Using Formal Concept Analysis to Represent Chroma Systems

Tobias Schlemmer[1] and Moreno Andreatta[2]

[1] Technische Universität Dresden,
Fachrichtung Mathematik, 01069 Dresden, Germany
Tobias.Schlemmer@tu-dresden.de
http://www.math.tu-dresden.de/~schlemme/

[2] IRCAM/CNRS/UPMC,
1, place Stravinsky, F-75004 Paris, France
moreno.andreatta@ircam.fr
http://repmus.ircam.fr/moreno

Abstract. The article discusses the application of Formal Concept Analysis to the algebraic enumeration, classification and representation of musical structures. It focuses on the music-theoretical notion of the Tone System and its equivalent classes obtained either via an action of a given finite group on the collection of subsets of it or via an identification of Forte's corresponding interval vector and Lewin's interval function. The use of concept lattices, applied to a simple case such as the division of the octave into five equal parts and the associated Chroma System, clearly shows that these approaches are conceptually different. The same result is obtained for a given subsystem of the traditional Tone System, as we will show by analysing the case of the pentatonic system. This opens a window towards generic tone systems that can be used as starting point for the structural analysis of other finite chroma systems.

Keywords: Formal Concept Analysis, Galois Correspondence, Lattice, Interval Vector, Interval Function, Interval Structure, Partition.

1 Introduction

Formal Concept Analysis (FCA) was introduced in the beginning of the 1980s by Rudolf Wille as an attempt at reconstructing Lattice Theory [1,2].[1] Although music is a major inspirational field for applying formal concept analysis [5,6], this approach did not establish itself as a paradigm in mathematical music theory. This paper is a first attempt toward a better understanding of ordered structures in music theory.

[1] Similar constructions have also been proposed, independently, by Marc Barbut and Louis Frey [3]. See [4] for an interesting discussion on the mutual influences between the Darmstadt school on Formal Concept Analysis and the French tradition on "Treillis de Galois".

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This musical structure theory relies on an Extensional Standard Language for Music Theory [7] that was originally developed by Rudolf Wille in parallel to FCA and applied in the Mutabor language [8].

The current paper provides a short introduction into both theories and illustrates them through the development of a new method for structure analysis of tone systems based on the notion of a chroma system.

2 Preliminaries

Formal Concept Analysis [9] is based on a formalisation of a philosophical notion of a concept. A concept is usually defined by a set of examples and/or by a set of properties we attribute to it. Nevertheless, for each concept there are many more objects that are examples and many more properties that can be attributed to it. Furthermore, all examples share the same set of attributes that belong to the concept (some by definition, some by logical implications, and some others). On the other hand, each property that is attributed to a concept holds for each example of the concept. The latter relation is used to define a formal concept. Let G be a set whose elements we call *objects*, M a set whose elements we call *attributes* and $I \subseteq G \times M$ a binary relation. Then, the triplet $\mathbb{K}(G, M, I)$ is called a *formal context*. It is usually visualised as a cross table (e.g., Fig. 2) where the objects denote rows and attributes denote columns. Let $A \subseteq G$ be a set of objects and $B \subseteq M$ be a set of attributes. We use two derivation operators (both denoted by the same sign) as follows:

$$A' := \{m \in M \mid \forall g \in A : g\ I\ m\}, \quad \text{and} \quad B' := \{g \in G \mid \forall m \in B : g\ I\ m\} \quad (1)$$

In other words: A' is the set of all attributes that are shared by all objects of A, while B' is the set of all objects that share all attributes of B. These two operators form a Galois correspondence between the power sets $\mathfrak{P}(G)$ and $\mathfrak{P}(M)$. A formal concept is now defined as a pair $(A, B) \in \mathfrak{P}(G) \times \mathfrak{P}(M)$ such that $A' = B$ and $B' = A$ hold. These formal concepts form an algebraic lattice which is denoted by $\mathfrak{B}(G, M, I)$.

Throughout the article we use the terminology of the so called "Extensional Standard Language of Music" as proposed by Rudolf Wille and fine tuned by Wilfried Neumaier [10,11,12,13] as it has been exemplified for n-tone equal tempered tone systems in [14]. This theory provides a good insight into the different levels of the basic constructions of mathematical music theory. Its notions of tone structures, 2-chords, 2-chordal forms, 2-harmonies and 2-harmonic forms provide answers to some questions recently raised by Dmitry Tymoczko [15], since the mentioned 2-forms, in particular, are an example of a structure having both an intervallic and a set-theoretic component. For philosophical reasons that cannot be discussed here, and in order to provide a universal and easy to use theory, all entities are considered as objects rather than certain aspects of them, e.g., pitch is considered to be a parameter of tones and chromas may have other aspects than being equivalence classes.

A triple $\mathrm{T} = (T, \delta, \mathrm{I})$ is called an *(algebraic) tone system* if T is a set, $\mathrm{I} = (I, +, -, 0)$ is an Abelian group and $\delta : T \times T \to I$ is a map such that for all $t_1, t_2, t_3 \in T$ the following equations hold:

$$\delta(t_1, t_2) + \delta(t_2, t_3) = \delta(t_1, t_3) \quad \text{and} \quad \delta(t_1, t_2) = 0 \text{ iff } t_1 = t_2. \tag{2}$$

The elements of the set T are called *tones* and each subset of T is called a *chord.* The empty chord is considered a *pause* here. The elements of I are considered as *intervals.* The signature of a tone system can be enriched by additional tone parameters. These are mappings from the set of tones into some set of values. The most prominent ones are frequency, pitch and loudness. For further examples we refer to Mazzola [16].

We call T *homogeneous* or *transposable* if it is a Generalised Interval System (GIS) as described by David Lewin [17].

In the following we consider the tone system $\mathrm{T} = (\mathbb{Z}, \delta, \mathbb{Z})$ with $\delta(s, t) = t - s$, and we fix a positive integer $\mathcal{O} \in \mathbb{Z}_+$, which we consider as an interval called *octave.* Let $\mathbb{Z}_\mathcal{O}$ denote the residue ring of integers modulo $\mathcal{O}$ and let $\mathrm{T}_\mathcal{O} := (\mathbb{Z}_\mathcal{O}, \delta_\mathcal{O}, \mathbb{Z}_\mathcal{O})$ be the algebraic tone system (where $\delta_\mathcal{O}(x, y)$ denotes the difference $y - x$ in $\mathbb{Z}_\mathcal{O}$). This system has different mathematical and musical properties than the underlying tone system T. Following the language of musicology, the elements of $\mathrm{T}_\mathcal{O}$ and are referred to as *chromas.*[2] For the whole system $\mathrm{T}_\mathcal{O}$ the article [14] had to introduce the new term a *chroma system.*[3] More specifically, we will refer to $\mathrm{T}_\mathcal{O}$ as *$\mathcal{O}$-tone equal tempered* chroma system, in short $\mathcal{O}$-tet. The most commonly used of these are the 12-tet (T_{12}) and the 7-tet (T_7).

The canonical group homomorphism $\phi_\mathcal{O} : \mathbb{Z} \to \mathbb{Z}_\mathcal{O}$ (which maps every integer x to its residue modulo $\mathcal{O}$, denoted by $x_\mathcal{O}$) maps every chord X in T to the chord $X_\mathcal{O} := \{x_\mathcal{O} \mid x \in X\}$ in $\mathrm{T}_\mathcal{O}$, which will be called the *harmony of X*.

Two chords or harmonies are equivalent if they are related by a transposition. The corresponding equivalence classes we refer to as *chordal forms* or *harmonic forms*, respectively. In short: transpositional equivalence classes of chords are chordal forms, which are mapped by octave identification to harmonic forms. On the other hand octave-identified chords are harmonies, which are mapped by transpositional identification to harmonic forms. If the signature of the tone system contains some pitch parameter, each chroma get a set of pitches assigned, the pitch classes. The latter ones are studied by Allen Forte [19]. A more complete and slightly generalised notion of his interval vector has been used in [14] in order to circumvent the combinatoric explosion that occurs while working with the order as discussed by Rudolf Wille (see below). One of the resulting lattices is shown in Fig. 1 on the right-hand side.

Consider two harmonic forms F_1 and F_2 of the same chroma system. We call F_1 a *harmonic subform* of F_2 iff there exist two harmonies $H_1 \in F_1$ and $H_2 \in F_2$ with $H_1 \subseteq H_2$. We denote this fact with $F_1 \sqsubseteq F_2$. This hierarchical

[2] The original German term introduced by Neumaier was *Tonigkeit.* The translation "chroma" is here used according to the terminology proposed in [18].

[3] Naming this concept got necessary by the parallel consideration of different chroma systems, which has been introduced in that article.

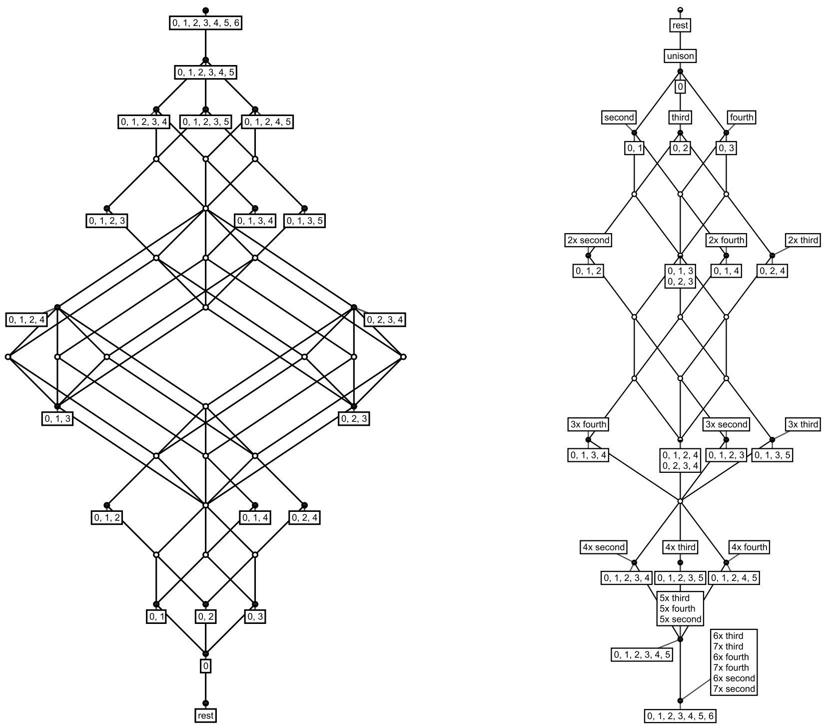

Fig. 1. Concept lattice of the order of harmonic forms (left) and concept lattice of the interval vectors (right) in the chroma system T_7

order has been studied by Rudolf Wille and other authors using FCA [5]. The corresponding concept lattice $\mathfrak{B}\big(H(T_7), H(T_7), \sqsubseteq\big)$ for the 7-tet T_7 can be seen in Fig. 1 on the left hand side. Besides the 20 nodes of harmonic forms it contains 22 additional nodes. These arise from the so called Dedekind-MacNeille completion [20], which transforms an ordered set (i.e., a set M together with a binary relation $\leq$ that is reflexive, transitive and antisymmetric) into a complete lattice (i.e., an ordered set where each subset has a least upper bound and greatest lower bound). The embedding of ordered sets into complete lattices allows us to make use of the much richer structure theory of the latter ones.

David Lewin [17] defines the interval function (*IFUNC*) to be another generalisation of Forte's interval vector. It maps a pair of chords or harmonies to the multiset of intervals between the individual tones or chromas between them. This function is invariant under the same transposition applied to both of its arguments. Thus, it is well-defined if both arguments denote the same harmonic form. This gives us a similar order principle as Forte's interval vector. Though our analysis is restricted to the harmonic properties of both approaches (in contrast to [21]), their difference has an impact on the concept lattices. An example is provided in the next section.

The fourth idea that will be discussed in this paper has been published by Reckziegel [22] and many others. The Mexican composer Julio Estrada [23] uses this scheme for his d1 theory. These authors partition the set of harmonic forms according to a description of them that is based on partitions of the octave.

	Pause	0	0, 1	0, 2	0, 1, 2	0, 1, 3	0, 1, 2, 3	0, 1, 2, 3, 4
Pause	×	×	×	×	×	×	×	×
0		×	×	×	×	×	×	×
0, 1			×		×	×	×	×
0, 2				×	×	×	×	×
0, 1, 2					×		×	×
0, 1, 3						×	×	×
0, 1, 2, 3							×	×
0, 1, 2, 3, 4								×

	unison (d = 0)	second (d = 1)	third (d = 2)
Pause			
0	1		
0, 1	2	1	
0, 2	2		1
0, 1, 2	3	2	1
0, 1, 3	3	1	2
0, 1, 2, 3	4	3	3
0, 1, 2, 3, 4	5	5	5

Fig. 2. Contexts of the chroma system T_5. Left: Set theoretic order, Right: Many-valued context describing chroma interval multisets of harmonic forms.

3 Border Case 1: Equally-Spaced Chroma Systems

In order to compare the different approaches we should agree upon a certain comparable set of chroma systems. It turns out that the 5-chroma-systems are sufficiently small, but large enough to show the main effects.

Let us start with the chroma system T_5. At first we should set up the necessary formal contexts. The classical order of harmonic forms must be described as well as the partition order by the order relation as it is shown in the left context of Fig. 2. The right context in this figure classifies harmonic forms according to the multiplicities of chroma intervals that are spanned between the chromas of the corresponding harmonic forms, i.e., we consider interval multisets.

The methods of Formal Concept Analysis are defined for binary relations. Nevertheless, it has been proved that any many-valued context can be represented by a one-valued context. The transformation from a many-valued into a single-valued context is called *scaling* [9, Chapter 1.3]. A set of attributes is replaced by the attribute set of a certain context, called *scale*. A scale is a context that describes how the incidence relation should be adapted. Its object set is the set of all possible value combinations of the attributes that shall be replaced. In the scaled context, the incidence relation is formed by the rows of the scale. Each value combination is replaced by the crosses of the corresponding row of the scale. Here, we use the two scales that are shown in Fig. 3: one simple scale that translates the multiset view into a simple set view (left) and one so-called ordinal scale that is the context of a linear order (right).

Forte's interval vectors count 2-harmonic forms. Each chroma system contains one 1-harmonic form and one 0-harmonic form (pause). Thus we scale the prime column with the left scale from Fig. 3 while the other columns are ordinally

Fig. 3. Scales used for scaling the chroma interval multiset context

	column name
0	
1	×
2	×
3	×
4	×
5	×

	1x	2x	3x	4x	5x
0					
1	×				
2	×	×			
3	×	×	×		
4	×	×	×	×	
5	×	×	×	×	×

	unison (d = 0)	second (d = 1)	2x second (d = 1)	3x second (d = 1)	4x second (d = 1)	5x second (d = 1)	third (d = 2)	2x third (d = 2)	3x third (d = 2)	4x third (d = 2)	5x third (d = 2)
Pause											
0	×										
0, 1	×	×									
0, 2	×						×				
0, 1, 2	×	×	×				×				
0, 1, 3	×	×					×	×			
0, 1, 2, 3	×	×	×	×			×	×	×		
0, 1, 2, 3, 4	×	×	×	×	×	×	×	×	×	×	×

	unison (d = 0)	2x unison (d = 0)	3x unison (d = 0)	4x unison (d = 0)	5x unison (d = 0)	second (d = 1)	2x second (d = 1)	3x second (d = 1)	4x second (d = 1)	5x second (d = 1)	third (d = 2)	2x third (d = 2)	3x third (d = 2)	4x third (d = 2)	5x third (d = 2)
Pause															
0	×														
0, 1	×	×				×									
0, 2	×	×									×				
0, 1, 2	×	×	×			×	×				×				
0, 1, 3	×	×	×			×					×	×			
0, 1, 2, 3	×	×	×	×		×	×	×			×	×	×		
0, 1, 2, 3, 4	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×

Fig. 4. Scaled contexts of the right hand context of Fig. 2. On the left: Ordinally scaled according to the interval vector [19]. On the right: Partly clarified and ordinally scaled according to the interval function [17].

scaled. The result is shown in the first context of Fig. 4. The second one in this figure describes Lewin's interval function applied to each of the harmonic forms. Since the count of an interval is always the same as the count of the complementary interval only one of the two is considered here. That does not change the structure that is generated in the concept lattice as redundant attributes can be clarified [9].

Fig. 5 shows the concept lattices of the order of harmonic forms (left) and the concept lattice of the interval vectors (middle) and the concept lattice of the interval function in this system (right). The partition lattice is the same as the order lattice in this special case. In contrast to Fig. 1, these concept lattices are very similar. If we consider the rightmost two of Fig. 5, we can see that counting the prime interval changes the structure of the lattice, although at first glance, it does not introduce new information. Obviously in all lattices, the k-harmonic forms structure the lattice somehow into layers, and so this information is implicitly available in all of them.

On the other hand, the separate chroma count introduces for each number k a top node, beneath which a copy of the supremum-reducible part of the layer above is introduced. Supremum-irreducible nodes are united with their lower neighbours. The reason is simple: every concept that can be described as union of certain concepts of the current layers is the concept that is described by the intersection of the attribute sets of these concepts. One of the common attributes of a layer is the count of chromas in each harmonic form whose object concept falls in that layer. Thus, for each set of concepts of that layer, the intersection of the attribute sets of the contained concepts shares the count attribute, even if the chroma set of the resulting harmonic form is actually smaller. Thus it must be a different concept.

These shadow copies also store another kind of information. Though the chromas of a harmony can be chosen arbitrarily, the interval vectors cannot. The structure of the chroma system imposes additional restrictions on them. Some of them are visualised when we introduce the count attributes. If the lower level without its object concepts is not a copy of the upper one there are implications

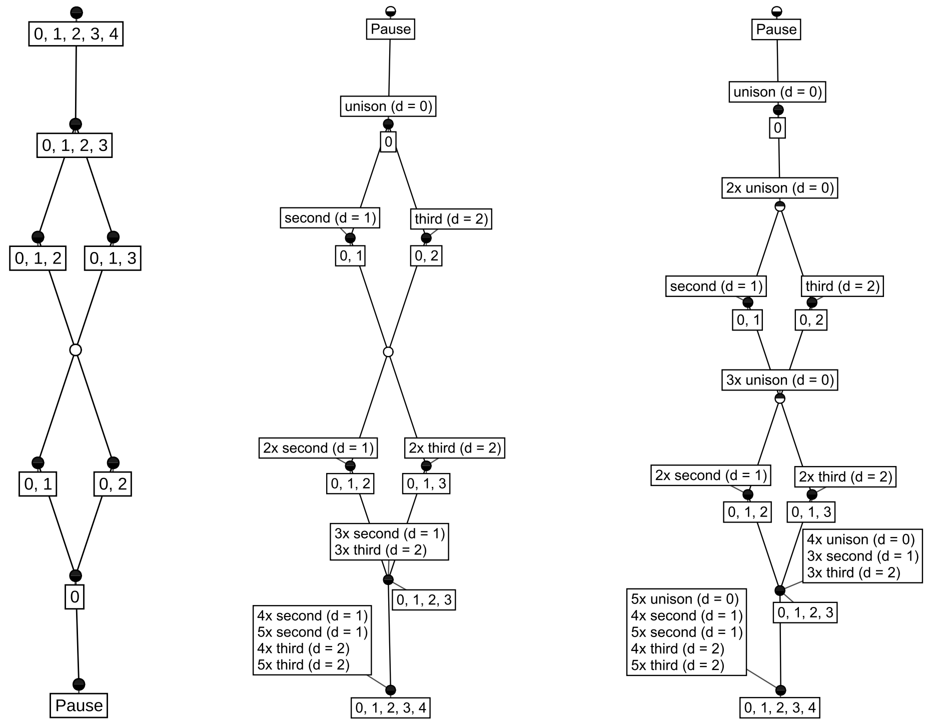

Fig. 5. Concept lattice of the order of harmonic forms (left) and concept lattice of the interval vectors (middle) and concept lattice of the interval function in the chroma system T_5. The left one is isomorphic to the lattice generated by the partition order.

of the following form: Interval vector x implies interval vector y where $y - x$ is a vector with only positive entries.

At some level k, an attribute concept of a count attribute exists which is beneath some concepts that have certain interval sets as subsets of their attribute sets (e.g., $3\times$unison in the right lattice of Fig. 5). In short: The attribute concept $k \times$ unison is not part of a separated chain as the earlier ones are. That tells us that every k-harmonic form contains at least one of the interval multisets above it (e.g., every 3-harmonic form contains at least one second or one third).

Despite of all these differences, the interval vector lattice can be embedded as an infimum-semilattice into the interval function lattice while object concepts can be mapped to object concepts and attribute concepts to attribute concepts. Similarly, there is a supremum-semilattice homomorphism from the lattice of the interval function to the one of interval vectors. Thus, the induced orders of the harmonic forms are structurally equivalent while both functions are conceptually different.

4 Non-equally-Spaced Chroma Systems

Though equally-spaced chroma systems and their obvious subsystems seem to be the most important ones, a complete theory of music must also be able to discuss

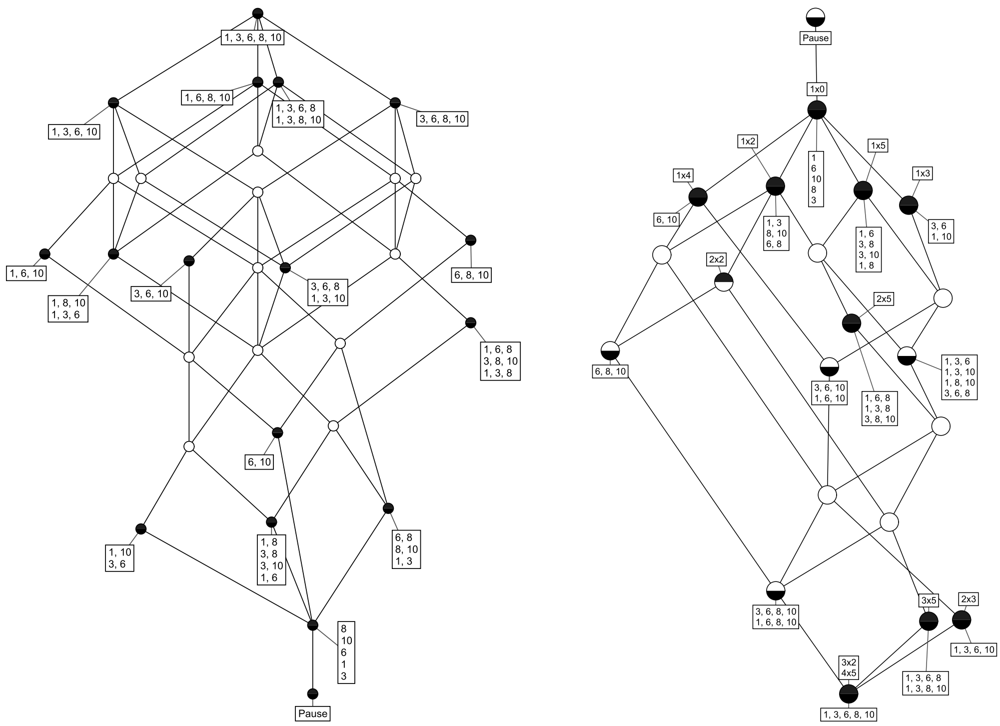

Fig. 6. Concept lattices of the pentatonic chroma subsystem of the equally-spaced 12 chroma system. Left: order, Right: interval vector lattice.

exotic tone systems that cannot be based on a sufficiently simple homogeneous tone system. Handling the general case is out of scope of this article, but before we head over to the other edge of finite chroma systems, we will provide a simple example of a more common non-homogeneous chroma system.

Fig. 6 shows two concept lattices of the pentatonic subsystem of the 12-tone chroma system T_{12}. As its construction is well-known to the community, we skip the construction of the underlying tone system. Instead of the harmonic forms, all harmonies are used. They are notated using the number of scale steps of the black keys in a 12-tone chroma system. Because the different harmonic forms do not share an object concept in the order lattice, representatives of the corresponding harmonic form are shown in each label. In this system, we have not only 2 but 4 different intervals, as can be seen at the nodes of the different 2-harmonic forms.

On the left hand side, we see the order of the harmonic forms. The right hand side shows the interval vector lattice. The interval function lattice of this system is visible in Fig. 7. To emphasise its structure the chroma count attribute concepts have been ordered separately. In this case, the level of 3-harmonic forms resembles the level of 2-harmonic forms very well with one exception: Every 3-harmonic form which contains a minor third (3) also contains a fourth (5).

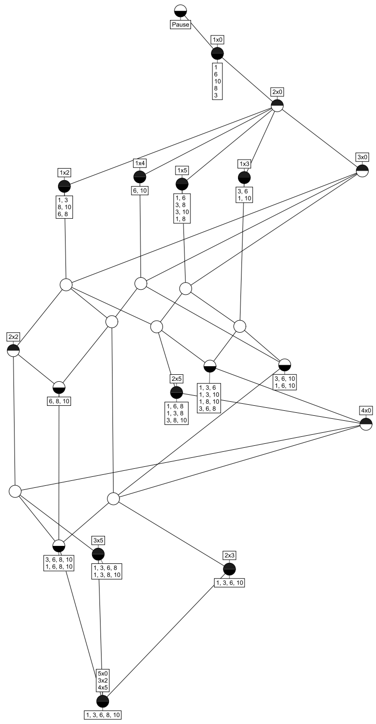

Fig. 7. Interval function concept lattice of the pentatonic chroma subsystem of the equally-spaced 12 chroma system

5 Border Case 2: Totally Asymmetric Chroma Systems

Although the usual approach to harmonic forms starts with equally-spaced chroma systems, we have a wide range of non-equally-spaced chroma systems available. If we want to get a better understanding of the structure of harmonic forms of certain chroma systems, we should risk a view to the other end of the playground: chroma systems with the maximum diversity of chroma intervals. On the one hand, this is further step towards the discussion of arbitrary tone systems. On the other hand, it opens a door to discuss links and differences between tone systems and chroma systems as discussed in Sect. 6.

Here, we leave the scope of well-known tone systems. Our first task is to find a tone system that generates a chroma system with the desired properties. We can construct finite totally-asymmetric chroma systems in the following way:

For any natural number $n \in \mathbb{N}$, let the mapping $p_n : \mathbb{Z} \to \mathbb{Z}$ and the mapping $\beta_n : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ be defined by

$$p_n(t) := (2^n - 1)\left\lfloor \frac{t}{n} \right\rfloor + 2^{t-n\lfloor \frac{t}{n} \rfloor} - 1 \quad \text{and} \quad \beta_n(t_1, t_2) := p(t_2) - p(t_1) \; . \qquad (3)$$

Then we get $\beta_n(n-1, n) = 2^n - 1 - (2^{(n-1)} - 1) = 2^{n-1}$. Obviously, for all integers $t_1, t_2, t_3 \in \mathbb{Z}$, the conditions $\beta_n(t_1, t_1) = 0$ and $\beta_n(t_1, t_2) + \beta_n(t_2, t_3) = \beta_n(t_1, t_3)$ hold. Thus, $\mathrm{B}_n := (\mathbb{Z}, \beta_n, \mathbb{Z})$ is a tone system. Let's call it the *binary number tone system*.

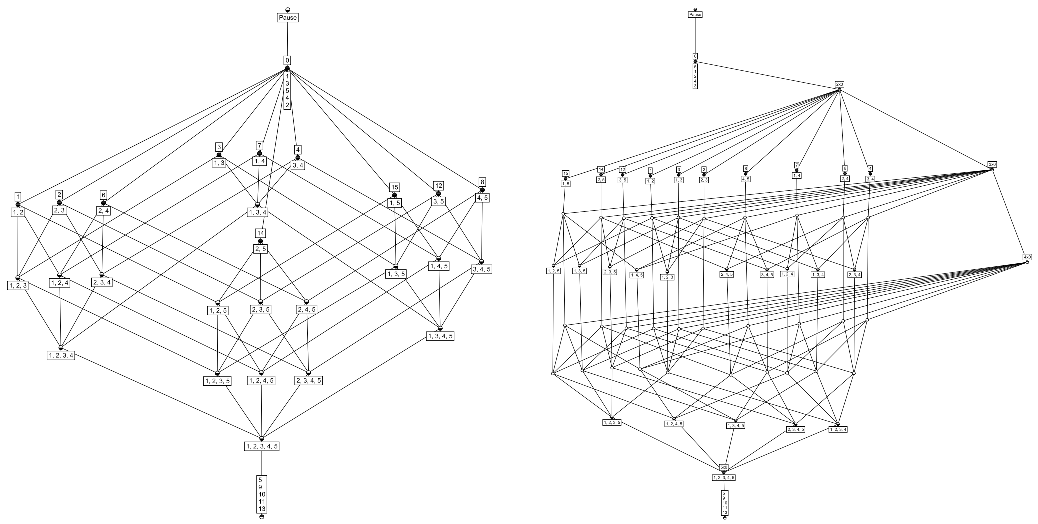

Fig. 8. Interval vector lattice of the harmonic forms of the chroma system of B_5 (left) and interval function lattice of the same system (right)

Obviously $\beta_n(t, t + n \cdot k) = k(2^n - 1)$ for any tone $t \in \mathbb{Z}$. Thus, the interval $2^n - 1$ can serve as an octave in B_n. Then the chromas and chroma intervals have the form

$$[t] = t + n\mathbb{Z} = \{t + kn \mid k \in \mathbb{Z}\} \text{ and} \tag{4}$$

$$[\beta_n(t_1, t_2)] = \beta_n\big[[t_1], [t_2]\big] = \beta_n(t_1, t_2) + (2^n - 1)\mathbb{Z}\ . \tag{5}$$

A straightforward calculation shows that for each integer $0 \leq k < n$, the formula $[\beta_n(k, k+1)] = 2^k + (2^n - 1)\mathbb{Z}$. Thus, summing up neighbouring intervals shows that $[\beta_n(t_1, t_2)] = [\beta_n(t_3, t_4)]$ iff $[t_1] = [t_3]$ and $[t_2] = [t_4]$. This means that no harmony can be transposed into a different one if it contains more than one chroma. As the set of harmonies is the power set of the chroma set, the set of harmonic forms is a Boolean lattice whose atoms are collapsed into one node.

As all multiplicities of non-zero chroma intervals are never larger than 1, each harmonic form can be uniquely described by the chroma intervals it spans. Thus, the interval content lattice is anti-isomorphic (isomorphic according to the dual order) to the order of the harmonic forms of B_n. For the same reason, the partition lattice is isomorphic to the interval vector lattice. Both can be seen in Fig. 8.

6 Conclusion and Further Research Topics

For an arbitrary chroma system, each chroma can be transposed into any other chroma, as every tone of the first chroma can be transposed to every tone of the second chroma by an interval. As we have seen, the mapping that assigns harmonies of the tone system B_n to harmonic forms of the same system is injective on the subset of the harmonies that do not consist of exactly one chroma. Thus, for any tone system T with n different chromas, there exists a mapping from the set of the chromas of B_n to the chromas of T that maps the set of harmonic forms of B_n to the set of harmonic forms of T.

As the cardinalities of chromas of B_n and any tone system T with n chromas coincide, we can use a conceptually different way to construct harmonic forms: First the harmonies of T are mapped to those of B_n and then the harmonic forms of the latter system are mapped back to T. This approach allows for splitting the last mentioned mapping into parts (e.g., identifying intervals step by step), provided that for each mapping φ, the following conditions are met:

1. The mapping φ acts on tones, mapping chromas to chromas.
2. The implied action of φ on chords and harmonic forms commutes with the octave identification and transposition such that for a subset A of the tones of the domain, the harmonic form of the range of φ that is generated by the chordal form of A coincides with the harmonic form that is formed by the image of φ of the harmony of A.
3. There is a corresponding mapping ψ between the interval groups of the tone systems, such that for any two tones t_1, t_2 of the domain tone system, the equation $\left[\psi\big(\delta(t_1,t_2)\big)\right] = \left[\delta\big(\varphi(t_1),\varphi(t_2)\right]$ holds.

We suggest calling such mappings *harmonic homomorphisms*. These harmonic homomorphisms are linked to the underlying tone systems. This very well matches musical reality where tones and pitches are easily performed while chromas and pitch classes are hard to generate. It is an open question to describe the generic harmonic homomorphisms, i.e., homomorphisms that cannot be split into two parts such that neither of them is a harmonic isomorphism. The changes of the concept lattice types that have been discussed in this paper under such generic harmonic homomorphisms are expected to give new insights into the structure of lattices which are generated by arbitrary chroma systems.

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