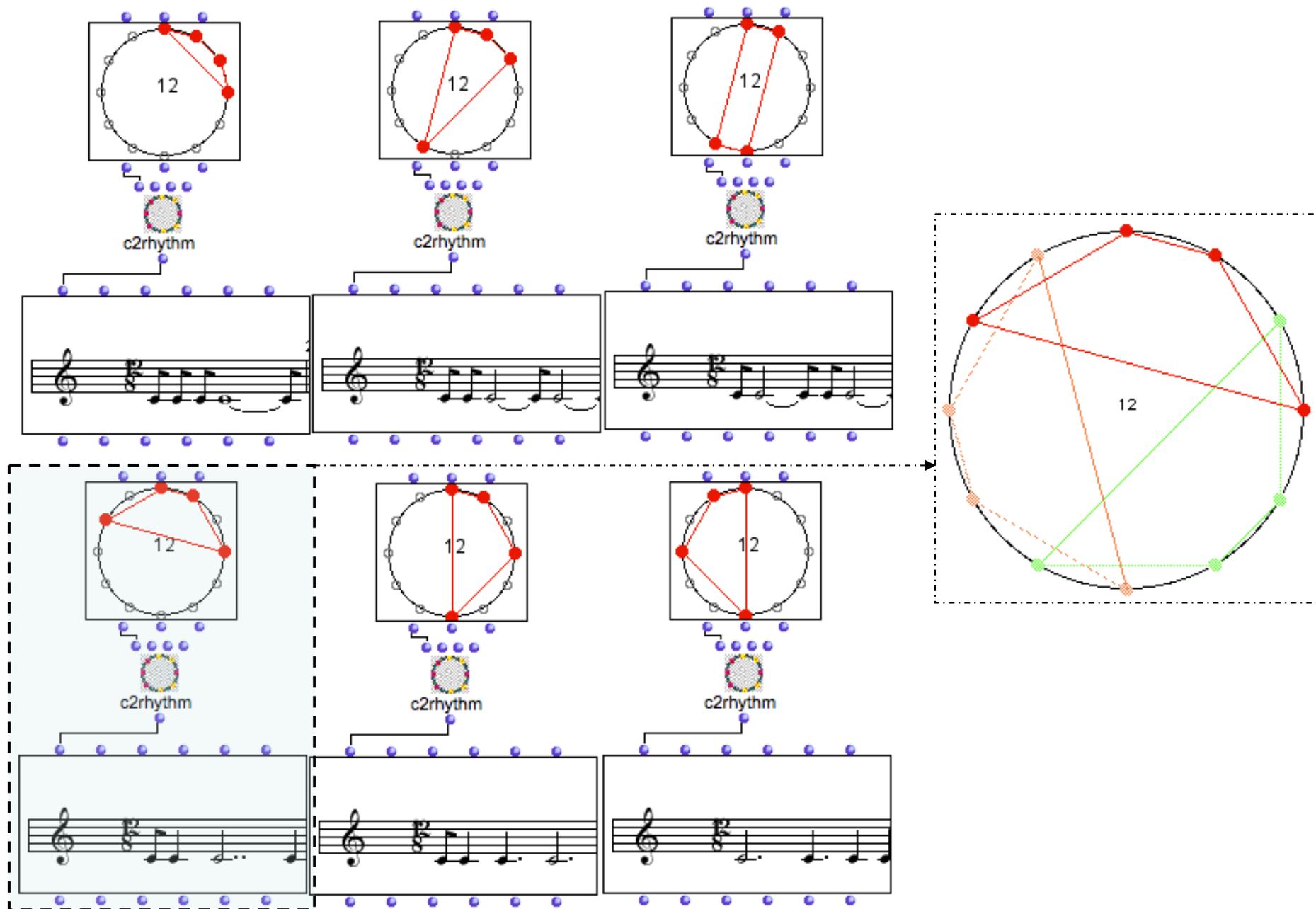
**Top Row:**

- Column 1:** A large circle of 12 with a box labeled  $(0\ 1\ 2\ 7)$  above it. A vertical line from the box points to a red dot at the 12 o'clock position. A 'c2rhythm' icon is below the circle. A musical staff below shows a sequence of notes: quarter, eighth, eighth, quarter, quarter, quarter, quarter, quarter, quarter, quarter, quarter, quarter.
- Column 2:** A circle of 12 with a red dot at 12 o'clock and another at 2 o'clock. A 'c2rhythm' icon is below. A musical staff below shows notes: quarter, eighth, eighth, quarter, quarter, quarter, quarter, quarter, quarter, quarter, quarter, quarter.
- Column 3:** A circle of 12 with red dots at 12, 2, and 7 o'clock. A 'c2rhythm' icon is below. A musical staff below shows notes: quarter, eighth, eighth, quarter, quarter, quarter, quarter, quarter, quarter, quarter, quarter, quarter.

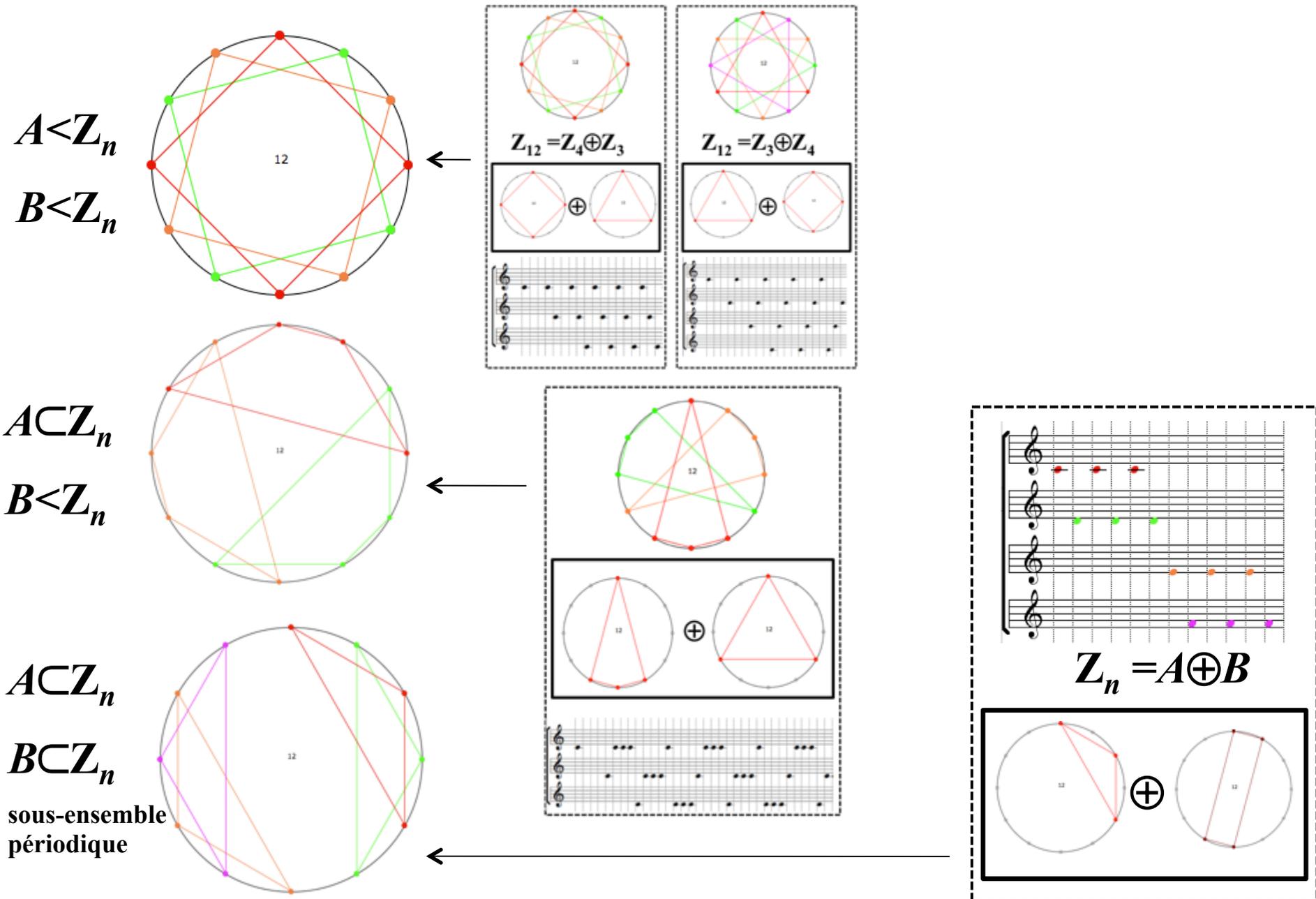
**Bottom Row:**

- Column 1:** A circle of 12 with red dots at 12, 2, and 7 o'clock. A 'c2rhythm' icon is below. A musical staff below shows notes: quarter, eighth, eighth, quarter, quarter, quarter, quarter, quarter, quarter, quarter, quarter, quarter.
- Column 2:** A circle of 12 with red dots at 12, 2, and 7 o'clock. A 'c2rhythm' icon is below. A musical staff below shows notes: quarter, eighth, eighth, quarter, quarter, quarter, quarter, quarter, quarter, quarter, quarter, quarter.
- Column 3:** A circle of 12 with red dots at 12, 2, and 7 o'clock. A 'c2rhythm' icon is below. A musical staff below shows notes: quarter, eighth, eighth, quarter, quarter, quarter, quarter, quarter, quarter, quarter, quarter, quarter.

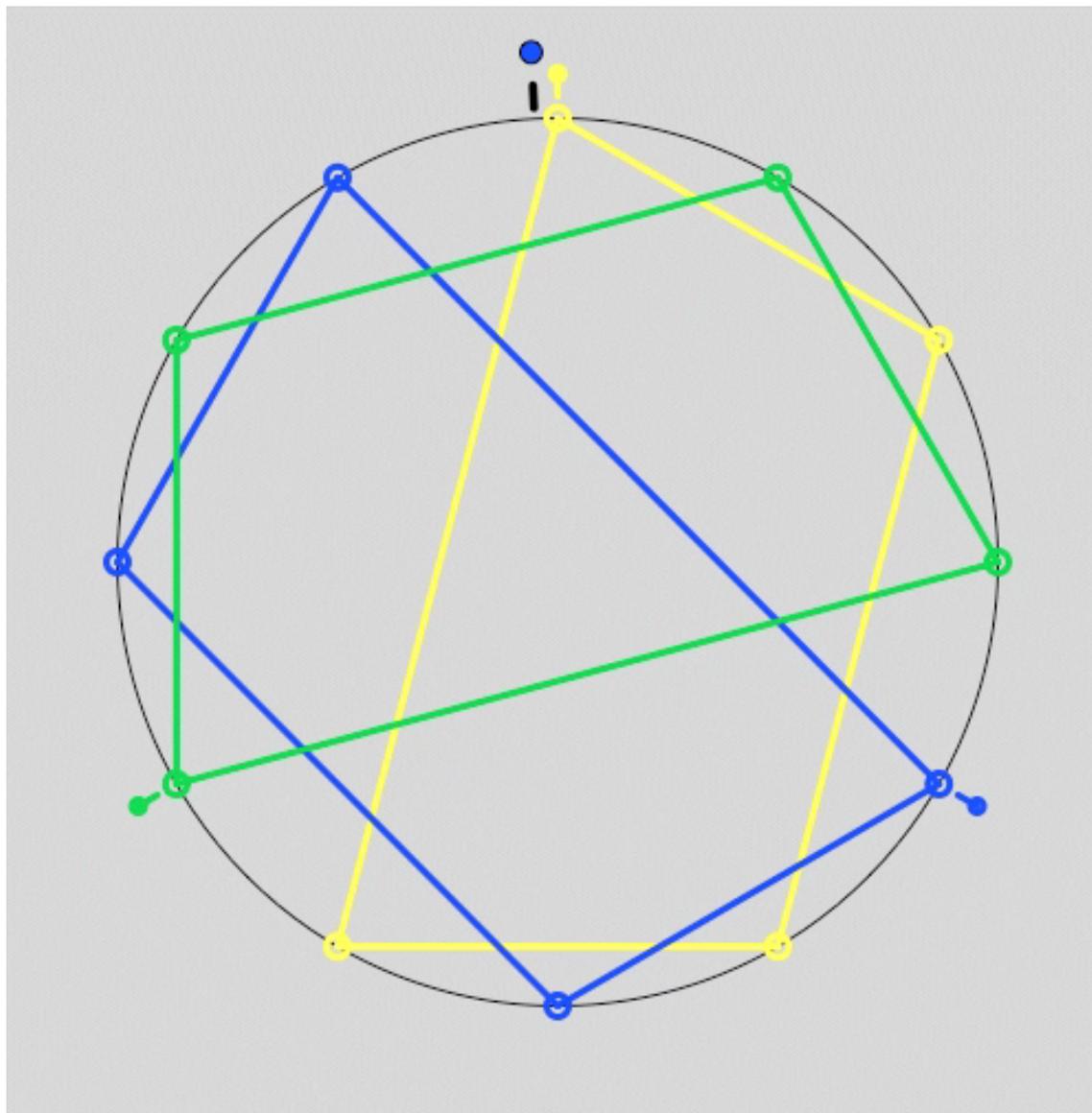
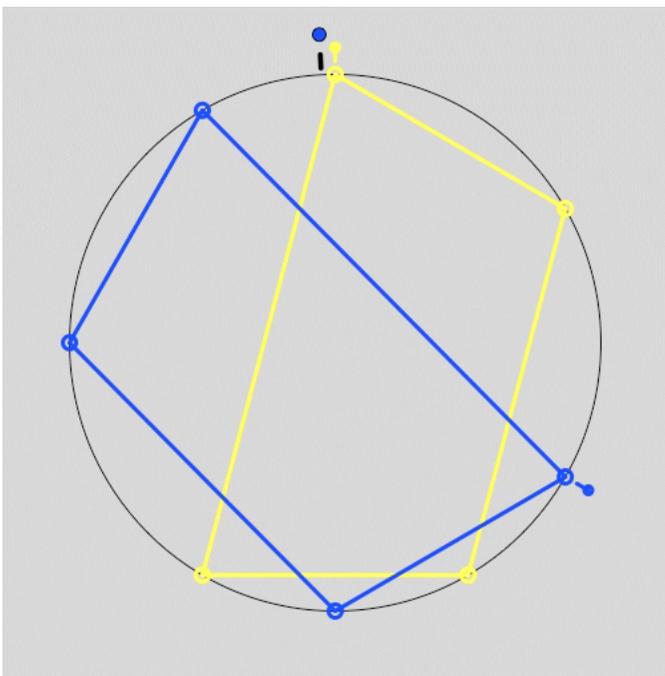
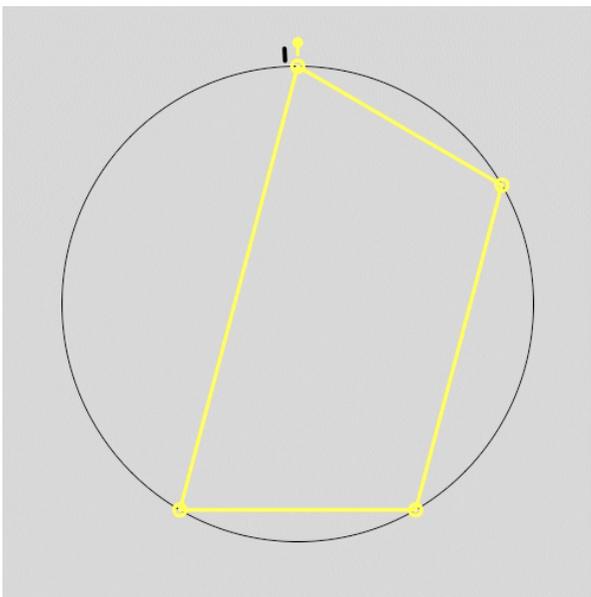
# Rythmes 3-asymétriques et pavage



# Factorisation de groupes et périodicités des facteurs

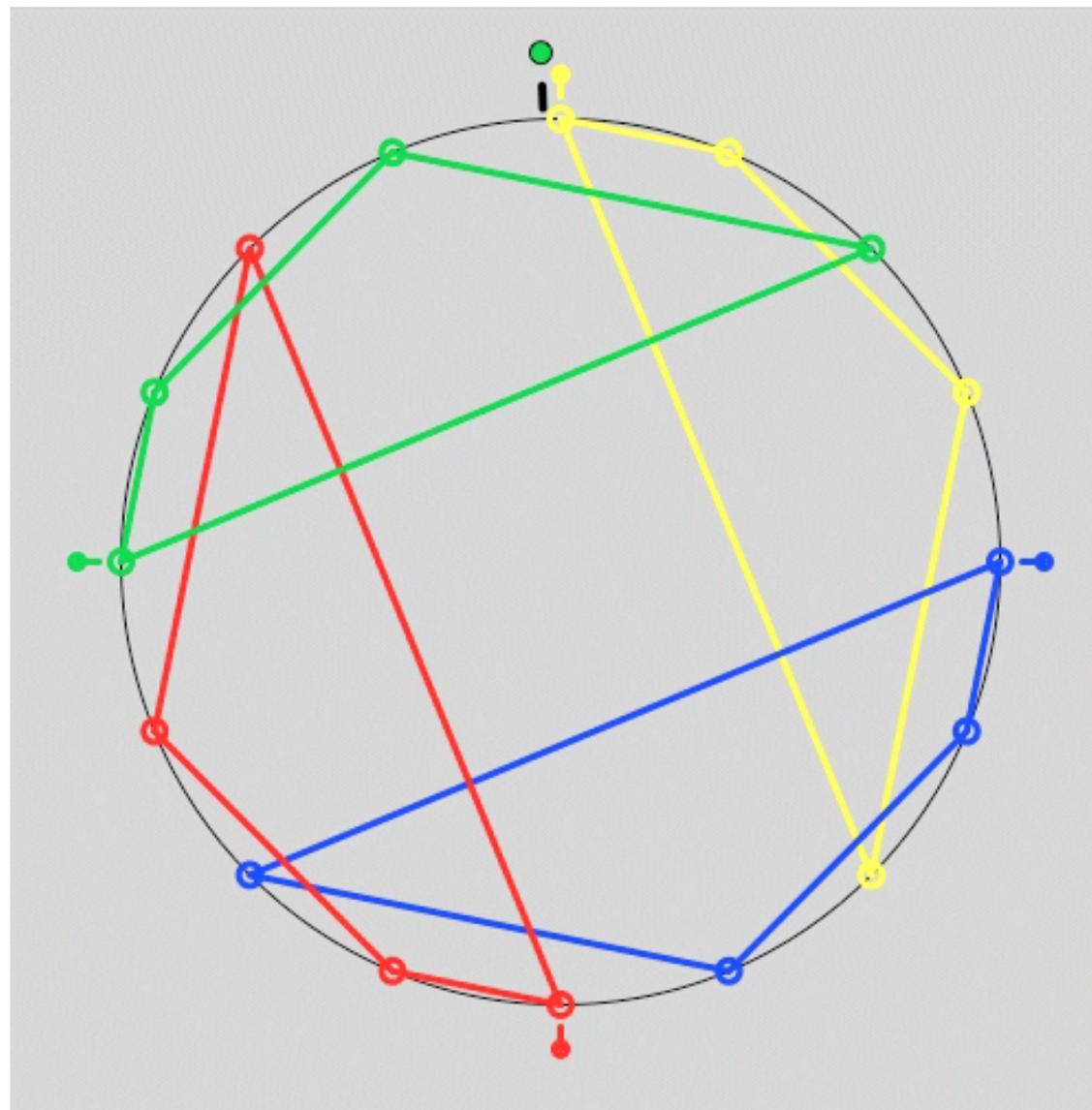
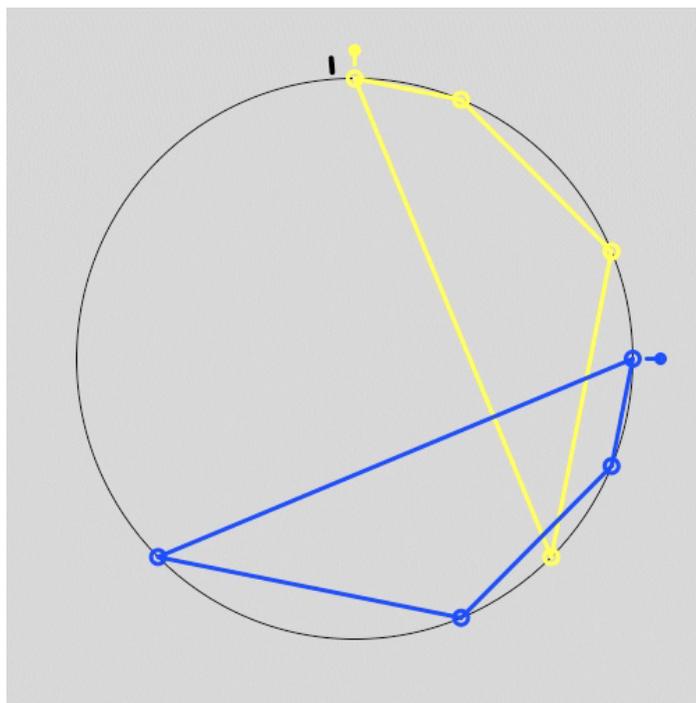
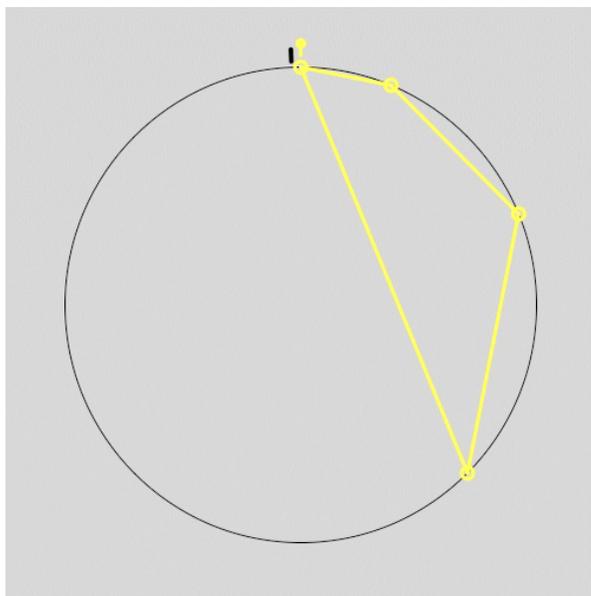


# Premiers exemples de canons rythmiques mosaïques



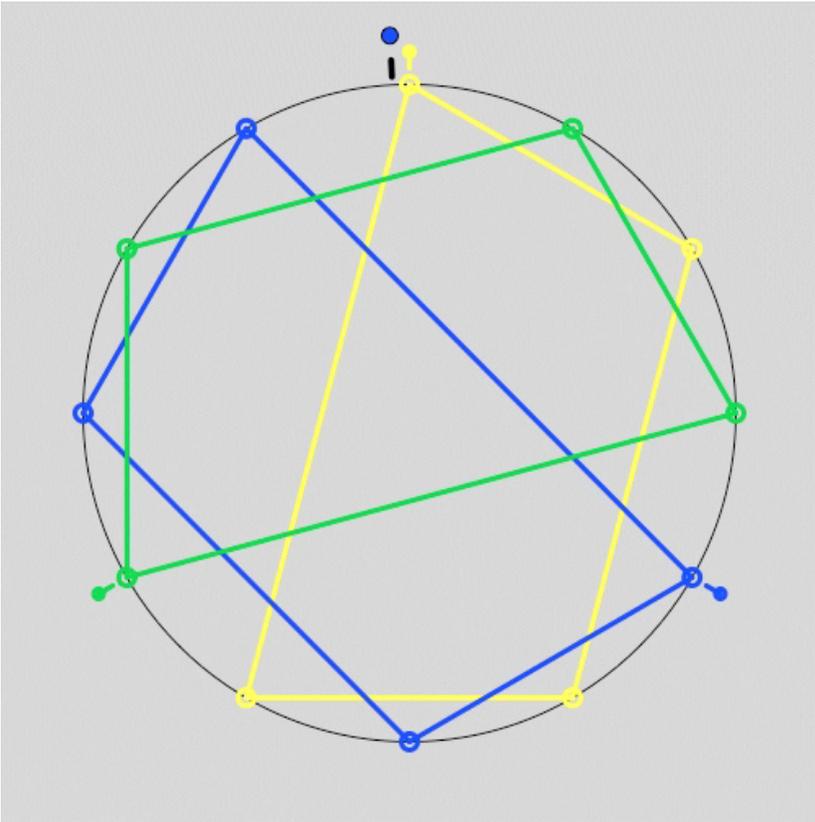
# Canons mélodico-rythmiques mosaïques

---



# Un canon rythmique mosaïque est une factorisation

The diagram illustrates the construction of a 12-measure rhythmic canon. At the top, a circle with 12 points is divided into two overlapping triangles of 6 points each, colored red and green. The number '12' is written in the center. Below this, a solid black box contains two identical 6-measure canons, each with a red triangle and the number '12' in the center, separated by a plus sign in a circle. At the bottom, a musical score with two staves shows the rhythmic patterns corresponding to the canons.

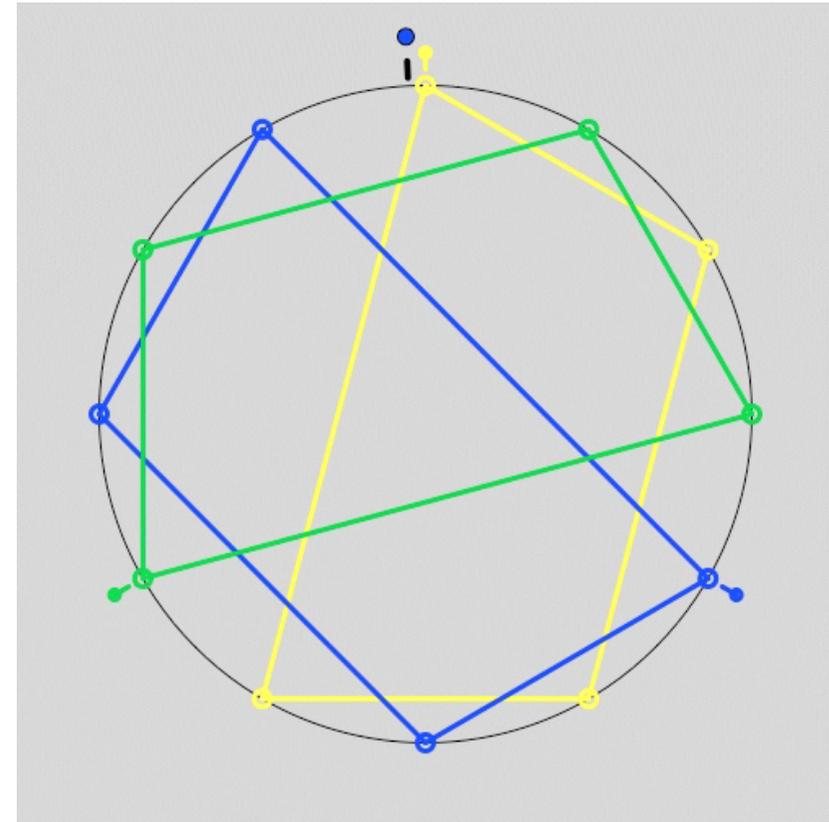
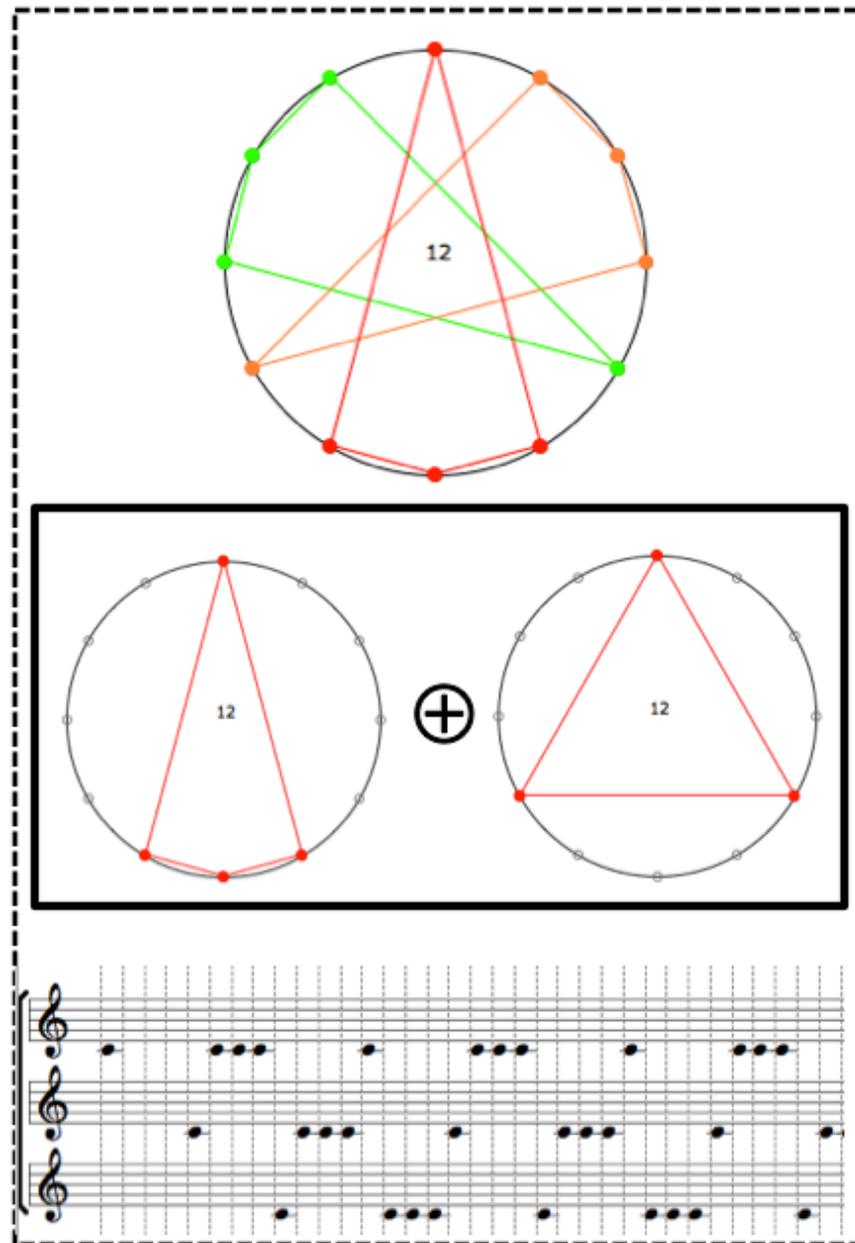


$$0 = 0 + 0$$

$$1 = 5 + 8$$

$$2 = ? + ?$$

# Un canon rythmique mosaïque est une factorisation



$$\begin{aligned} 0 &= 0 + 0 \\ 1 &= 5 + 8 \\ 2 &= 6 + 8 \end{aligned}$$

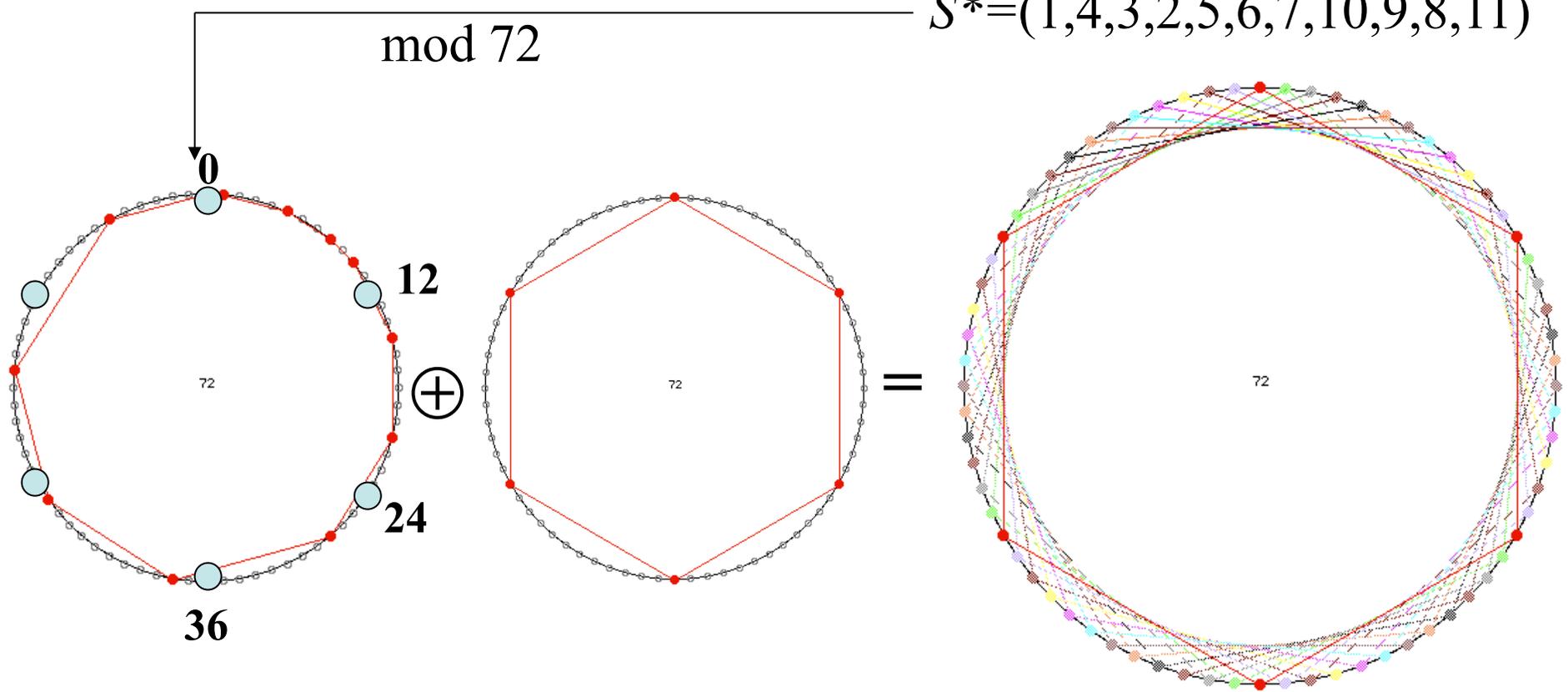
# Canons rythmiques mosaïques et séries tous-intervalles

...  
...  
36  
24 25  
12 13

The musical staff shows a sequence of notes with various accidentals. An arrow points from the staff to a circular diagram representing a 72-note scale. The diagram features a vertical dashed line and several chords (triads and dyads) connected by lines, illustrating the intervallic structure of the scale.

$S = \{0, 1, 3, 6, 7, 9, 8, 10, 11, 2, 4, 5\}$

$S^* = (1, 4, 3, 2, 5, 6, 7, 10, 9, 8, 11)$



# Construction explicite d'un rythme $k$ -asymétrique

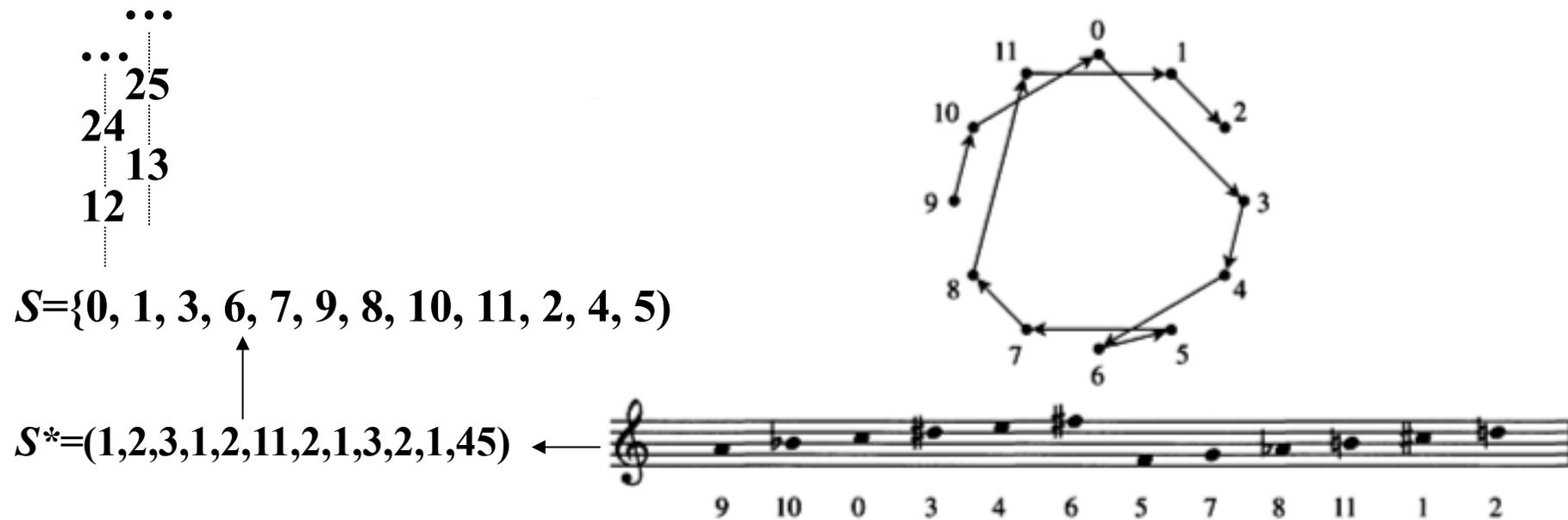
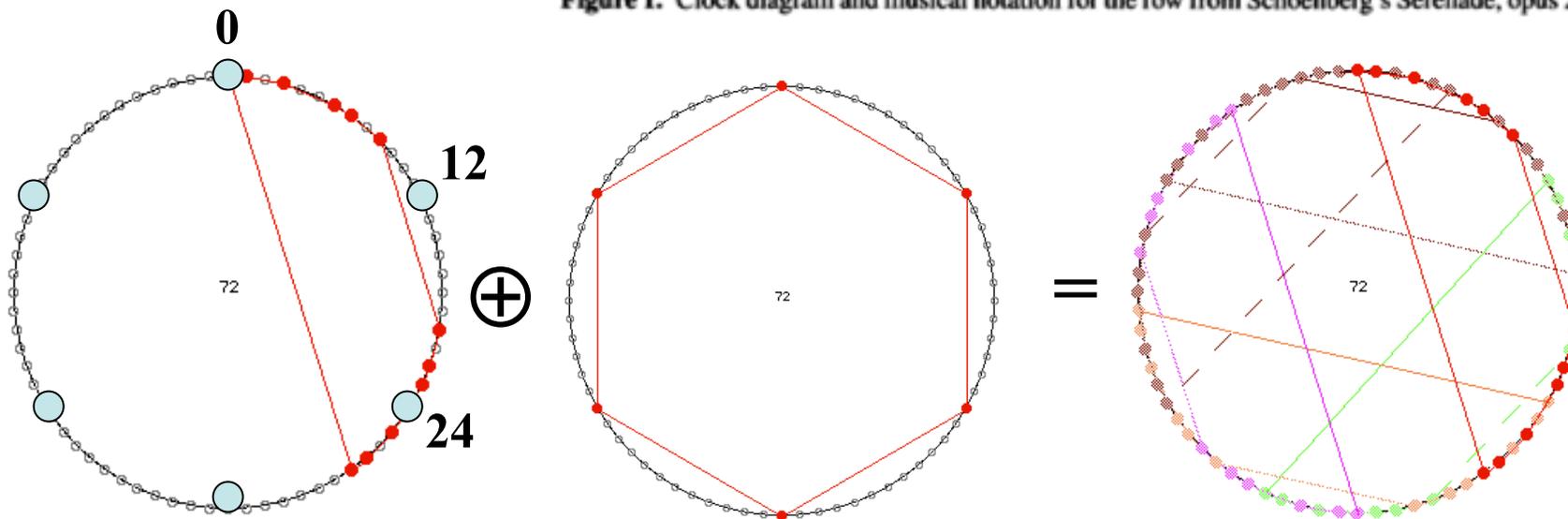
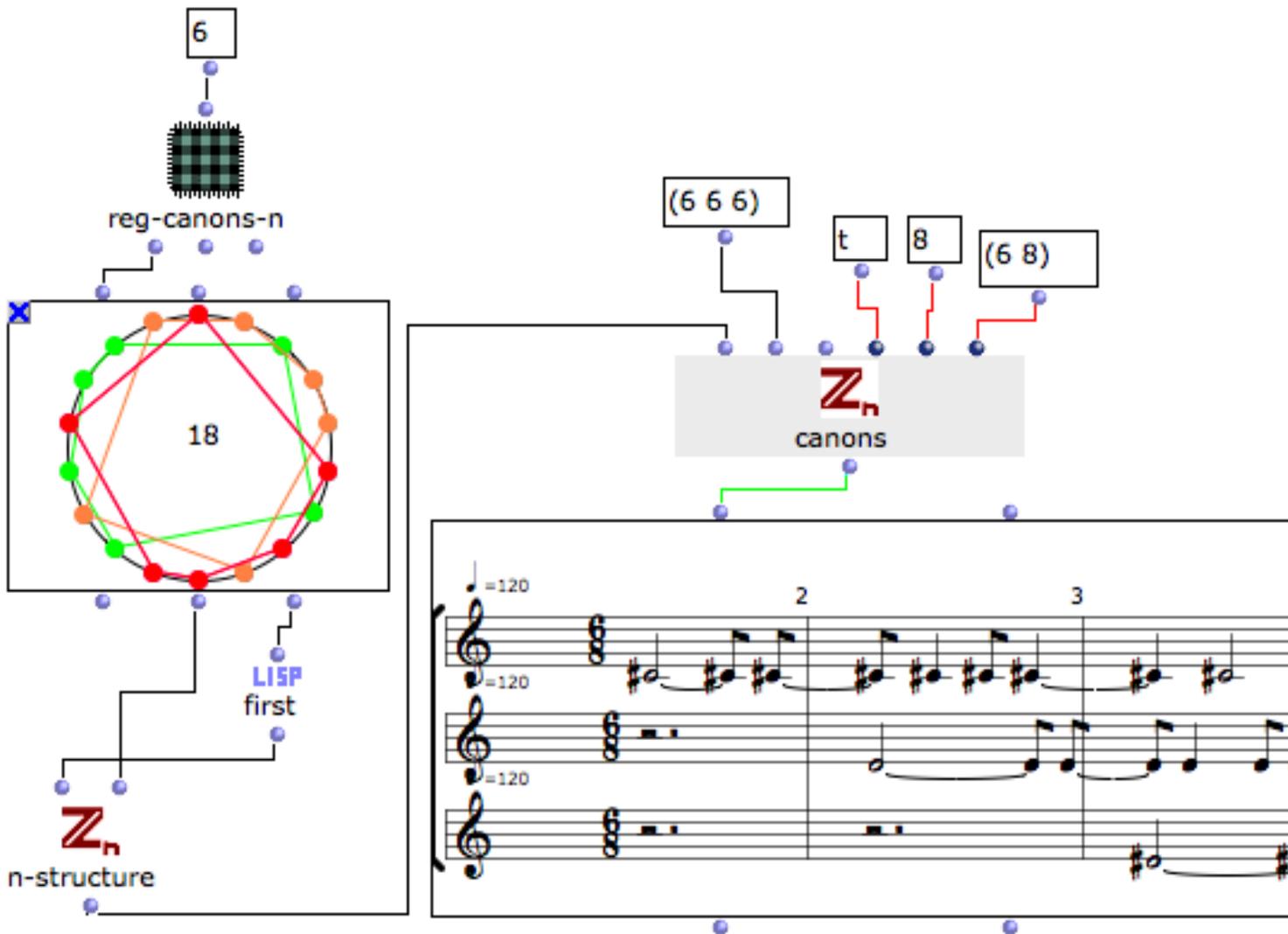


Figure 1. Clock diagram and musical notation for the row from Schoenberg's Serenade, opus 24, movement 5.



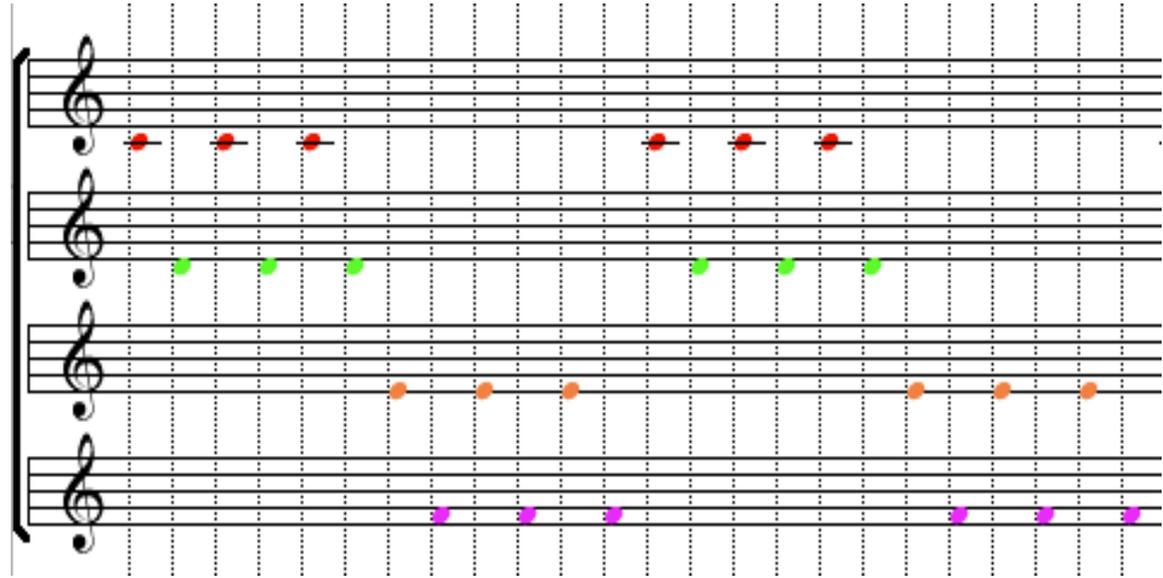
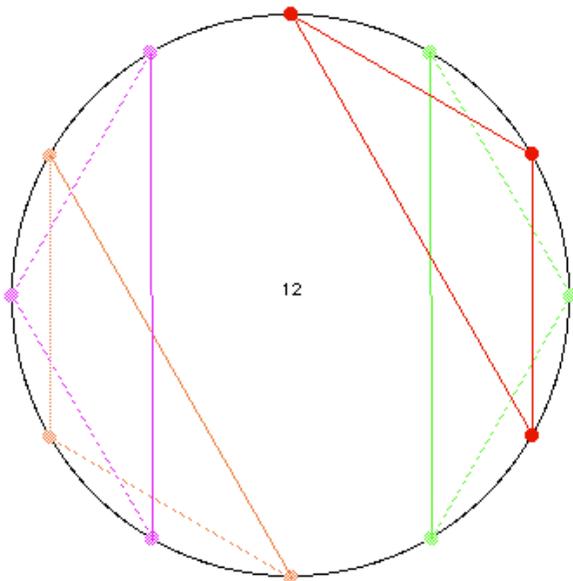
Pattern 8-asymétrique

# Construction explicite d'un rythme $k$ -asymétrique



16-k-asymmetry-canons

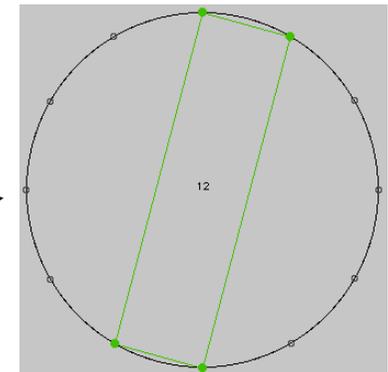
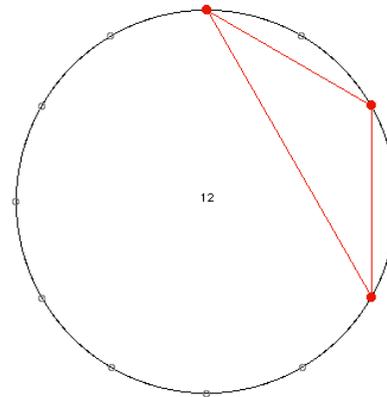
# Canons mosaïques avec symétrie transpositionnelle



$$\mathbf{Z}_{12} = \mathbf{A} \oplus \mathbf{B}$$

$$\mathbf{A} = \{0, 2, 4\}$$

$$\mathbf{B} = \{0, 1, 6, 7\}$$



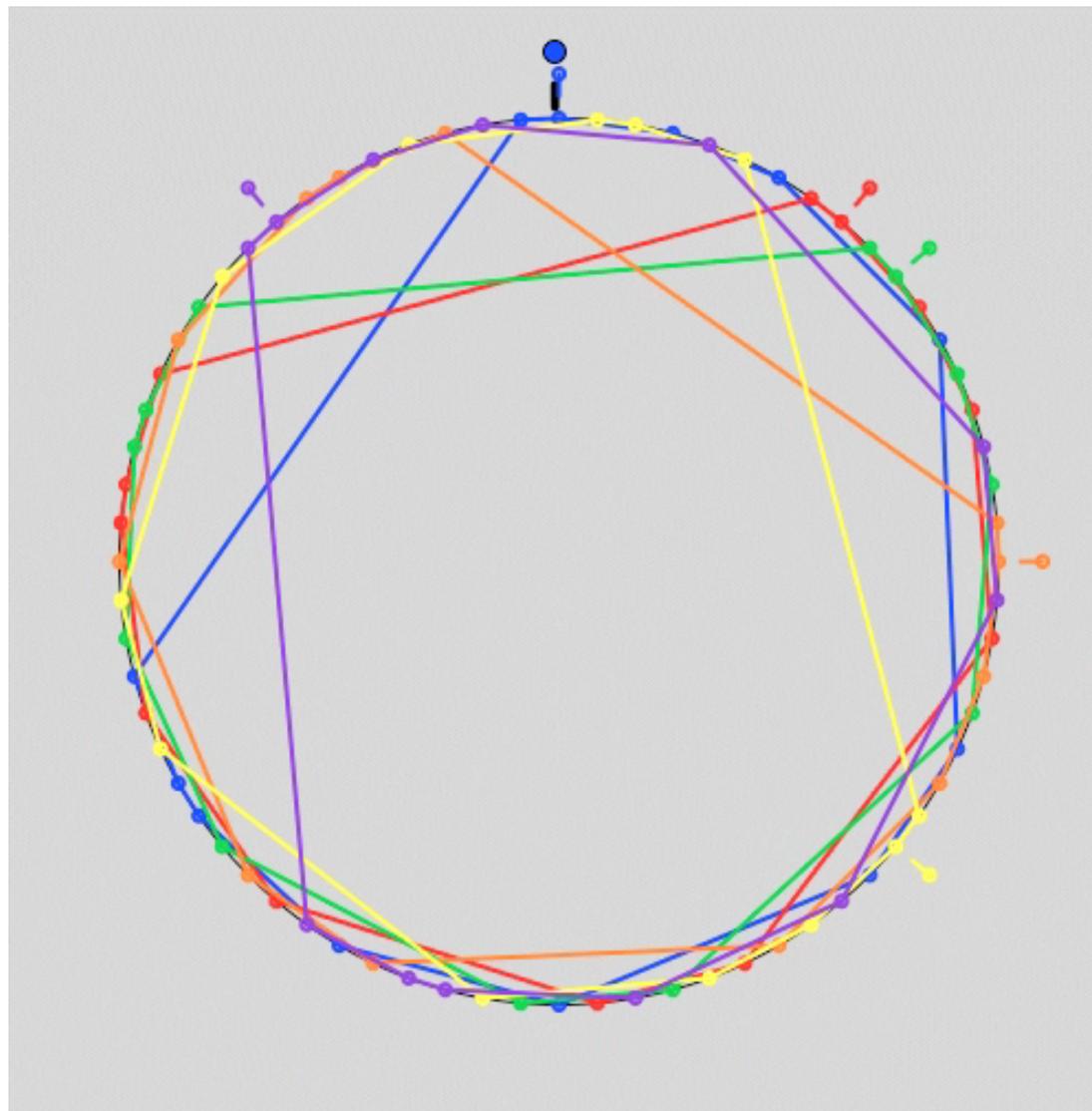
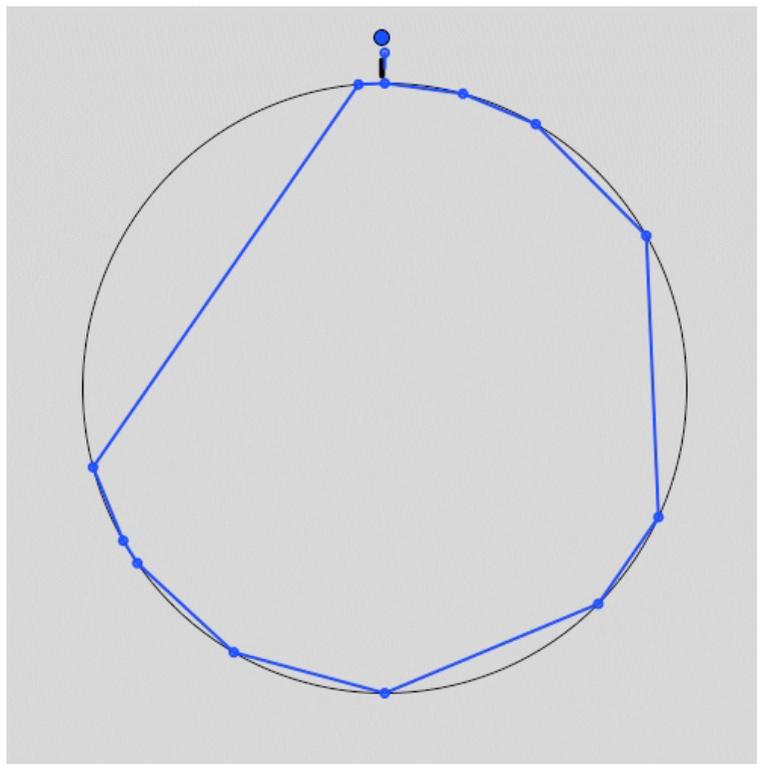
# Canons rythmiques mosaïques sans périodicité interne

**Dan Vuza**

**Anatol Vieru**

# Canon mélodico-rythmique mosaïque sans périodicité interne

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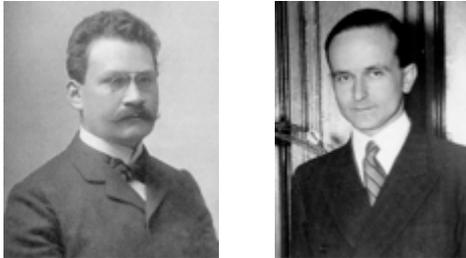
**Dan Tudor Vuza**



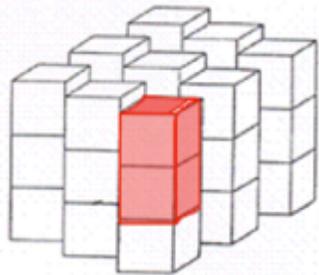
**Anatol Vieru**

# Les canons mosaïques comme problème « mathémusical »

## Le problème de Minkowski/Hajos



Dans un pavage simple [*simple lattice tiling*] d'un espace à  $n$  dimensions par des cubes unités, il y a au moins un couple de cubes qui ont en commun une face entière de dimension  $n-1$ .



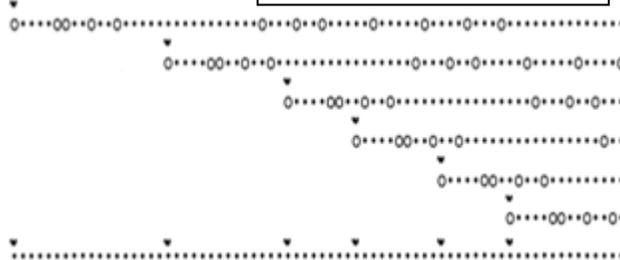
1907-1942

## Les canons mosaïques de Vieru/Vuza



Un canon de Vuza est une factorisation d'un groupe cyclique en somme directe de deux sous-ensembles non-périodiques

$$\mathbb{Z}/n\mathbb{Z} = R \oplus S$$



1991

## Lien entre Minkowski et Vuza (Andreatta, 1996)

### Groupes de Hajós (*good groups*)

$\mathbb{Z}/n\mathbb{Z}$  avec  $n \in \{p^\alpha, p^\alpha q, pqr, p^2q^2, p^2qr, pqrs\}$  où  $p, q, r, s$ , sont des nombres premiers distincts



### Groupes non-Hajós (*bad groups*)

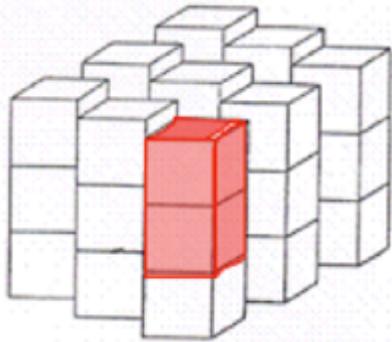
72  
 108 120 144 168 180  
 200 216 240 252 264 270 280 288  
 300 312 324 336 360 378 392 396  
 400 408 432 440 450 456 468 480  
 500 504 520 528 540 552 560 576 588 594  
 600 612 616 624 648 672 675 680 684 696  
 700 702 720 728 744 750 756 760 784 792  
 800 810 816 828 864 880 882 888...



1996

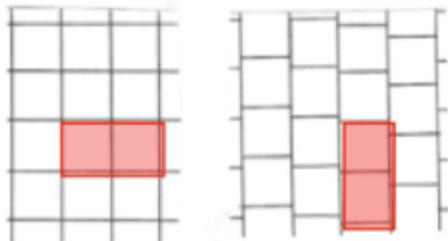
M. Andreatta, « Constructing and Formalizing Tiling Rhythmic Canons: a historical Survey of a “Mathemusical” Problem », Special Issue « Tiling Problems in Music », *Perspectives of New Music*, J. Rahn (ed.), University of Washington, Seattle (2011).

# De la Conjecture de Minkowski aux groupes de Hajos



## Conjecture de Minkowski (1896/1907)

Dans un pavage simple [simple lattice tiling] d'un espace à  $n$  dimensions par des cubes unités, il y a au moins un couple de cubes qui ont en commun une face entière de dimension  $n-1$ .



(Cf. S. Stein, S. Szabó : *Algebra and Tiling*, 1994)

## *Théorème de Hajós (1942)*

Soit  $G$  un groupe abélien fini et soient  $a_1, a_2, \dots, a_n$   $n$  éléments de  $G$ . Si l'on suppose que le groupe admet comme factorisation la somme directe des sous-ensembles  $A_1 \dots A_n$

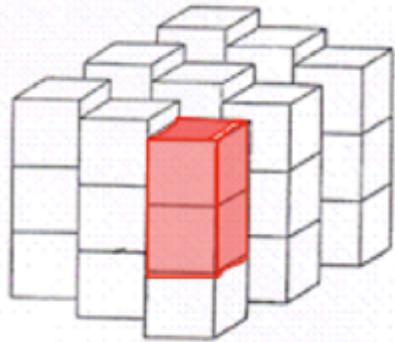
$$A_1 = \{1, a_1, \dots, a_1^{m_1-1}\}, A_2 = \{1, a_2, \dots, a_2^{m_2-1}\}, \dots, A_n = \{1, a_n, \dots, a_n^{m_n-1}\}$$

avec  $m_i > 0$  pour tout  $i=1, 2, \dots, n$ , alors un des  $A_i$  est un groupe

## *Groupes de Hajos (good groups)*

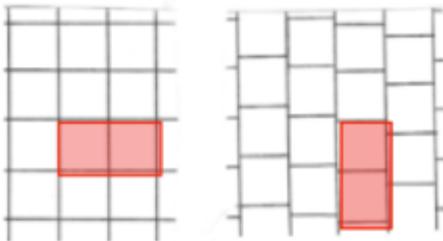
Rédei 1947	$(p, p)$	Sands 1962	$(p, 3, 3)$
Hajós 1950	$\mathbb{Z}$		$(p, 2^2, 2)$
De Brujin 1953	$\mathbb{Z}/n\mathbb{Z}$ avec $n=p^\alpha$		$(p, 2, 2, 2, 2)$
	$(p^\alpha, q)$		$(p^2, 2, 2, 2)$
	$(p, q, r)$		$(p^3, 2, 2)$
Sands 1957	$(p^2, q^2)$		$(p, q, 2, 2)$
	$(p^2, q, r)$		
	$(p, q, r, s)$	Sands 1964 Q	$\mathbb{Z} + \mathbb{Z}/p\mathbb{Z}$
Sands 1959	$(2^2, 2^2)$		$\mathbb{Q} + \mathbb{Z}/p\mathbb{Z}$
	$(3^2, 3)$		
	$(2^n, 2)$		

# Quelques versions faibles de la conjecture de Minkowski



## Conjecture de Minkowski (1896/1907)

Dans un pavage simple [simple lattice tiling] d'un espace à  $n$  dimensions par des cubes unités, il y a au moins un couple de cubes qui ont en commun une face entière de dimension  $n-1$ .



- *Les quatre conditions de la Conjecture de Minkowski*
  - [1] Les cubes sont tous obtenus par translation
  - [2] Les vecteurs de translations forment un réseau [lattice]
  - [3] Les parties internes des cubes sont disjointes
  - [4] Tout point de l'espace qui n'est pas dans le bord d'un cube est contenu exactement dans un cube
  
- *Conjecture de Keller (1930) = Minkowski – [2]*
  - Vraie pour  $n \leq 6$  (Perron, 1940)
  - Fausse pour  $n \geq 10$  (Lagarias et Shor, 1992)
  - Fausse pour  $n=8$  et  $n=9$  (Mackey, 2000)
  - Ouverte pour  $n=7$
  
- *Conjecture de Furtwangler = Minkowski – [3 et 4] + nouvelle condition :*
  - [4'] Tout point de l'espace qui n'est pas dans le bord de chaque cube est contenu dans exactement  $k$  cubes
  - La conjecture est vraie ssi  $n \leq 3$  (Hajos 1941)
  
- *Conjecture semi-périodique de Hajos (1950) : toute factorisation d'un groupe  $G = A+B$  est semipériodique i.e.  $B=B_1, \dots, B_m$  et s'il existe un sousgroupe  $G'=\{g_1, \dots, g_m\}$  telle que  $A+B_i=g_i+A+B_1$ .*

# Groupes non Hajós et Canons de Vuza

## Groupes non-Hajós (bad groups)

72  
 108 120 144 168 180  
 200 216 240 252 264 270 280 288  
 300 312 324 336 360 378 392 396  
 400 408 432 440 450 456 468 480  
 500 504 520 528 540 552 560 576 588 594  
 600 612 616 624 648 672 675 680 684 696  
 700 702 720 728 744 750 756 760 784 792  
 800 810 816 828 864 880 882 888...

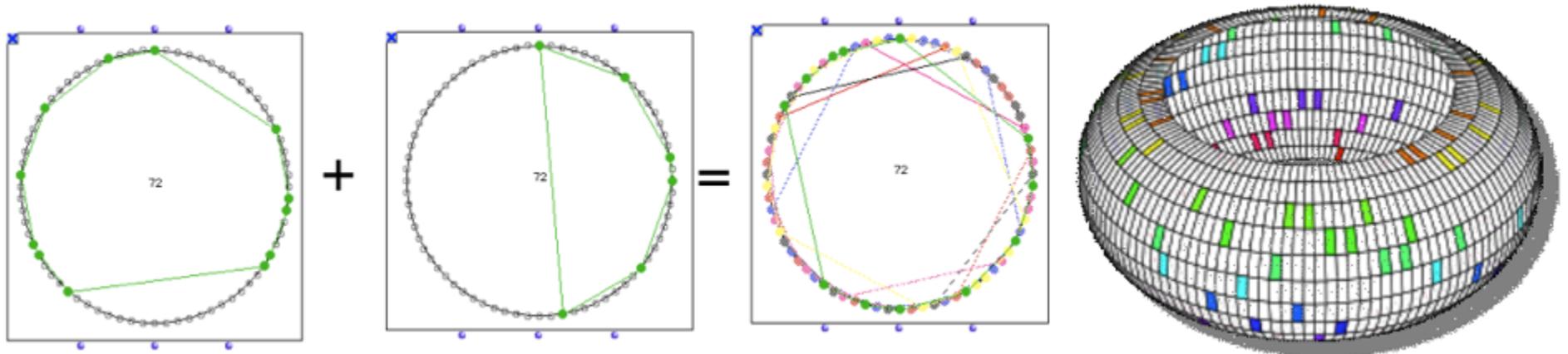
## Groupes Hajós (good groups)

$\mathbb{Z}/n\mathbb{Z}$  avec

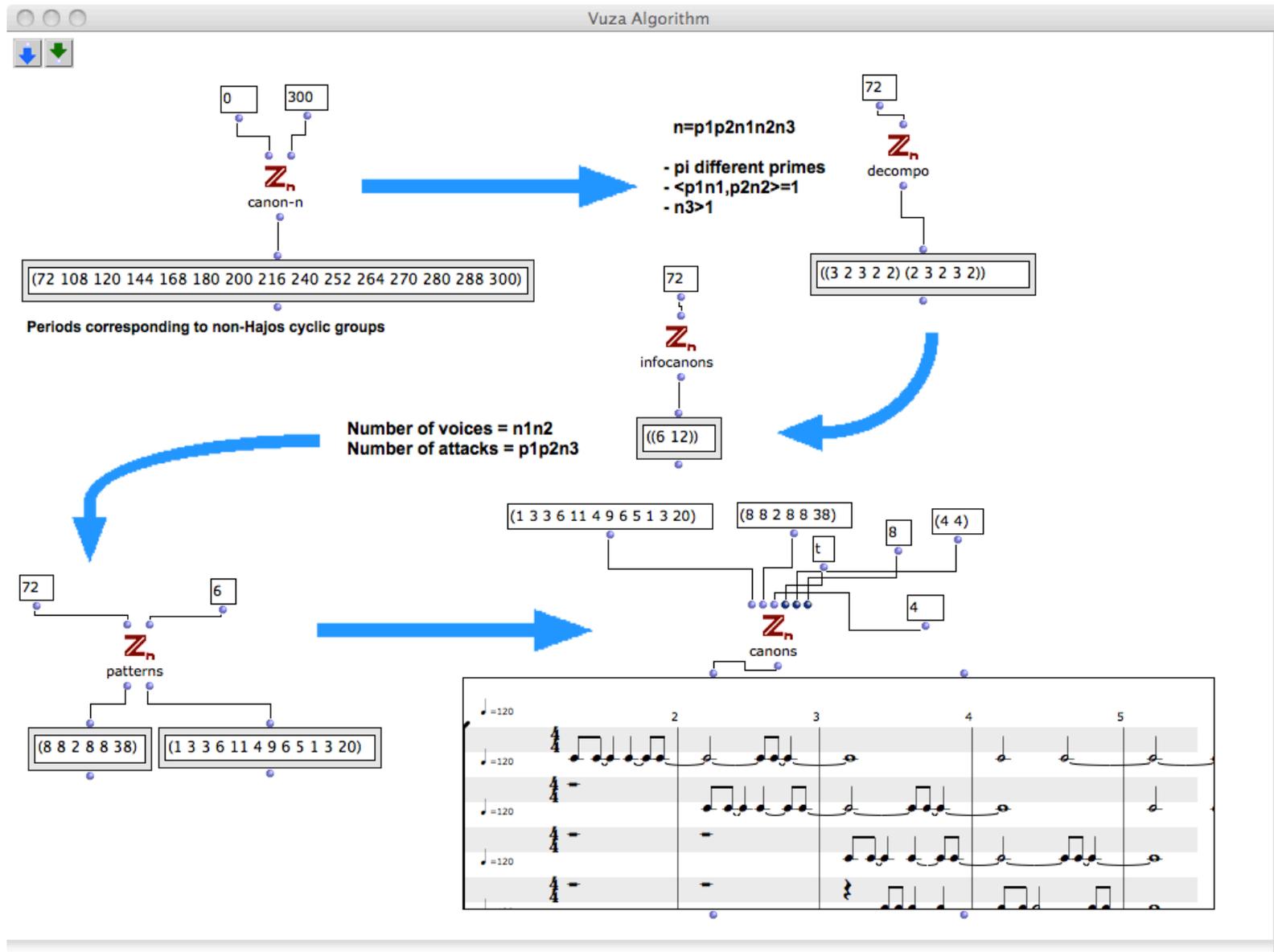
$n \in \{p^\alpha, p^\alpha q, pqr, p^2q^2, p^2qr, pqrs\}$

où  $p, q, r, s$ , sont des nombres premiers distincts

**Problème ouvert : Trouver un algorithme qui permet d'obtenir toutes les factorisations d'un groupe cyclique non-Hajós en somme directe de deux sous-ensembles non périodiques (i.e. classifier tous les Canons de Vuza).**



# Algorithme de Vuza et implémentation en OpenMusic



17-vuza algorithm

# Algorithme de Vuza (*Perspectives of New Music*)

---

1.  $\langle n, m \rangle = 1$  (i.e.,  $n$  and  $m$  are relatively prime);
2.  $n = n_1 n_2, m = m_1 m_2$ ;
3.  $n_1, n_2, m_1, m_2, k$  are all greater than 1.

In order to apply the construction proposed by de Bruijn to  $72 = n \cdot m \cdot k = 4 \cdot 9 \cdot 2$ , one needs the following subsets:

1.  $R_k = \mathbb{Z}_N / k \mathbb{Z}_N$  (i.e.,  $R_k = 0, 1, 2, \dots, k-1$ , for every  $k \in \mathbb{N}$ );
2.  $C_1 = k_{n_1} R_{n_2} \oplus k_{n_1 n_2 m_1} \mathbb{Z}_N$ ;
3.  $C_2 = k_{m_1} R_{m_2} \oplus k_{m_1 m_2 n_1} \mathbb{Z}_N$ .

Since  $n_1 = n_2 = 2$  and  $m_1 = m_2 = 3$ , one obtains the following factorization of  $\mathbb{Z}/72\mathbb{Z}$  into non-periodic subsets  $S$  and  $T$ :

$$\begin{aligned}
 S &= C_2 \cup \{1, 2, \dots, k-1\} \oplus C_1 \\
 &= k_{m_1} R_{m_2} \oplus k_{m_1 m_2 n_1} \mathbb{Z}_N \cup \{1, 2, \dots, k-1\} \oplus k_{n_1} R_{n_2} \oplus k_{n_1 n_2 m_1} \mathbb{Z}_N \\
 &= 6\{0, 1, 2\} \oplus 36\mathbb{Z}/72\mathbb{Z} \cup \{1\} \oplus 4\{0, 1\} \oplus 24\mathbb{Z}/72\mathbb{Z} \\
 &= \{0, 6, 12, 36, 42, 48\} \cup \{1\} \oplus \{0, 4, 24, 28, 48, 52\} \\
 &= \{0, 1, 5, 6, 12, 25, 29, 36, 42, 48, 49, 53\}
 \end{aligned}$$

$$\begin{aligned}
 T &= k_{m_1 m_2} R_{n_1} \oplus k_{n_1 n_2} R_{m_1} \\
 &= 18\{0, 1\} \oplus 8\{0, 1, 2\} \\
 &= \{0, 8, 16, 18, 26, 34\}
 \end{aligned}$$

# Classification « paradigmatique » des canons mosaïques de Vuza

Résultat : uniquement deux « types » de canons différents (à une transformation affine près, i.e.  $f: \mathbb{Z}_{72} \rightarrow \mathbb{Z}_{72}$  t.q.  $f(x)=ax+b$  avec  $a \in (\mathbb{Z}_{72})^*$  et  $b \in \mathbb{Z}_{72}$ )



• R. Tijdeman: “Decomposition of the Integers as a direct sum of two subsets”, *Number Theory*, Cambridge University Press, 1995. The fundamental Lemma:  $R$  pave  $\mathbb{Z}_n \Rightarrow aR$  pave  $\mathbb{Z}_n$   $\langle a, n \rangle = 1$

$\{Z_n\}$   
R  
(1 3 3 6 11 4 9 6 5 1 3 20)  
(20 3 1 5 6 9 4 11 6 3 3 1)  
(1 4 1 19 4 1 6 6 7 4 13 6)  
(6 13 4 7 6 6 1 4 19 1 4 1)  
(1 5 15 4 5 6 6 3 4 17 3 3)  
(3 3 17 4 3 6 6 5 4 15 5 1)

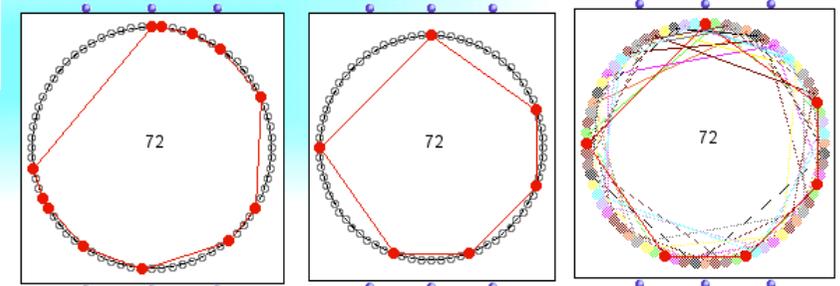
S  
(8 8 2 8 8 38)  
(16 2 14 2 16 22)  
(14 8 10 8 14 18)

$\{O_n\}$   
R  
(1 3 3 6 11 4 9 6 5 1 3 20)  
(1 4 1 19 4 1 6 6 7 4 13 6)  
(1 5 15 4 5 6 6 3 4 17 3 3)

S  
(8 8 2 8 8 38)  
(16 2 14 2 16 22)  
(14 8 10 8 14 18)

$\{Af_n\}$   
R  
(1 3 3 6 11 4 9 6 5 1 3 20)  
(1 4 1 19 4 1 6 6 7 4 13 6)

S  
(14 8 10 8 14 18)



$\mathbb{Z}/72\mathbb{Z} = R \oplus S$

Collection « Musique/ Sciences » (dir. J.-M. Bardez & M. Andreatta)



F. Lévy



G. Bloch



M. Lanza



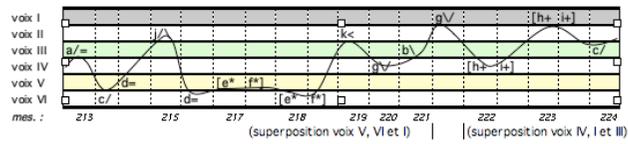
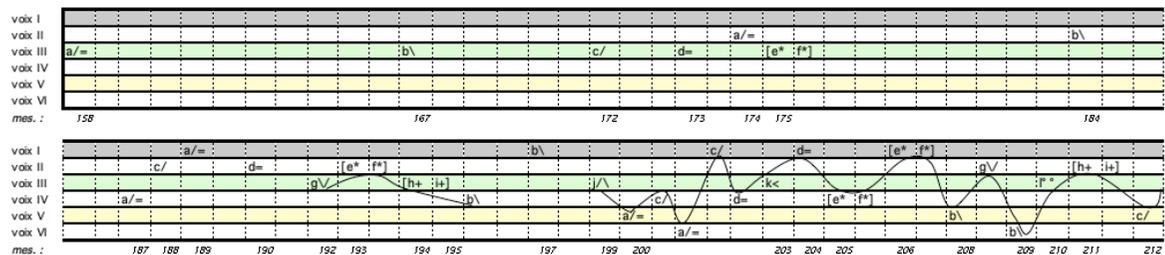
T. Johnson

1999



18-Catalogue-Z72

# Quelques applications compositionnelles des canons de Vuza



a/= : montée vers accord puis "mise en pulsation"  
 b\ : "mise en pulsation" superposé à un gliss. descendant  
 c/ : montée vers accord (tête de a/=)  
 d= : "mise en pulsation" en diminuenudo (fin de a/=)  
 [e\* f\*] : accord mis en "cross rythm I" (durée double)  
 gV : gliss. descendant puis ascendant  
 [h+ i+] : accord mis en "cross rythm II" (durée double)  
 jV : gliss. ascendant puis descendant avec accent  
 k< : "son à l'envers"  
 !\* : deux impacts brefs et piano



F. Lévy

## Coïncidences (1999)

Coincidence - Fabien Levy : déroulement du canon (mes. 158 à 226)  
 (chaque impact fait 3 temps)



M. Lanza

**La bataille de caresse et de charnage**  
 (pour violoncelle et accompagnement, 2012)



**A piece based on Monk (2007)**  
 (« Well You Need'nt »)



G. Bloch




**La notte poco prima della foresta**  
 (opéra de chambre pour acteur, mezzo-soprano, baryton, ensemble et électronique, 2009)



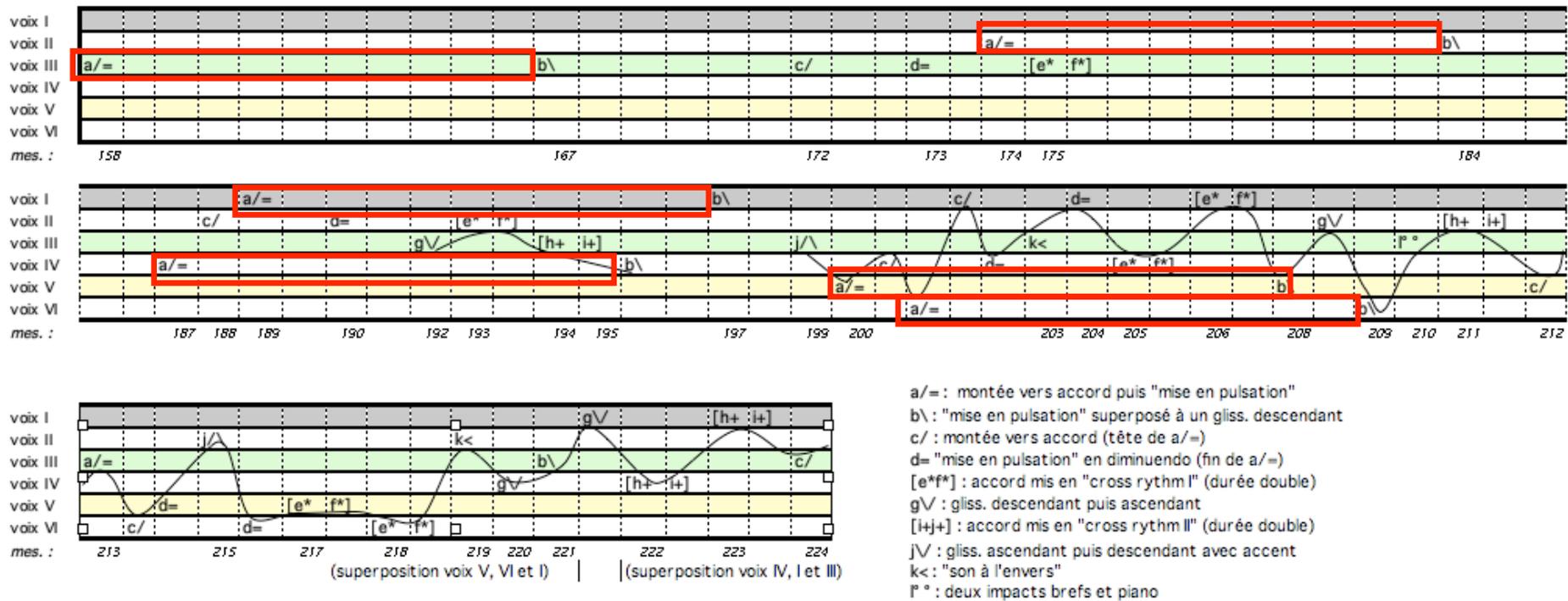
D. Ghisi

# Fabien Lévy

## Première utilisation des canons de Vuza



• *Coïncidences* (pour 33 musiciens, 1999-2007)



Coïncidences - Fabien Lévy : déroulement du canon (mes. 158 à 226)  
 (chaque impact fait 3 temps)



Interprètes : Tokyo Symphony Orchestra, Dir.: Kazuyoshi Akiyama, 05/09/2007, Suntory Hall, Tokyo, Japon

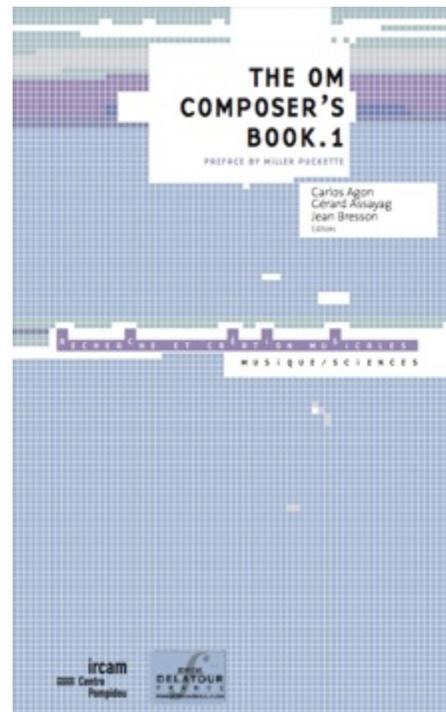
# Georges Bloch

## Stratégies compositionnelles à partir d'un modèle formel



- Organisation métrique d'un canon mosaïque
- Réduction d'un canon par auto-similarité
- Modulation métrique entre canons
- Transformation d'un canon dans une texture
- Canons mosaïques et IAO (*OMax*)

- *Projet Beyeler* (2001)
- *Projet Hitchcock*
- *Visite des tours de la cathédrale de Reims*
- *Noël des Chasseurs*
- *Canons à marcher*
- *Canon à eau*
- *Harawun* (2004)
- *L'Homme du champ* (2005)
- *A piece based on Monk* (2007)
- *Peking Duck Soup* (2008)

A musical score for six voices (V1-V6) in bass clef. The score is divided into two systems. The first system includes staves V1, V2, and V3. The second system includes staves V4, V5, and V6. The music features various dynamics such as *mp*, *pp*, *mf*, and *f*. The notation includes notes, rests, and bar lines.

- *A piece based on Monk* (2007)  
(« Well You Need'nt »)

# Mauro Lanza

## Canons de Vuza et périodicités locales



- *La descrizione del diluvio* (Ricordi, 2007-2008)

Canon à 14 voix sur le pattern rythmique :

(1 3 25 27 1 3 11 14 27 1 3 25 27 4 25 27 1 3 25 14 13 1 3 25 27 1 3 52)

No. 1 "Aria"

Local Dynamics :

Elettronica

Soprano

Mezzo

Alto

Tenore

Baritono

Basso

General Dynamic: pppp - pp

poco a poco crescendo fino a misura 40 (ppp - mf)

*6 voix sont en live et 8 dans l'électronique. L'unité est la double-croche de triolet. Le choix des notes et des durées est fait en cherchant à souligner certaines **quasi-périodicités** du canon de Vuza, et cela donne à chaque voix un caractère beaucoup plus "redondant".*

# Mauro Lanza

## Canons de Vuza et périodicités locales

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- *La descrizione del diluvio* (Ricordi, 2007-2008)

*[...] Le choix des notes et des durées est fait en cherchant à souligner certaines quasi-périodicités du canon de Vuza, et cela donne à chaque voix un caractère beaucoup plus “redondant”.*

(1 3 25 27 1 3 11 14 27 1 3 25 27 4 25 27 1 3 25 14 13 1 3 25 27 1 3 52)

(1 3 25 27 1 3 11 14 27 1 3 25 27 4 25 27 1 3 25 14 13 1 3 25 27 1 3 52)

(1 3 25 27 1 3 11 14 27 1 3 25 27 4 25 27 1 3 25 14 13 1 3 25 27 1 3 52)

(1 3 25 27 1 3 11 14 27 1 3 25 27 4 25 27 1 3 25 14 13 1 3 25 27 1 3 52)

(1 3 25 27 1 3 11 14 27 1 3 25 27 4 25 27 1 3 25 14 13 1 3 25 27 1 3 52)

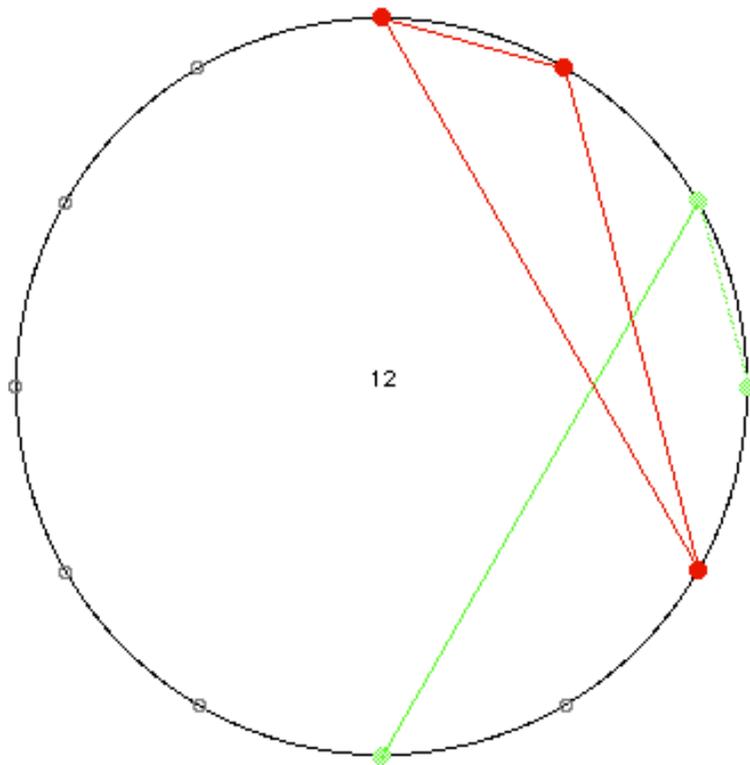




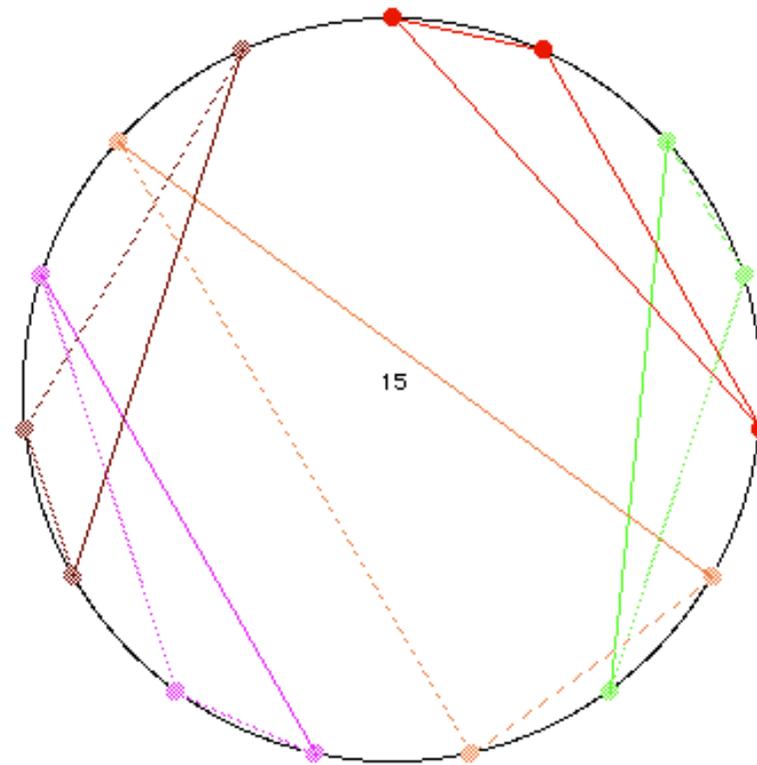
# Tiling the line and/or circle with augmentations

- **Tom Johnson (2001): tiling the line with a given rhythmic pattern**
  - ex. (0 1 4). Does it tile? With augmentations? With which period?

• **Theorem (Amiot, 2002) : Any tiling of the line with the pattern (0 1 4) and its augmentations is periodic and the period is equal to a multiple of 15**

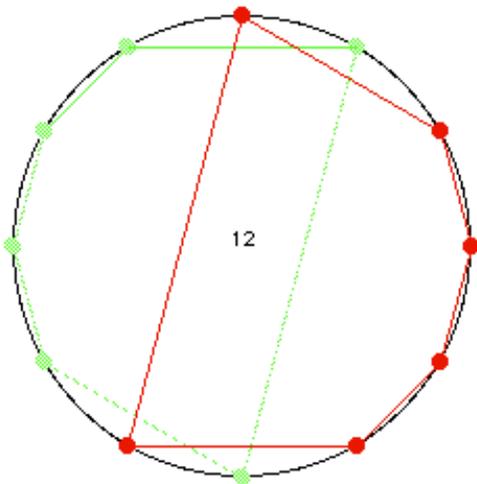
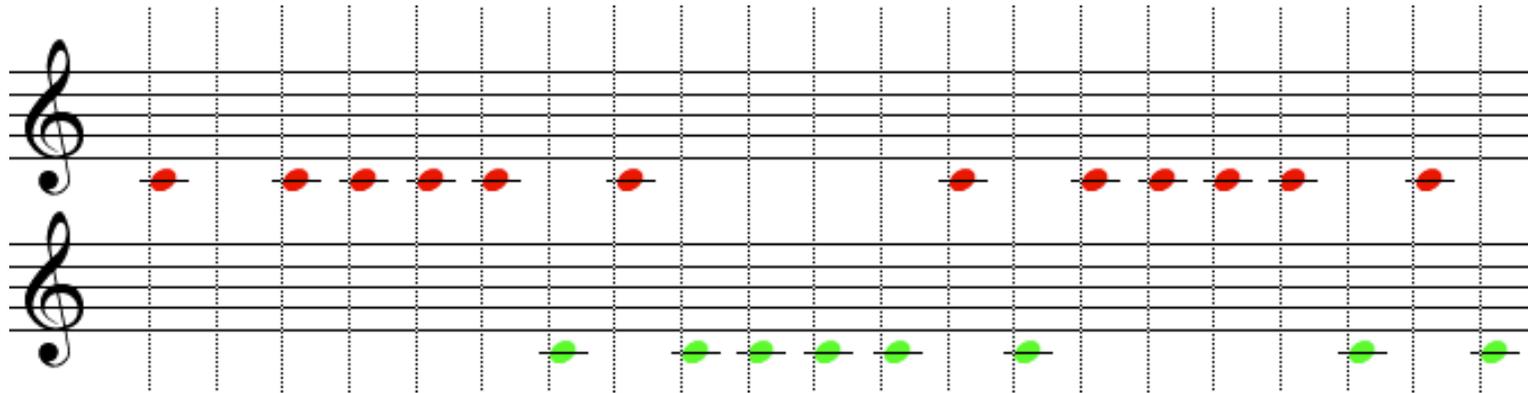


$n = 12$



$n = 15$

# Canons mosaïques par translation et augmentation

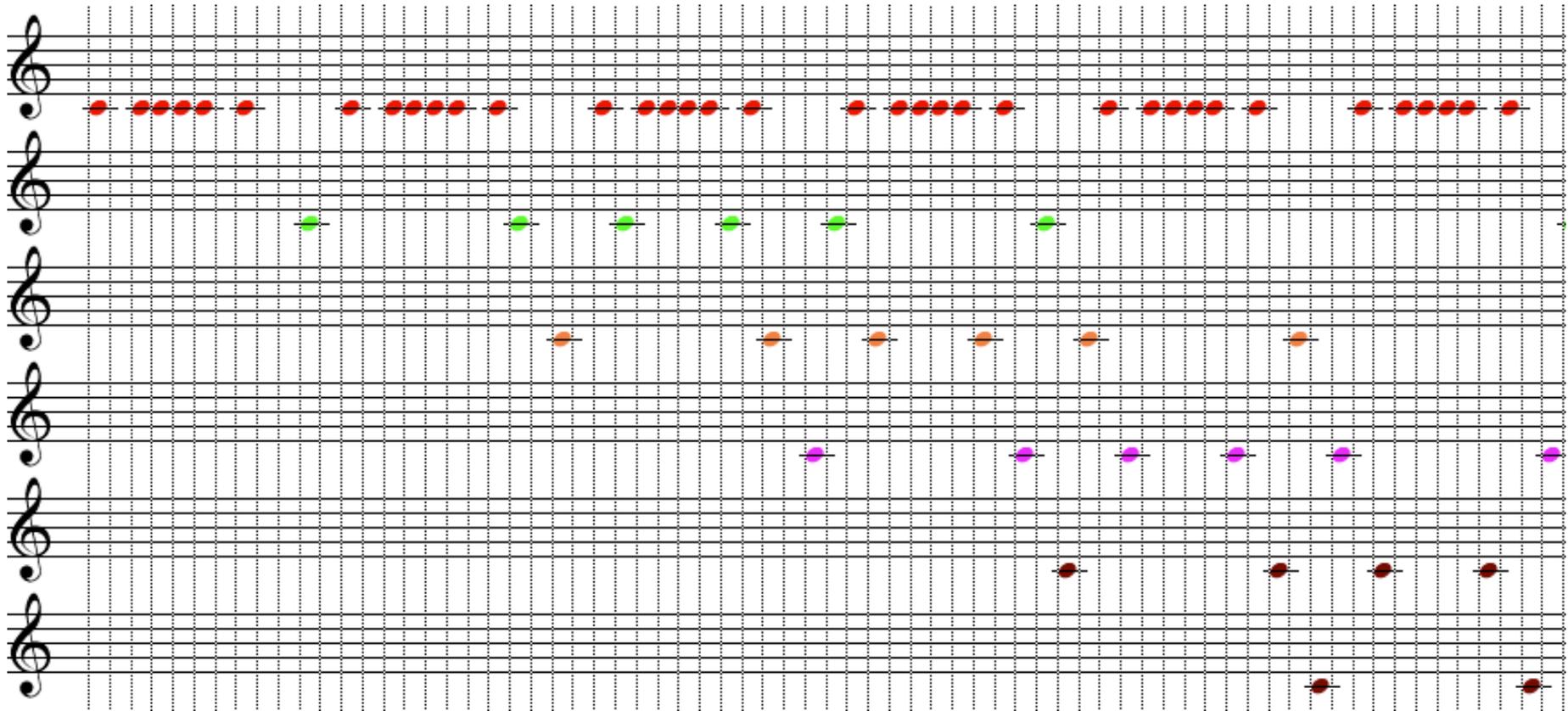


- ((0 1 2 3 4 6) ((1 11)))
- ((0 1 2 3 4 5) ((1 11) (1 1)))
- ((0 1 2 3 5 7) ((1 11) (1 7)))
- ((0 1 3 4 7 8) ((1 5)))
- ((0 1 2 3 6 7) ((1 11)))
- ((0 1 3 4 6 9) ((1 11) (1 5)))
- ((0 1 3 6 7 9) ((1 11) (1 5)))
- ((0 1 2 6 7 8) ((1 11) (1 7) (1 5) (1 1)))
- ((0 1 4 5 8 9) ((1 11) (1 7) (1 5) (1 1)))
- ((0 1 2 5 6 7) ((1 7) (1 5)))
- ((0 2 3 4 5 7) ((1 11) (1 7) (1 5) (1 1))) ←
- ((0 1 4 5 6 8) ((1 11) (1 7)))
- ((0 1 2 4 5 7) ((1 5)))
- ((0 1 3 4 5 8) ((1 5) (1 1)))
- ((0 1 2 4 5 8) ((1 11)))
- ((0 1 2 4 6 8) ((1 11) (1 7)))
- ((0 2 3 4 6 8) ((1 11)))
- ((0 2 4 6 8 10) ((1 11) (1 7) (1 5) (1 1)))



# Augmented Tiling Canons ou l'action du groupe affine

(en collaboration avec Thomas Noll)

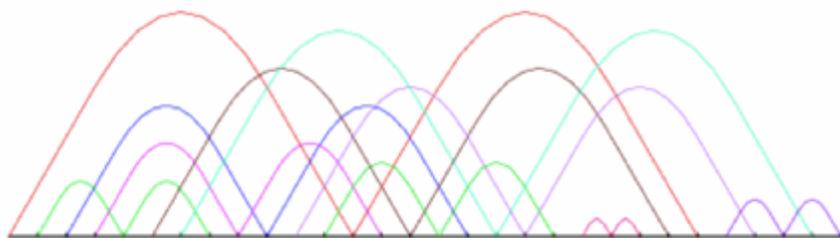
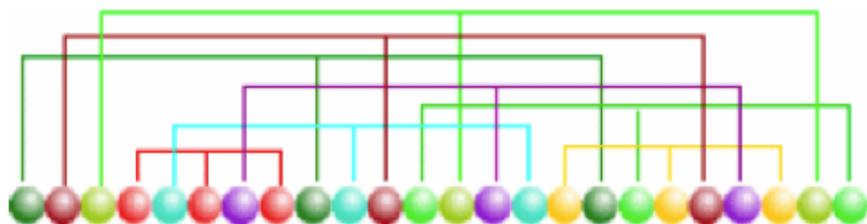
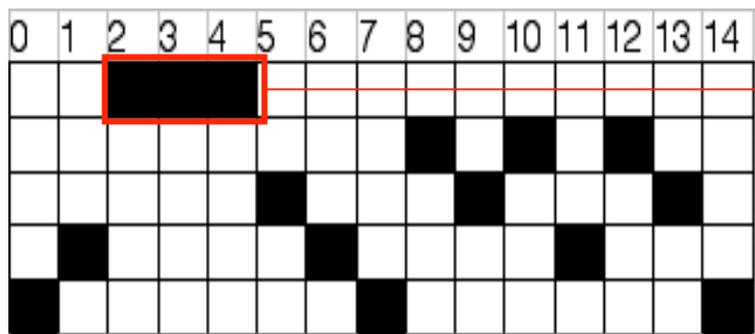


22-Augmented-Canons

# Tom Johnson's Perfect Tilings

## Tilework for Piano

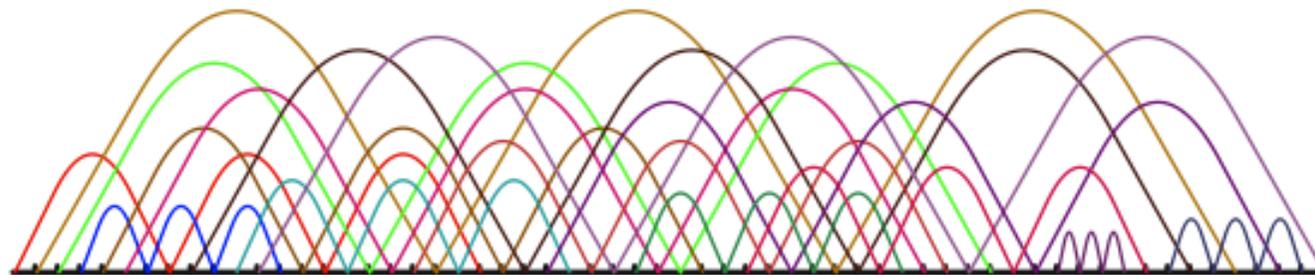
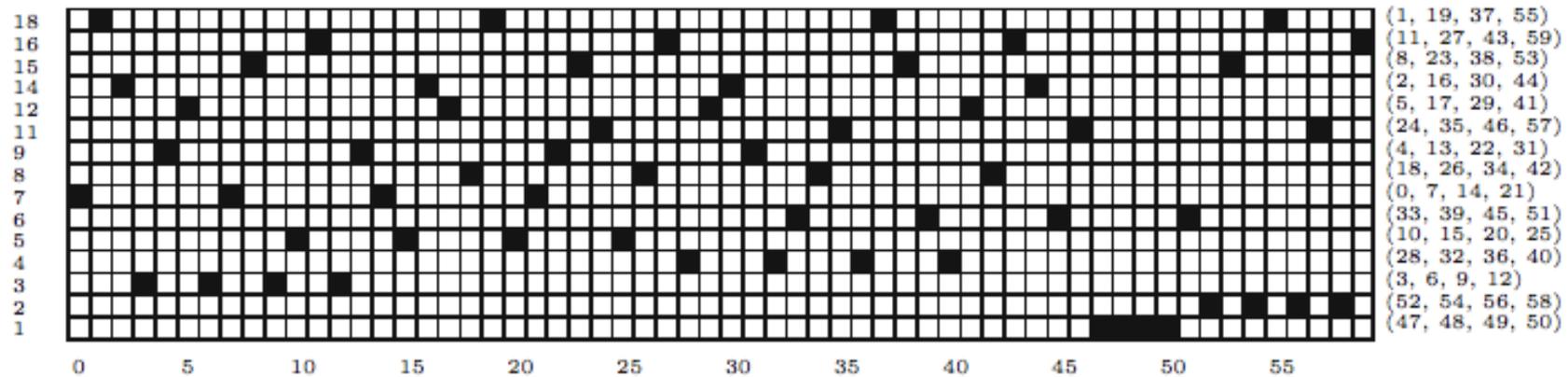
perfect triplet tilings, 5th order  
with thanks to Jon Wild and Erich Neuwirth



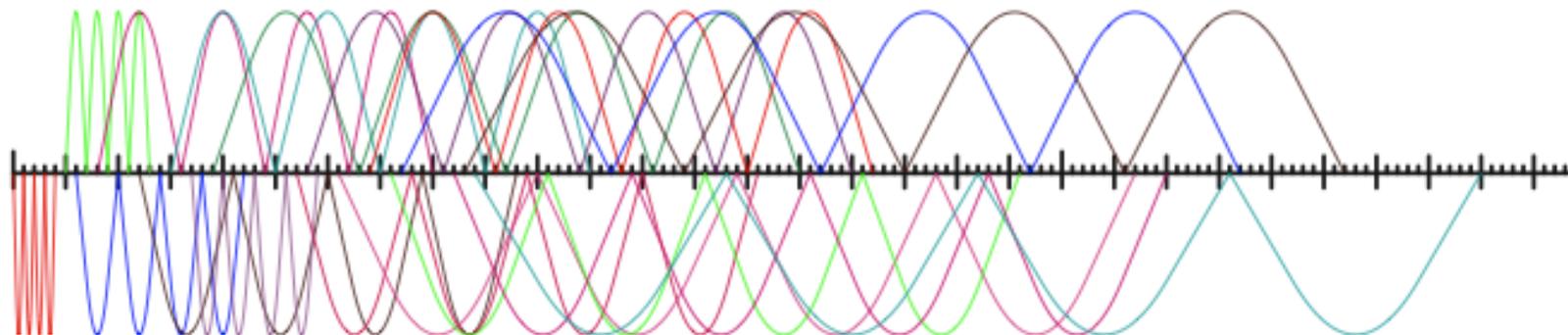
Jean-Paul Davalan, « Perfect Rhythmic Tilings » (to appear in *Tiling Problems in Music*, M. Andreatta & C. Agon eds., Collection « Musique/Sciences », 2008)



# Perfect Rhythmic Tilings and open problems



Does it exist a quintuple perfect tiling canon?



# Un exercice minimaliste sur les permutations circulaires

CLAPPING MUSIC  
FOR TWO PERFORMERS

*J=144-168*

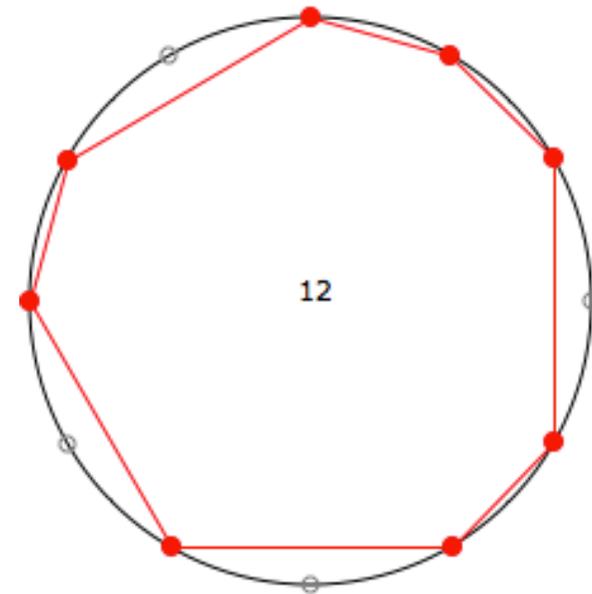
CLAP 1  
CLAP 2

Repeat bar 1, then end.

The performance begins and ends with both performers in unison at bar 1. The number of repeats of each bar should be fixed at twelve repeats per bar. Since the first performer's part does not change, it is up to the second performer to move from one bar to the next. The second performer should try to keep his or her downbeat where it is written, i.e. on the first beat of each measure (not on the first beat of the group of three claps), so that his downbeat always falls on a new beat of his or her changing pattern.

The choice of a particular clapping sound, i.e. with cupped or flat hands is left up to the performers. Whichever timbre is chosen, both performers should try to get the same one so that their two parts will blend to produce one overall resulting pattern.

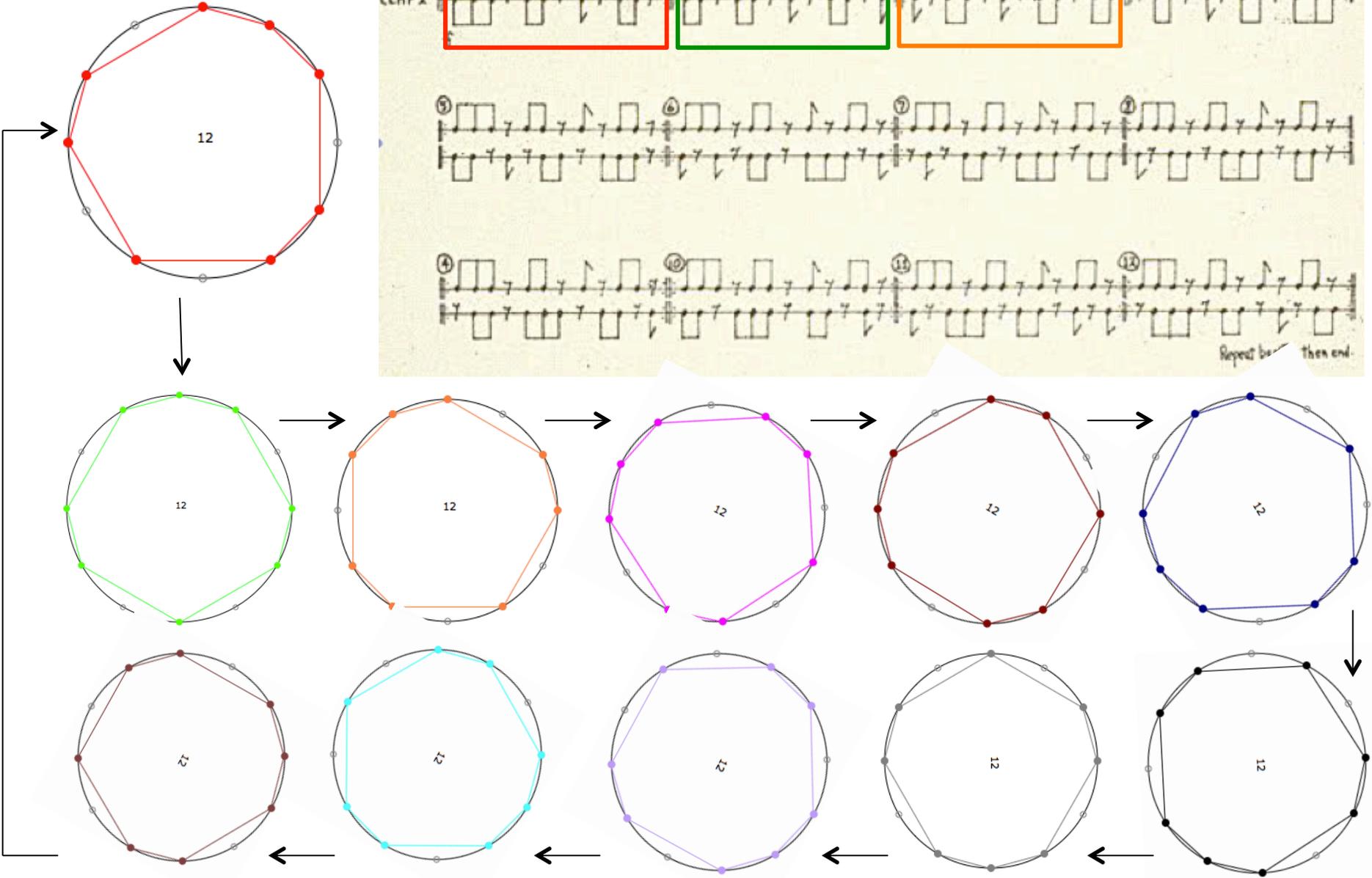
*Steve Reich 12/72  
re-copied 1/78*



Clapping Music (1972)

# Un exercice minimaliste sur les permutations circulaires

Handwritten musical score for two claps, CLAP 1 and CLAP 2, with measures 1-15. Three measures (1, 2, 3) are highlighted with colored boxes: red, green, and orange. The score includes a tempo marking  $\text{♩} = 144-168$  and a note "Repeat before then end."



# Un exercice minimaliste sur les permutations circulaires

CLAPPING MUSIC  
FOR TWO PERFORMERS

*J=144-168*

CLAP 1  
CLAP 2

① ② ③ ④ ⑤ ⑥ ⑦ ⑧ ⑨ ⑩ ⑪ ⑫

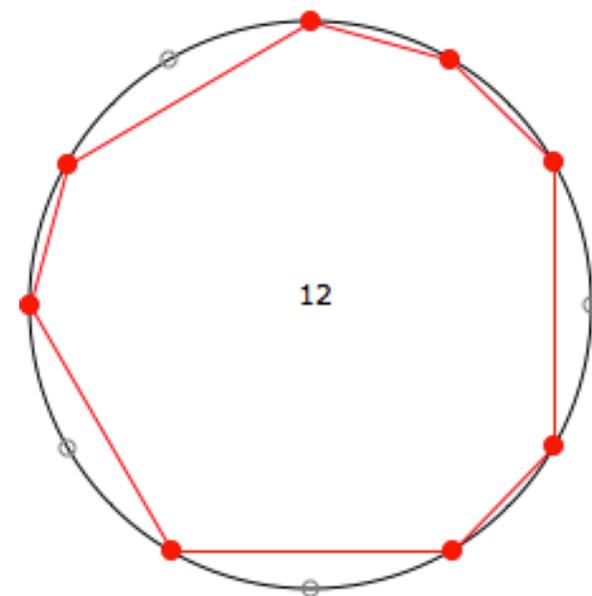
Repeat bar ①, then end.

The performance begins and ends with both performers in unison at bar ①. The number of repeats of each bar should be fixed at twelve repeats per bar. Since the first performer's part does not change, it is up to the second performer to move from one bar to the next. The second performer should try to keep his or her downbeat where it is written, i.e. on the first beat of each measure (not on the first beat of the group of three claps), so that his downbeat always falls on a new beat of his or her changing pattern.

The choice of a particular clapping sound, i.e. with cupped or flat hands is left up to the performers. Whichever timbre is chosen, both performers should try to get the same one so that their two parts will blend to produce one overall resulting pattern.

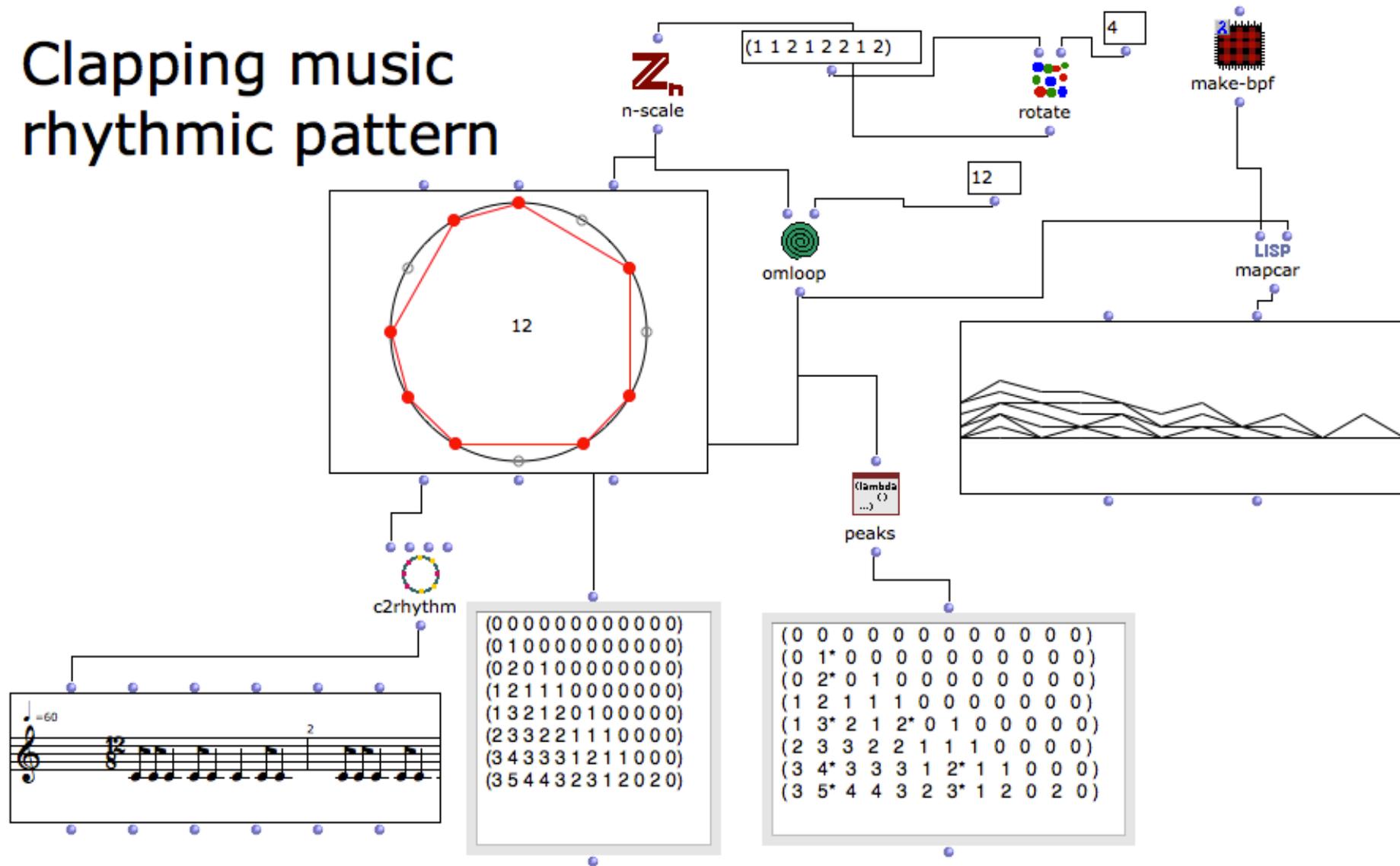
*New York 12/72  
re-copied 1/78*

**Clapping Music (1972)**

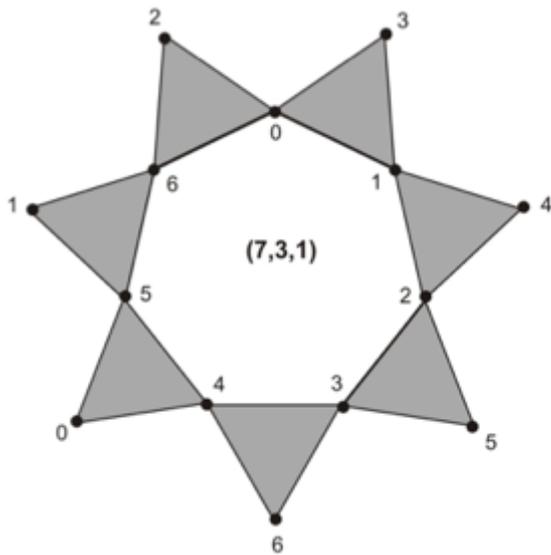
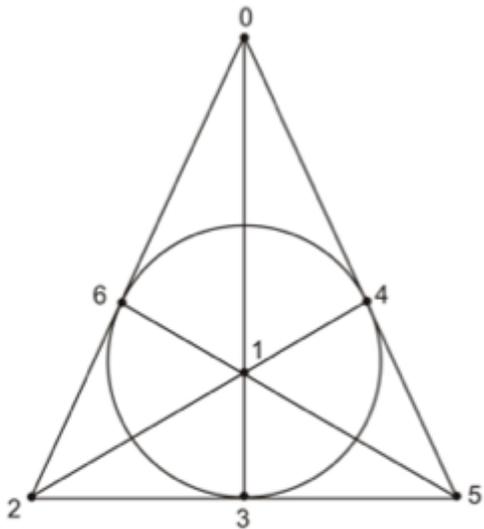


# Modélisation via le *Unfolding Rhythmic Interval Vector*

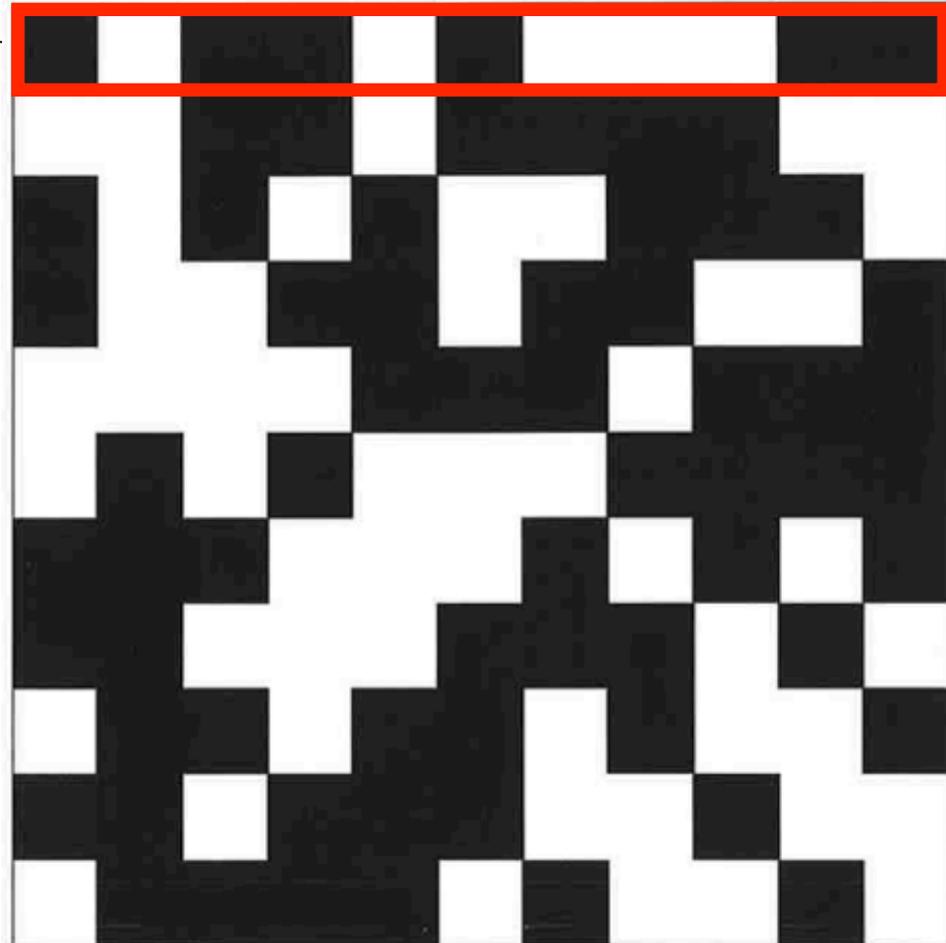
## Clapping music rhythmic pattern



# Théorie des *Block-Designs*

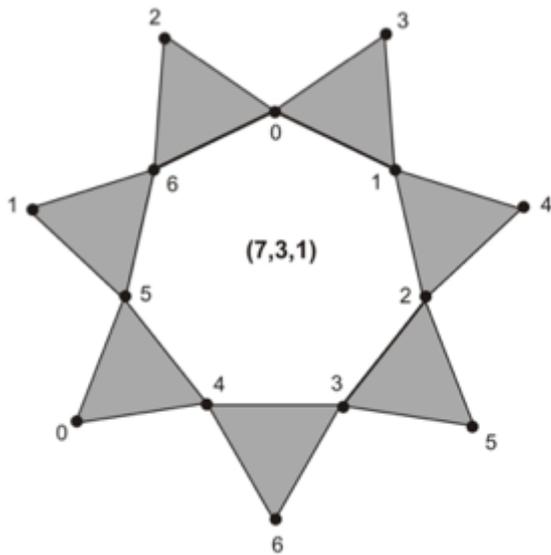
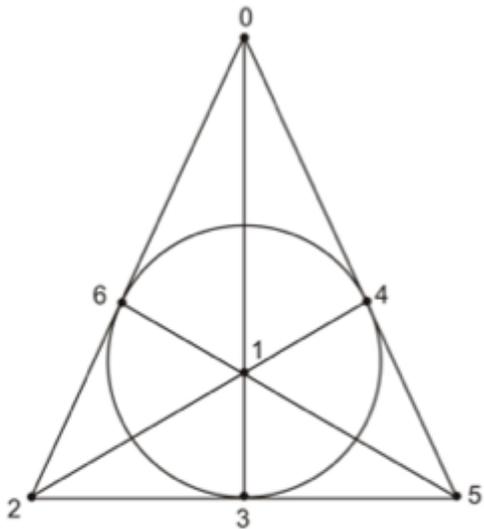
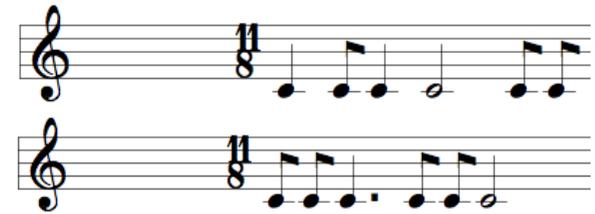


**Plan de Fano = (7,3,1)**

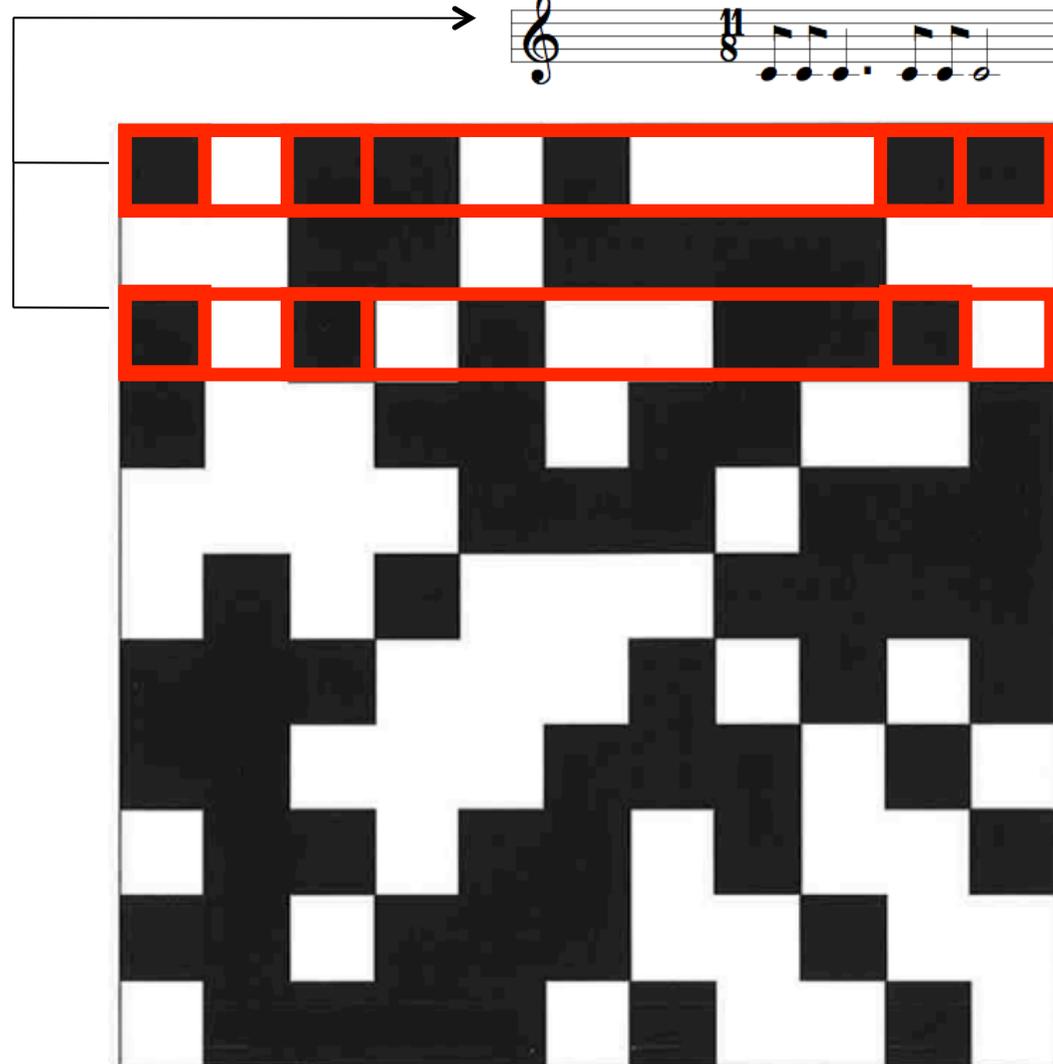


**Block-design (11,6,3)**

# Théorie des *Block-Designs*



**Plan de Fano = (7,3,1)**

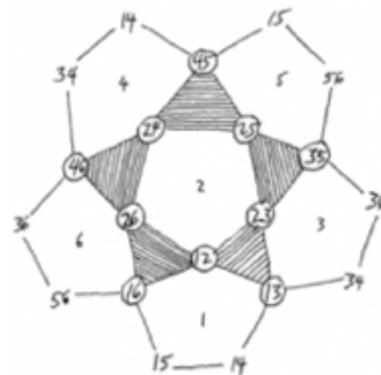


**Block-design (11,6,3)**

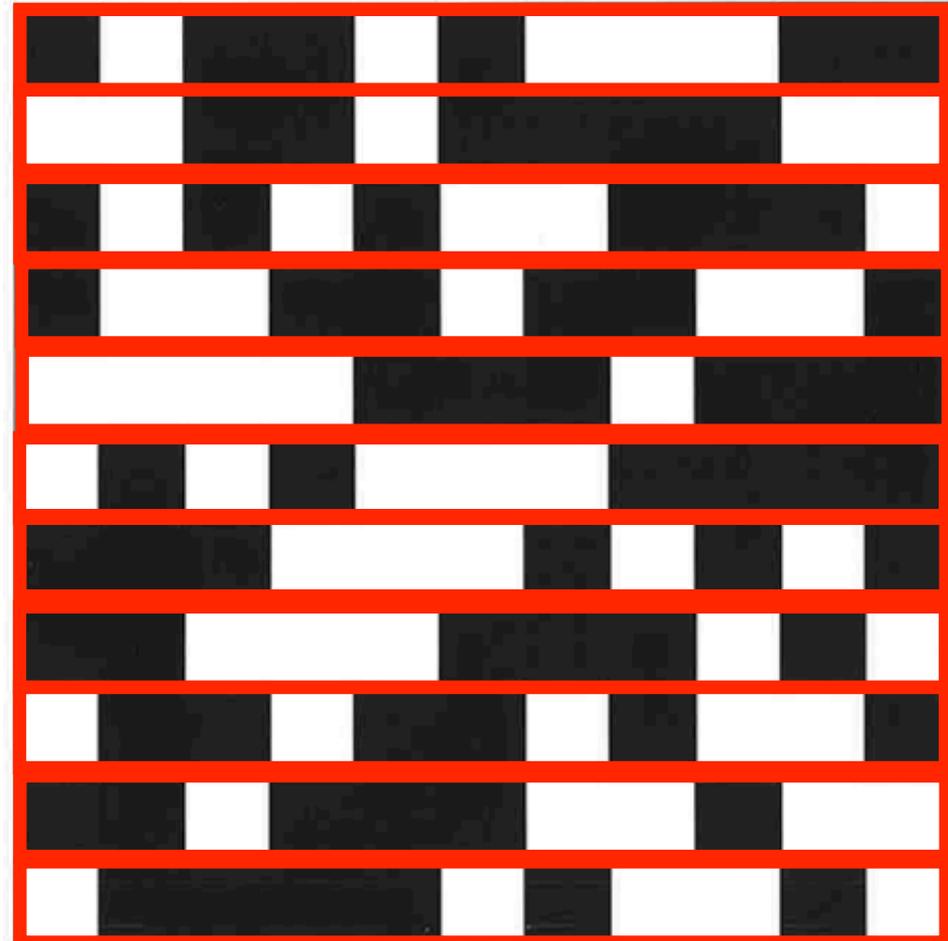
# *Block-Designs* et musique algorithmique



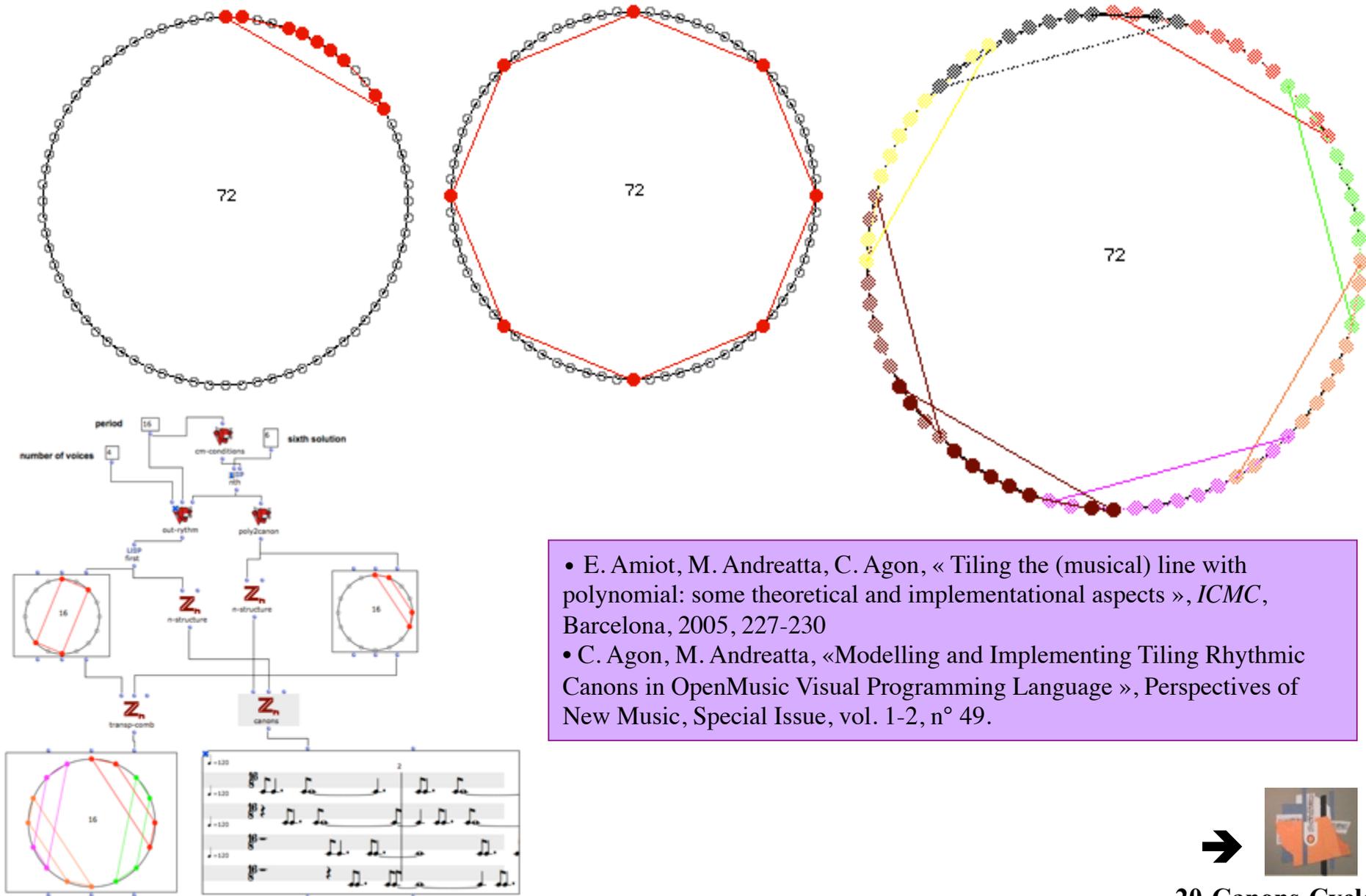
**Tom Johnson : Vermont Rhythms  
(Ensemble Klang, 2010)**



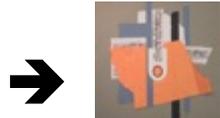
**Block-design (11,6,3)**



# La famille des « canons cyclotomiques »



- E. Amiot, M. Andreatta, C. Agon, « Tiling the (musical) line with polynomial: some theoretical and implementational aspects », *ICMC*, Barcelona, 2005, 227-230
- C. Agon, M. Andreatta, «Modelling and Implementing Tiling Rhythmic Canons in OpenMusic Visual Programming Language », *Perspectives of New Music*, Special Issue, vol. 1-2, n° 49.



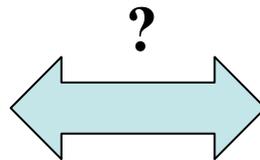
# Conjecture spectrale et problème de Minkowski/Hajos

## La conjecture de Fuglede



Un sous-ensemble de  $\mathbb{R}^n$  pave par translation ssi il est spectral (i.e. il admet une décomposition hilbertienne d'exponentiels complexes)  
*J. Func. Anal.* 16, 1974.

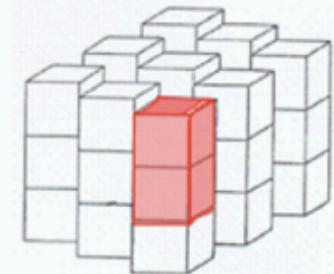
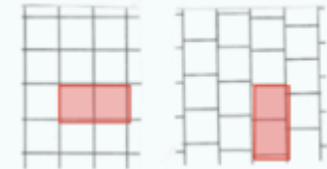
Fausse en dim.  $n \geq 3$   
 Ouverte en dim. 1 et 2



## Le problème de Minkowski/Hajos



Dans un pavage simple [simple lattice tiling] d'un espace à  $n$  dimensions par des cubes unités, il y a au moins un couple de cubes qui ont en commun une face entière de dimension  $n-1$ .



**DEFINITION 6** A subset  $A$  of some vector space (say  $\mathbb{R}^n$ ) is spectral iff it admits a Hilbert base of exponentials, i.e. if any map  $f \in L^2(A)$  can be written

$$f(x) = \sum f_k \exp(2i\pi \lambda_k \cdot x)$$

for some fixed family of vectors  $(\lambda_k)_{k \in \mathbb{Z}}$  where the maps  $e_k : x \mapsto \exp(2i\pi \lambda_k \cdot x)$  are mutually orthogonal (i.e.  $\int_A \bar{e}_k e_j = 0$  whenever  $k \neq j$ ).



(M. Andreatta et C. Agon, eds 2009)

# Conjecture spectrale et canons de Vuza

## La conjecture de Fuglede



Un sous-ensemble de  $\mathbb{R}^n$  pave par translation ssi il est spectral (i.e. il admet une décomposition hilbertienne d'exponentiels complexes)

*J. Func. Anal.* 16, 1974.

Fausse en dim.  $n \geq 3$

Ouverte en dim. 1 et 2

## Canons de Vuza de période $n$

- $n = p_1 p_2 n_1 n_2 n_3$
- $\langle p_1 n_1, p_2 n_2 \rangle = 1$
- $n_3 > 1$

72

108 120 144 168 180

200 216 240 252 264 270 280 288

300 312 324 336 360 378 392 396

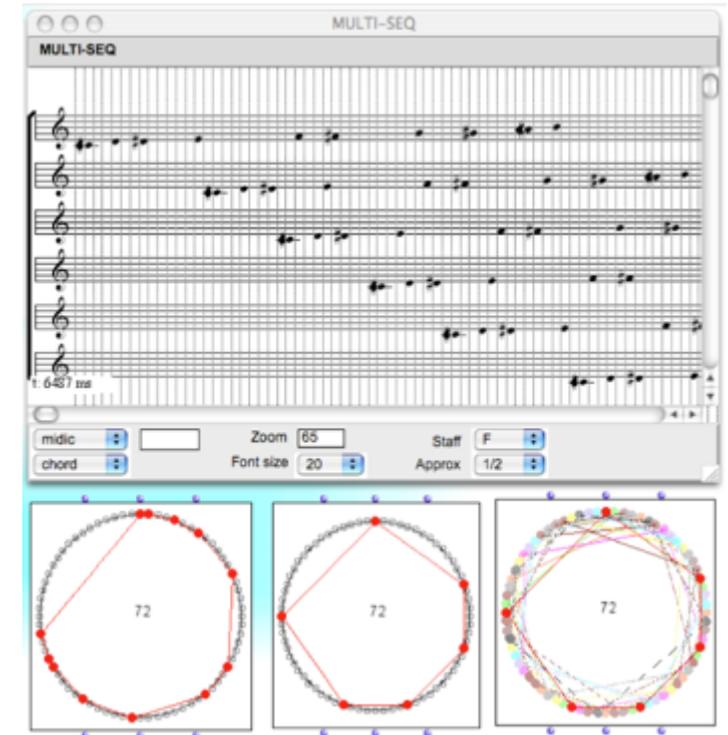
400 408 432 440 450 456 468 480

500 504 520 528 540 552 560 576 588 594

600 612 616 624 648 672 675 680 684 696

700 702 720 728 744 750 756 760 784 792

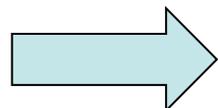
800 810 816 828 864 880 882 888...



## Résultat (Amiot 2009) :

Si  $A$  pave mais il n'est pas spectral

$\Rightarrow A$  est le rythme d'un canon de Vuza



*Problème ouvert : Trouver un algorithme qui permet d'obtenir toutes les factorisations d'un groupe cyclique non-Hajos en somme directe de deux sous-ensembles non périodiques*

# Les conjectures de Minkowski/Fuglede et les canons rythmiques

- Minkowski's conjecture and Hajos algebraic solution
- The classification of Hajos groups (Hajos, de Bruijn, Sands, ...) and polynomial representations (Redei)
- Vuza Canons and their enumeration (Vuza, Friperntinger, ...)
- The « Fundamental Lemma » in group factorization (Tijdeman)
- Classification of factorizations for non-Hajos groups (Vuza, Andreatta, Agon, Amiot, Friperntinger, ...)
- Computational model (Andreatta, Agon, Amiot, Noll, Jedrzejewski...)
- The Tiling of the line problem and Fuglede's Conjecture (Tijdeman, Lagarias, ...)
- Link between C&M conditions and Fuglede Conjecture (Laba, Kolountzakis; ...)
- Given a finite set that tiles  $\mathbf{Z}$ , what will be the period (Kolountzakis, Steinberger, ...)
- Fuglede's Conjecture and Vuza's Canons (Amiot, Matolcsi, Kolountzakis, ...)
- ...

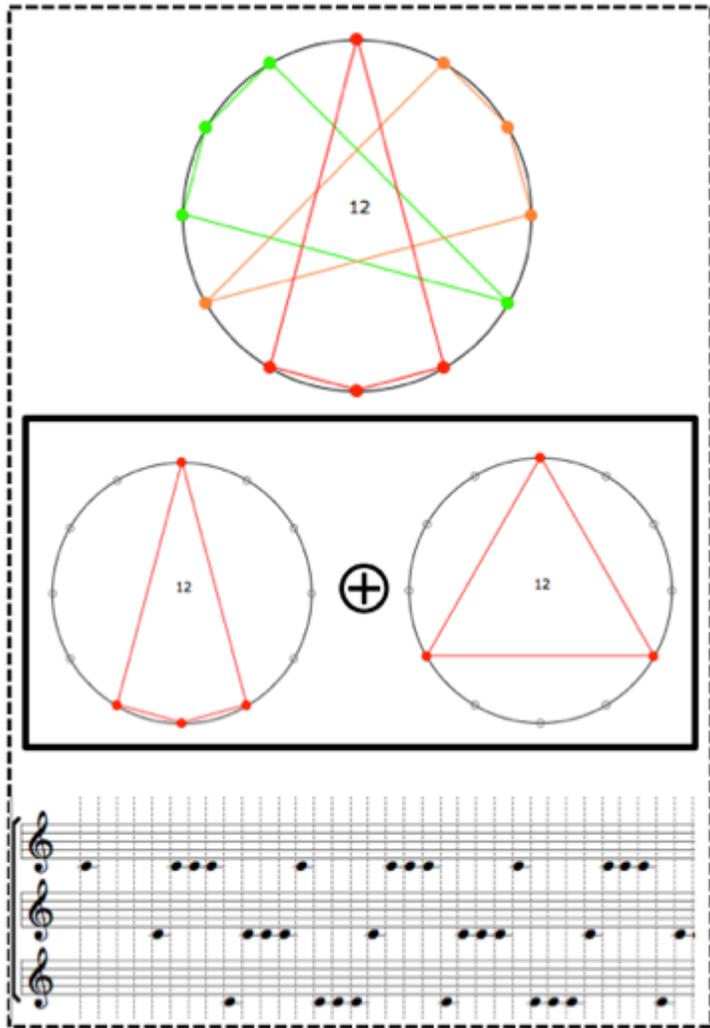
• R. Tijdeman: "Decomposition of the Integers as a direct sum of two subsets", *Number Theory*, Cambridge University Press, 1995. The fundamental Lemma:  
 $A \text{ tiles } \mathbf{Z}_n \Rightarrow pA \text{ tiles } \mathbf{Z}_n \text{ when } \langle p, n \rangle = 1$

• E. Coven & A. Meyerowitz: "Tiling the integers with translates of one finite set", *J. Algebra*, 212, pp.161-174, 1999  
 $T_1 + T_2 \Rightarrow \text{tile}$   
 $\text{Tile} \Rightarrow T_1$

• I. Laba : "The spectral set conjecture and multiplicative properties of roots of polynomials", *J. Lond Math Soc*, 2002  
 $T_1 + T_2 \Rightarrow \text{spectral}$   
 $T_2 \Rightarrow \text{spectral}$   
 $\text{spectral} \Rightarrow T_1$

• E. Amiot : "A propos des canons rythmiques", *Gazette des Mathématiciens*, n°106, Octobre 2005.  
 if  $A$  tiles with period  $n$  and  $\mathbf{Z}_n$  is Hajos  
 $\Rightarrow A$  has  $T_2$  ( $\Rightarrow A$  is spectral)

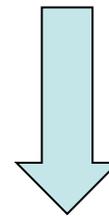
# Representation polynômiale



$$Z_n \longleftrightarrow 1 + X + X^2 + \dots + X^{n-1}$$

$$A = \{0, 5, 6, 7\} \longleftrightarrow A(X) = 1 + X^5 + X^6 + X^7$$

$$B = \{0, 4, 8\} \longleftrightarrow B(X) = 1 + X^4 + X^8$$



$$\Delta_{12} = 1 + X + \dots + X^{11} = A(X) \times B(X) \pmod{X^{12}-1}$$

# Racines de l'unité et polynômes cyclotomiques

---

Racines  $n$ -ièmes de l'unité :  $z^n = 1$

$$n=3 \longrightarrow \left\{ 1, \frac{-1 + i\sqrt{3}}{2}, \frac{-1 - i\sqrt{3}}{2} \right\}$$

$$n=4 \longrightarrow \{1, +i, -1, -i\}$$

Les racines  $n$ -ièmes de l'unité peuvent s'écrire sous la forme :

$$e^{\frac{2k\pi i}{n}} = \cos\left(\frac{2k\pi}{n}\right) + i \sin\left(\frac{2k\pi}{n}\right) \quad (k, n \in \mathbb{N} \text{ et } 0 \leq k < n)$$

Elles sont exactement les racines du polynôme :  $P(X) = X^n - 1$

Les racines  $n$ -ièmes primitives de l'unité :  $e^{\frac{2ki\pi}{n}} \quad (n,k)=1$

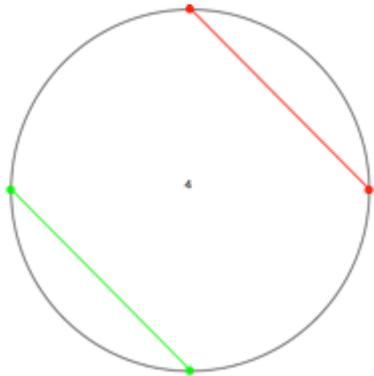
Elles sont exactement les racines du polynôme cyclotomique :

$$\Phi_n(X) = \prod_{k=1}^{\varphi(n)} (X - z_k) \longleftrightarrow X^n - 1 = \prod_{d|n} \Phi_d(X).$$

# Pavage de la ligne et polynômes cyclotomiques

$$\Phi_n(X) = \prod_{k=1}^{\varphi(n)} (X - z_k) \longleftrightarrow X^n - 1 = \prod_{d|n} \Phi_d(X).$$

$\Phi_1(X) = X - 1$	$\longleftrightarrow$	?
$\Phi_2(X) = 1 + X$	$\longleftrightarrow$	$\{0,1\}$
$\Phi_3(X) = 1 + X + X^2$	$\longleftrightarrow$	$\{0,1,2\}$
$\Phi_4(X) = 1 + X^2$	$\longleftrightarrow$	$\{0,2\}$
$\Phi_5(X) = 1 + X + X^2 + X^3 + X^4$	$\longleftrightarrow$	$\{0,1,2,3,4\}$
$\Phi_6(X) = 1 - X + X^2$	$\longleftrightarrow$	?



$$\Delta_n = 1 + X + X^2 + \dots + X^{n-1} = \prod_{\substack{d|n \\ d \neq 1}} \Phi_d(X)$$

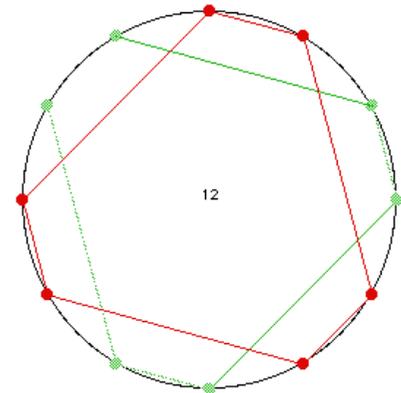
$$\Delta_4 = 1 + X + X^2 + X^3 = \Phi_2(X) \times \Phi_4(X)$$

$$A(x) \times B(x) = (A \oplus B)(x) \equiv 1 + x + \dots + x^{n-1} \pmod{X^n - 1}$$

# Bonnes et mauvaises factorisations

$$\Delta_n = 1 + X + X^2 + \dots + X^{n-1} = \prod_{\substack{d \mid n \\ d \neq 1}} \Phi_d(X)$$

$\Phi_2(X) = 1 + X$	←-----→	$(1, 1)$
$\Phi_3(X) = 1 + X + X^2$	←-----→	$(1, 1, 1)$
$\Phi_4(X) = 1 + X^2$	←-----→	$(1, 0, 1)$
$\Phi_6(X) = 1 - X + X^2$	←-----→	$(1, -1, 1)$



$$\Delta_{12} = 1 + X + \dots + X^{11} = \boxed{\Phi_2 \times \Phi_3} \times \boxed{\Phi_4} \times \boxed{\Phi_6 \times \Phi_{12}}$$

$$A(X) = \Phi_2 \times \Phi_3 \times \Phi_6 \times \Phi_{12} = 1 + X + X^4 + X^5 + X^8 + X^9 \quad \leftarrow$$

$$B(X) = \Phi_4 = 1 + X^2$$

$$S = \{0, 2\}$$

$$R = \{0, 1, 4, 5, 8, 9\}$$

$$\Phi_2(X) = 1 + X$$

Conditions de C& M

$$\Phi_3(X) = 1 + X + X^2$$

$$(T1) A(1) = 6 = \Phi_2(1) \times \Phi_3(1) = 2 \times 3$$

$$(T2) \Phi_2 \mid A(X) \text{ et } \Phi_3 \mid A(X) \Rightarrow \Phi_{2 \times 3} \mid A(X)$$

# Les conditions de Coven-Meyerowitz

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- E. Coven & A. Meyerowitz : “Tiling the integers with translates of one finite set”, *J. Algebra*, 212, pp.161-174, 1999

There is no loss of generality in restricting attention to translates of a finite set  $A$  of *nonnegative* integers. Then  $A(x) = \sum_{a \in A} x^a$  is a polynomial such that  $\#A = A(1)$ . Let  $S_A$  be the set of prime powers  $s$  such that the  $s$ -th cyclotomic polynomial  $\Phi_s(x)$  divides  $A(x)$ . Consider the following conditions on  $A(x)$ .

(T1)  $A(1) = \prod_{s \in S_A} \Phi_s(1)$ .

(T2) If  $s_1, \dots, s_m \in S_A$  are powers of distinct primes, then  $\Phi_{s_1 \dots s_m}(x)$  divides  $A(x)$ .

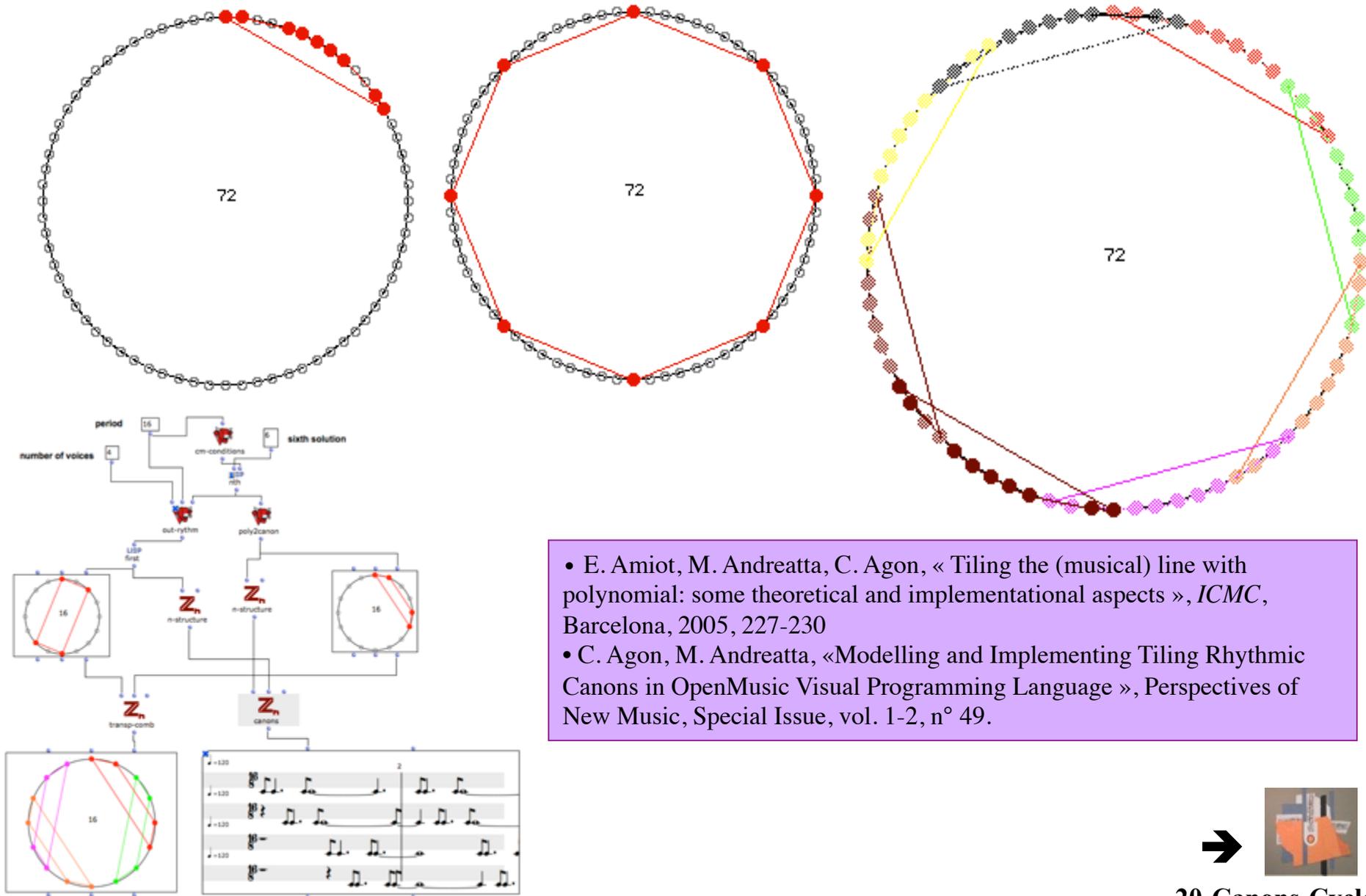
**Theorem A.** *If  $A(x)$  satisfies (T1) and (T2), then  $A$  tiles the integers.*

**Theorem B1.** *If  $A$  tiles the integers, then  $A(x)$  satisfies (T1).*

**Theorem B2.** *If  $A$  tiles the integers and  $\#A$  has at most two prime factors, then  $A(x)$  satisfies (T2).*

**Corollary.** *If  $\#A$  has at most two prime factors, then  $A$  tiles the integers if and only if  $A(x)$  satisfies (T1) and (T2).*

# La famille des « canons cyclotomiques »



- E. Amiot, M. Andreatta, C. Agon, « Tiling the (musical) line with polynomial: some theoretical and implementational aspects », *ICMC*, Barcelona, 2005, 227-230
- C. Agon, M. Andreatta, «Modelling and Implementing Tiling Rhythmic Canons in OpenMusic Visual Programming Language », *Perspectives of New Music*, Special Issue, vol. 1-2, n° 49.



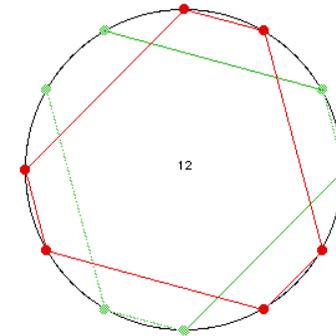
# Les conditions de C&M, pavage et spectralité

(T1)  $A(1) = \prod_{s \in S_A} \Phi_s(1)$ .

(T2) If  $s_1, \dots, s_m \in S_A$  are powers of distinct primes, then  $\Phi_{s_1 \dots s_m}(x)$  divides  $A(x)$ .

**C&M  
(1999)**

- T1 + T2  $\Rightarrow$  pave
- pave  $\Rightarrow$  T1
- pave  $Z_n$  avec  $n = p^\alpha q^\beta \Rightarrow$  T1 + T2



$$A^*(X) = \Phi_2 \times \Phi_3 \times \Phi_{12} = 1 + 2X + 2X^2 - X^3 - X^4 + X^5 + 2X^6 + X^7$$

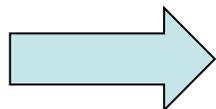
$$A^*(1) = 7 \neq \Phi_2(1) \times \Phi_3(1) = 6$$

**Laba  
(2002)**

- T1 + T2  $\Rightarrow$  spectral
- T2  $\Rightarrow$  spectral
- spectral  $\Rightarrow$  T1

**Amiot  
(2009)**

- pave avec périodicités internes  $\Rightarrow$  T2



**La condition T2 est-elle nécessaire ?**

# Transformée de Fourier discrète et pavage

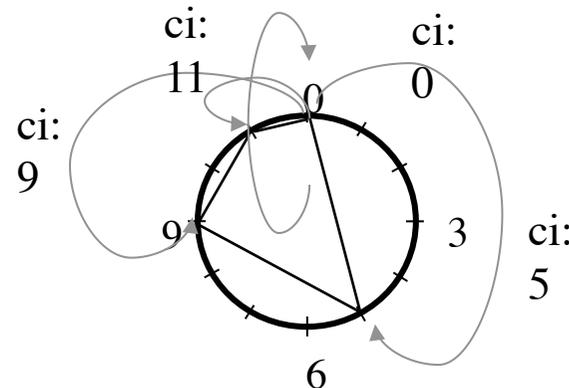
## TILING

Let  $Z_A = \{ t \in \mathbb{Z}_c, F_A(t)=0 \}$

$A$  tiles  $\mathbb{Z}_c$  when equivalently:

- There exists  $B$ ,  $A \oplus B = \mathbb{Z}_c$
- $1_A \star 1_B = 1$
- $F_A \times F_B(t) = 1 + e^{-2i\pi t/c} + \dots + e^{-2i\pi t(c-1)/c}$  (0 unless  $t=0$ )
- $Z_A \cup Z_B = \{1, 2 \dots c-1\}$  AND  $\text{Card } A \times \text{Card } B = c$
- $IC_A \star IC_B = IC(\mathbb{Z}_c) = c$  and  $\text{Card } A \times \text{Card } B = c$

$$\mathcal{F}_A : t \mapsto \sum_{k \in A} e^{-2i\pi kt/c}$$



$$A = \{0, 5, 9, 11\}$$

$$IC_A(k) = 1 \quad \forall k = 1 \dots 11$$

$$IC_A(k) = \text{Card}\{(x, y) \in A \times A \mid x + k = y\}$$

$$IC_A(k) = (1_A \star 1_{-A})(k)$$



**E. Amiot, « New Perspectives on Rhythmic Canons and the Spectral Conjecture », *Journal of Mathematics and Music*, 2009.**

# Homométrie et canons rythmiques mosaïques

## TILING

Let  $Z_A = \{ t \in \mathbb{Z}_c, F_A(t)=0 \}$

*A tiles  $\mathbb{Z}_c$  when equivalently:*

- There exists B,  $A \oplus B = \mathbb{Z}_c$
- $1_A \star 1_B = 1$
- $F_A \times F_B(t) = 1 + e^{-2i\pi t/c} + \dots + e^{-2i\pi t(c-1)/c}$  (0 unless  $t=0$ )
- $Z_A \cup Z_B = \{1, 2 \dots c-1\}$  AND  $\text{Card } A \times \text{Card } B = c$
- $IC_A \star IC_B = IC(\mathbb{Z}_c) = c$  and  $\text{Card } A \times \text{Card } B = c$

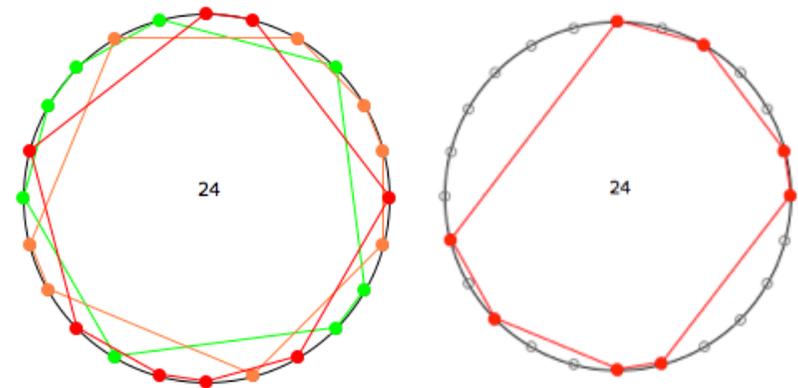
$$\mathcal{F}_A : t \mapsto \sum_{k \in A} e^{-2i\pi kt/c}$$

$$IC_A(k) = (1_A \star 1_{-A})(k)$$

## *A musical offering:*

### • *Theorem:*

If A tiles with B and A' has the same IC, then A' tiles with B, too.



# Pavages et homométrie

## TILING

Let  $Z_A = \{ t \in \mathbb{Z}_c, F_A(t)=0 \}$

$A$  tiles  $\mathbb{Z}_c$  when equivalently:

- There exists  $B$ ,  $A \oplus B = \mathbb{Z}_c$
- $1_A \star 1_B = 1$
- $F_A \times F_B(t) = 1 + e^{-2i\pi t/c} + \dots + e^{-2i\pi t(c-1)/c}$  (0 unless  $t=0$ )
- $Z_A \cup Z_B = \{1, 2 \dots c-1\}$  AND  $\text{Card } A \times \text{Card } B = c$
- $IC_A \star IC_B = IC(\mathbb{Z}_c) = c$  and  $\text{Card } A \times \text{Card } B = c$

$$\mathcal{F}_A : t \mapsto \sum_{k \in A} e^{-2i\pi kt/c}$$

$$IC_A(k) = (1_A \star 1_{-A})(k)$$

## *A musical offering:*

### • *Theorem:*

If  $A$  tiles with  $B$  and  $A'$  has the same IC, then  $A'$  tiles with  $B$ , too.

