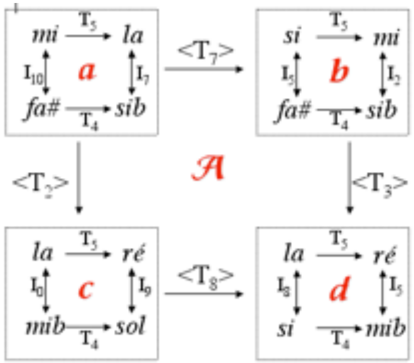
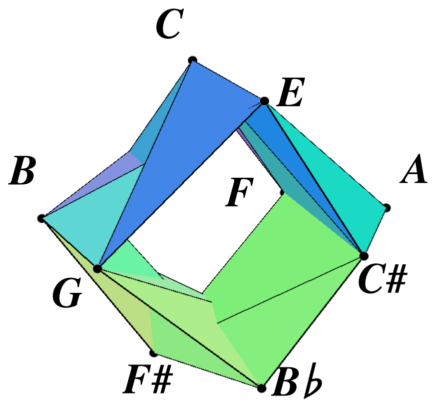
Exploring the “mathemusical” dynamics: some theoretical and philosophical aspects of a musically driven mathematical practice





Institut Henri Poincaré

November 2nd - 4th 2015

Moreno Andreatta

Music Representations Team

IRCAM/CNRS/UPMC

http://www.ircam.fr/repmus.html

The interplay between algebra and geometry in music

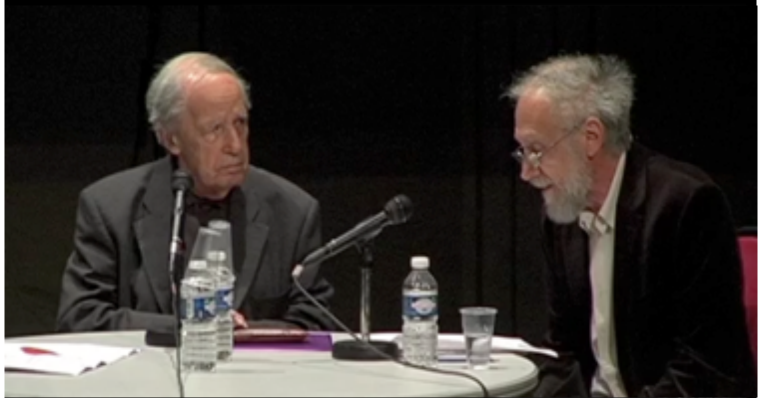
MATH / MUSiC MEETiNGS

Creativity in Music and Mathematics

Pierre Boulez & Alain Connes

Encounter with two major figures of musical creation and contemporary mathematical research: Pierre Boulez and Alain Connes.

What is the role of intuition in mathematical reasoning and in artistic activities? Is there an aesthetic dimension to mathematical activity? Does the notion of elegance of a mathematical demonstration or of a theoretical construction in music play a role in creativity?



Gérard Assayag, director of the CNRS/IRCAM Laboratory for The Science and Technology of Music and Sound, will lead this dialogue on invention in the two disciplines.

Photo: Pierre Boulez © Jean Radel

Wednesday, June 15, 2011, 6:30pm / IRCAM, Espace de projection

"Concerning **music**, it takes place in **time**, like **algebra**. In **mathematics**, there is this fundamental duality between, on the one hand, **geometry** – which corresponds to the visual arts, an immediate intuition – and on the other hand **algebra**. This is not visual, it has a temporality. This fits in time, it is a computation, something that is very close to the language, and which has its diabolical precision. [...] **And one only perceives the development of algebra through music**" (A. Connes).

➔ `http://videotheque.cnrs.fr/`

➔ `http://agora2011.ircam.fr`

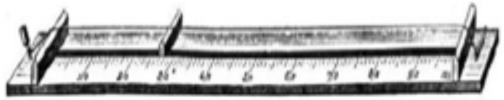
ircam Centre Pompidou AGORA FESTiVAL 8-18 JUiN 2011

Société Mathématique de France SMF

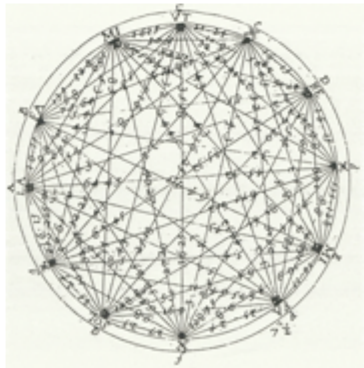
Music & maths: parallel destiny or mutual influences?

I. Xenakis

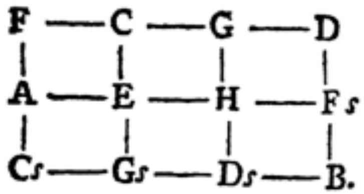
MUSIC	MATHS
500 B.C. Pitches and lengths of strings are related. Here **music** gives a marvelous thrust to **number theory** and **geometry**.	Discovery of the fundamental importance of **natural numbers** and the invention of **fractions**.
No correspondence in music.	Positive irrational numbers [...]
300 B.C. [...] Music theory highlights the discovery of the **isomorphism between the logarithms** (musical intervals) and **exponentials** (string lengths) more than 15 centuries before their discovery in mathematics; also a **premonition of group theory** is suggested by Aristoxenos.	No reaction in mathematics. [...]
1000 A.D. Invention of the **two-dimensional spatial representation of pitches** linked with time by means of staves and points [...] seven centuries (1635-37) before the magnificent **analytical geometry** of Fermat and Descartes.	*No parallel in mathematics.*
1500 *No response or development of the preceding concepts.*	Zero and negative numbers are adopted. Construction of the set of rationals.
1600 *No equivalence, no reaction.*	The sets of real numbers and of logarithms are invented.
1648 Invention of **musical combinatorics** by Marin Mersenne (*Harmonicorum Libri*)	Probability theory by Bernoulli (*Ars Conjectandi*, 1713)
1700 [...] The **fugue**, for example, is an **abstract automaton** used two centuries before the birth of the science of automata. Also, there is an **unconscious manipulation of finite groups** (Klein group) in the four variations of a melodic line used in counterpoint.	Number theory is ahead of but has no equivalent yet in temporal structures. [...]
1773 A first **geometric** and **graph-theoretic** representation of pitches (*Speculum Musicum*)	Invention of graph theory
1900 Liberation from the tonal yoke. First acceptance of the neutrality of chromatic totality (Loquin [1895], Hauer, Schoenberg).	The infinite and transfinite numbers (Cantor). Peano axiomatics. [...] The beautiful measure theory (Lebesque, ...)
1920 First radical formalization of macrostructures through the serial system of Schoenberg.	No new development of the number theory.
1929 and **1937-1939** Susanne K. Langer and Ernst Krenek on the role of axioms in music	David Hilbert, *Die Grundlage der Geometrie* (1899)
1946 Milton Babbitt on group theory and integral serialism	Rudolf Carnap, *The Logical Syntax of Language* (1937)



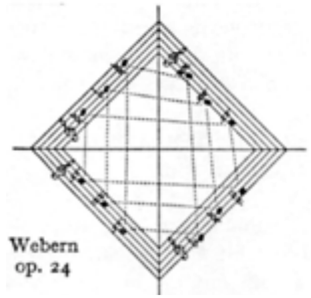
Pythagoras and the monochord, VIth-Vth Century B.C.

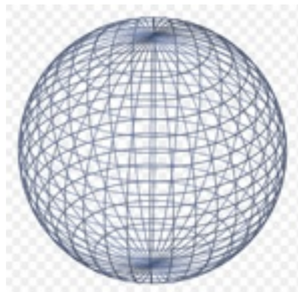


Mersenne and the 'musical clock', 1648



Euler and the *Speculum musicum*, 1773





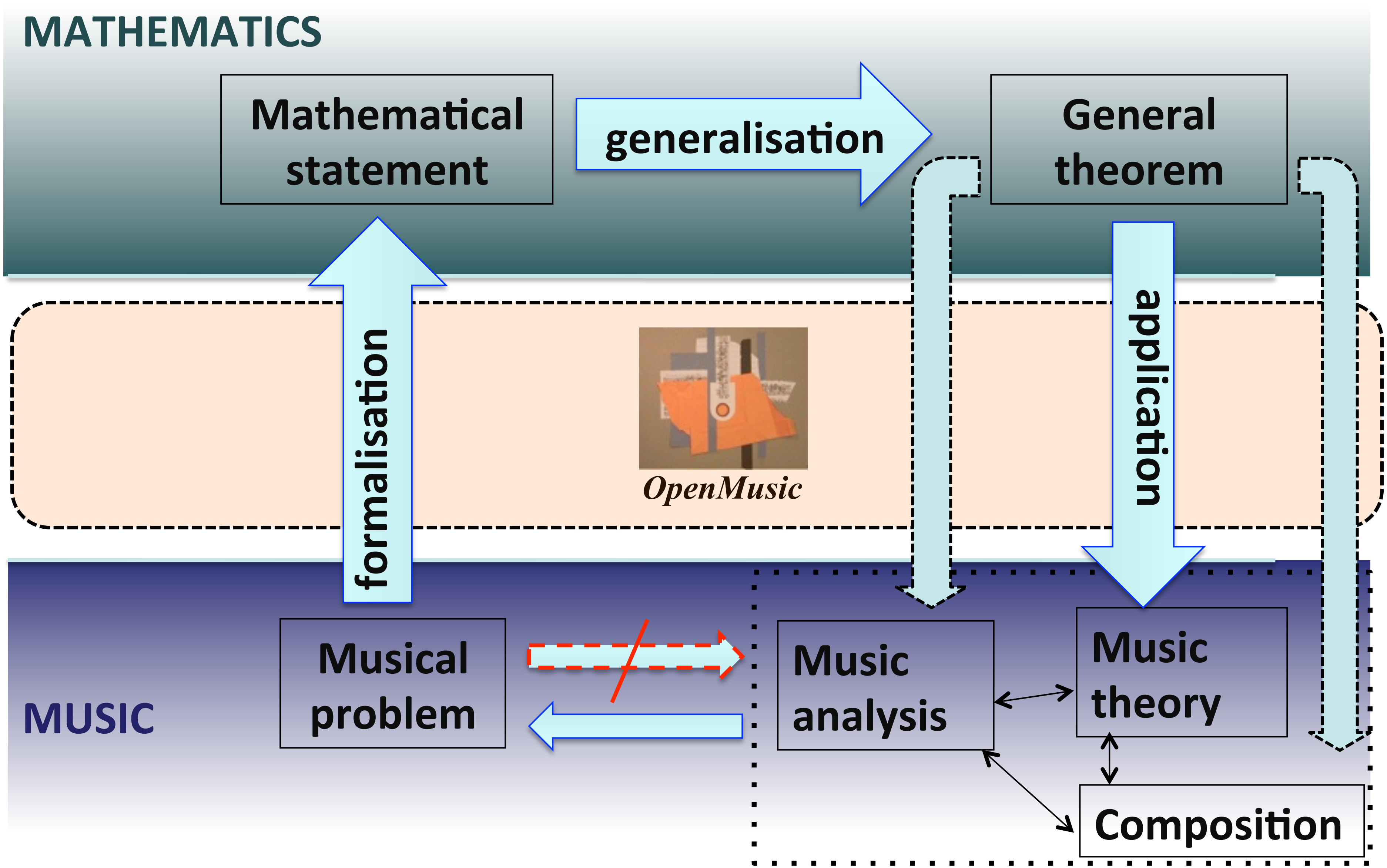
Maths/music: a (very) recent perspective

- **1999: 4e Forum Diderot (Paris, Vienne, Lisbonne), *Mathematics and Music* (G. Assayag, H.G. Feichtinger, J.F. Rodrigues, Springer, 2001)**
- **2000-2001: *MaMuPhi Seminar, Penser la musique avec les mathématiques ?* (Assayag, Mazzola, Nicolas éds., Coll. 'Musique/Sciences', Ircam/Delatour, 2006)**
- **2000-2003: International Seminar on *MaMuTh* (*Perspectives in Mathematical and Computational Music Theory* (Mazzola, Noll, Luis-Puebla eds, epOs, 2004)**
- **2003: *The Topos of Music* (G. Mazzola et al.)**
- **2001-…: *MaMuX* Seminar at Ircam**
- **2004-…: *mamuphi* Seminar (Ens/Ircam)**
- **2006: Collection 'Musique/Sciences' (Ircam/Delatour France)**
- **2007: Journal of Mathematics and Music (Taylor & Francis) and SMCM**
- **2007: First MCM 2007 (Berlin) and Proceedings by Springer**
- **2007-…: AMS Special Session on Mathematical Techniques in Musical Analysis**
- **2009: *Computational Music Science* (eds: G. Mazzola, M. Andreatta, Springer)**
- **2009: MCM 2009 (Yale University) and Proceedings by Springer**
- **2010: Mathematics Subject Classification : 00A65 Mathematics and music** ←
- **2011: MCM 2011 (Ircam, 15-17 June 2011) and Proceedings LNCS Springer**
- **2013: MCM 2013 (McGill University, Canada, 12-14 June 2013) - Springer**
- **2015: MCM 2015 (Queen Mary University, Londres, 22-25 June 2013) - Springer**

American Mathematical Society, Founded 1888

Société Mathématique de France SMF

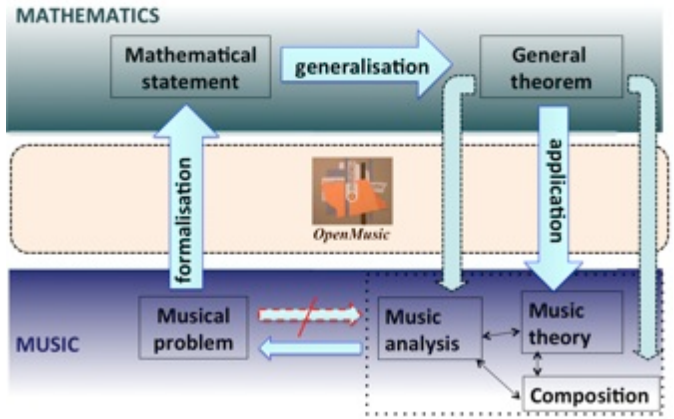
The double movement of a 'mathemusical' activity

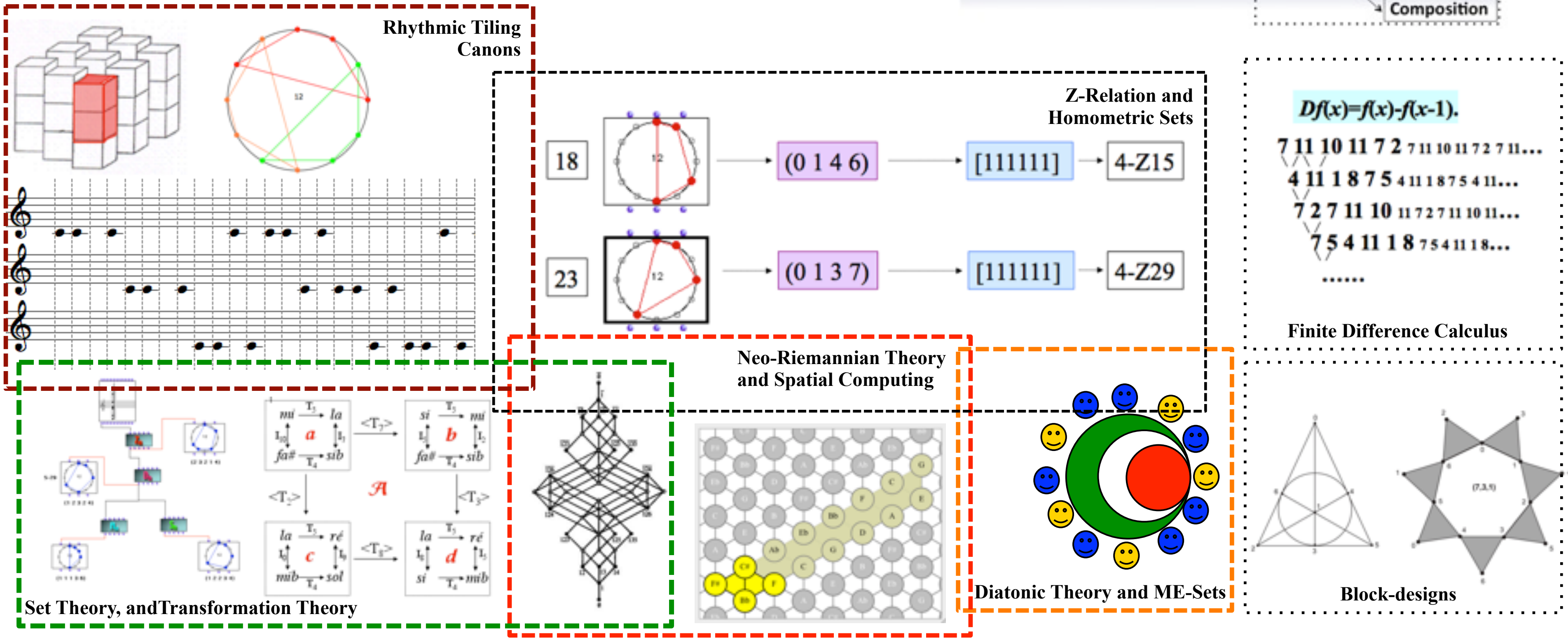


Some examples of 'mathemusical' problems

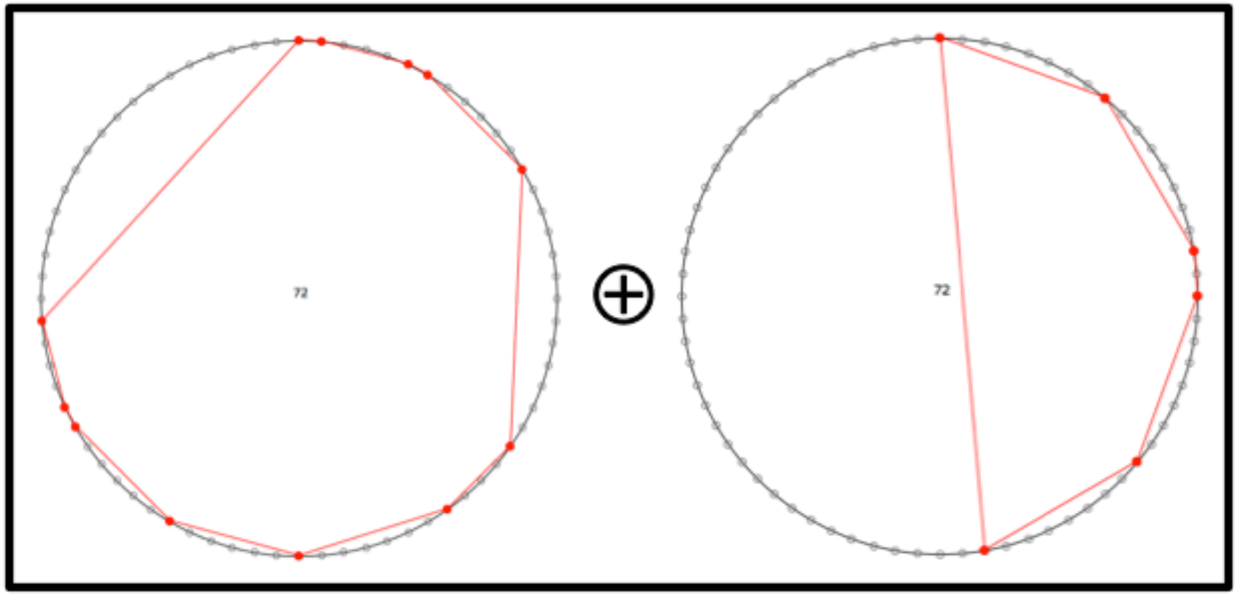
M. Andreatta : *Mathematica est exercitium musicae*, Habilitation Thesis, IRMA University of Strasbourg, 2010

- The construction of Tiling Rhythmic Canons
- The *Z* relation and the theory of homometric sets
- *Set Theory* and Transformational Theory
- Neo-Riemannian Theory, Spatial Computing and FCA
- Diatonic Theory and Maximally-Even Sets
- Periodic sequences and finite difference calculus
- Block-designs and algorithmic composition



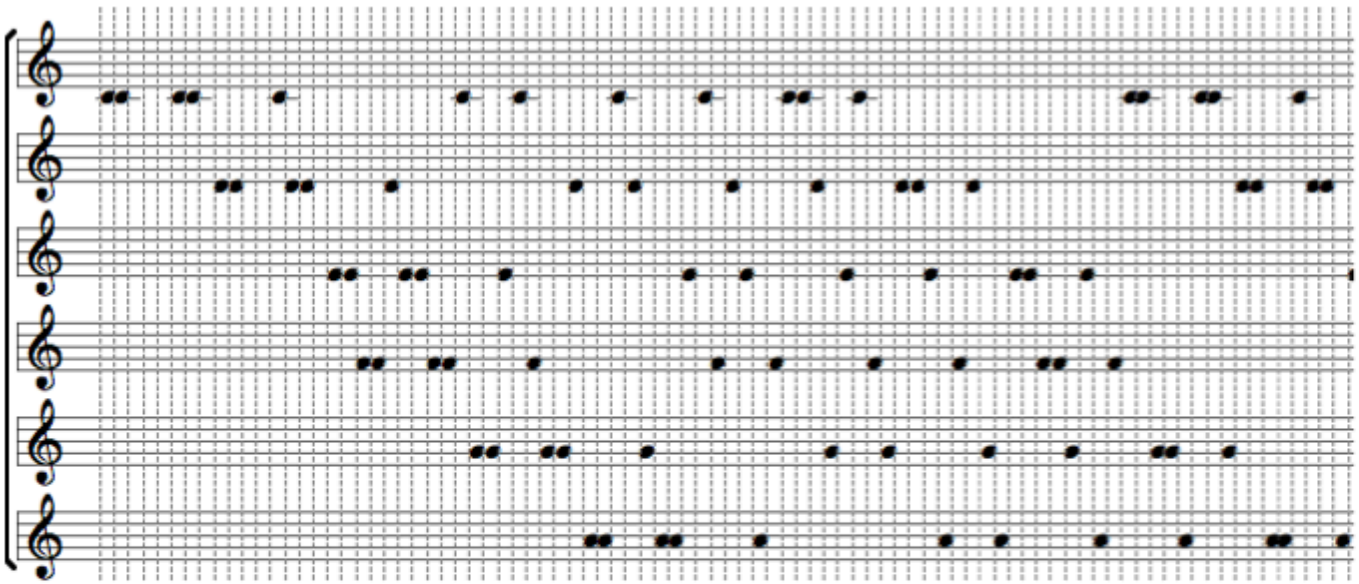


Aperiodic Rhythmic Tiling Canons (Vuza Canons)



Dan Vuza

Anatol Vieru

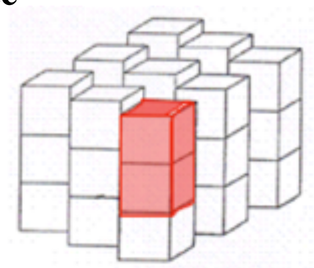


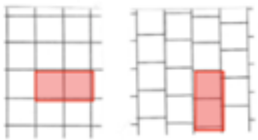
[ ]

Tiling Rhythmic Canons as a 'mathemusical' problem

Minkowski/Hajós Problem (1907-1941)

In any simple lattice tiling of the *n*-dimensional Euclidean space by unit cubes, some pair of cubes must share a complete (*n*-1)-dimensional face



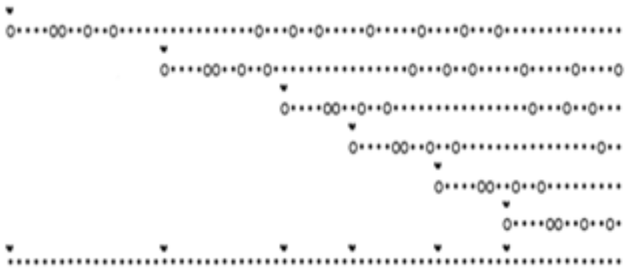


S. Stein, S. Szabó:
Algebra and Tiling,
Carus Math. Mon. 1994

Vieru's problem and Vuza's formalization (PNM, 1991)

A Vuza Canon is a factorization of a cyclic group in a direct sum of two non-periodic subsets

$$\mathbf{Z}/n\mathbf{Z} = R \oplus S$$



Link between Minkowski problem and Vuza Canons (Andreatta, 1996)

Hajós groups (*good groups*)

Z/*n*Z with *n* $\in \{p^{\alpha}, p^{\alpha}q, pqr, p^2q^2, p^2qr, pqrs\}$ **where *p*, *q*, *r*, *s*, are distinct prime numbers**

Non-Hajós group (*bad groups*)

72
108 120 144 168 180
200 216 240 252 264 270 280 288
300 312 324 336 360 378 392 396
400 408 432 440 450 456 468 480
500 504 520 528 540 552 560 576 588 594
600 612 616 624 648 672 675 680 684 696
700 702 720 728 744 750 756 760 784 792
800 810 816 828 864 880 882 888…

(Sloane's sequence A102562)

Fuglede Spectral Conjecture — **A subset of the *n*-dimensional Euclidean space tiles by translation *iff* it is *spectral*.**
(*J. Func. Anal.* 16, 1974)

Fuglede Spectral Conjecture

A subset of the n-dimensional Euclidean space tiles by translation *iff* it is *spectral*.
(*J. Func. Anal.* 16, 1974)

➔False in dim. $n \geq 3$
(Tao, Kolountzakis, Matolcsi, Farkas and Mora)

➔ Open in dim. 1 et 2

DEFINITION 6 *A subset A of some vector space (say $\mathbb{R}^n$) is spectral iff it admits a Hilbert base of exponentials, i.e. if any map $f \in L^2(A)$ can be written*

$$f(x) = \sum f_k \exp(2i\pi\lambda_k.x)$$

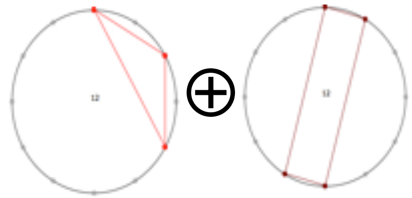
for some fixed family of vectors $(\lambda_k)_{k\in\mathbb{Z}}$ where the maps $e_k : x \mapsto \exp(2i\pi\lambda_k.x)$ are mutually orthogonal (i.e. $\int_A \overline{e_k} e_j = 0$ whenever $k \neq j$).

↓ ***n*=1**

DEFINITION 8. *A subset $A \in \mathbb{Z}$ is* spectral *if there exists a spectrum $\Lambda \subset [0,1]$ (i.e., a subset with the same cardinality as A) such that $e^{2i\pi(\lambda_i - \lambda_j)}$ is a root of $A(X)$ for all distinct $\lambda_i, \lambda_j \in \Lambda$.*

Example:
$A=\{0,1,6,7\}$ ➔ $\Lambda=\{0,1/12,1/2,7/12\}$
since exp(πi), exp(πi/6), exp(-πi/6), exp(5πi/6), exp(-5πi/6) are the roots of the associated polynomial
$A(X)=1+X+X^6+X^7$

$\mathbf{Z}_n = B \oplus A$



Fuglede Spectral Conjecture and Vuza Canons

A subset of the *n*-dimensional Euclidean space tiles by translation *iff* it is *spectral*.
(*J. Func. Anal.* 16, 1974)

➔**False in dim. *n*≥3**
(Tao, Kolountzakis, Matolcsi, Farkas and Mora)

➔ **Open in dim. 1 et 2**

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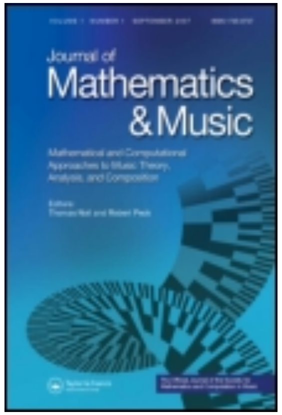
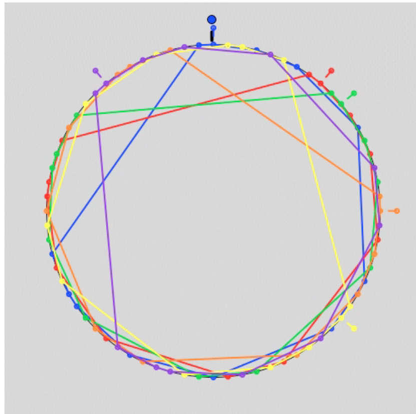
Theorem (Amiot, 2009)

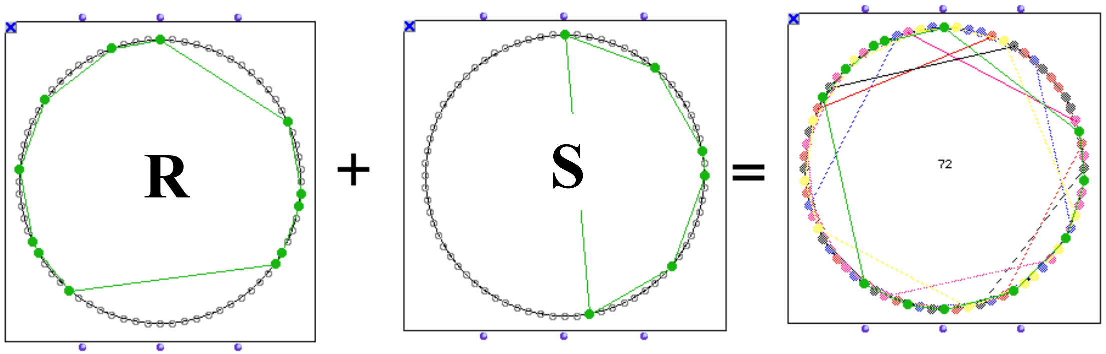
- *All non-Vuza canons are spectral.*
- *Fuglede Conjecture is true (or false) iff it is true (or false) for Vuza Canons*

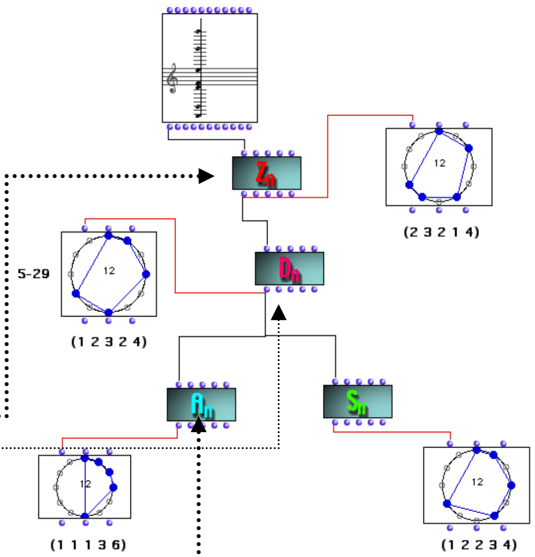




M. Andreatta & C. Agon (eds), « Tiling Problems in Music », Special Issue of the *Journal of Mathematics and Music*, Vol. 3, Number 2, July 2009 (with contributions by E. Amiot, F. Jedrzejewski, M. Kolountzakis and M. Matolcsi)

Group-based paradigmatic classification of Vuza Canons





Z_n

R

(1 3 3 6 11 4 9 6 5 1 3 20)
(20 3 1 5 6 9 4 11 6 3 3 1)
(1 4 1 19 4 1 6 6 7 4 13 6)
(6 13 4 7 6 6 1 4 19 1 4 1)
(1 5 15 4 5 6 6 3 4 17 3 3)
(3 3 17 4 3 6 6 5 4 15 5 1)

S

(8 8 2 8 8 38)
(16 2 14 2 16 22)
(14 8 10 8 14 18)

D_n

R

(1 3 3 6 11 4 9 6 5 1 3 20)
(1 4 1 19 4 1 6 6 7 4 13 6)
(1 5 15 4 5 6 6 3 4 17 3 3)

S

(8 8 2 8 8 38)
(16 2 14 2 16 22)
(14 8 10 8 14 18)

A_n

R

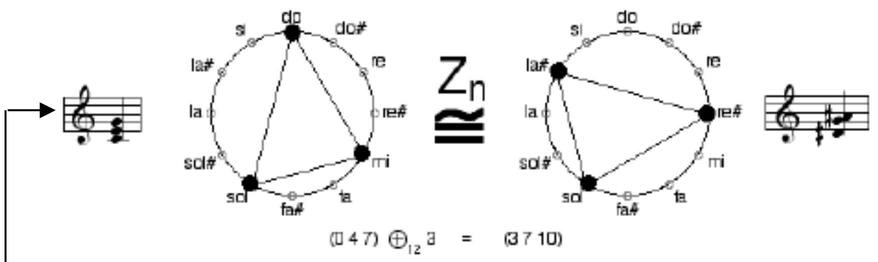
(1 3 3 6 11 4 9 6 5 1 3 20)
(1 4 1 19 4 1 6 6 7 4 13 6)

S

(14 8 10 8 14 18)

Tijdeman's
'Fundamental Lemma' (1995)
***R* tiles Z_n => *aR* tiles Z_n $\langle a,n \rangle=1$**

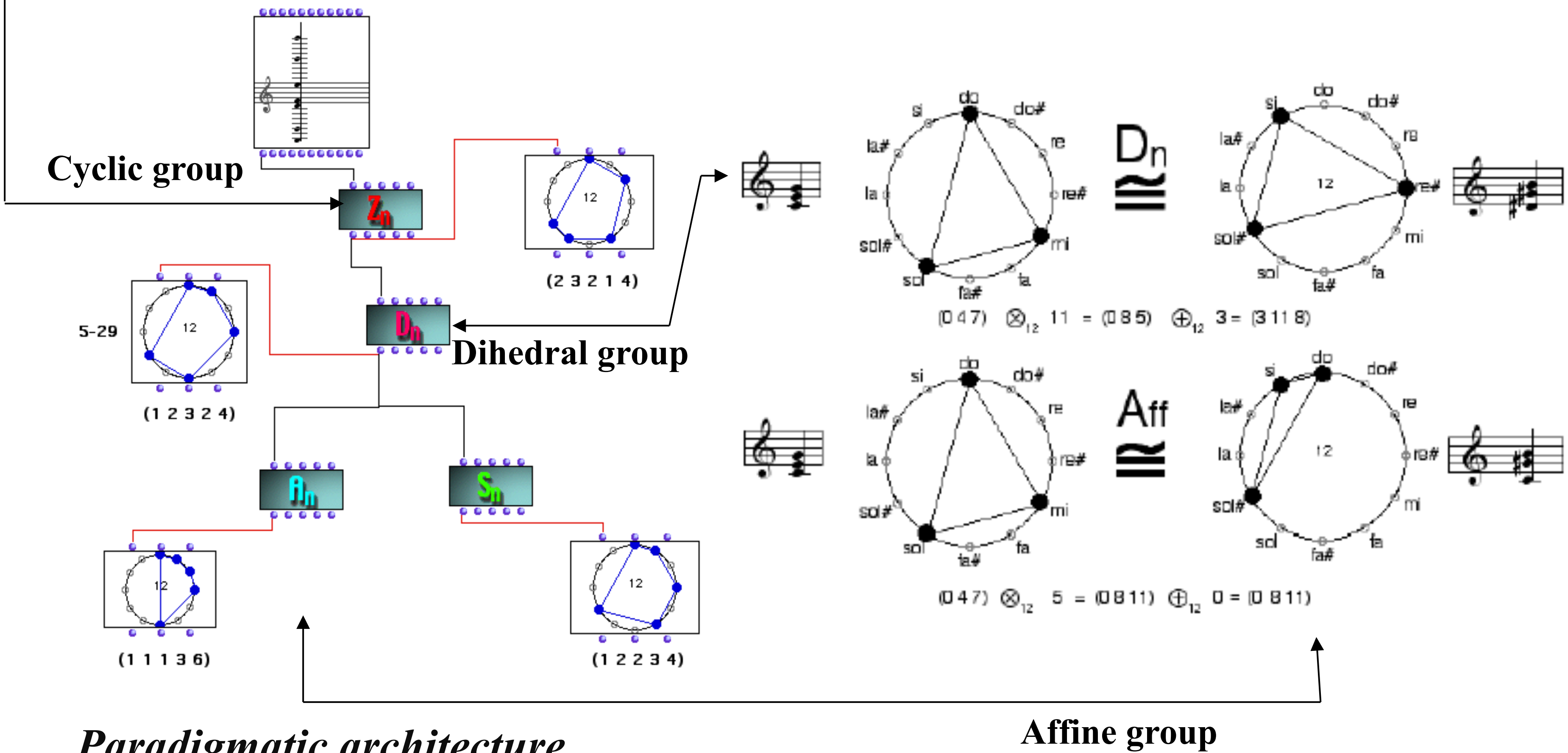
Equivalence classes of musical structures up to a group action (see Definitions by Abstraction)



$$\mathbf{Z_{12}} = \langle T_k \mid (T_k)^{12} = T_0 \rangle$$

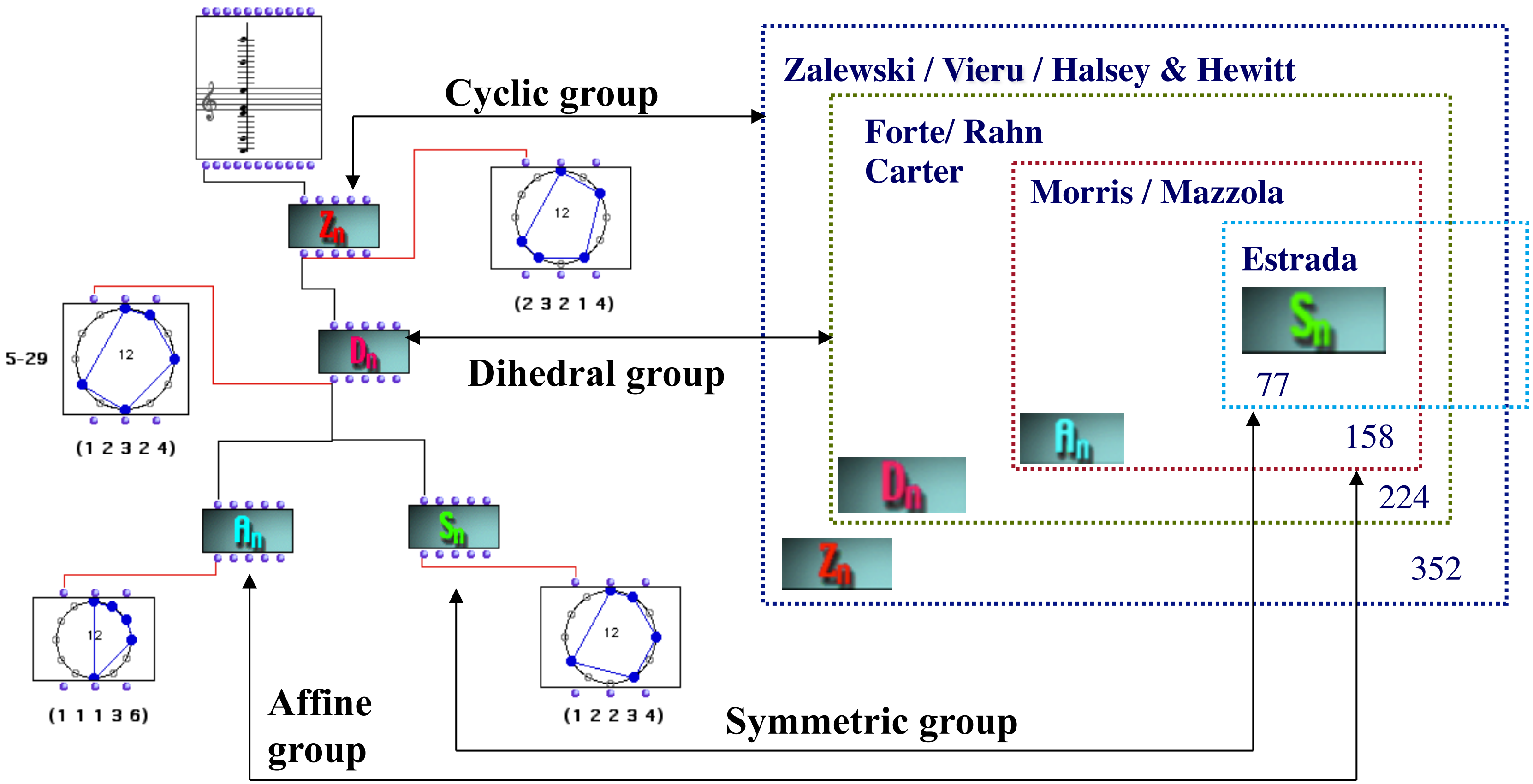
$$\mathbf{D_{12}} = \langle T_k , I \mid (T_k)^{12} = I^2 = T_0 , ITI = I(IT)^{-1} \rangle$$

$$\mathbf{Aff} = \{ f \mid f(x) = ax+b,\ a \in (\mathbf{Z_{12}})^*,\ b \in \mathbf{Z_{12}} \}$$



Paradigmatic architecture

Group actions and the classification of musical structures




F. Klein


W. Burnside

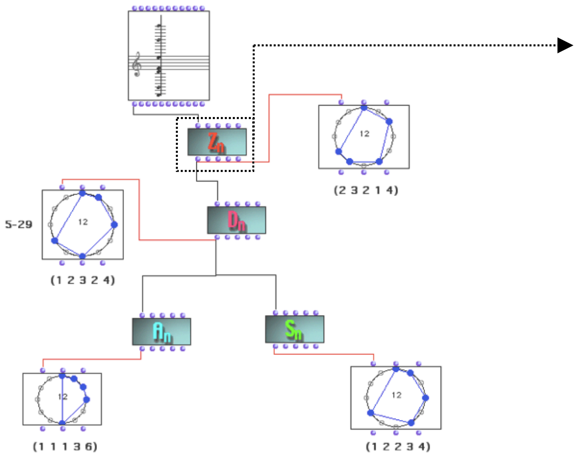

G. Polya

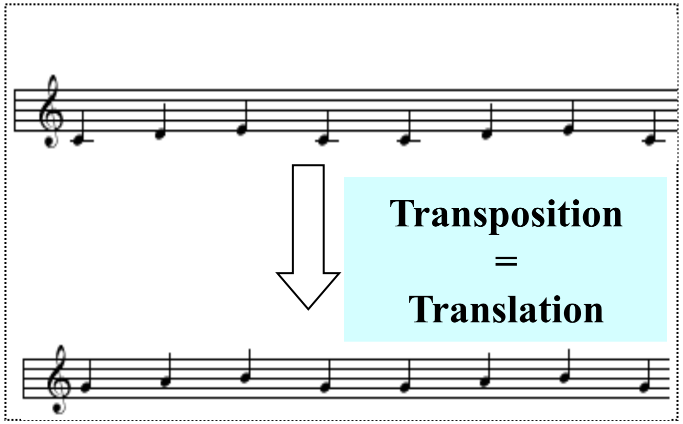

E. Cassirer

The nature of a given geometry is […] defined by the *reference* to a determinate group and the way in which spatial forms are related within that type of geometry. [*Cf. Felix Klein Erlangen Program - 1872*][…] We may raise the question whether there are any concepts and principles that are, although in different ways and different degrees of distinctness, necessary conditions for both the *constitution* of the perceptual world and the construction of the universe of geometrical thought. It seems to me that the **concept of group** and the **concept of invariance** are such principles.

E. Cassirer, "The concept of group and the theory of perception", 1944

The algebraic-geometric genealogy of structuralism






F. Klein


E. Cassirer


G.-G. Granger


J. Piaget

« [C'est la **notion de groupe** qui] donne un sens précis à l'idée de **structure** d'un ensemble [et] permet de déterminer les éléments efficaces des transformations en réduisant en quelque sorte à son **schéma opératoire** le domaine envisagé. […]L'objet véritable de la science est le **système des relations** et non pas les termes supposés qu'il relie. […] Intégrer les résultats - symbolisés - d'une expérience nouvelle revient […] à créer un canevas nouveau, un **groupe de transformations** plus complexe et plus compréhensif »

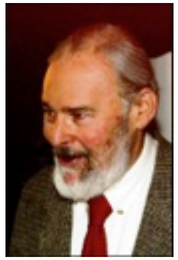
G.-G. Granger, "Pygmalion. Réflexions sur la pensée formelle", 1947

« La **théorie des catégories** est une théorie des constructions mathématiques, qui est macroscopique, et procède d'étage en étage. Elle est un bel exemple d'**abstraction réfléchissante**, cette dernière reprenant elle-même un principe constructeur présent dès le stade sensori-moteur. Le **style catégoriel** qui est ainsi à l'image d'un aspect important de la genèse des facultés cognitives, est un style adéquat à la description de cette genèse »

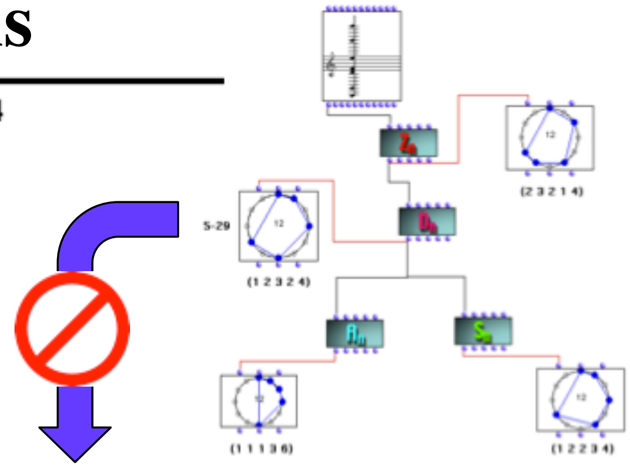
Jean Piaget, Gil Henriques et Edgar Ascher, *Morphismes et Catégories. Comparer et transformer*, 1990

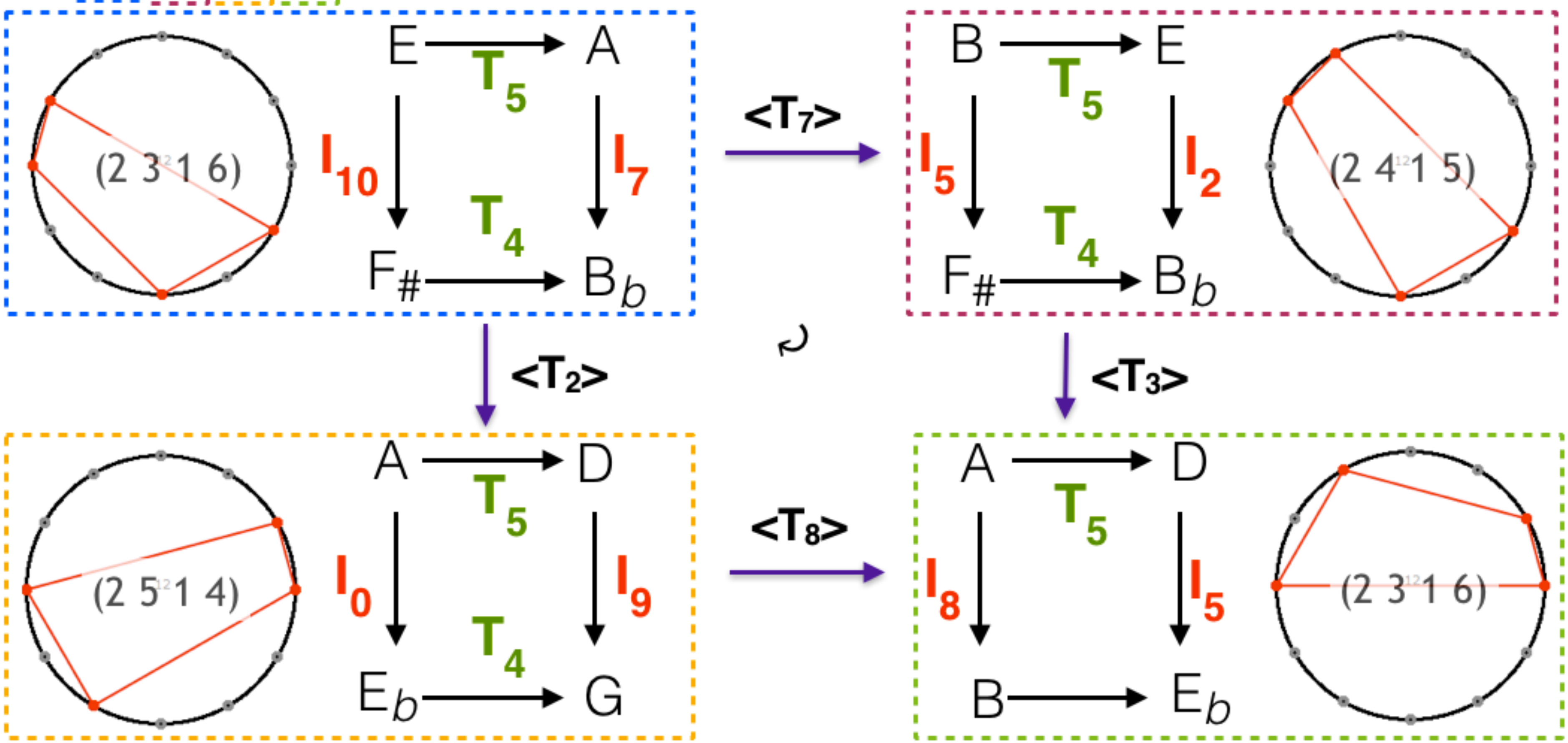
CT and transformational music analysis

D. Lewin, "A Tutorial on K-nets using the Chorale in Schoenberg's Op.11, N°2 », *JMT*, 1994

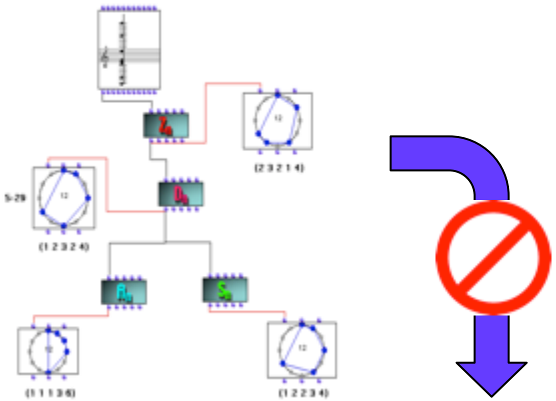


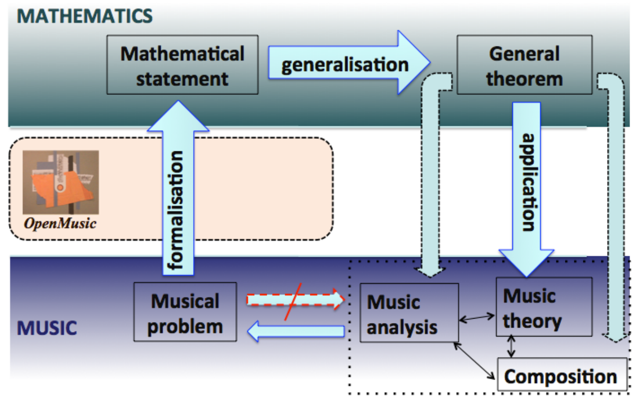
D. Lewin

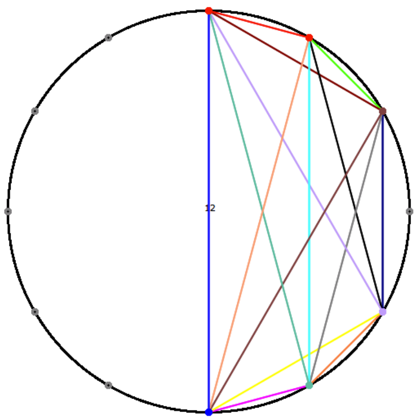




Musical Z-relation and Homometry Theory







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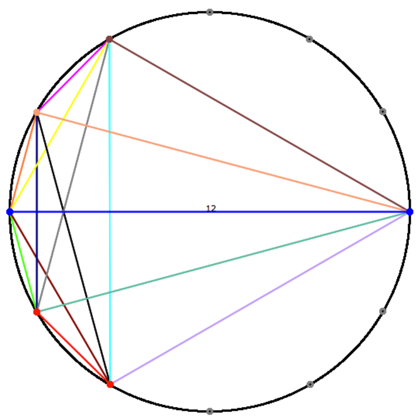
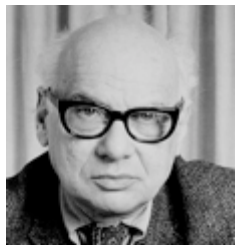
Hexachord Theorem

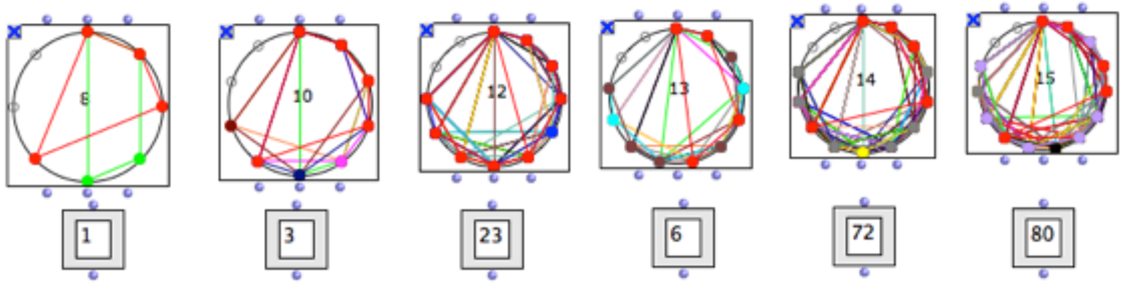




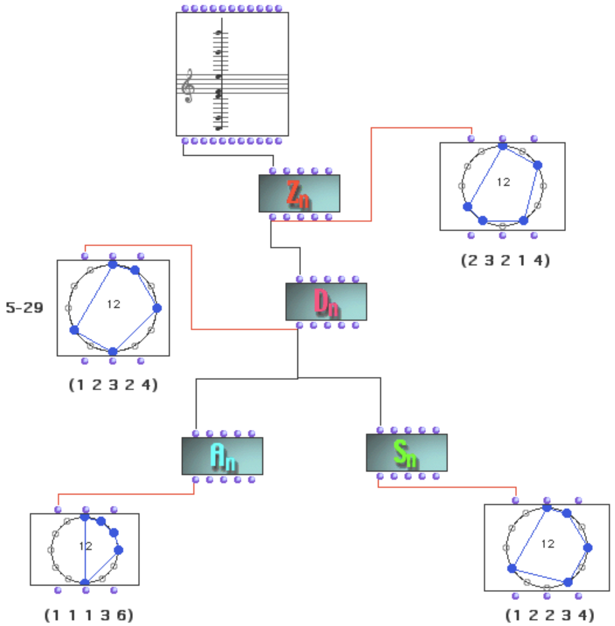
M. Babbitt

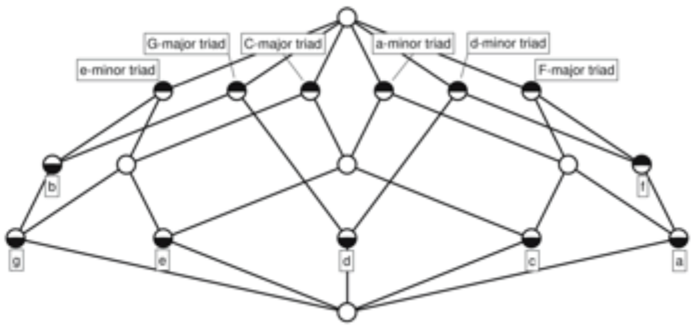
• Mandereau J., D. Ghisi, E. Amiot, M. Andreatta, C. Agon, (2011), « Z-relation and homometry in musical distributions », *Journal of Mathematics and Music*, vol. 5, n° 2, p. 83-98.

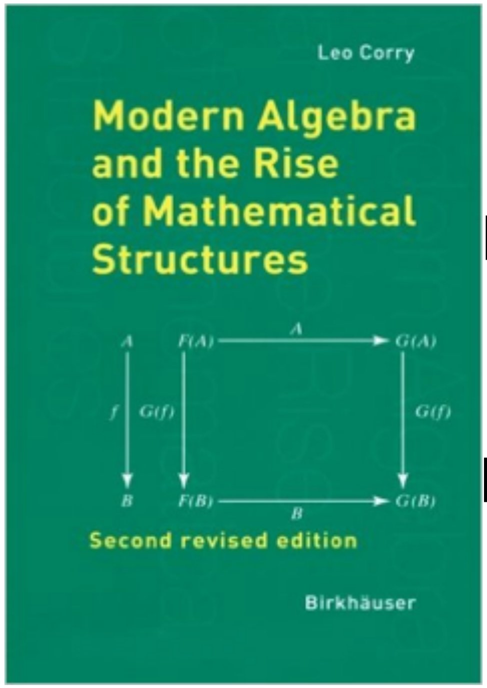
• Mandereau J, D. Ghisi, E. Amiot, M. Andreatta, C. Agon (2011), « Discrete phase retrieval in musical structures », *Journal of Mathematics and Music*, vol. 5, n° 2, p. 99-116.



Music and the Rise of the Structural Approach in Maths



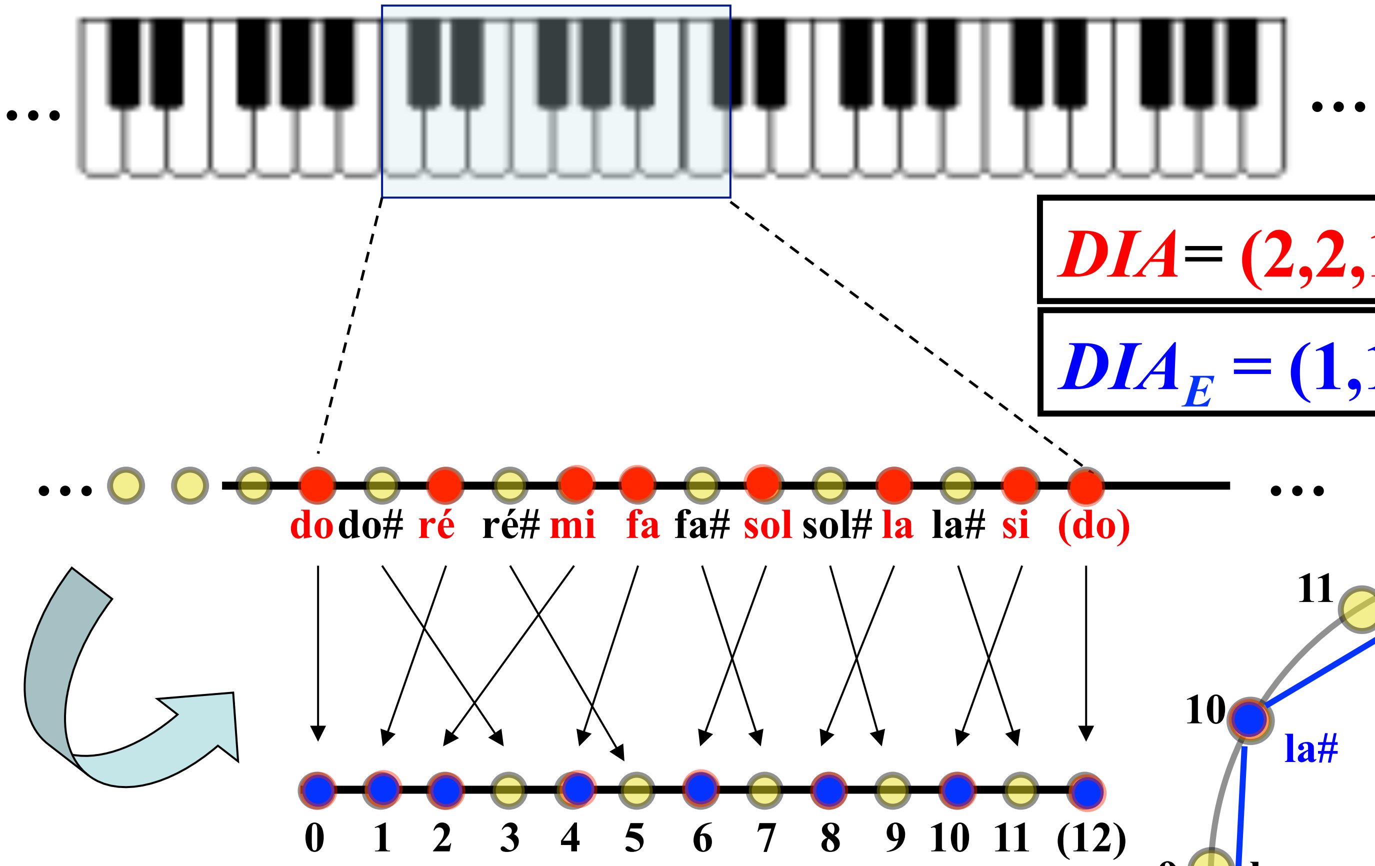




The concept of *group*

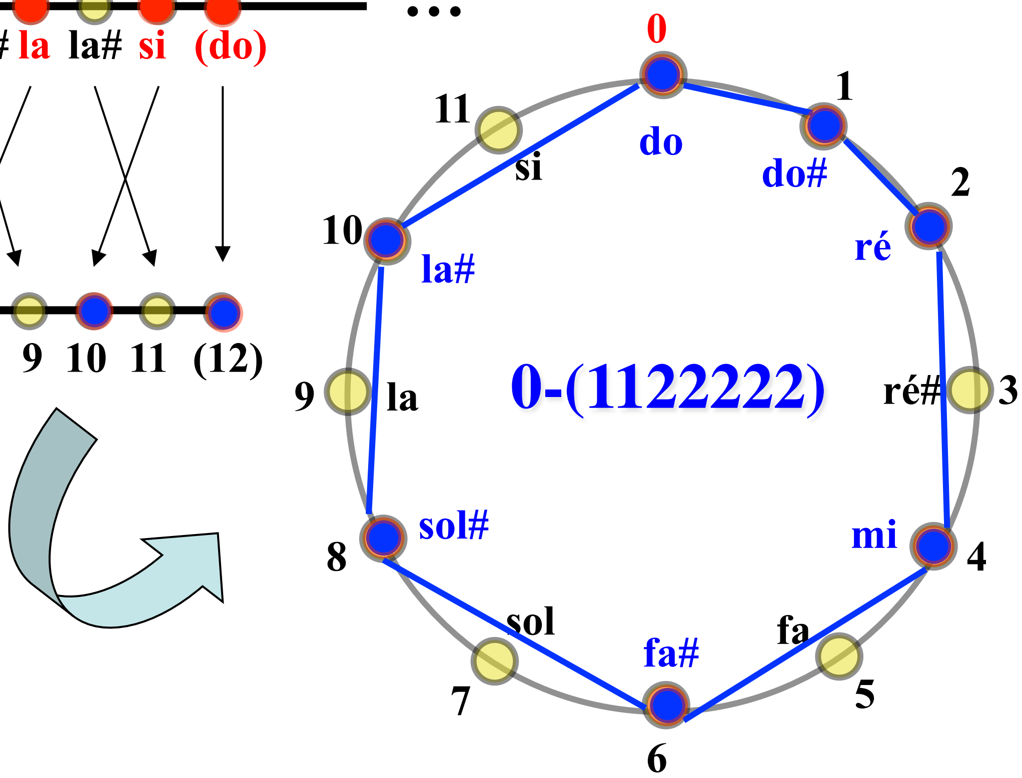
The concept of *lattice*

The partition lattice of musical structures



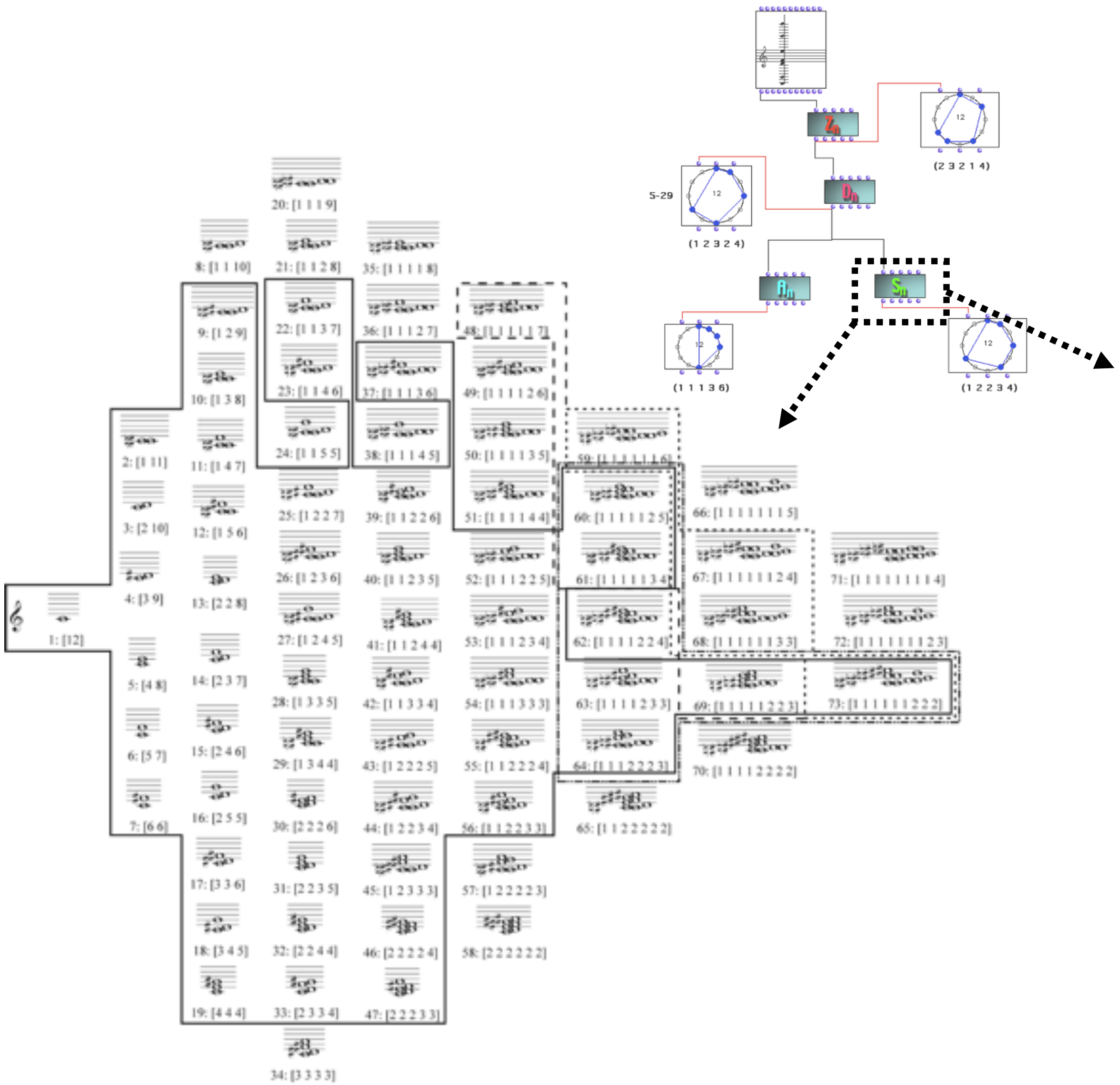
$DIA = (2,2,1,2,2,2,1)$

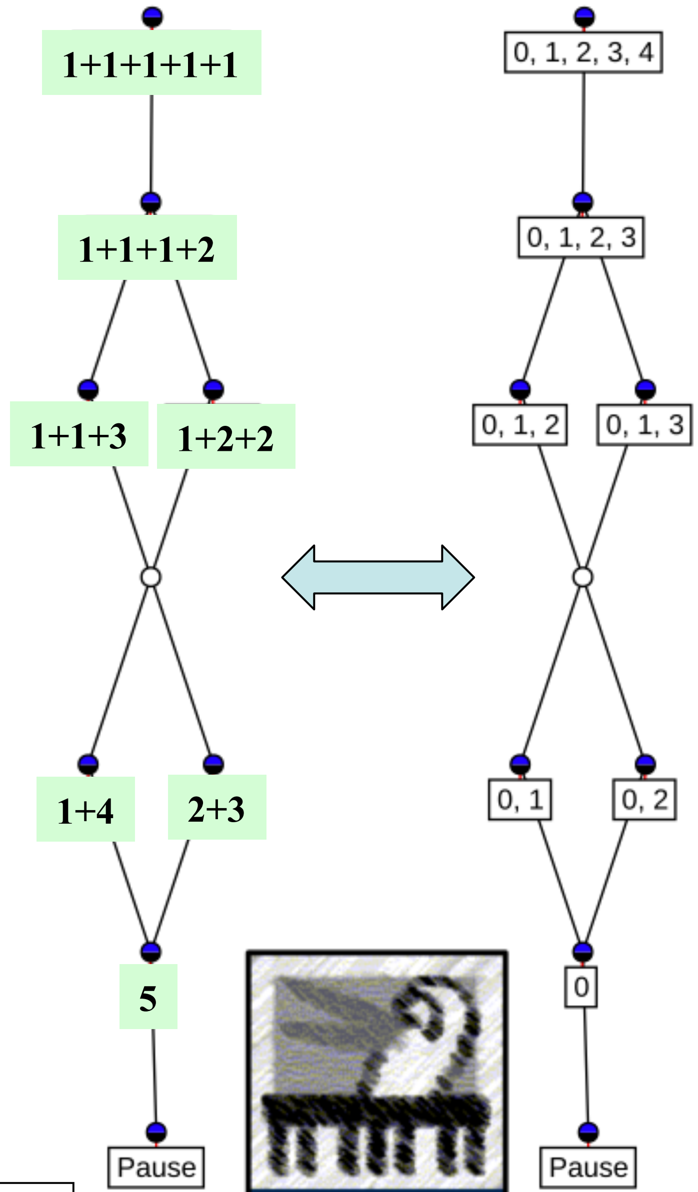
$DIA_E = (1,1,2,2,2,2,2)$



Julio Estrada, *Théorie de la composition : discontinuum – continuum*, université de Strasbourg II, 1994

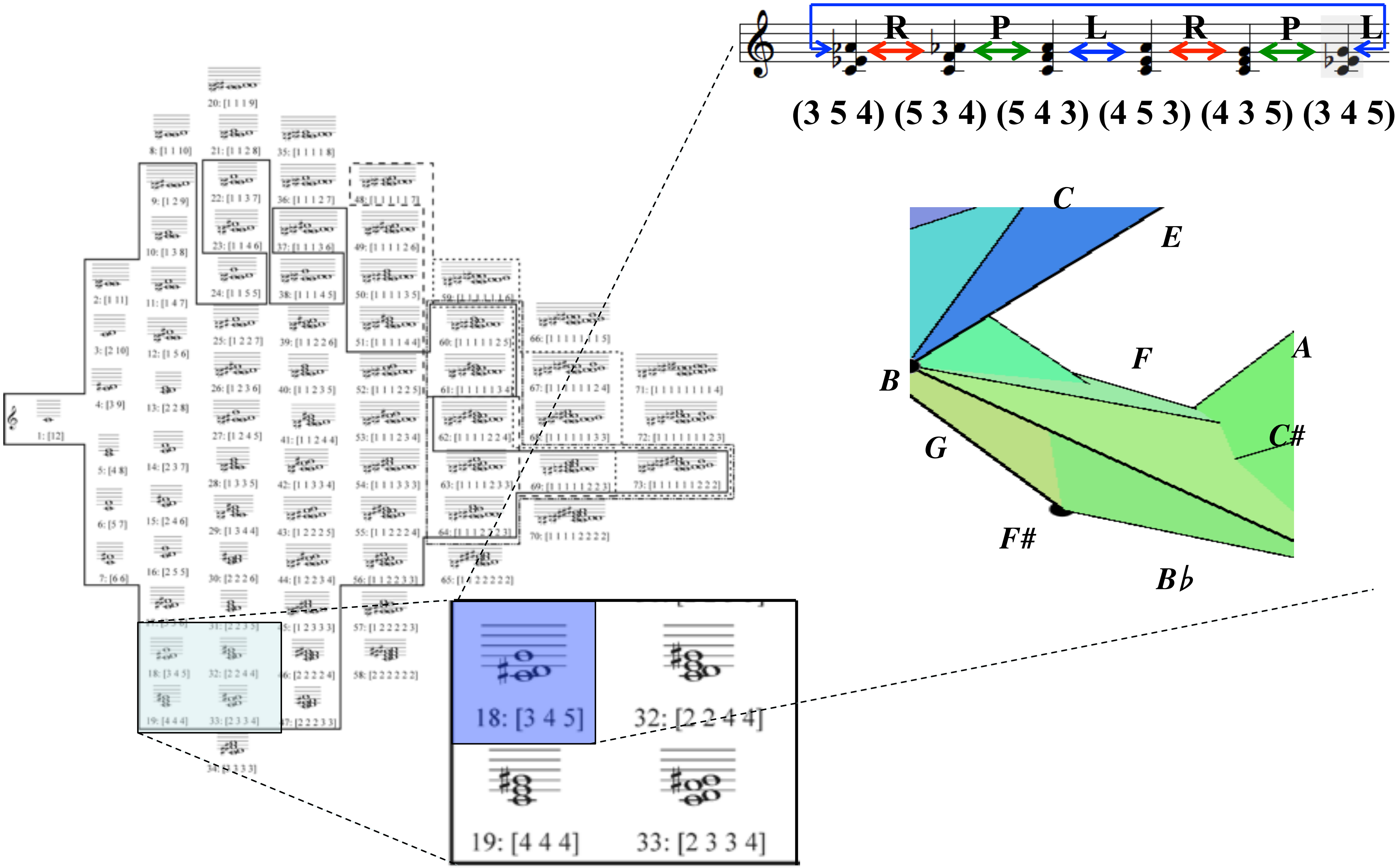
The permutohedron as a lattice of formal concepts



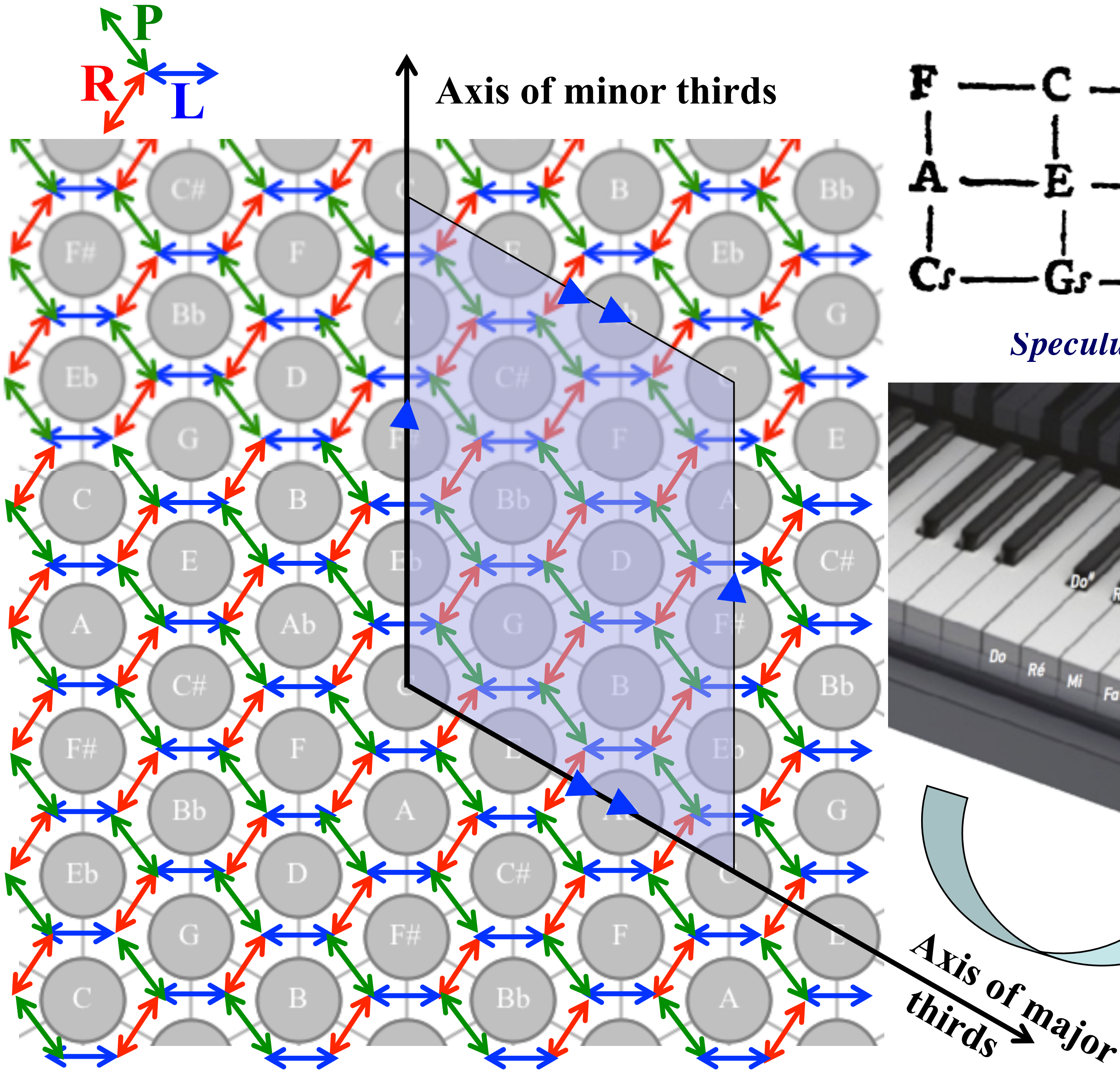


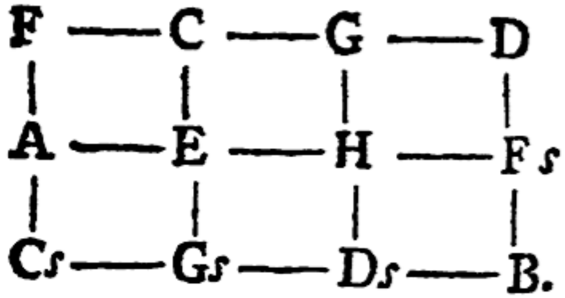
• T. Schlemmer, M. Andreatta, « Using Formal Concept Analysis to represent Chroma Systems », MCM 2013, McGill Univ., Springer, LNCS.

Permutohedron and a topological structural inclusion

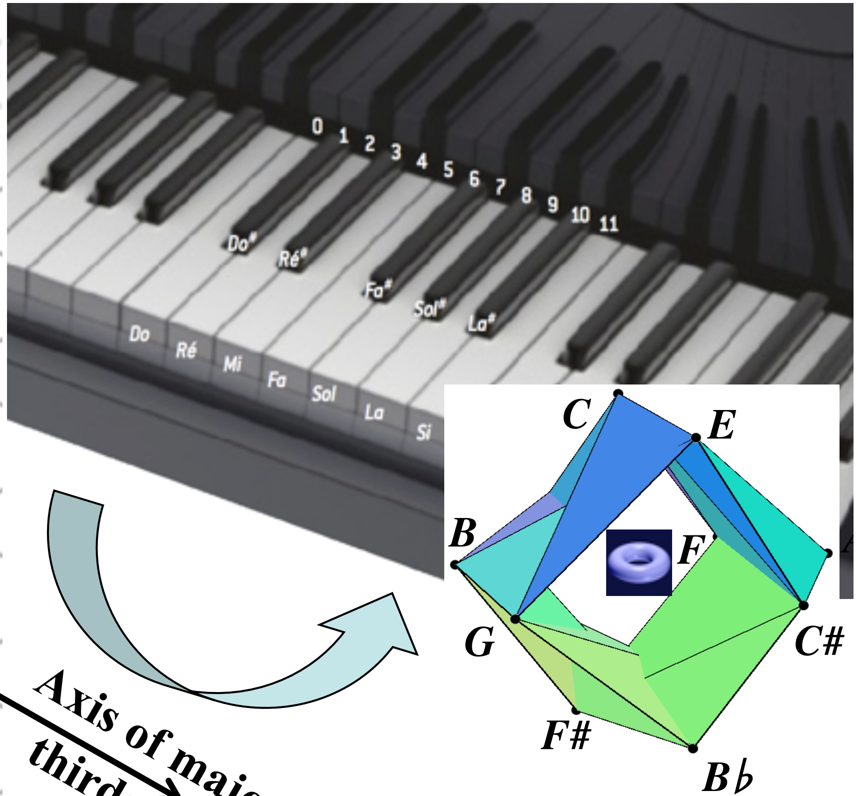


The *Tonnetz* as a topological structure





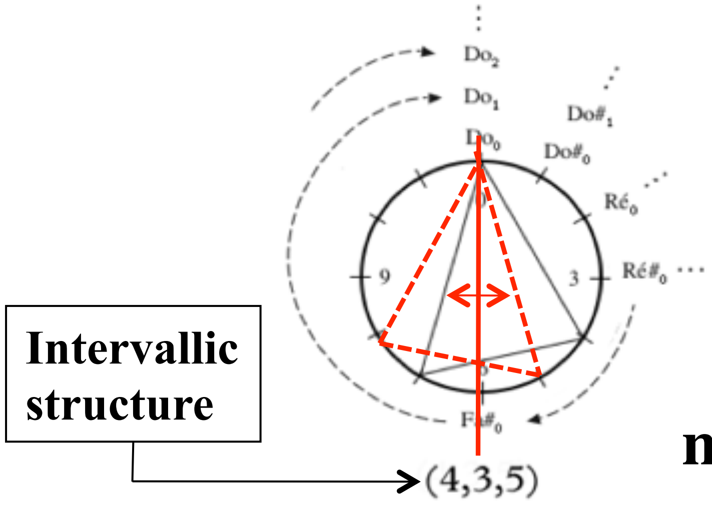
Speculum Musicum (Euler, 1773)

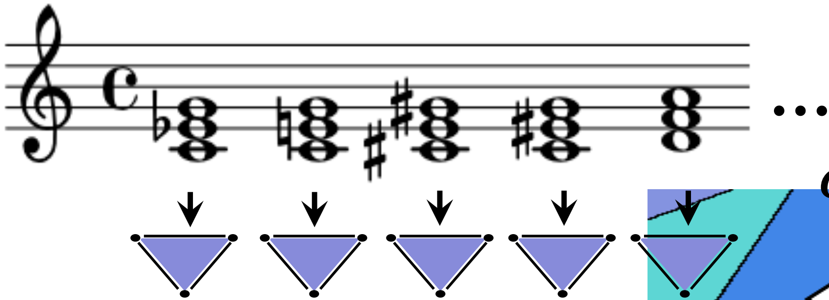


The *Tonnetz* as a simplicial complex

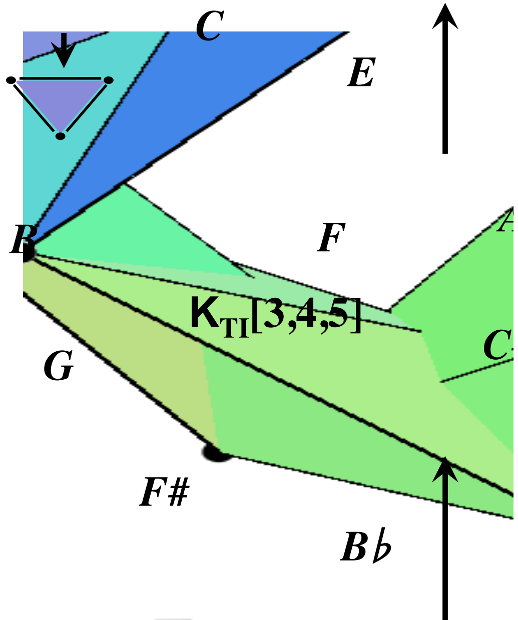
L. Bigo, *Représentation symboliques musicales et calcul spatial*, PhD, Ircam / LACL, 2013

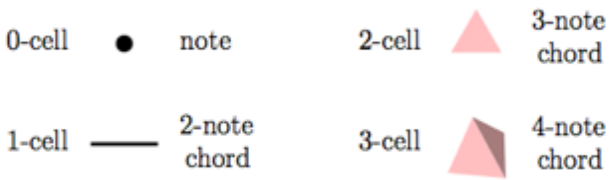
- Assembling chords related by some equivalence relation
  - Transposition/inversion: Dihedral group action on $\mathrm{P}(\mathrm{Z}_n)$

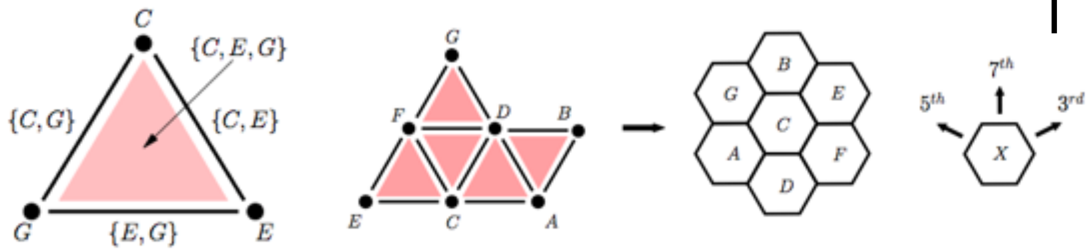




major/minor triads

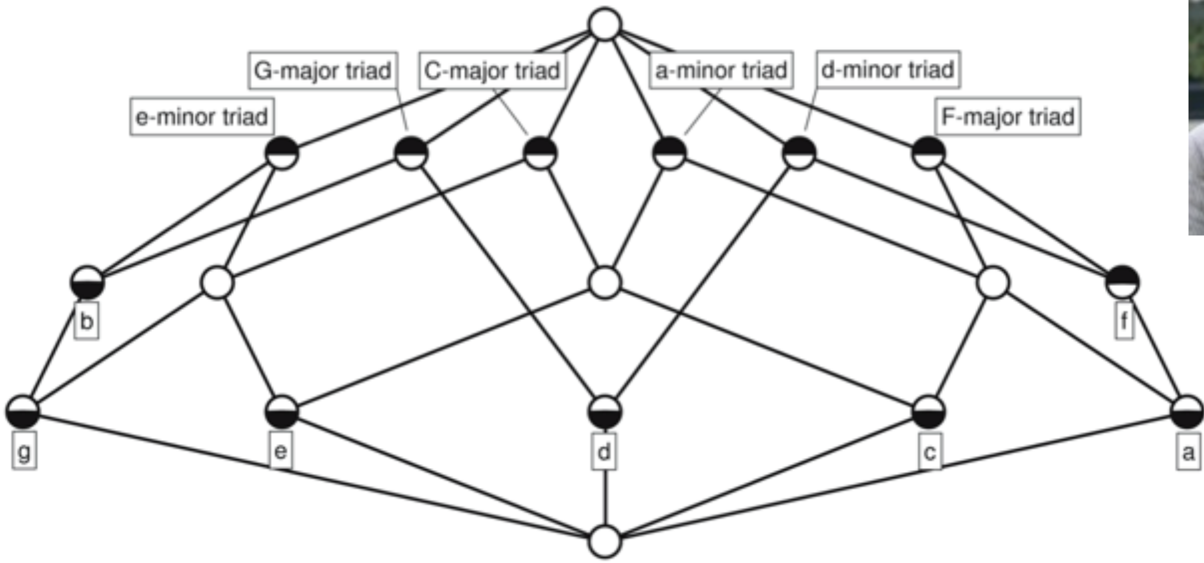


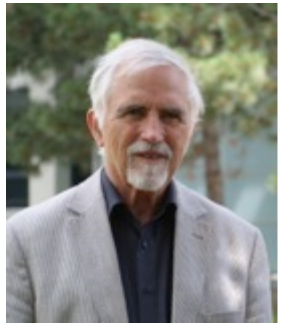




Formal Concept Analysis and topology in music theory

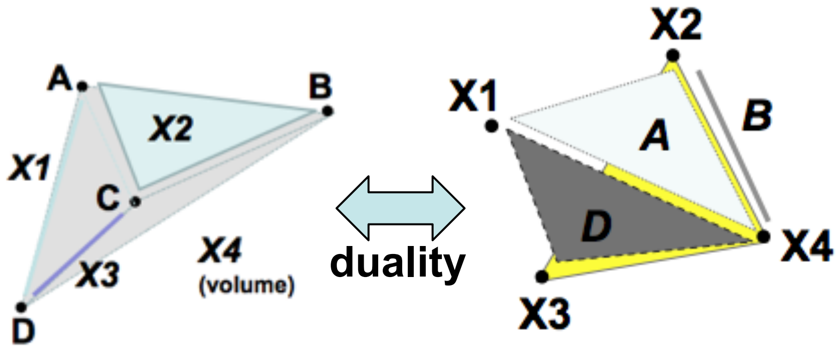
	C-major triad	d-minor triad	e-minor triad	F-major triad	G-major triad	a-minor triad
c	X			X		X
d		X			X	
e	X		X			X
f		X		X		
g	X		X		X	
a		X		X		X
b			X		X	





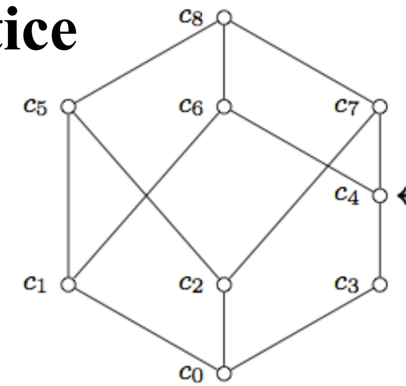
Rudolf Wille

	A	B	C	D
X1	1	0	0	1
X2	1	1	1	0
X3	0	0	1	1
X4	1	1	1	1



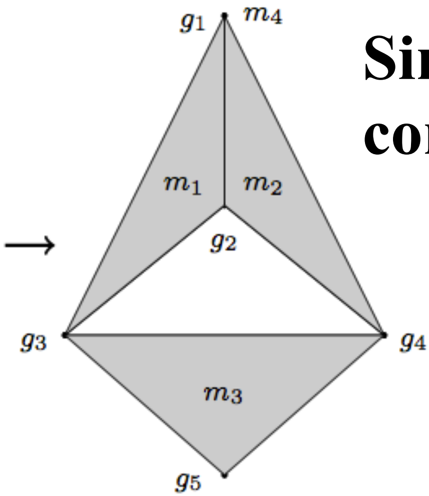
Concept lattice *vs* simplicial complex

Lattice

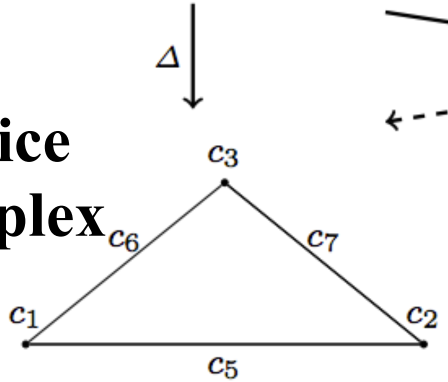


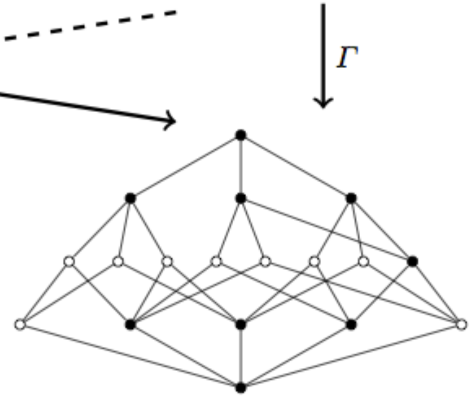
	m_1	m_2	m_3	m_4
g_1	×	×		×
g_2	×	×		
g_3	×		×	
g_4		×	×	
g_5			×	

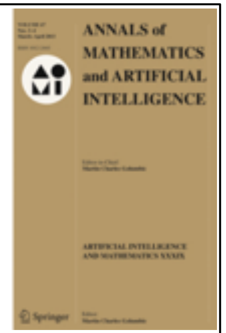
Simplicial complex



Lattice complex



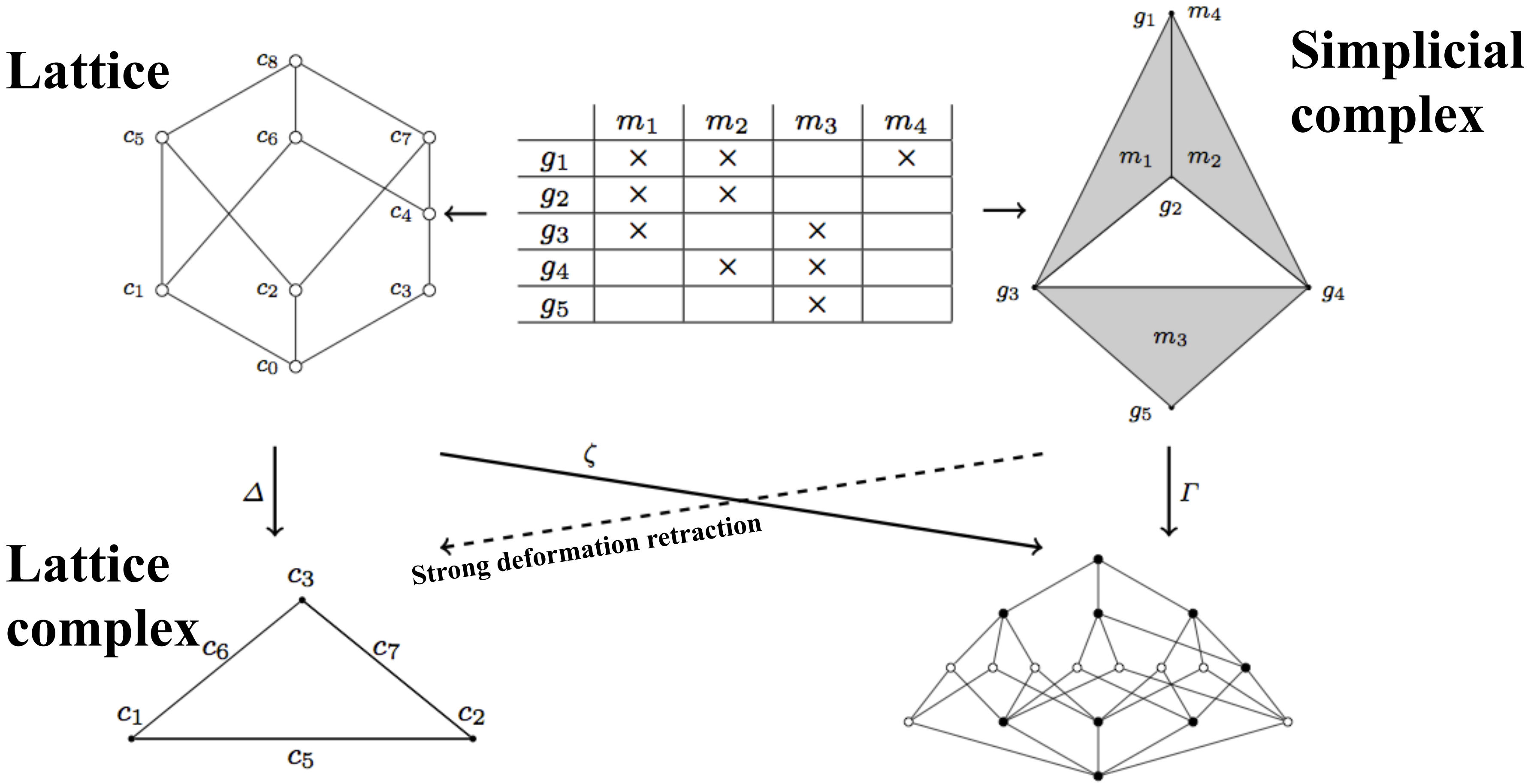




Lattice-based and Topological Representations of Binary Relations with an Application to Music

Anton Freund · Moreno Andreatta · Jean-Louis Giavitto

Concept lattice *vs* simplicial complex

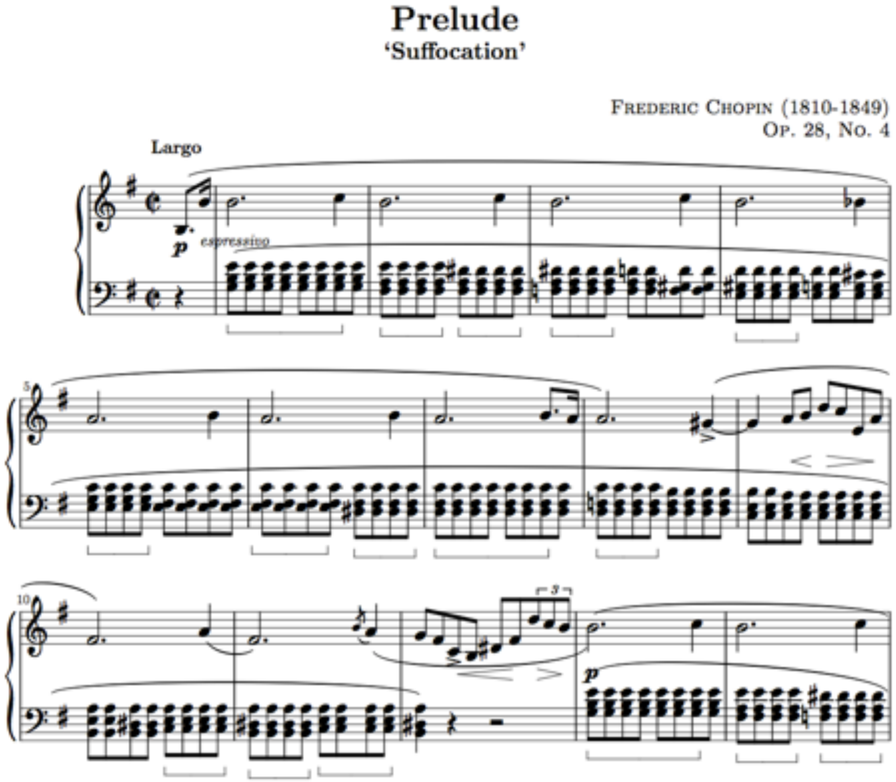


	m_1	m_2	m_3	m_4
g_1	×	×		×
g_2	×	×		
g_3	×		×	
g_4		×	×	
g_5			×	

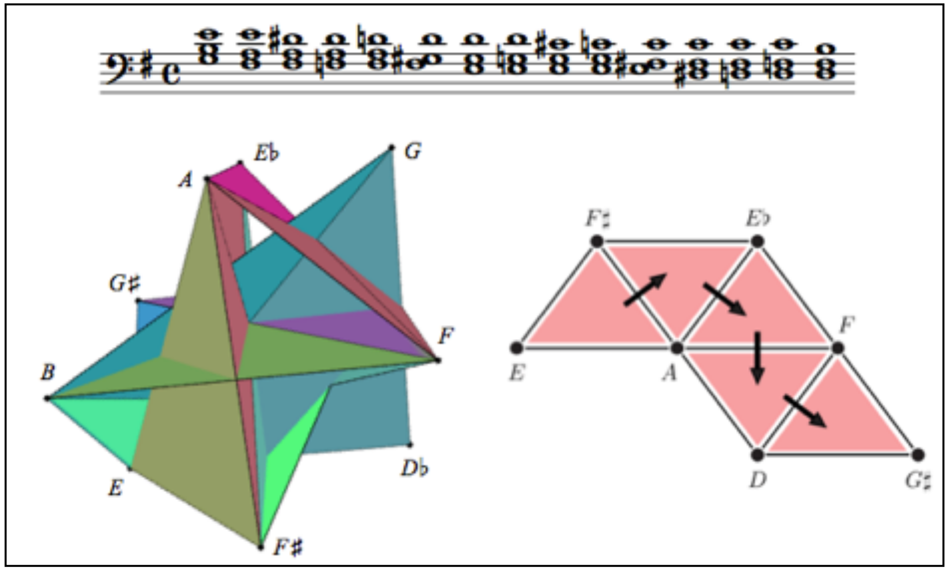
Conclusions:

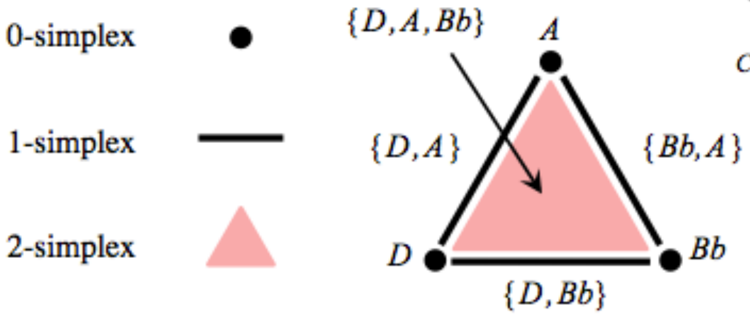
- The concept lattice alone cannot be fully reconstructed from the simplicial complex
- The simplicial complex cannot be fully determined from the concept lattice alone
- The concept lattice alone allows to determine the homotopy type of the simplicial complex

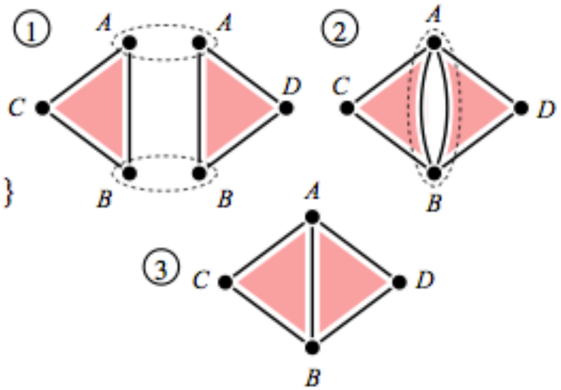
The simplicial complex of a Chopin's *Prelude*



➔ Hexachord
(by Louis Bigo, 2013)

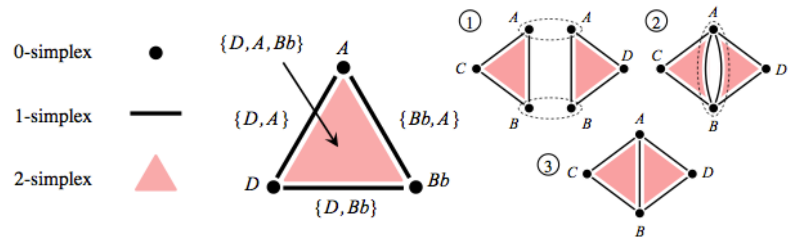




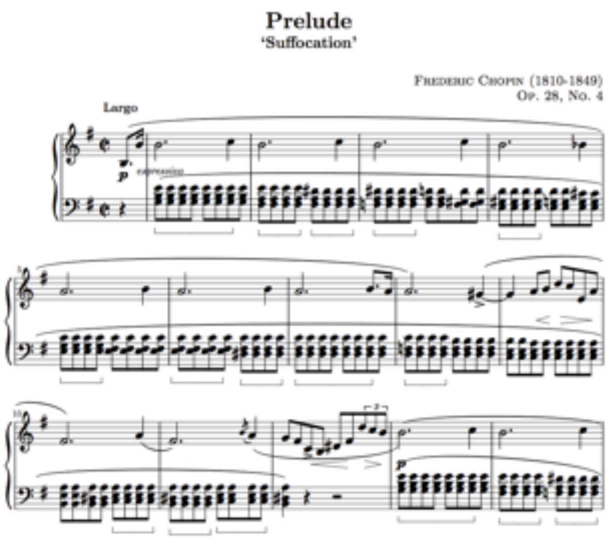


Towards a topological signature of a musical piece
A structural approach in Music Information Retrieval

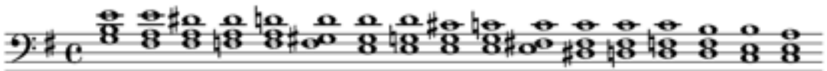
The simplices and their self-assembly



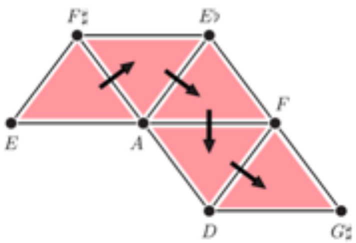
The score



Score reduction



The simplicial complex generated by the piece



A specific trajectory in the complex

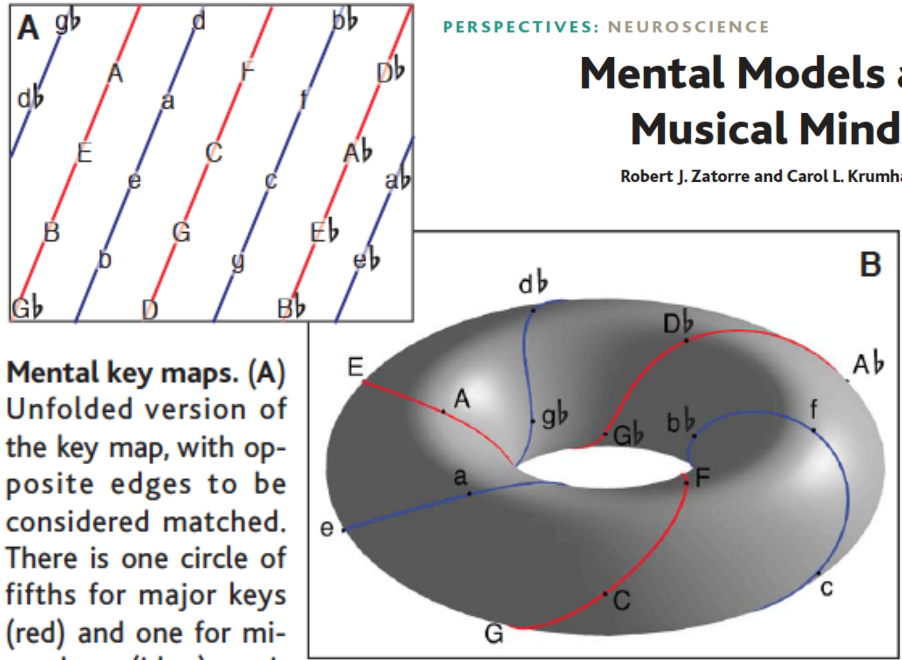
Topological signature?

Neurosciences and Tonnetz

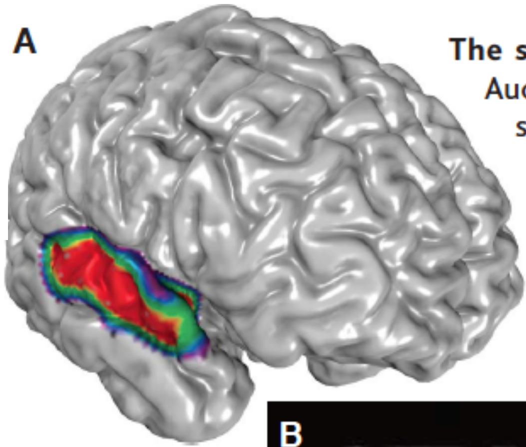
PERSPECTIVES: NEUROSCIENCE

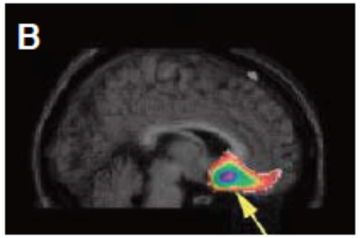
Mental Models and Musical Minds

Robert J. Zatorre and Carol L. Krumhansl

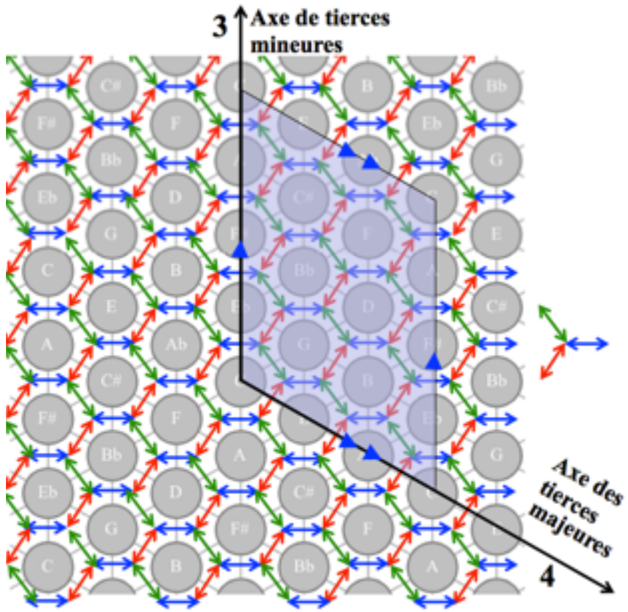


Mental key maps. (A) Unfolded version of the key map, with opposite edges to be considered matched. There is one circle of fifths for major keys (red) and one for minor keys (blue), each wrapping the torus three times. In this way, every major key is flanked by its relative minor on one side (for example, C major and a minor) and its parallel minor on the other (for example, C major and c minor). **(B)** Musical keys as points on the surface of a torus.





The sensation of music. (A) Auditory cortical areas in the superior temporal gyrus that respond to musical stimuli. Regions that are most strongly activated are shown in red. **(B)** Metabolic activity in the ventromedial region of the frontal lobe increases as a tonal stimulus becomes more consonant.

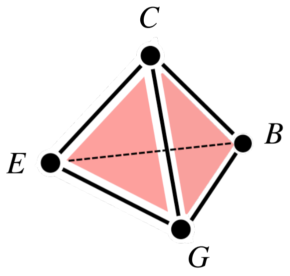


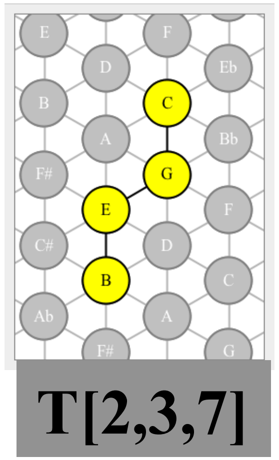
Acotto E. et M. Andreatta (2012), « Between Mind and Mathematics. Different Kinds of Computational Representations of Music », *Mathematics and Social Sciences*, n° 199, 2012(3), p. 9-26.

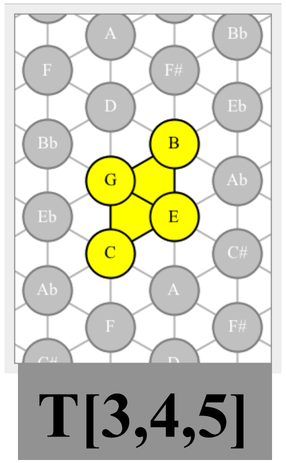
European Mathematical Psychology Group

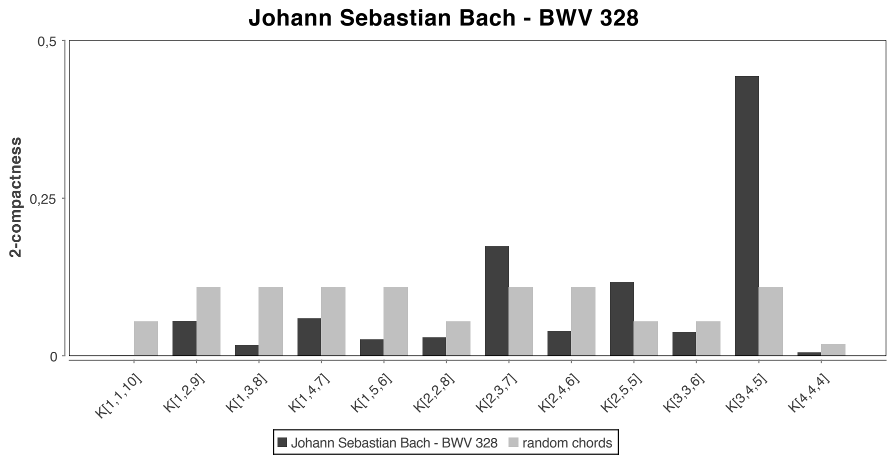
The spatial character of the « musical style »

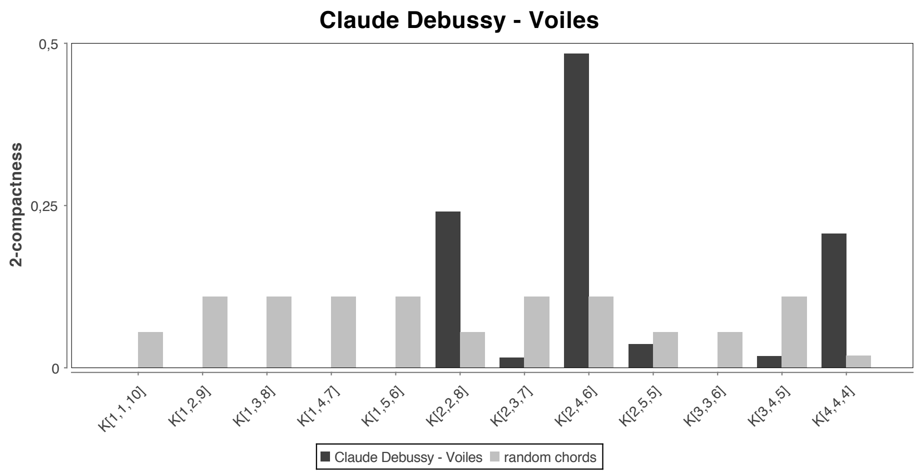
Bigo L., M. Andreatta, « Musical analysis with simplicial chord spaces », in D. Meredith (ed.), *Computational Music Analysis*, Springer (in press)

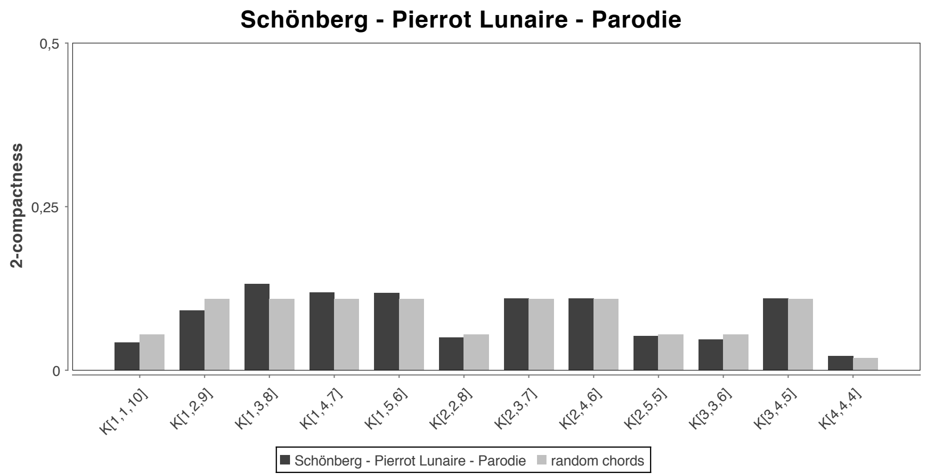












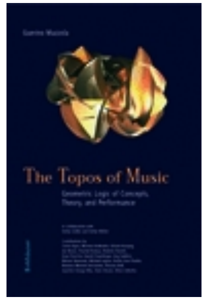
Towards a categorical theory of creativity (in music, cognition and discourse)

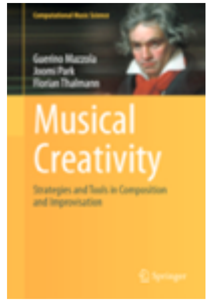
Abstract

This article presents a first attempt at establishing a **category-theoretical model of creative processes**. The model, which is applied to musical creativity, discourse theory, and cognition, suggests the relevance of the notion of "colimit" as a unifying construction in the three domains as well as the central role played by the Yoneda Lemma in the categorical formalization of creative processes.

MCM 2013

MATHEMATICS AND COMPUTATION IN MUSIC

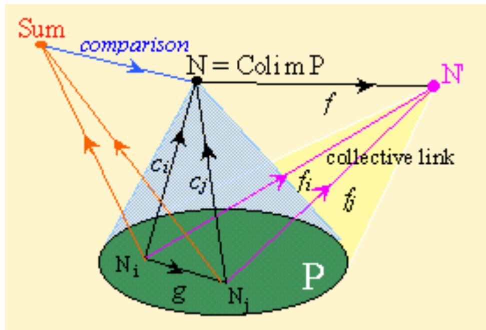




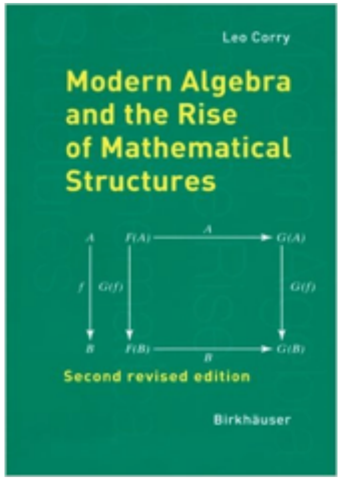
MAMUPHI

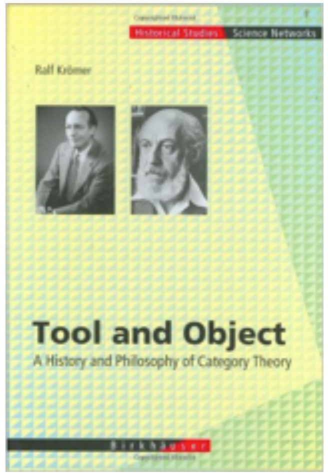
Séminaire MaMux

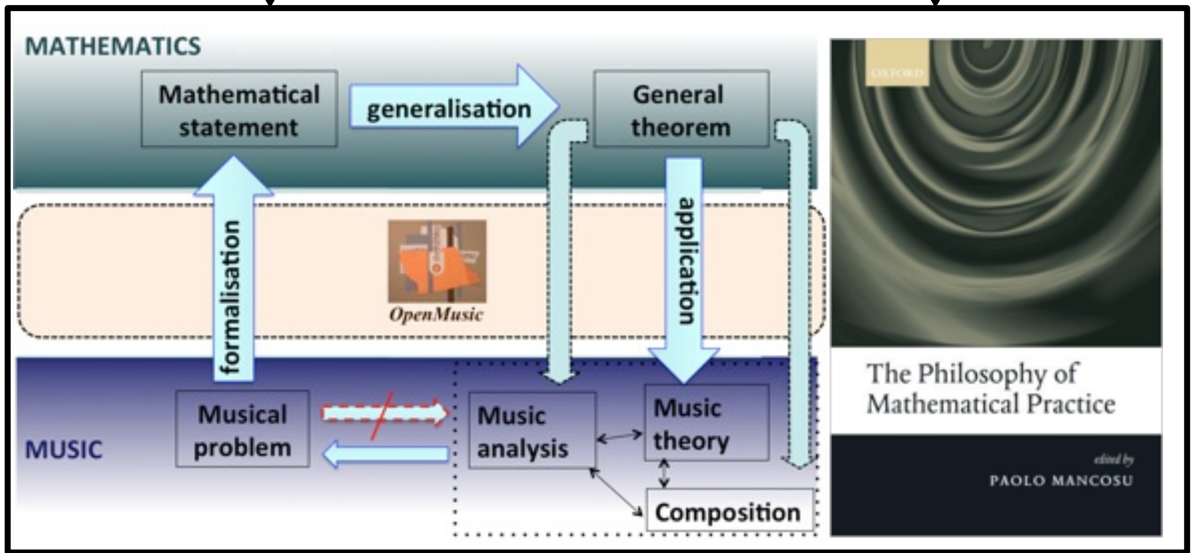
Andreatta M., A. Ehresmann, R. Guitart, G. Mazzola, « Towards a categorical theory of creativity », Fourth International Conference, MCM 2013, McGill University, Montreal, June 12-14, 2013, Springer, 2013.

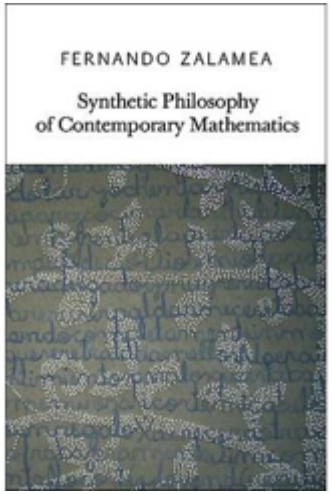


Which type of philosophy for the *mathemusical* practice?

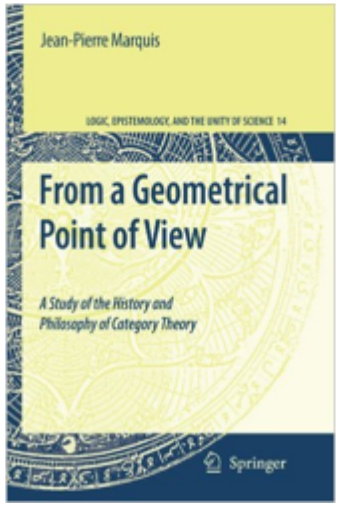


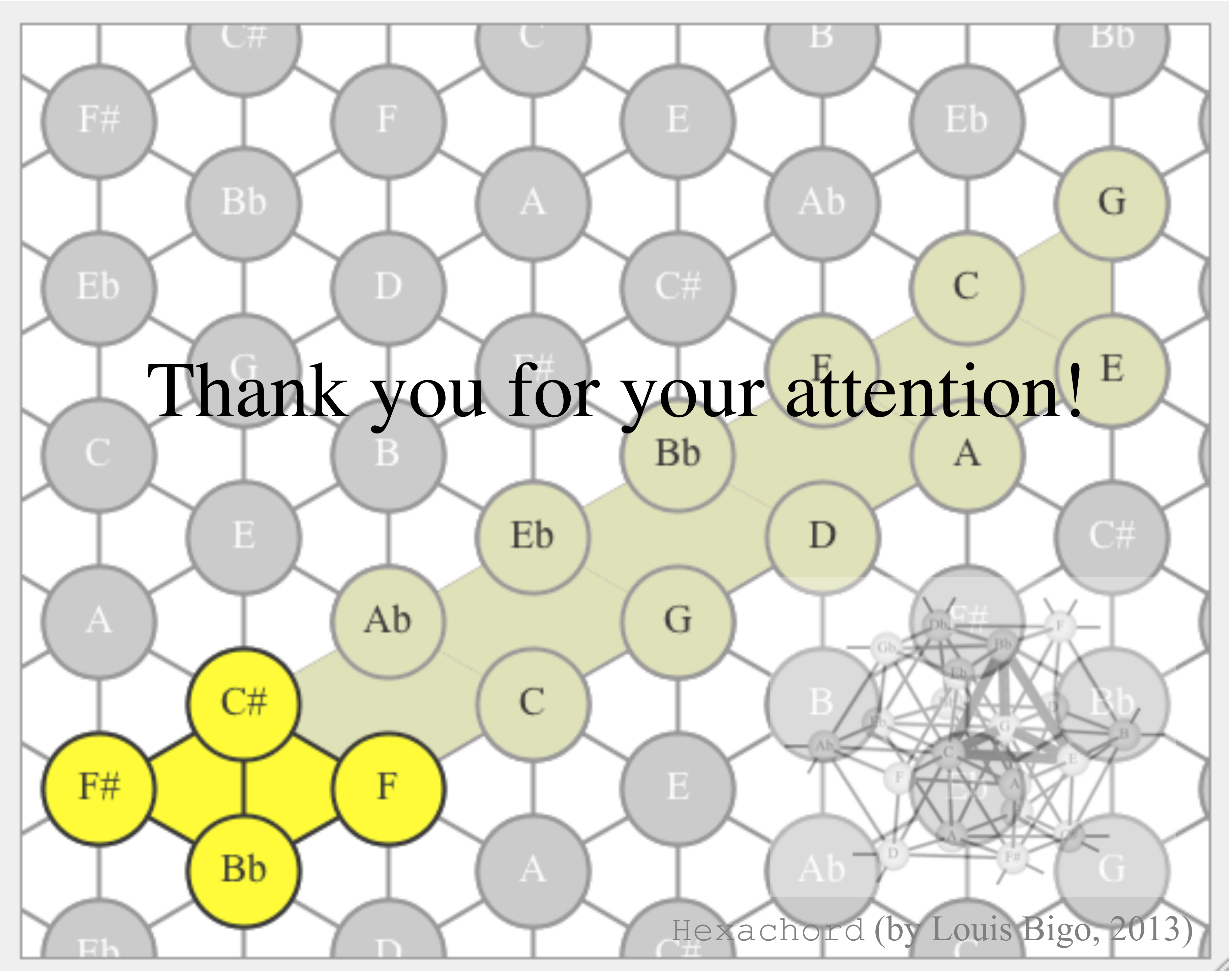






*A **synthetic** vision allows us to link together apparently distant strata of mathematics and culture, helping us to break down many artificial barriers. Not only can today's mathematics be appreciated through epistemic, ontic, phenomenological and aesthetic modes, but in turn, it should help to transform **philosophy.***





Thank you for your attention!

Hexachord (by Louis Bigo, 2013)