Exploring the “mathemusical” dynamics: some theoretical and philosophical aspects of a musically driven mathematical practice

Institut Henri Poincaré
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http://www.ircam.fr/repmus.html
The interplay between algebra and geometry in music

“Concerning music, it takes place in time, like algebra. In mathematics, there is this fundamental duality between, on the one hand, geometry — which corresponds to the visual arts, an immediate intuition — and on the other hand algebra. This is not visual, it has a temporality. This fits in time, it is a computation, something that is very close to the language, and which has its diabolical precision. […] And one only perceives the development of algebra through music” (A. Connes).

http://videotheque.cnrs.fr

### Music & maths: parallel destiny or mutual influences?

#### MUSIC

<table>
<thead>
<tr>
<th>Year</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>500 B.C.</td>
<td>Pitches and lengths of strings are related. Here music gives a marvelous thrust to number theory and geometry. No correspondence in music.</td>
</tr>
<tr>
<td>300 B.C.</td>
<td>Music theory highlights the discovery of the isomorphism between the logarithms (musical intervals) and exponentials (string lengths) more than 15 centuries before their discovery in mathematics; also a premonition of group theory is suggested by Aristoxenos.</td>
</tr>
<tr>
<td>1000 A.D.</td>
<td>Invention of the two-dimensional spatial representation of pitches linked with time by means of staves and points; seven centuries (1635-37) before the magnificent analytical geometry of Fermat and Descartes. No parallel in mathematics.</td>
</tr>
<tr>
<td>1500</td>
<td>No response or development of the preceding concepts.</td>
</tr>
<tr>
<td>1600</td>
<td>No equivalence, no reaction.</td>
</tr>
<tr>
<td>1648</td>
<td>Invention of musical combinatorics by Marin Mersenne (Harmonicorum Libri). Probability theory by Bernoulli (Ars Conjectandi, 1713).</td>
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<tr>
<td>1700</td>
<td>The fugue, for example, is an abstract automaton used two centuries before the birth of the science of automata. Also, there is an unconscious manipulation of finite groups (Klein group) in the four variations of a melodic line used in counterpoint. Number theory is ahead of but has no equivalent yet in temporal structures.</td>
</tr>
<tr>
<td>1773</td>
<td>A first geometric and graph-theoretic representation of pitches (Speculum Musicum). Invention of graph theory.</td>
</tr>
<tr>
<td>1900</td>
<td>Liberation from the tonal yoke. First acceptance of the neutrality of chromatic totality (Loquin [1895], Hauer, Schoenberg). The infinite and transfinite numbers (Cantor). Peano axiomatics. The beautiful measure theory (Lebesque, ...)</td>
</tr>
<tr>
<td>1920</td>
<td>First radical formalization of macrostructures through the serial system of Schoenberg. No new development of the number theory.</td>
</tr>
<tr>
<td>1929 and 1937-1939</td>
<td>Susanne K. Langer and Ernst Krenek on the role of axioms in music. David Hilbert, Die Grundlage der Geometrie (1899)</td>
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#### MATHS

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<tbody>
<tr>
<td>500 B.C.</td>
<td>Discovery of the fundamental importance of natural numbers and the invention of fractions. Positive irrational numbers [...]</td>
</tr>
<tr>
<td>300 B.C.</td>
<td>No reaction in mathematics. [...]</td>
</tr>
<tr>
<td>1000 A.D.</td>
<td>No parallel in mathematics.</td>
</tr>
<tr>
<td>1500</td>
<td>Zero and negative numbers are adopted. Construction of the set of rationals.</td>
</tr>
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<td>1600</td>
<td>The sets of real numbers and of logarithms are invented.</td>
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Maths/music: a (very) recent perspective

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<tr>
<th>Year</th>
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<tr>
<td>2000-2003</td>
<td>International Seminar on <em>MaMuTh (Perspectives in Mathematical and Computational Music Theory)</em> (Mazzola, Noll, Luis-Puebla eds, epOs, 2004)</td>
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<tr>
<td>2003</td>
<td><em>The Topos of Music</em> (G. Mazzola et al.)</td>
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<tr>
<td>2001-…</td>
<td><em>MaMuX Seminar at Ircam</em></td>
</tr>
<tr>
<td>2004-…</td>
<td><em>mamuphi Seminar (Ens/Ircam)</em></td>
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<td>2006</td>
<td>Collection ‘Musique/Sciences’ (Ircam/Delatour France)</td>
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<tr>
<td>2007</td>
<td><em>Journal of Mathematics and Music</em> (Taylor &amp; Francis) and SMCM</td>
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<td>2007</td>
<td>First MCM 2007 (Berlin) and Proceedings by Springer</td>
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<tr>
<td>2007-…</td>
<td>AMS Special Session on Mathematical Techniques in Musical Analysis</td>
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<tr>
<td>2009</td>
<td>MCM 2009 (Yale University) and Proceedings by Springer</td>
</tr>
<tr>
<td>2010</td>
<td>Mathematics Subject Classification : 00A65 Mathematics and music</td>
</tr>
<tr>
<td>2011</td>
<td>MCM 2011 (Ircam, 15-17 June 2011) and Proceedings LNCS Springer</td>
</tr>
<tr>
<td>2013</td>
<td>MCM 2013 (McGill University, Canada, 12-14 June 2013) - Springer</td>
</tr>
<tr>
<td>2015</td>
<td>MCM 2015 (Queen Mary University, Londres, 22-25 June 2013) - Springer</td>
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</tbody>
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The double movement of a ‘mathemusical’ activity

MATHEMATICS

Mathematical statement → generalisation

MUSIC

Musical problem → formalisation

Music analysis → application

Music theory → Composition

OpenMusic
Some examples of ‘mathemusical’ problems


- The construction of Tiling Rhythmic Canons
- The Z relation and the theory of homometric sets
- Set Theory and Transformational Theory
- Neo-Riemannian Theory, Spatial Computing and FCA
- Diatonic Theory and Maximally-Even Sets
- Periodic sequences and finite difference calculus
- Block-designs and algorithmic composition
Aperiodic Rhythmic Tiling Canons (Vuza Canons)
Tiling Rhythmic Canons as a ‘mathemusical’ problem

**Minkowski/Hajós Problem (1907-1941)**

In any simple lattice tiling of the $n$-dimensional Euclidean space by unit cubes, some pair of cubes must share a complete $(n-1)$-dimensional face.

**Vieru’s problem and Vuza’s formalization (PNM, 1991)**

A Vuza Canon is a factorization of a cyclic group in a direct sum of two non-periodic subsets.

$$\mathbb{Z}/n\mathbb{Z} = R \oplus S$$

**Link between Minkowski problem and Vuza Canons (Andreatta, 1996)**

Hajós groups (good groups)

$\mathbb{Z}/n\mathbb{Z}$ with $n \in \{p^a, p^aq, pqr, p^2q^2, p^2qr, pqrs\}$ where $p, q, r, s$ are distinct prime numbers.

Non-Hajós group (bad groups)

72
108 120 144 168 180
200 216 240 252 270 280 288
300 312 324 336 360 378 392 396
400 408 432 440 450 456 468 480
500 504 520 528 540 552 560 576 588 594
600 612 616 624 648 672 675 680 684 696
700 702 720 728 744 750 756 760 784 792
800 810 816 828 864 880 882 888...

(Sloane’s sequence A102562)

**Fuglede Spectral Conjecture**

A subset of the $n$-dimensional Euclidean space tiles by translation iff it is spectral.

*(J. Func. Anal. 16, 1974)*

Fuglede Spectral Conjecture

DEFINITION 6. A subset $A$ of some vector space (say $\mathbb{R}^n$) is spectral iff it admits a Hilbert base of exponentials, i.e. if any map $f \in L^2(A)$ can be written

$$f(x) = \sum f_k \exp(2i\pi \lambda_k x)$$

for some fixed family of vectors $(\lambda_k)_{k \in \mathbb{Z}}$ where the maps $e_k : x \mapsto \exp(2i\pi \lambda_k x)$ are mutually orthogonal (i.e. $\int_A \overline{e_k} e_j = 0$ whenever $k \neq j$).

$\Rightarrow$ False in dim. $n \geq 3$
(Tao, Kolountzakis, Matolcsi, Farkas and Mora)

$\Rightarrow$ Open in dim. 1 et 2

DEFINITION 8. A subset $A \subset \mathbb{Z}$ is spectral iff there exists a spectrum $\Lambda \subset [0,1]$ (i.e., a subset with the same cardinality as $A$) such that $e^{2i\pi (\lambda_i - \lambda_j)}$ is a root of $A(X)$ for all distinct $\lambda_i, \lambda_j \in \Lambda$.

Example:
$A = \{0, 1, 6, 7\} \Rightarrow \Lambda = \{0, 1/12, 1/2, 7/12\}$
since $\exp(\pi i), \exp(\pi i/6), \exp(-\pi i/6), \exp(5\pi i/6), \exp(-5\pi i/6)$ are the roots of the associated polynomial $A(X) = 1 + X + X^6 + X^7$

$\mathbb{Z}_n = \mathbb{B} \oplus A$
Fuglede Spectral Conjecture and Vuza Canons

A subset of the $n$-dimensional Euclidean space tiles by translation iff it is spectral.

(\textit{J. Func. Anal.} 16, 1974)

\[ \Rightarrow \text{False in dim. } n \geq 3 \]

(Tao, Kolountzakis, Matolcsi, Farkas and Mora)

\[ \Rightarrow \text{Open in dim. } 1 \text{ et } 2 \]

\textbf{Definition 6} A subset $A$ of some vector space (say $\mathbb{R}^n$) is spectral iff it admits a Hilbert base of exponentials, i.e. if any map $f \in L^2(A)$ can be written

\[ f(x) = \sum f_k \exp(2i\pi \lambda_k \cdot x) \]

for some fixed family of vectors $(\lambda_k)_{k \in \mathbb{Z}}$ where the maps $e_k : x \mapsto \exp(2i\pi \lambda_k \cdot x)$ are mutually orthogonal (i.e. $\int_A \overline{e_k} e_j = 0$ whenever $k \neq j$).

\[ \Downarrow n=1 \]

\textbf{Definition 8}. A subset $A \subseteq \mathbb{Z}$ is spectral if there exists a spectrum $\Lambda \subseteq [0,1]$ (i.e., a subset with the same cardinality as $A$) such that $e^{2i\pi \sum_{\Lambda} \lambda_i \cdot x}$ is a root of $A(X)$ for all distinct $\lambda_i, \lambda_j \in \Lambda$.

\textbf{Theorem (Amiot, 2009)}

- All non-Vuza canons are spectral.
- Fuglede Conjecture is true (or false) iff it is true (or false) for Vuza Canons

Group-based paradigmatic classification of Vuza Canons

\[
\begin{align*}
R & \quad + \quad S = \quad \mathbb{Z}_n \\
(1 & \quad 3 & \quad 6 & \quad 11 & \quad 4 & \quad 9 & \quad 6 & \quad 5 & \quad 1 & \quad 3 & \quad 20) \\
(20 & \quad 3 & \quad 1 & \quad 5 & \quad 6 & \quad 9 & \quad 4 & \quad 11 & \quad 6 & \quad 3 & \quad 1) \\
(1 & \quad 4 & \quad 1 & \quad 9 & \quad 4 & \quad 1 & \quad 6 & \quad 6 & \quad 7 & \quad 4 & \quad 13 & \quad 6) \\
(6 & \quad 13 & \quad 4 & \quad 7 & \quad 6 & \quad 1 & \quad 4 & \quad 19 & \quad 1 & \quad 4 & \quad 1) \\
(1 & \quad 5 & \quad 15 & \quad 4 & \quad 5 & \quad 6 & \quad 6 & \quad 3 & \quad 4 & \quad 17 & \quad 3 & \quad 3) \\
(3 & \quad 3 & \quad 17 & \quad 4 & \quad 3 & \quad 6 & \quad 6 & \quad 5 & \quad 4 & \quad 15 & \quad 5 & \quad 1) \\
S & \quad + \quad R = \quad \mathbb{Z}_n \\
(8 & \quad 8 & \quad 2 & \quad 8 & \quad 8 & \quad 38) \\
(16 & \quad 2 & \quad 14 & \quad 2 & \quad 16 & \quad 22) \\
(14 & \quad 8 & \quad 10 & \quad 8 & \quad 14 & \quad 18) \\
Tijdeman’s ‘Fundamental Lemma’ (1995)
R tiles \mathbb{Z}_n \Rightarrow aR tiles \mathbb{Z}_n <a,n> = 1
\end{align*}
\]
Equivalence classes of musical structures up to a group action (see Definitions by Abstraction)

\[ Z_{12} = \langle T_k \mid (T_k)^{12} = T_0 \rangle \]

\[ D_{12} = \langle T_k, I \mid (T_k)^{12} = I^2 = T_0, ITI=I(IT)^{-1} \rangle \]

\[ \text{Aff} = \{ f \mid f(x) = ax+b, a \in (Z_{12})^*, b \in Z_{12} \} \]
The nature of a given geometry is [...] defined by the reference to a determinate group and the way in which spatial forms are related within that type of geometry. [Cf. Felix Klein Erlangen Program - 1872][...] We may raise the question whether there are any concepts and principles that are, although in different ways and different degrees of distinctness, necessary conditions for both the constitution of the perceptual world and the construction of the universe of geometrical thought. It seems to me that the concept of group and the concept of invariance are such principles.

E. Cassirer, “The concept of group and the theory of perception”, 1944
« [C’est la notion de groupe qui] donne un sens précis à l’idée de structure d’un ensemble [et] permet de déterminer les éléments efficaces des transformations en réduisant en quelque sorte à son schéma opératoire le domaine envisagé. […] L’objet véritable de la science est le système des relations et non pas les termes supposés qu’il relie. […] Intégrer les résultats - symbolisés - d’une expérience nouvelle revient […] à créer un canevas nouveau, un groupe de transformations plus complexe et plus compréhensif »


« La théorie des catégories est une théorie des constructions mathématiques, qui est macroscopique, et procède d’étage en étage. Elle est un bel exemple d’abstraction réfléchissante, cette dernière reprenant elle-même un principe constructeur présent dès le stade sensori-moteur. Le style catégoriel qui est ainsi à l’image d’un aspect important de la genèse des facultés cognitives, est un style adéquat à la description de cette genèse »

Jean Piaget, Gil Henriques et Edgar Ascher, Morphismes et Catégories.Comparer et transformer, 1990
CT and transformational music analysis

Musical Z-relation and Homometry Theory

M. Babbitt

≈

Hexachord Theorem


Music and the Rise of the Structural Approach in Maths

The concept of group

The concept of lattice
The partition lattice of musical structures

\[ \text{DIA} = (2,2,1,2,2,2,1) \]

\[ \text{DIA}_E = (1,1,2,2,2,2,2,2) \]

The permutohedron as a lattice of formal concepts

Permutohedron and a topological structural inclusion

(3 5 4) (5 3 4) (5 4 3) (4 5 3) (4 3 5) (3 4 5)
The *Tonnetz* as a topological structure

**Axis of minor thirds**

**Speculum Musicum** (Euler, 1773)
Assembling chords related by some equivalence relation
- Transposition/inversion: Dihedral group action on $P(\mathbb{Z}_n)$

The Tonnetz as a simplicial complex
L. Bigo, Représentation symboliques musicales et calcul spatial, PhD, Ircam / LACL, 2013

Intervallic structure

major/minor triads

Intervallic structure major/minor triads
Formal Concept Analysis and topology in music theory

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
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<tbody>
<tr>
<td>X1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>X2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>X3</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>X4</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
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</table>
Concept lattice vs simplicial complex

Lattice

Lattice complex

Simplicial complex

Lattice-based and Topological Representations of Binary Relations with an Application to Music

Anton Freund · Moreno Andreatta · Jean-Louis Giavitto
Conclusions:

- The concept lattice alone cannot be fully reconstructed from the simplicial complex.
- The simplicial complex cannot be fully determined from the concept lattice alone.
- The concept lattice alone allows to determine the homotopy type of the simplicial complex.
The simplicial complex of a Chopin’s *Prelude*

**Prelude**

‘Suffocation’

**Frederic Chopin** (1810-1849)

Op. 28, No. 4

Hexachord

(by Louis Bigo, 2013)
Towards a topological signature of a musical piece

A structural approach in Music Information Retrieval

Score reduction

The simplices and their self-assembly

The score

A specific trajectory in the complex

Topological signature?

The simplicial complex generated by the piece
Neurosciences and Tonnetz

The spatial character of the « musical style »

Abstract
This article presents a first attempt at establishing a category-theoretical model of creative processes. The model, which is applied to musical creativity, discourse theory, and cognition, suggests the relevance of the notion of “colimit” as a unifying construction in the three domains as well as the central role played by the Yoneda Lemma in the categorical formalization of creative processes.

Towards a categorical theory of creativity (in music, cognition and discourse)

A synthetic vision allows us to link together apparently distant strata of mathematics and culture, helping us to break down many artificial barriers. Not only can today’s mathematics be appreciated through epistemic, ontic, phenomenological and aesthetic modes, but in turn, it should help to transform philosophy.
Thank you for your attention!