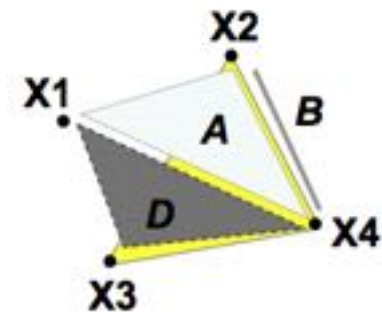
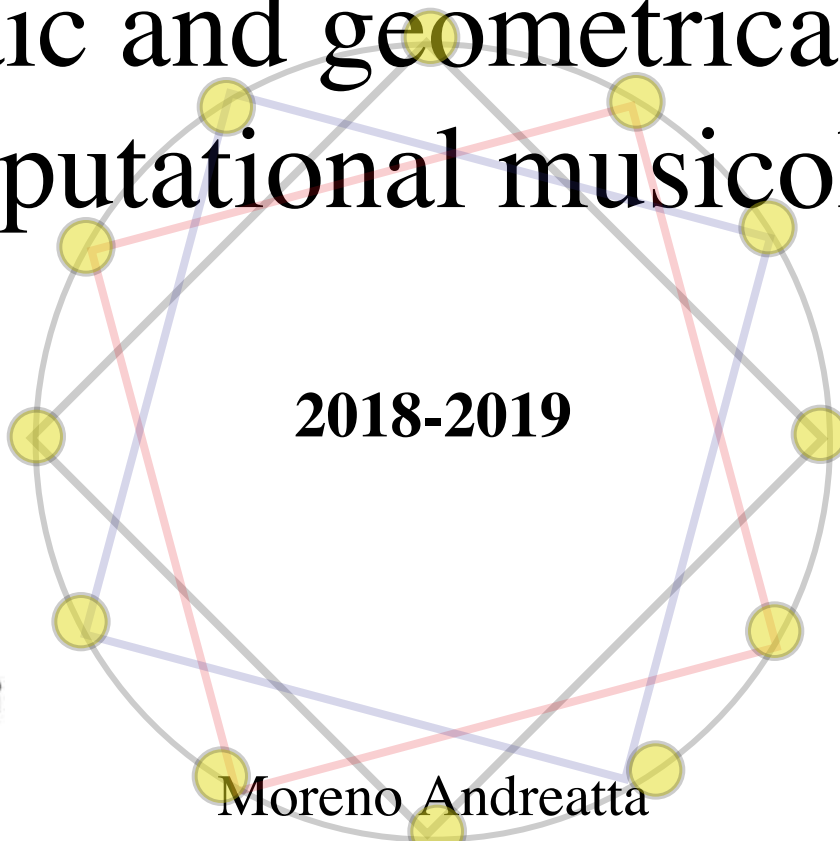
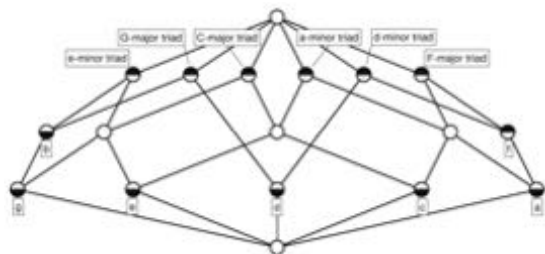




Algebraic and geometrical models in computational musicology (I)

2018-2019



Moreno Andreatta

Equipe Représentations Musicales
IRCAM/CNRS/UPMC

<http://www.ircam.fr/repmus.html>

The SMIR Project: Structural Music Information Research



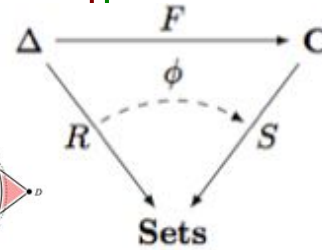
Signal-based
Music
Information
Retrieval



Oleg Berg

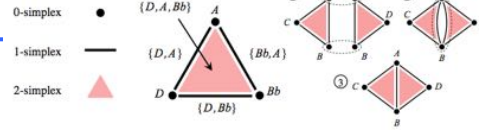
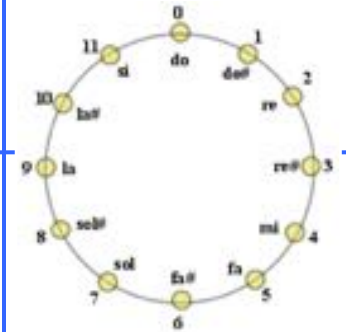
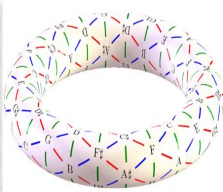
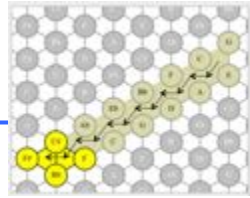
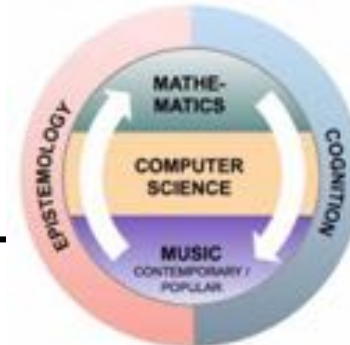


Algebraic
models



Topological
models

Categorical
models



Computational models

Cognitive models

Structural Symbolic Music
Information Research

<http://repmus.ircam.fr/moreno/smir>

Computational Musicology in academic research

Conferences of the SMCM:

- 2007 Technische Universität (Berlin, Allemagne)
- 2009 Yale University (New Haven, USA)
- 2011 IRCAM (Paris, France)
- 2013 McGill University (Canada)
- 2015 Queen Mary University (Londres)
- 2017 UNAM (Mexico City)
- **2019 Universidad Complutense de Madrid (Spain)**



Official Journal and MC code (00A65: Mathematics and Music)

- *Journal of Mathematics and Music*, Taylor & Francis
(Editors: Th. Fiore, C. Callender | Associate eds.: E. Amiot, J. Yust)



Books Series:

- *Computational Music Sciences Series*, Springer (G. Mazzola & M. Andreatta eds. – 12 books published (since 2009))
- *Collection Musique/Sciences*, Ircam-Delatour France (J.-M. Bardez & M. Andreatta dir. – 16 books published (since 2006))



Some examples of PhD on maths & music

- **Alessandro Ratoci**, *Vers l'hybridation stylistique assistée par ordinateur*, PhD in music **composition & research**, Sorbonne University / IRCAM (cosupervised with Laurent Cugny)
- **Sonia Cannas**, *Représentations géométriques et formalisations algébriques en musicologie computationnelle*, PhD in **maths** in cotutelle agreement, **University of Pavia** (L. Pernazza) / **Université de Strasbourg** (A. Papadopoulos & M. Andreatta), 2018.
- **Grégoire Genuys**, *Théorie de l'homométrie et musique*, PhD in **maths**, **Sorbonne University** / IRCAM (cosupervised with Jean-Paul Allouche), 2017.
- **Hélianthe Caure**, *Pavages en musique et conjectures ouvertes en mathématiques*, PhD in **computer science**, **Sorbonne University** (cosupervised with Jean-Paul Allouche), 2016.
- **Mattia Bergomi**, *Dynamical and topological tools for (modern) music analysis*, PhD in **maths** in a cotutelle agreement Sorbonne University / University of Milan (cosupervised with Goffredo Haus, 2015).
- **Charles De Paiva**, *Systèmes complexes et informatique musicale*, thèse de doctorat, Programme Doctoral International « Modélisation des Systèmes Complexes », PhD in **musicology** in a cotutelle agreement, **Sorbonne University** / **UNICAMP**, Brésil, 2016.
- **John Mandereau**, *Des systèmes d'Intervalles Généralisés aux Systèmes Evolutifs à Mémoire : aspects théoriques et computationnels*, thèse de doctorat en mathématiques, PhD in cotutelle agreement **University of Pisa** / **Sorbonne University** (cosupervised with F. Acquistapace). PhD in **maths** (aborted).
- **Louis Bigo**, *Représentation symboliques musicales et calcul spatial*, PhD in **computer science**, **University of Paris Est Créteil** / **IRCAM**, 2013 (cosupervised with Olivier Michel and Antoine Spicher)
- **Emmanuel Amiot**, *Modèles algébriques et algorithmiques pour la formalisation mathématique de structures musicales*, PhD in, **Sorbonne University** / **IRCAM**, 2010 (cosupervised with Carlos Agon) **computer science**
- **Yun-Kang Ahn**, *L'analyse musicale computationnelle*, PhD in **computer science**, **Sorbonne University** / **IRCAM**, 2009 (cosupervised with Carlos Agon)



UNIVERSITÀ DI PISA



The double movement of a 'mathemusal' activity

MATHEMATICS

Mathematical statement

generalisation

General theorem

formalisation

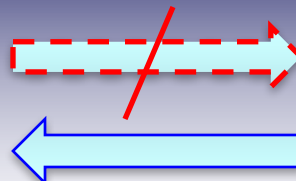


OpenMusic

application

MUSIC

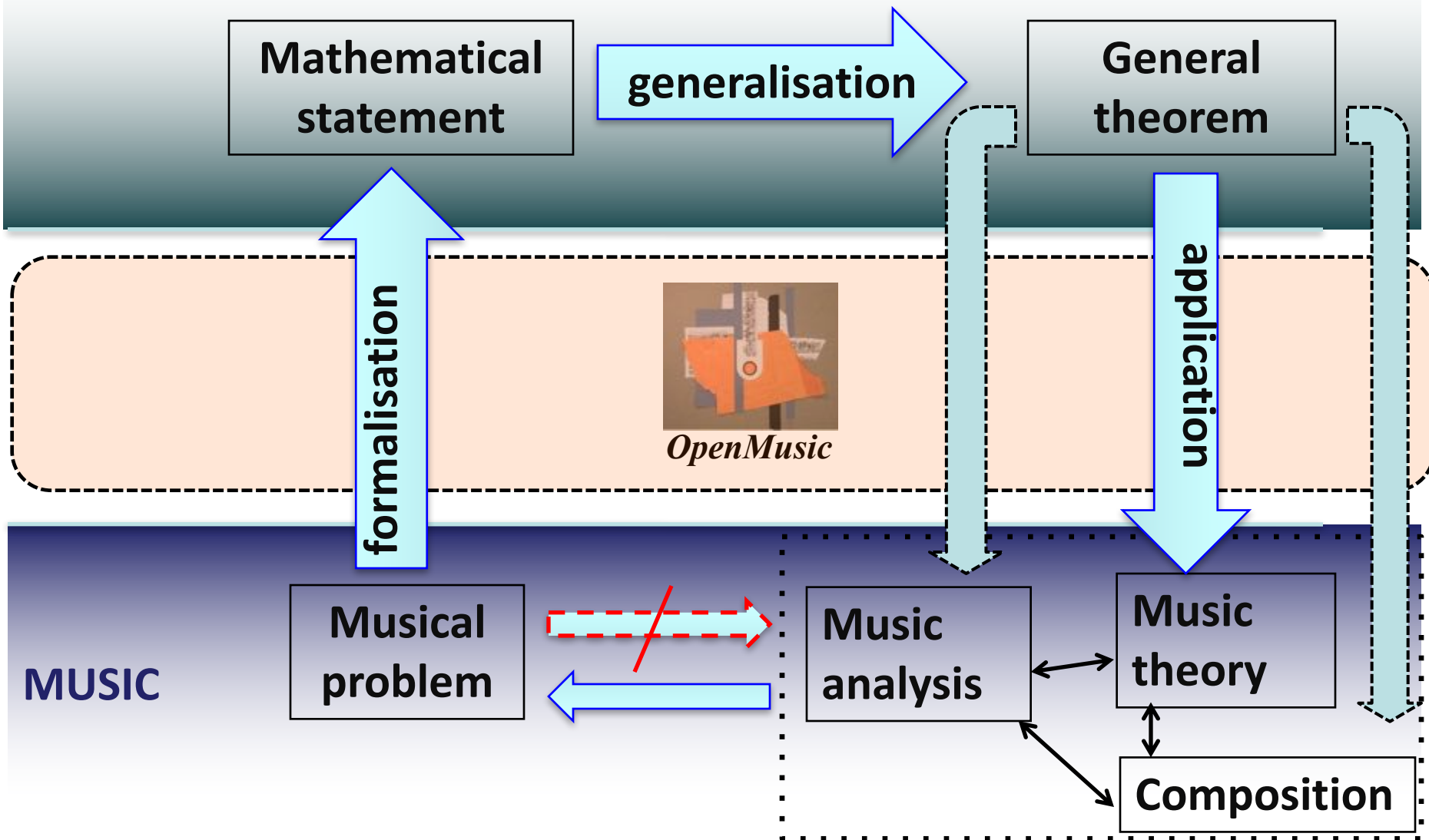
Musical problem



Music analysis

Music theory

Composition



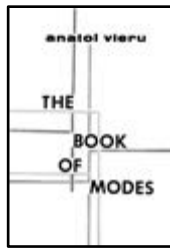
Periodic sequences and finite difference calculus

$$Df(x) = f(x) - f(x-1)$$

$$\begin{aligned}
 f &= 7 \ 11 \ 10 \ 11 \ 7 \ 2 \ 7 \ 11 \ 10 \ 11 \ 7 \ 2 \ 7 \ 11 \dots \\
 Df &= 4 \ 11 \ 1 \ 8 \ 7 \ 5 \ 4 \ 11 \ 1 \ 8 \ 7 \ 5 \ 4 \ 11 \dots \\
 D^2 f &= 7 \ 2 \ 7 \ 11 \ 10 \ 11 \ 7 \ 2 \ 7 \ 11 \ 10 \ 11 \dots \\
 D^3 f &= 7 \ 5 \ 4 \ 11 \ 1 \ 8 \ 7 \ 5 \ 4 \ 11 \ 1 \ 8 \dots \\
 D^k f &= \dots\dots
 \end{aligned}$$



Anatol Vieru



dalassima

mf mp pp mp p mf mp pp pp

V	0	3	8	7	11	0	11	10	6	9	0	9	1	2	9	8	4	3	6
VIII	0	0	0	0	3	3	7	2	0	0	0	6	3	3	3	4	8	0	0
IV	3	3	4	4	1	11	11	8	3	3	9	4	1	7	11	8	11	3	9
IX	0	0	0	0	0	3	6	[1]	3	3	3	3	9	0	3	6	[10]	6	6
IV	0	10	3	9	10	0	9	7	0	6	7	9	6	4	9	3	4	6	3

Anatol Vieru: *Zone d'oubli* pour alto (1973)

Reducible and reproducible sequences

$$\begin{array}{rcl}
 f & = & 11 \ 6 \ 7 \ 2 \ 3 \ 10 \ 11 \ 6 \ \dots \\
 Df & = & \begin{array}{c} \diagdown \ \diagup \\ 7 \ 1 \ 7 \ 1 \ 7 \ 1 \ 7 \ 1 \ \dots \end{array} \\
 D^2f & = & \begin{array}{c} \diagdown \ \diagup \\ 6 \ 6 \ 6 \ 6 \ 6 \ \dots \end{array} \\
 D^4f & = & \begin{array}{c} \diagdown \ \diagup \\ 0 \ 0 \ 0 \end{array}
 \end{array}$$

Reducible sequences:
 $\exists k \geq 1$ such that
 $D^k f = 0$

$$\begin{array}{rcl}
 f & = & 7 \ 11 \ 10 \ 11 \ 7 \ 2 \ 7 \ 11 \ \dots \\
 Df & = & \begin{array}{c} \diagdown \ \diagup \\ 4 \ 11 \ 1 \ 8 \ 7 \ 5 \ 4 \ 11 \ 1 \ \dots \end{array} \\
 D^2f & = & \begin{array}{c} \diagdown \ \diagup \\ 7 \ 2 \ 7 \ 11 \ 10 \ 11 \ 7 \ 2 \ 7 \ \dots \end{array} \\
 D^3f & = & \begin{array}{c} \diagdown \ \diagup \\ 7 \ 5 \ 4 \ 11 \ 1 \ 8 \ 7 \ 5 \ 4 \ 11 \ 1 \ 8 \ \dots \end{array} \\
 D^4f & = & 10 \ 11 \ 7 \ 2 \ 7 \ 11 \ 10 \ 11 \ \dots \\
 D^5f & = & 1 \ 8 \ 7 \ 5 \ 4 \ 11 \ 1 \ 8 \ \dots \\
 D^6f & = & 7 \ 11 \ 10 \ 11 \ 7 \ 2 \ 7 \ 11 \ \dots
 \end{array}$$

Reproducible sequences:
 $\exists k \geq 1$ such that
 $D^k f = f$

A decomposition property of any periodic sequence

$$Df(x) = f(x) - f(x-1).$$

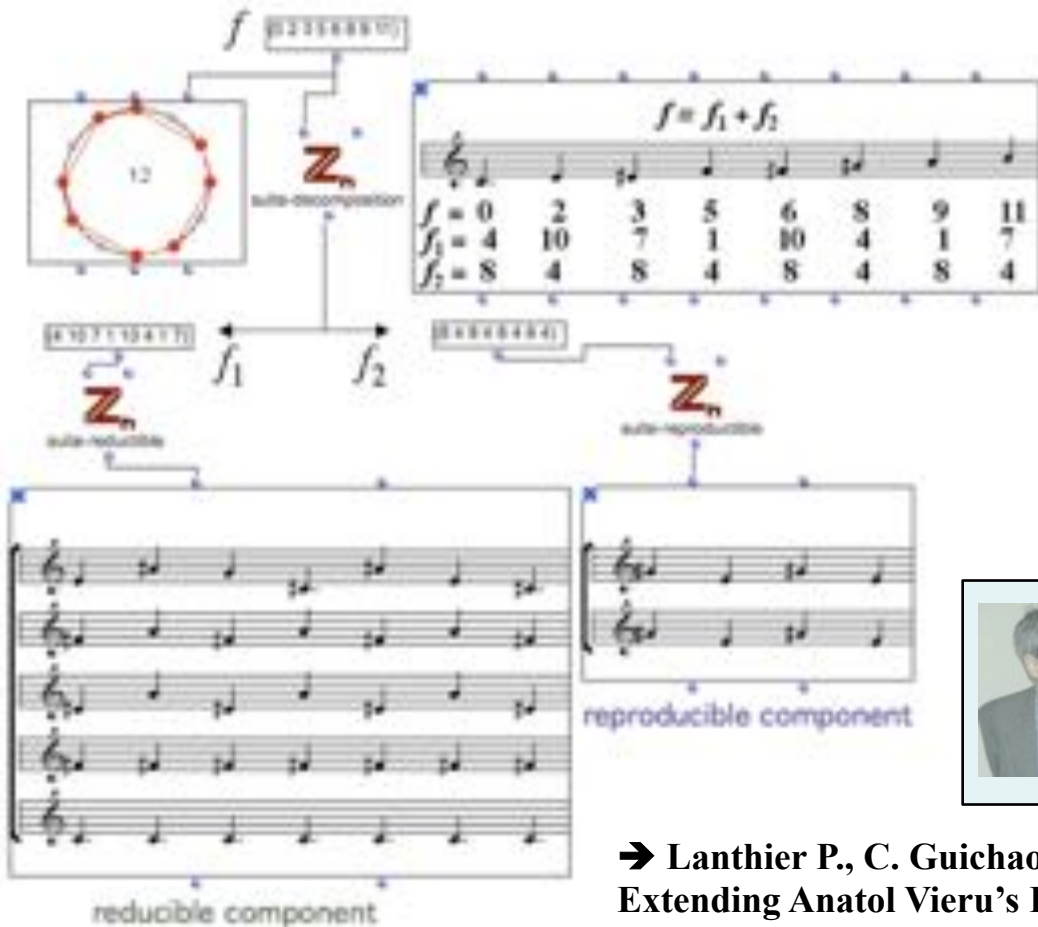
7 11 10 11 7 2 7 11 10 11 7 2 7 11...
 4 11 1 8 7 5 4 11 1 8 7 5 4 11...
 7 2 7 11 10 11 7 2 7 11 10 11...
 7 5 4 11 1 8 7 5 4 11 1 8...

V	0	3	8	7	11	0	11	10	6	9	0	9	8	4	3	6
VIII	0	0	0	0	0	3	7	2	6	0	0	4	4	8	3	0
IV	3	3	4	4	4	1	11	8	3	3	3	1	3	11	11	3
IX	0	0	4	4	1	0	11	8	3	3	3	1	3	11	11	3
IV	0	10	3	9	10	0	9	1	0	0	7	9	3	4	3	3

Anatol Vieru: *Zone d'oubli* for viola (1973)

Reducible sequences:
 $\exists k \geq 1$ such that $D^k f = 0$

Reproducible sequences:
 $\exists k \geq 1$ such that $D^k f = f$



• **Decomposition theorem**
 (Vuza & Andreatta, *Tatra M.*, 2001)
 Every periodic sequence f can be decomposed in a unique way as a sum $f_1 + f_2$ of a reducible sequence f_1 and a reproducible sequence f_2



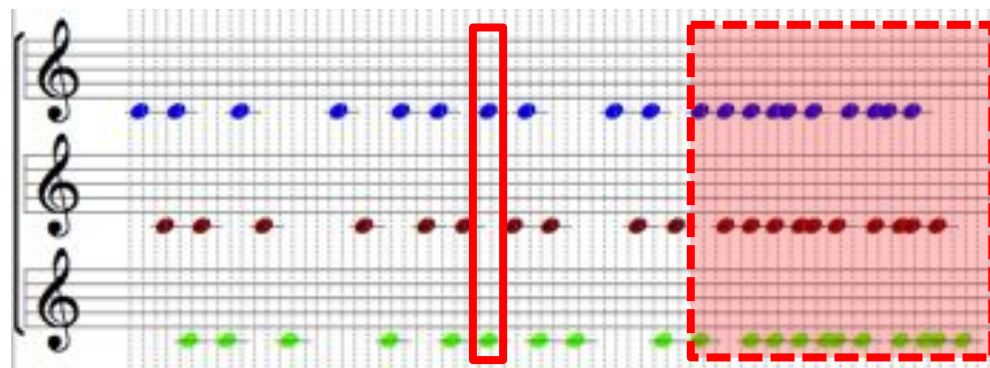
D. Vuza, M. Andreatta (2001), « On some properties of periodic sequences in Anatol Vieru's modal theory », *Tatra Mountains Mathematical Publications*, Vol. 23, p. 1-15

➔ Lanthier P., C. Guichaoua, M. Andreatta (2019), "Reinterpreting and Extending Anatol Vieru's Periodic Sequences Through the Cellular Automata Formalisms", in M. Montiel et al. (eds), *Proc. MCM 2019*, Springer.

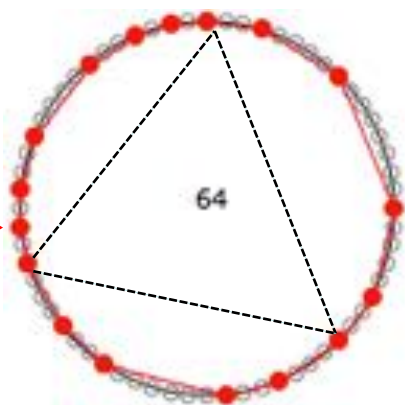
Periodic rhythmic sequences and tiling canons



Harawi (1945)



Harawi: rhythmic reduction



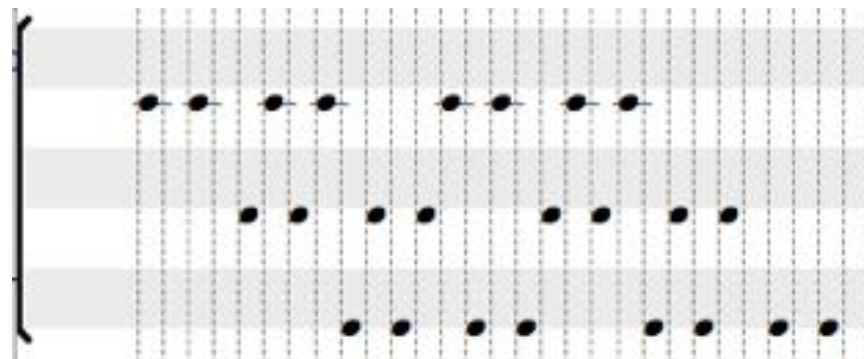
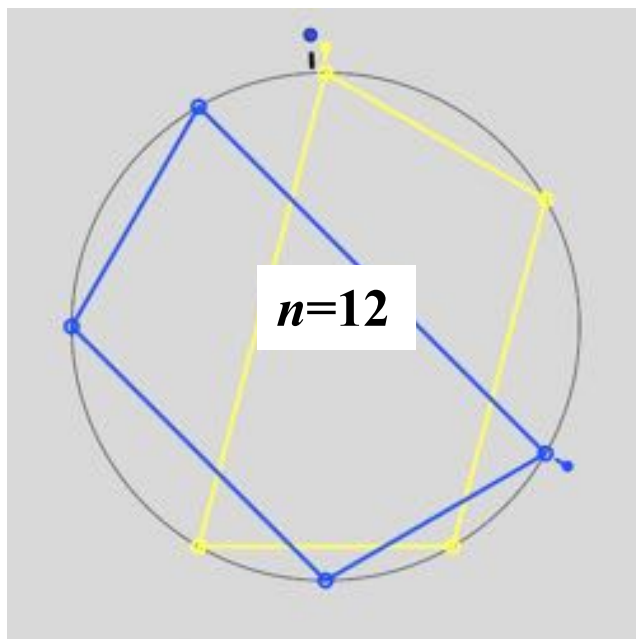
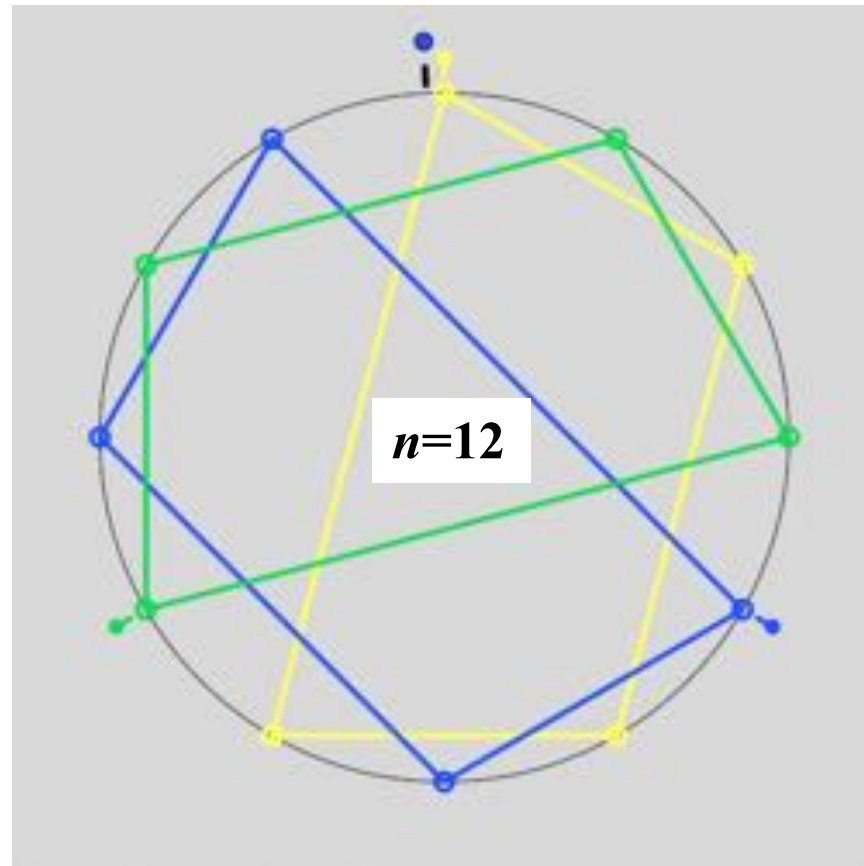
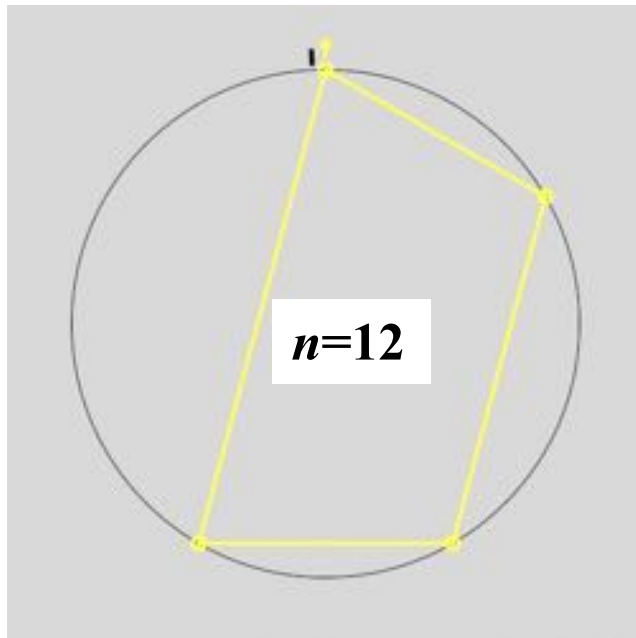
« ...il résulte de tout cela que les différentes sonorités se mélangent ou s'opposent de manières très diverses, **jamais au même moment ni au même endroit** [...]. C'est du désordre organisé »

O. Messiaen: *Traité de Rythme, de Couleur et d'Ornithologie*, tome 2, Alphonse Leduc, 1992.

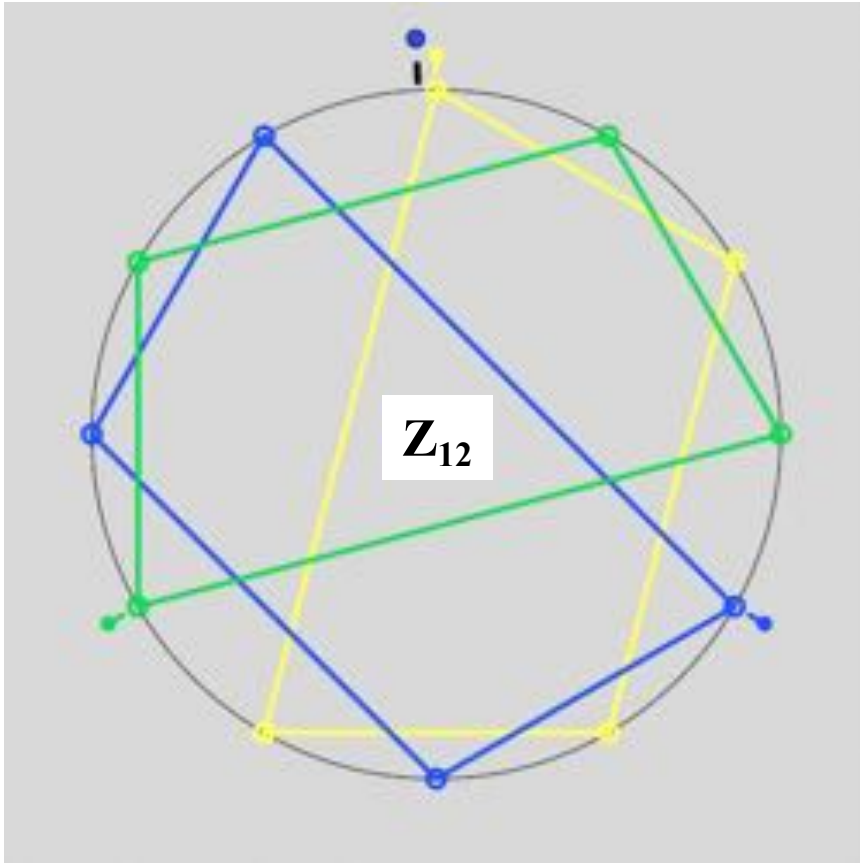


Olivier Messiaen

Tiling the time axis with translates of one tile



Formalizing the tiling process as a direct sum of subsets



$$A_1 = \{0, 2, 5, 7\}$$

$T_4 \downarrow$

$$A_2 = \{4, 6, 9, 11\}$$

$T_4 \downarrow$

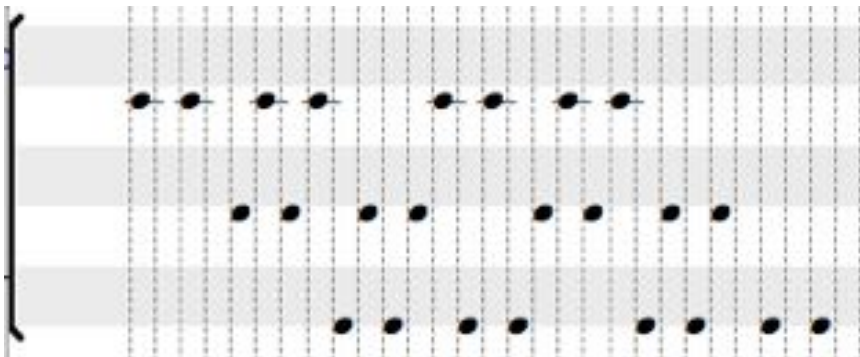
$$A_3 = \{8, 10, 1, 3\}$$

$$Z_{12} = A_1 \cup A_2 \cup A_3$$

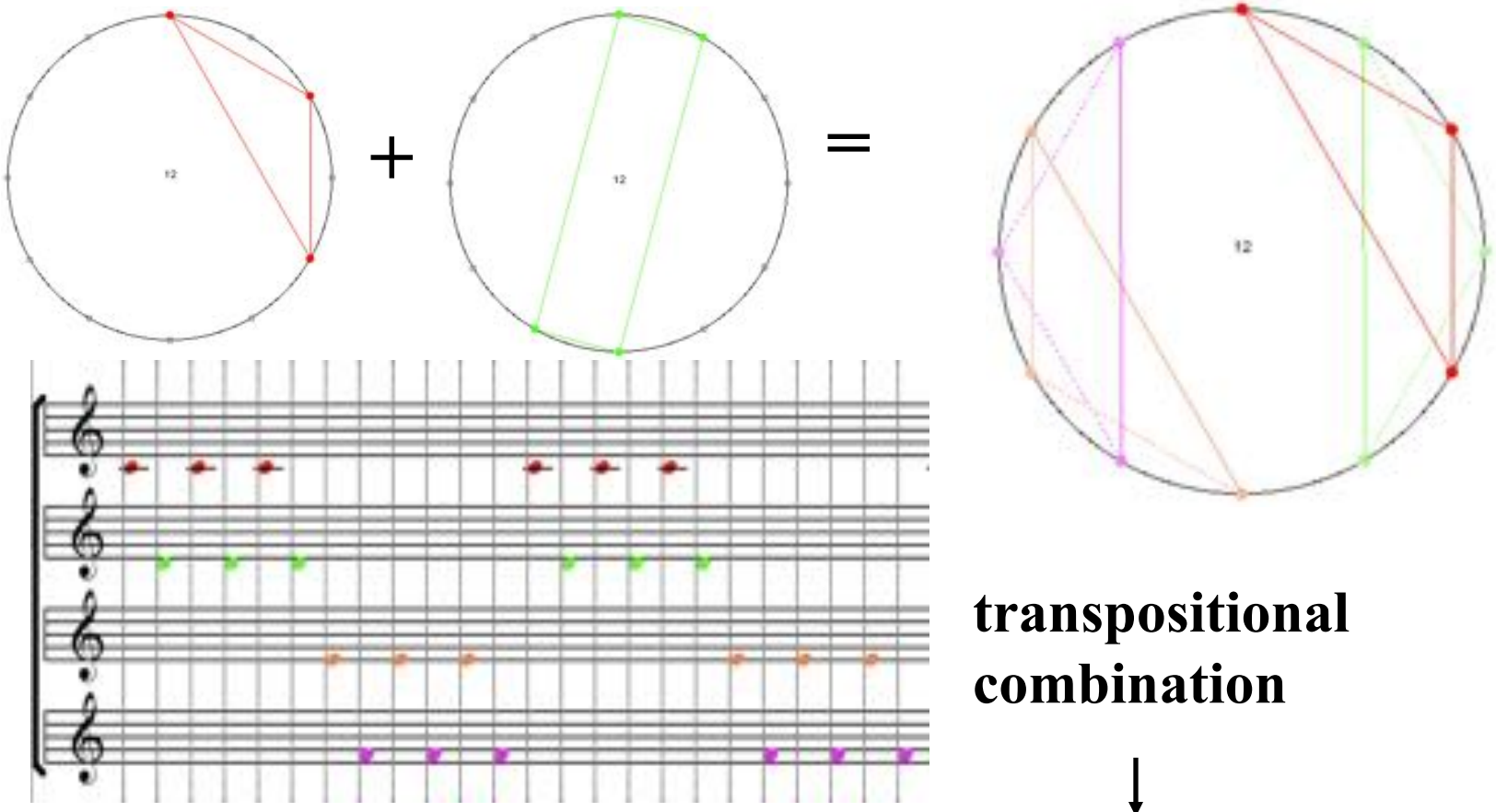
$$Z_{12} = A \oplus B$$

$$A = \{0, 2, 5, 7\}$$

$$B = \{0, 4, 8\}$$



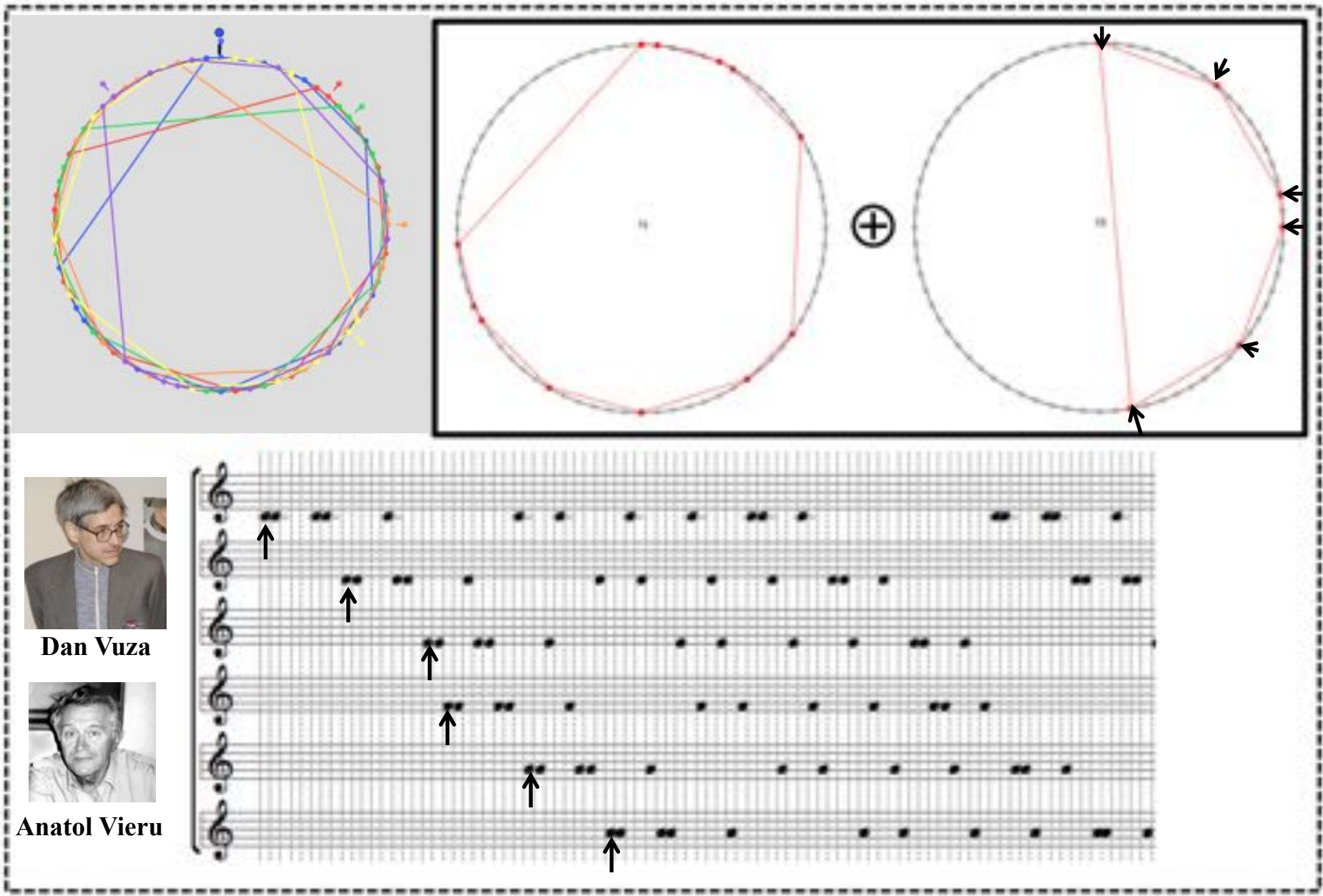
Rhythmic tiling canons with no regular entries



$$\{0,2,4\} \oplus \{0,1,6,7\} = \mathbb{Z}_{12} = (2 \ 2 \ 8) \bullet (1 \ 5 \ 1 \ 5)$$

One of the two factors is a Messiaen's mode of limited transposition

Aperiodic Rhythmic Tiling Canons (Vuza Canons)



The image illustrates the concept of Aperiodic Rhythmic Tiling Canons (Vuza Canons) through three main components:

- Graphical Representation:** A circle with multiple colored chords (red, green, blue, yellow, purple) connecting various points on its circumference, representing the structure of a canon.
- Direct Sum Representation:** Two circles are shown side-by-side, separated by a plus sign (\oplus). The left circle has a red chord, and the right circle has a red chord with arrows pointing to its endpoints, illustrating the direct sum of two canons.
- Musical Score:** A musical score consisting of five staves, each with a treble clef. The score shows a sequence of notes with upward-pointing arrows indicating the start of each canon.

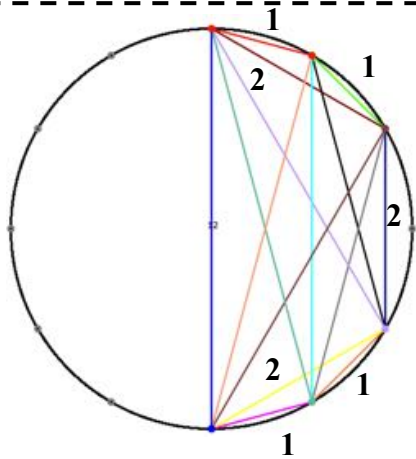
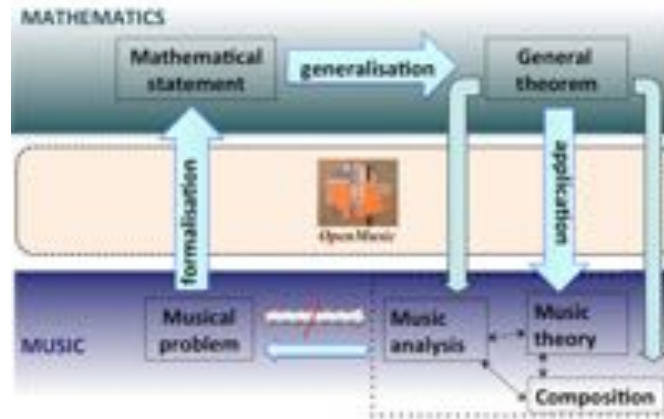
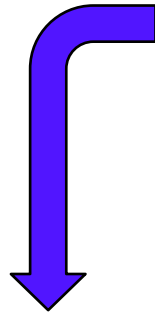


Dan Vuza

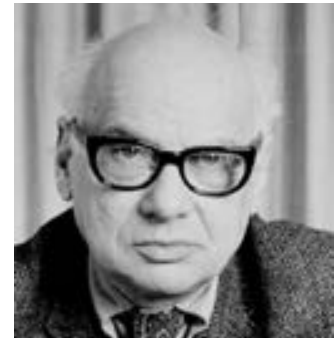
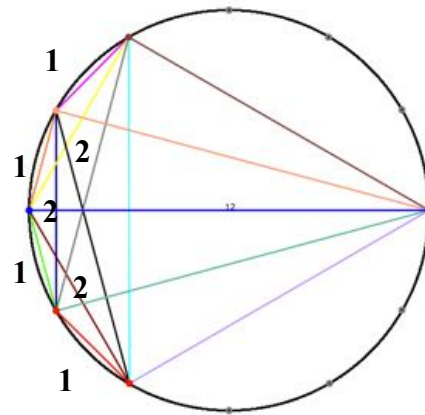


Anatol Vieru

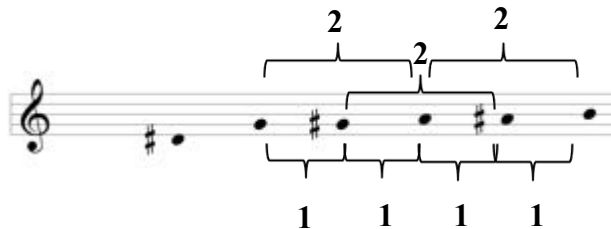
A historical example of “mathemusal” problem



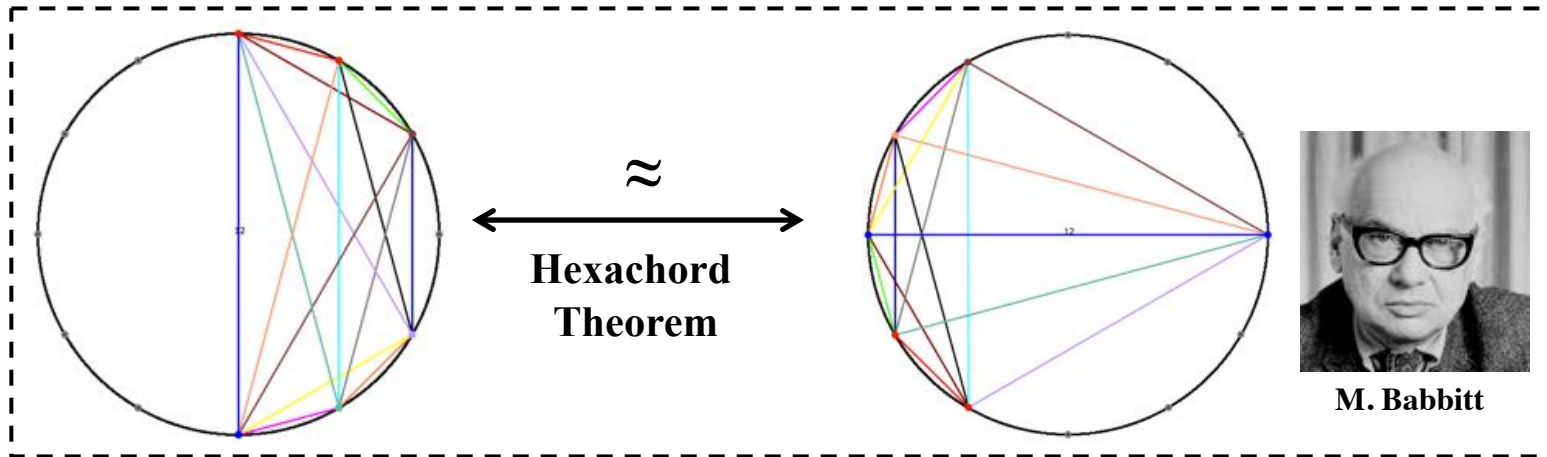
HOMOMETRY
←→
**Hexachord
Theorem**



M. Babbitt



The shortest proof of Babbitt's Theorem?



$$IC_A = [4, 3, 2, 3, 2, 1] = [4, 3, 2, 3, 2, 1] = IC_A,$$

$$IC_A(k) = (1_A \star 1_{-A})(k)$$

$$\mathcal{F}_A : t \mapsto \sum_{k \in A} e^{-2i\pi kt/c}$$

$$\forall k \mathcal{F}(IC_{\mathbb{Z}_c \setminus A})(k) = \mathcal{F}(IC_A)(k)$$



E. Amiot : « Une preuve élégante du théorème de Babbitt par transformée de Fourier discrète », *Quadrature*, 61, 2006.

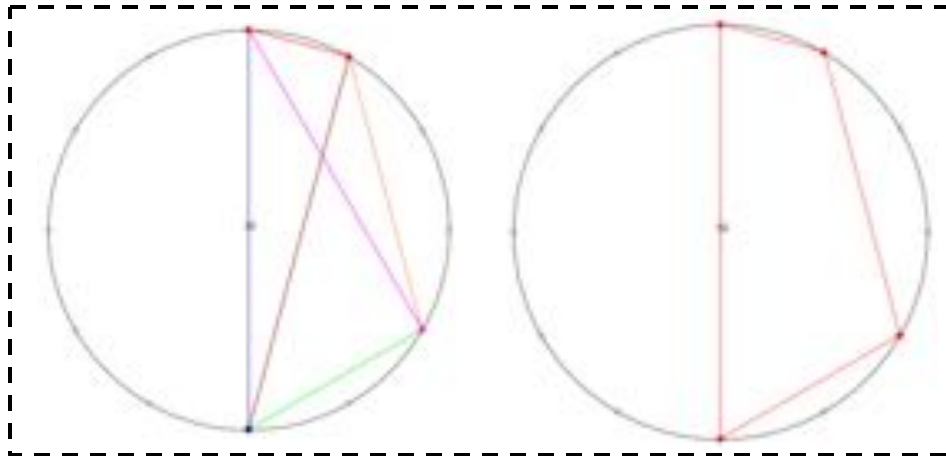
Z-relation, homometry and phase retrieval problem

- Two sets are Z-related if they have the same module of the DFT

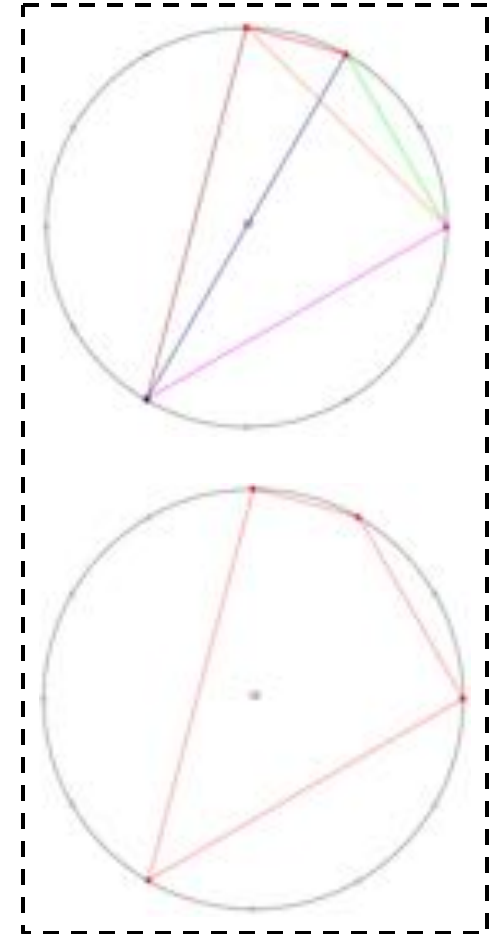
$$IC_A(k) = (1_A \star 1_{-A})(k)$$

$$\mathcal{F}_A : t \mapsto \sum_{k \in A} e^{-2i\pi kt/c}$$

$$\mathcal{F}(IC_A) = \mathcal{F}_A \times \mathcal{F}_{-A} = |\mathcal{F}_A|^2$$



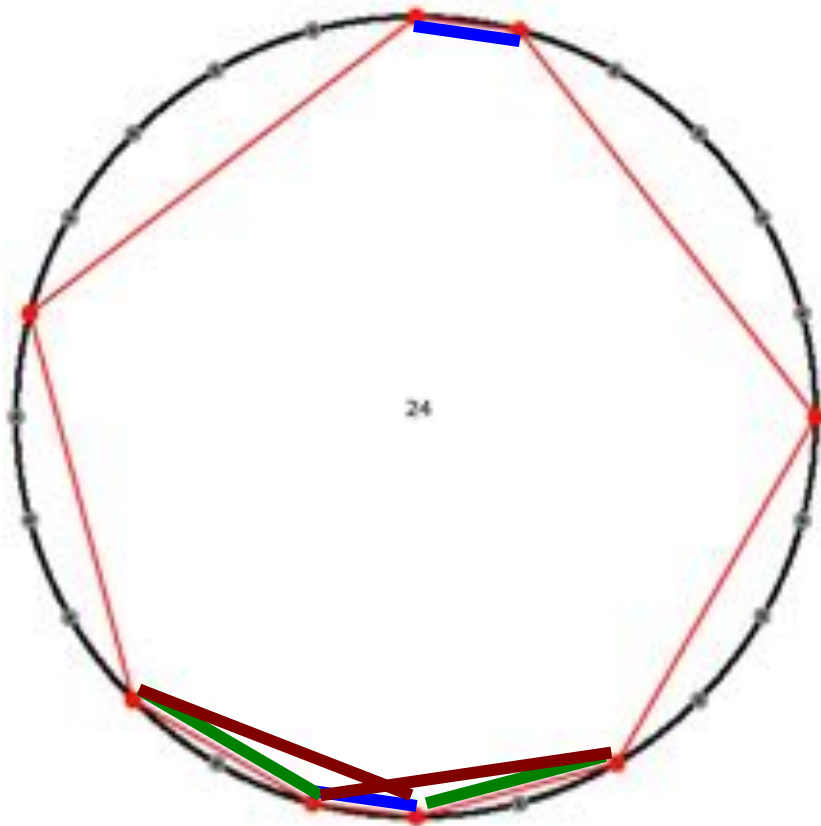
Z-relation
↔
homometry



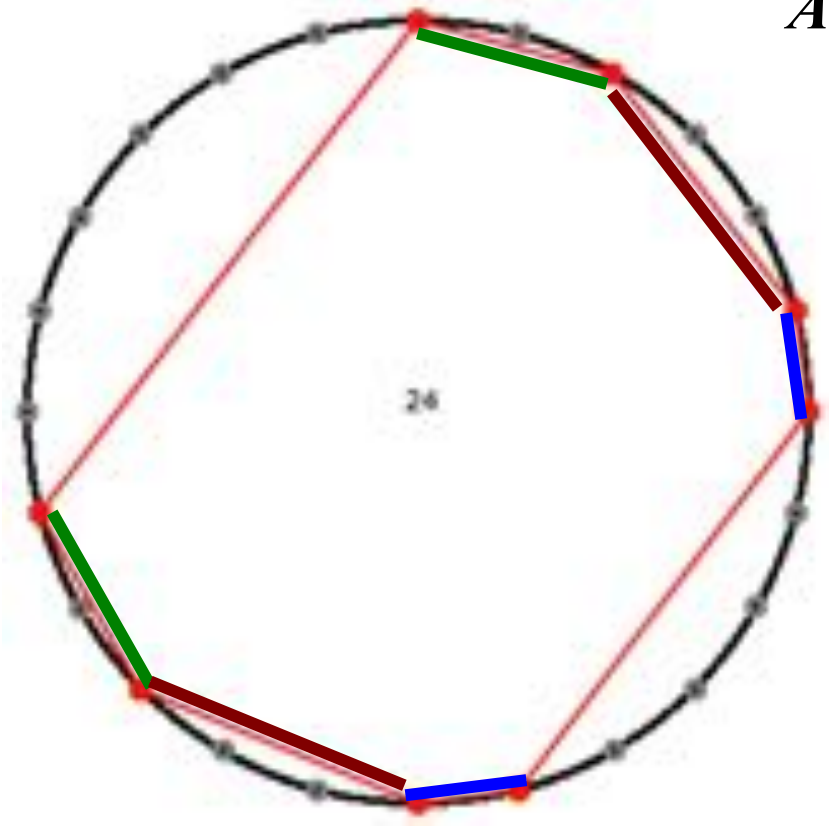
- Mandereau J., D. Ghisi, E. Amiot, M. Andreatta, C. Agon, (2011), « Z-relation and homometry in musical distributions », *Journal of Mathematics and Music*, vol. 5, n° 2, p. 83-98.
- Mandereau J, D. Ghisi, E. Amiot, M. Andreatta, C. Agon (2011), « Discrete phase retrieval in musical structures », *Journal of Mathematics and Music*, vol. 5, n° 2, p. 99-116.

Z-relation (music) and homometry (cristallography)

A



A'



$$IC_A(k) = (1_A \star 1_{-A})(k)$$

$A \sim A'$

Z-relation
 \longleftrightarrow
 Homometry

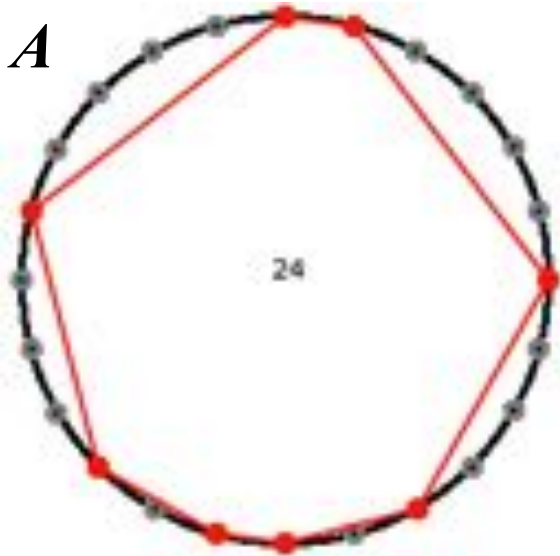
$$IC_A(k) = IC_{A'}(k)$$

\longleftrightarrow

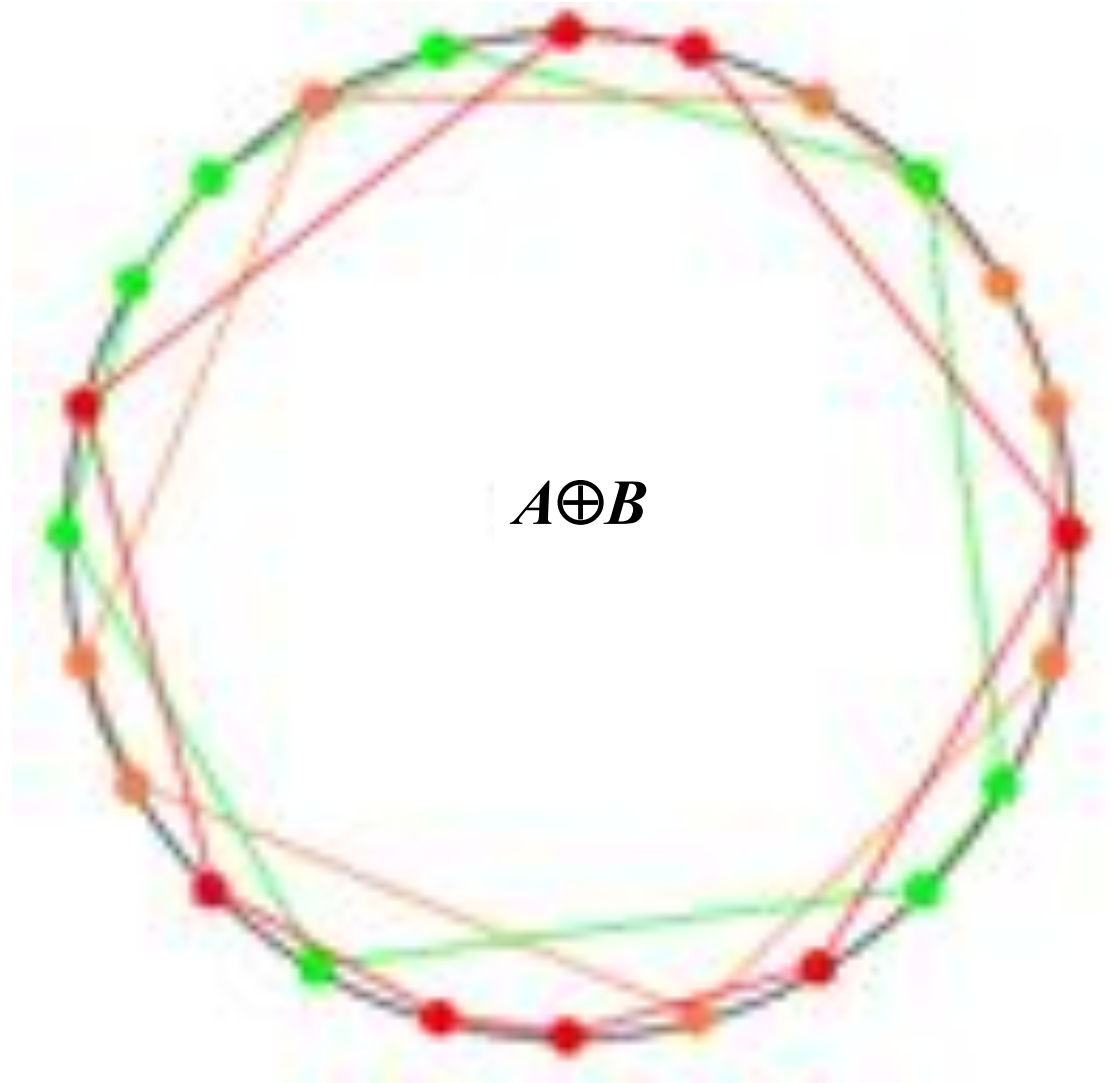
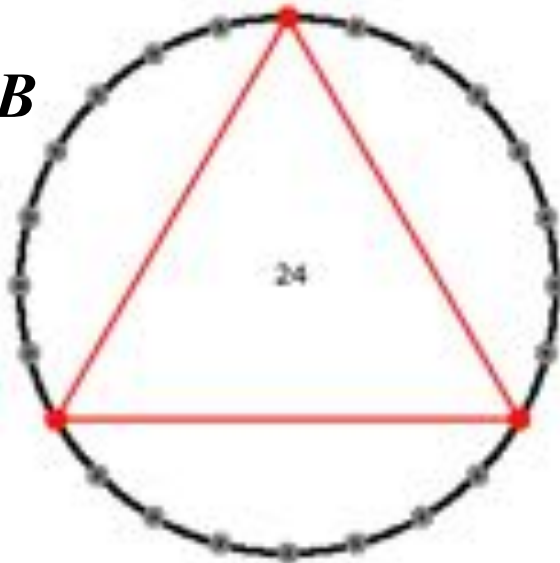
$$|F_A|^2 = |F_{A'}|^2$$

Tiling Rhythmic Canons and Homometry

A

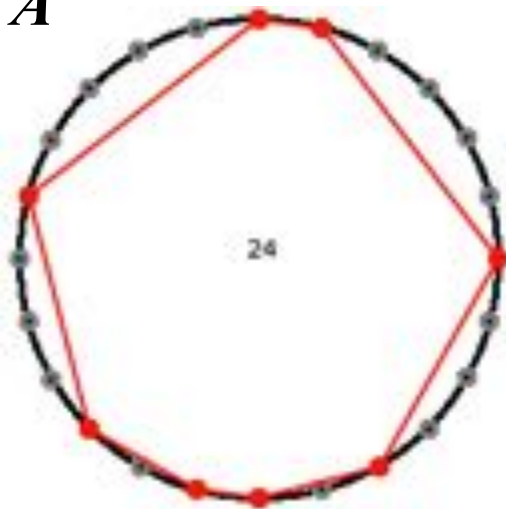


B

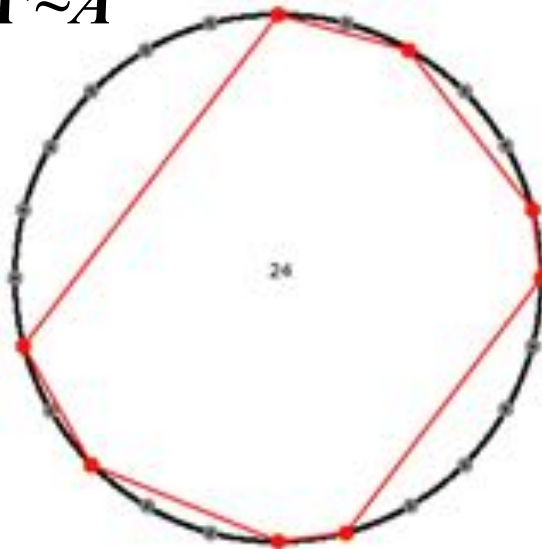


Tiling Rhythmic Canons and Homometry

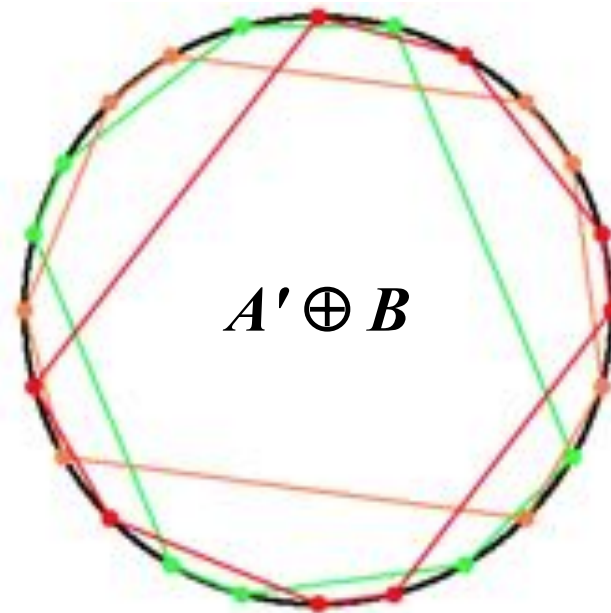
A



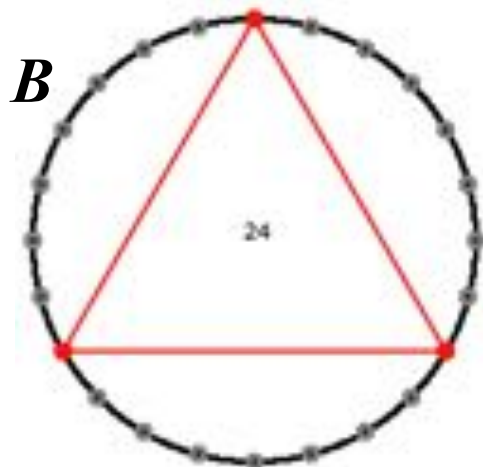
$A' \sim A$



$A' \oplus B$



B

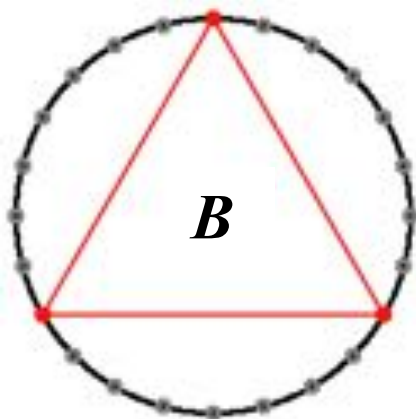
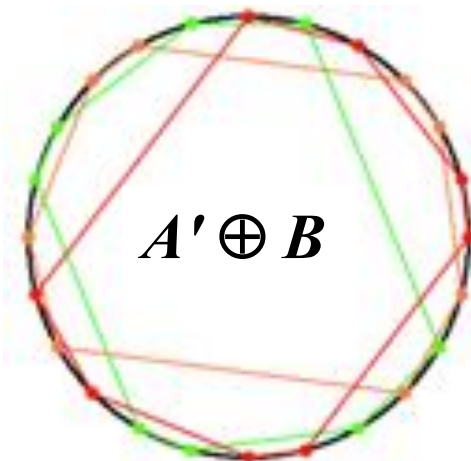
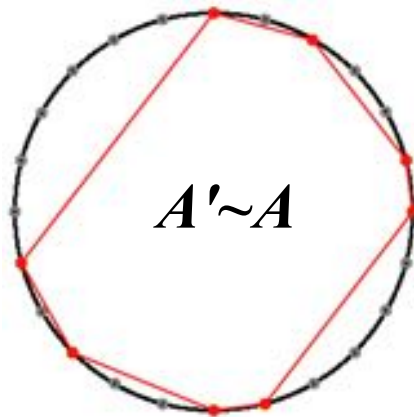
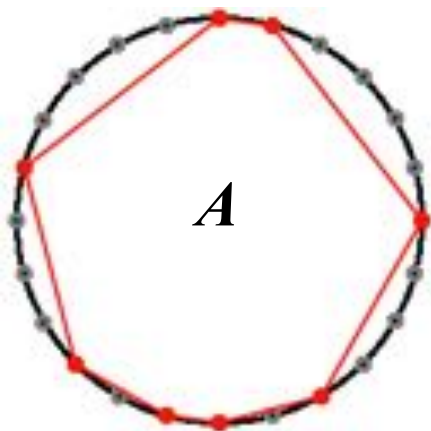


A musical offering:

● *Theorem:*

If *A* tiles with *B* and *A'* has the same IC, then *A'* tiles with *B*, too.

Tiling Rhythmic Canons and Homometry



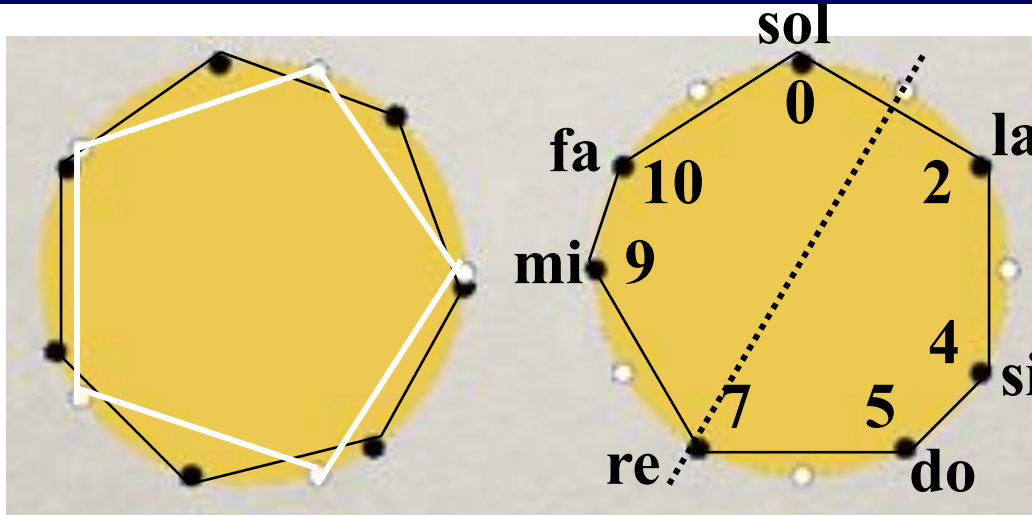
TILING

Let $Z_A = \{ t \in \mathbb{Z}_c \mid F_A(t) = 0 \}$

A tiles \mathbb{Z}_c when equivalently:

- ☉ There exists B, $A \oplus B = \mathbb{Z}_c$
- ☉ $l_A \star l_B = 1$
- ☉ $F_A \times F_B(t) = 1 + e^{-2i\pi t/c} + \dots + e^{-2i\pi t(c-1)/c}$ (0 unless $t=0$)
- ☉ $Z_A \cup Z_B = \{1, 2, \dots, c-1\}$ AND $\text{Card } A \times \text{Card } B = c$
- ☉ $IC_A \star IC_B = IC(\mathbb{Z}_c) = c$ and $\text{Card } A \times \text{Card } B = c$

Maximally Even Sets



Diatonic scale:
 $\{0, 2, 4, 5, 7, 9, 10\}$

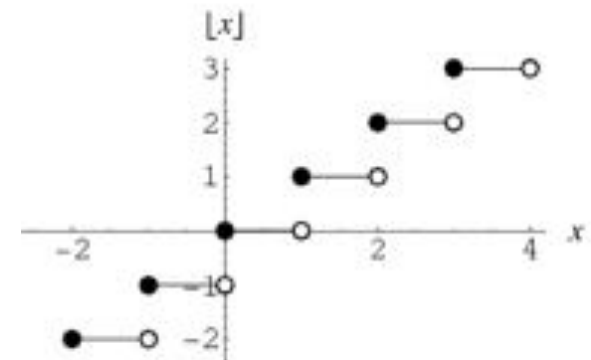
Pentatonic scale:
 $\{1, 3, 6, 8, 11\}$

Definition (Clough-Myerson-Douthett) A set A with cardinality d in a given equal tempered space \mathbf{Z}_c is maximally even if $A = \{a_k\}$

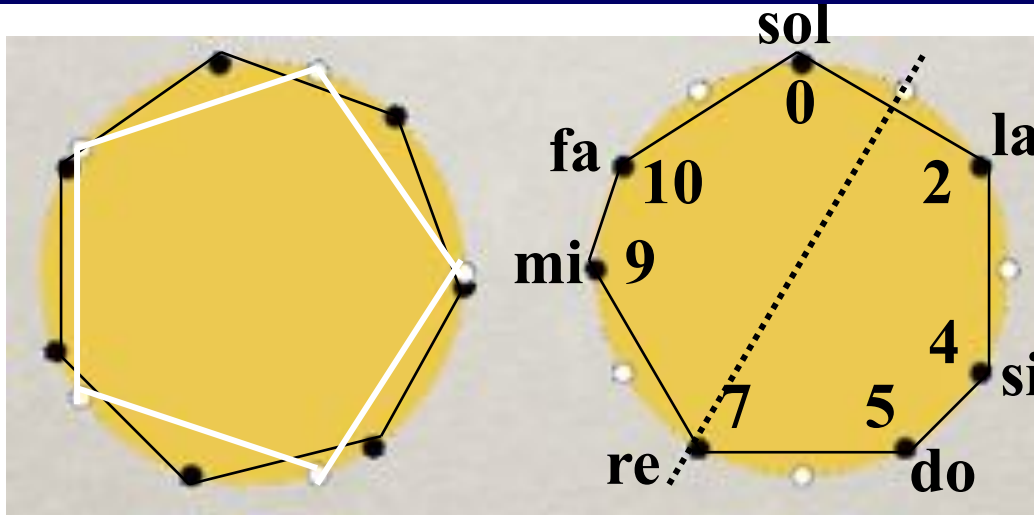
$$a_k = J_{c,d}^\alpha(k) = \left\lfloor \frac{kc + \alpha}{d} \right\rfloor$$

where $\alpha \in \mathbf{R}$
 $\lfloor x \rfloor$ is the integer part of x

$$J_{12,7}^5 = \left\{ \left\lfloor \frac{12k + 5}{7} \right\rfloor \right\}_{k=0}^6 = \{0, 2, 4, 5, 7, 9, 11\}$$



Maximally Even Sets



Diatonic scale:
 $\{0, 2, 4, 5, 7, 9, 10\}$

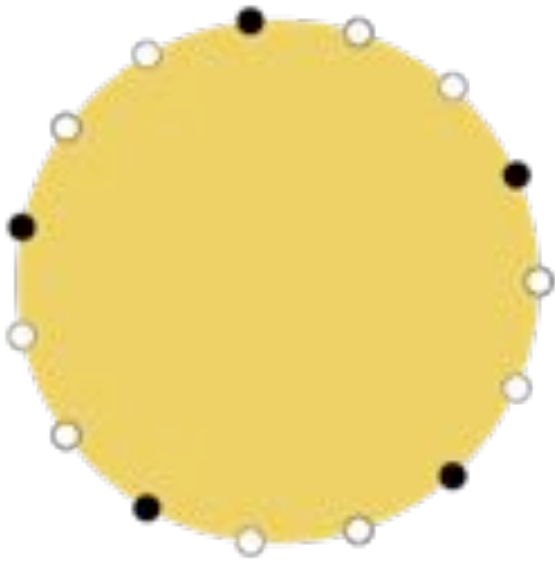
Pentatonic scale:
 $\{1, 3, 6, 8, 11\}$

Definition (Amiot, 2005) A set A with cardinality d given equal tempered space \mathbf{Z}_c is maximally even if $|F_A(d)| \geq |F_B(d)|$ for all subsets B of cardinality d in \mathbf{Z}_c .

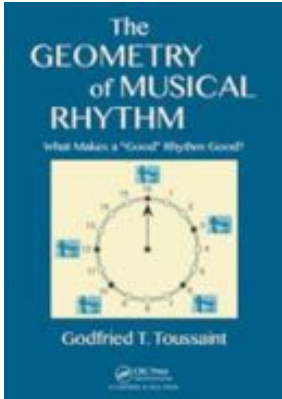
$$\text{where } F_{\text{set}}(t) := \sum_{k \in \text{set}} e^{2i\pi kt/12}$$

$$|F_A(5)| = 1+1+1+1+1 = 5$$

En général, $|F_A(t)| \leq \#A$

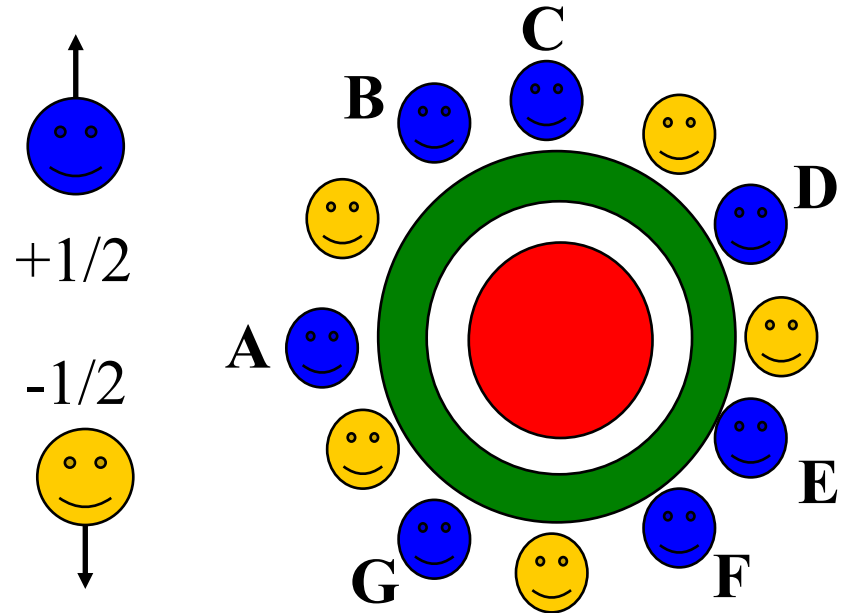
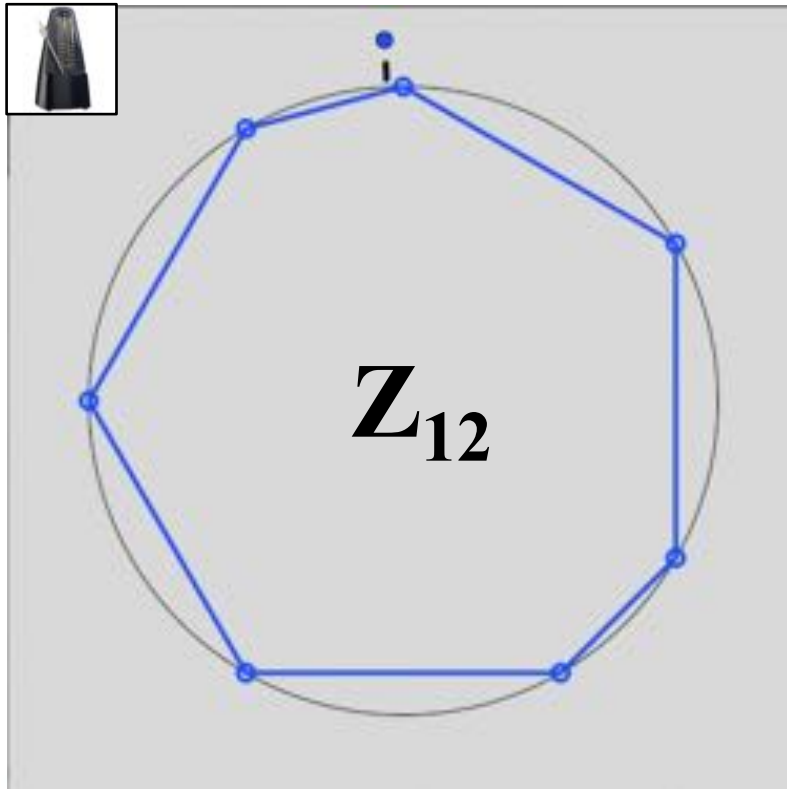


The pitch-rhythm isomorphic correspondence



“It is well known that there exists an isomorphic relation between pitch and rhythm, which several authors have pointed out from time to time.”

(G. Toussaint, *The geometry of musical rhythm. What makes a “Good” Rhythm Good?* CRC Press, 2013, p. xiii)



J. Douthett & R. Krantz, “Energy extremes and spin configurations for the one-dimensional antiferromagnetic Ising model with arbitrary-range interaction”, *J. Math. Phys.* 37 (7), July 1996

The pitch-rhythm *cognitive* isomorphic correspondence

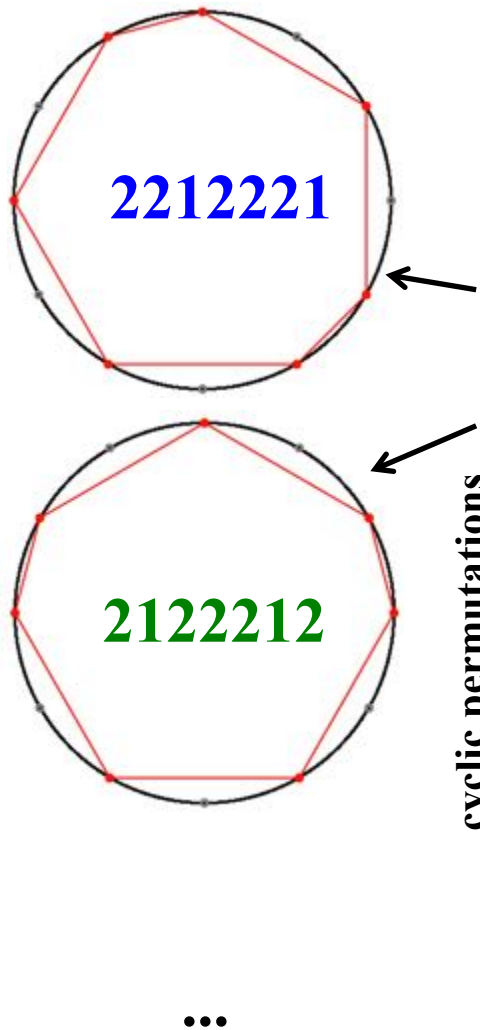
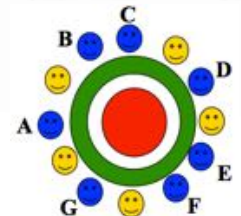


TABLE 1

Comparison of M = 7, L = 12 patterns for pitch (scales) and rhythm (time-lines)

pattern	pitch domain name and notation (in C)	rhythm domain notation	examples from West Africa	references
1. 2212221	major scale (Ionian) CDEFGAB	♪ ♪ ♫ ♪ ♪ ♪ ♪	Ewe (Atsiabek , Sogba, Atsia) also Yoruba	Jones (1959), C. K. Ladzekpo, S. K. Ladzekpo and Pantaleoni, Locke
2. 2122212	Dorian CDE ^b FGAB ^b	♪ ♪ ♪ ♪ ♪ ♪ ♪	Bemba—Northern Rhodesia	Jones (1965), (Ekwueme)
3. 1222122	Phrygian CD ^b E ^b FGA ^b B ^b	♪ ♪ ♪ ♪ ♪ ♪ ♪	—	—
4. 2221221	Lydian CDEF [#] GAB	♪ ♪ ♪ ♪ ♪ ♪ ♪	Ga-Adangme (common) also common Haitian pattern, Akan (Ab fo)	C. K. Ladzekpo, Combs (1974), R. Hill, Asiama
5. 2212212	Mixolydian CDEFGAB ^b	♪ ♪ ♪ ♪ ♪ ♪ ♪	Yoruba sacred music from Ekiti	King
6. 2122122	Aeolian CDE ^b FGA ^b B ^b	♪ ♪ ♪ ♪ ♪ ♪ ♪	Ashanti (Ab fo , Mpre)	Koetting
7. 1221222	Locrian CD ^b E ^b FG ^b A ^b B ^b	♪ ♪ ♪ ♪ ♪ ♪ ♪	Ghana*	Nketia (1963a)
8. 2121222	(#2 Locrian) CDE ^b FG ^b A ^b B ^b	♪ ♪ ♪ ♪ ♪ ♪ ♪	Ashanti (Asedua)	C. K. Ladzekpo
9. 2112123	— CDD [#] EF [#] GA	♪ ♪ ♪ ♪ ♪ ♪ ♪	Akan (juvenile song)	Nketia (1963b)

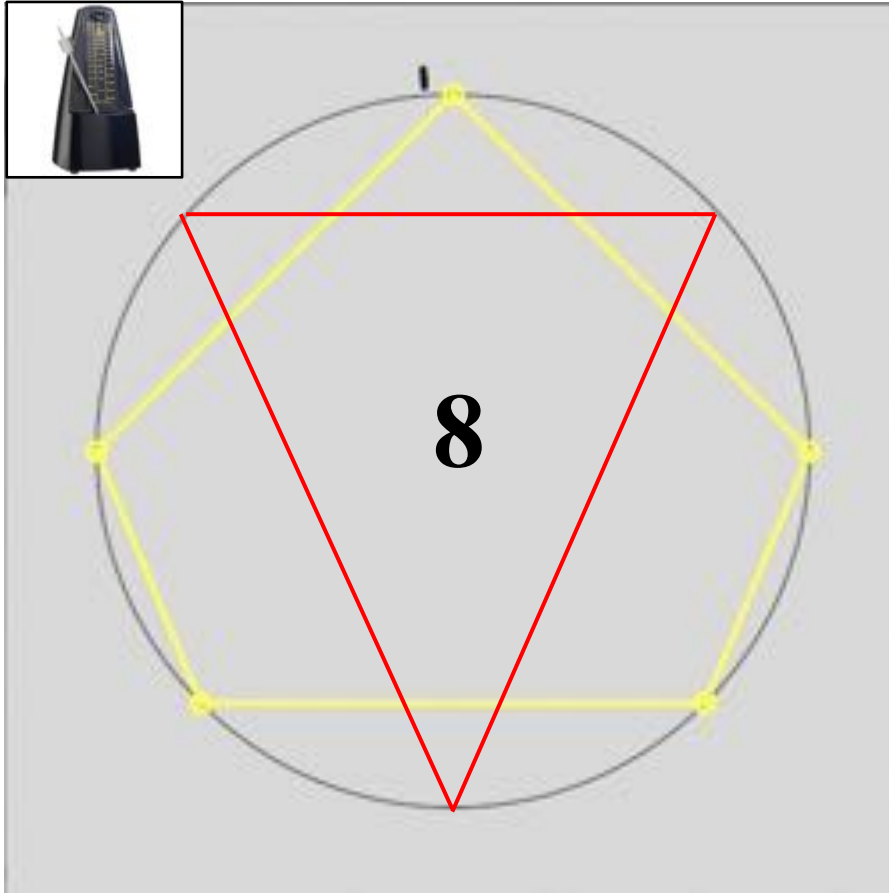
* clap pattern
† mute stroke on bell



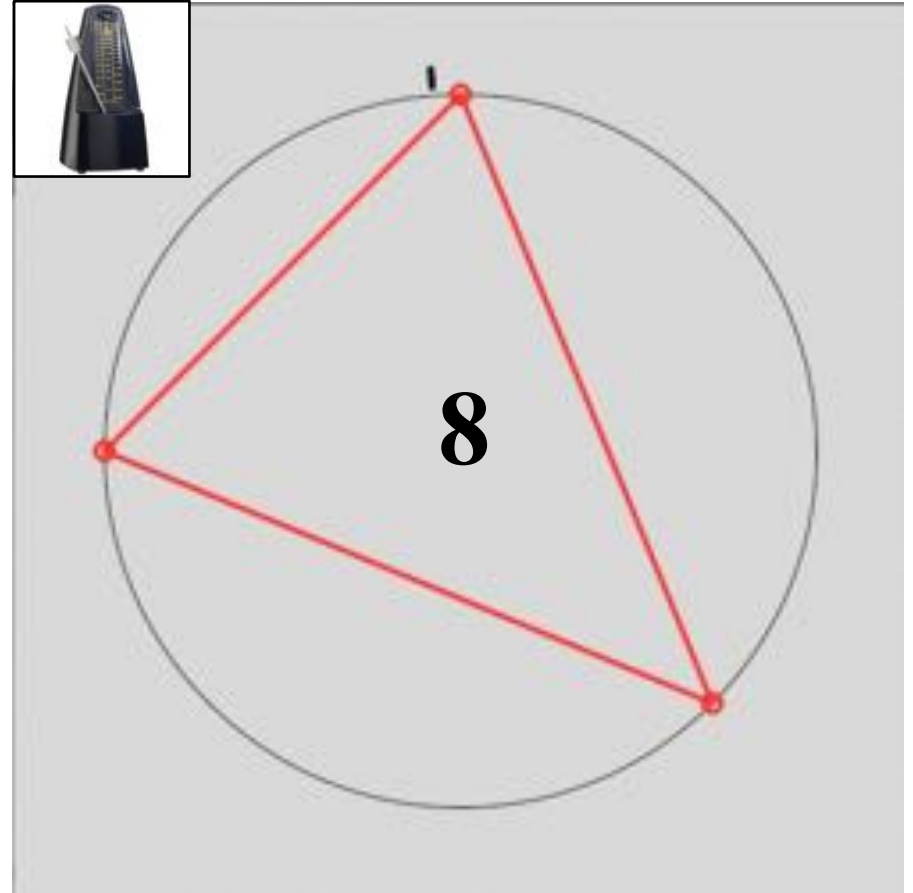
J. Pressing, “Cognitive isomorphisms between pitch and rhythm in world musics: West Africa, the Balkans and Western tonality”, *Studies in Music*, 17, p. 38-61

African-cuban ME-rhythms

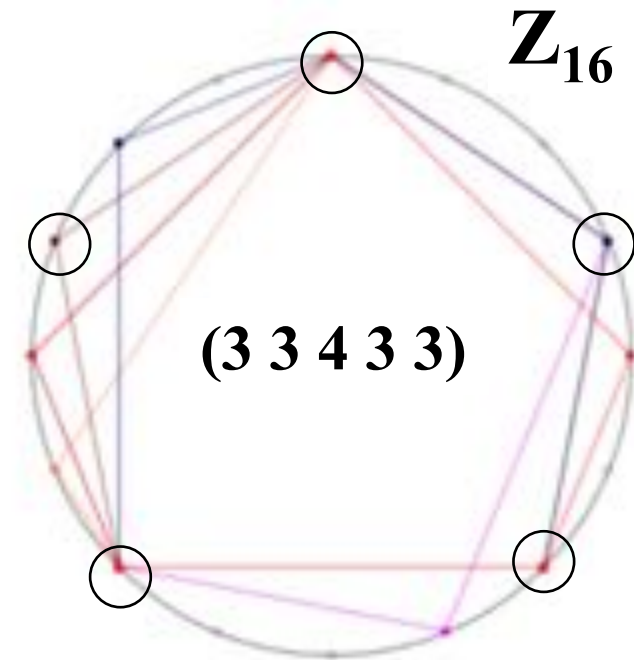
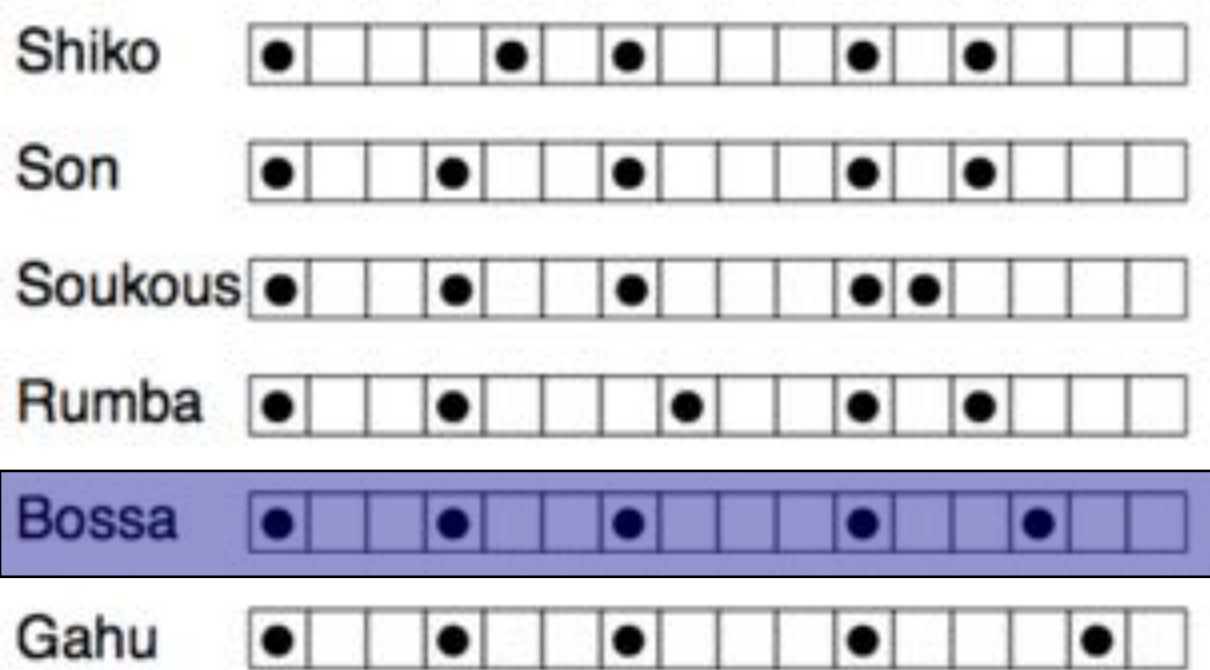
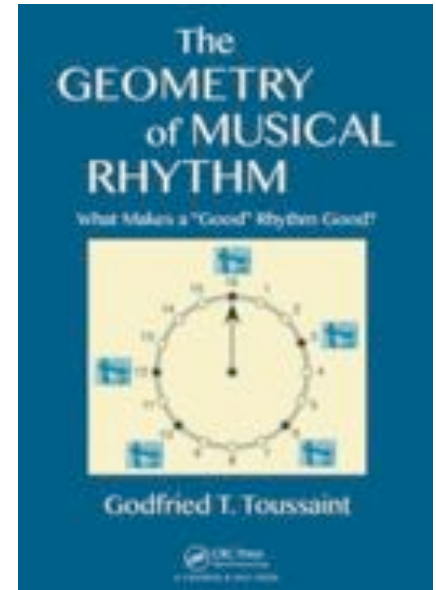
El cinquillo



El trecillo



The geometry of African-Cuban rhythms



Palindromic structures in Steve Reich's music

CLAPPING MUSIC

FOR TWO PERFORMERS

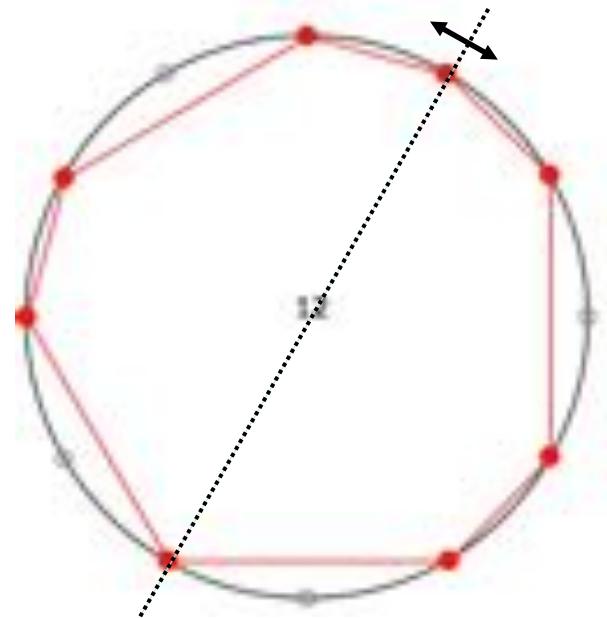
The image shows the musical score for Steve Reich's 'Clapping Music'. It consists of two staves, labeled 'CLAP 1' and 'CLAP 2'. The score is written in a rhythmic notation where vertical lines represent claps. The first measure is highlighted with a red box. The score is divided into measures by circled numbers 1 through 4. At the bottom right of the score, it says 'Repeat back, then end'.

The performers begin and end with both performers in unison at the ①. The number of repeats of each line should be fixed at twelve repeats per line. Since the first performer's part does not change, it is up to the second performer to vary how one line is to the next. The second performer should try to keep an even interval where it is written, i.e. on the first beat of each section (and on the last beat of the group of three steps), so that his direction always falls on a new beat of his own underlying pattern.

The choice of a particular clapping sound, or with cuffed or flat hands, is left up to the performer, whichever takes is chosen, both performers should try to get the same one so that their two parts will blend together and sound similar.

Clapping Music de Steve Reich (1972)

Steve Reich 1972
revised 1978



The circle and its 'canonic' rotations

CLAPPING MUSIC

FOR TWO PERFORMERS

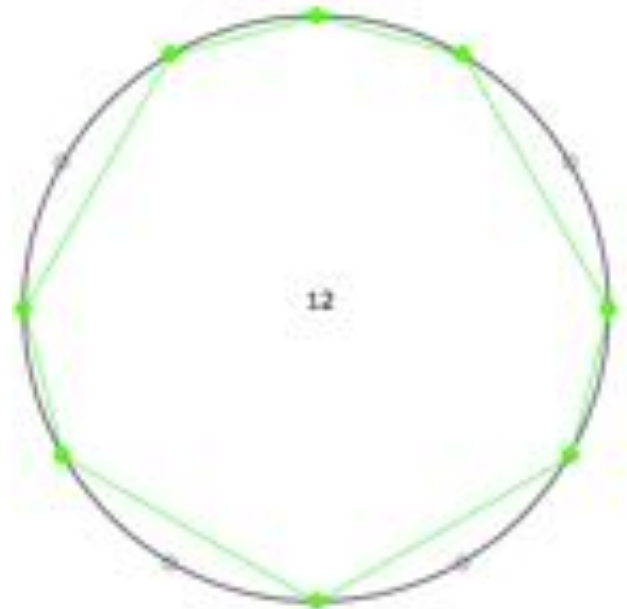
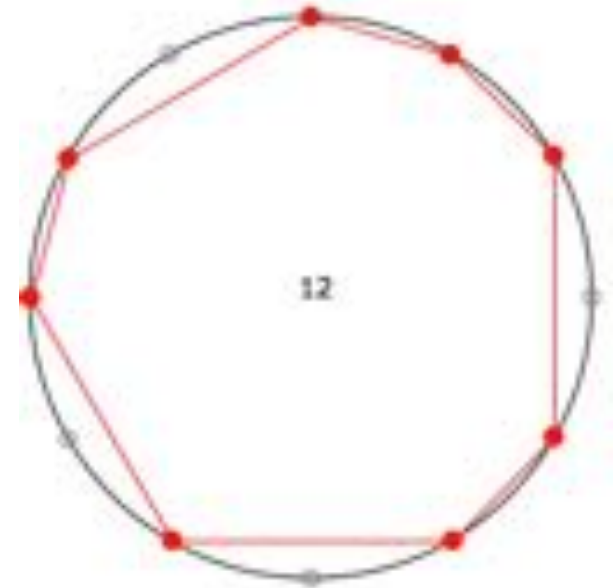
CLAP 1
CLAP 2

♩ = 1/4 = 100

Repeat last 4 measures

The performers begin and end with both performers in unison at the ①. The number of repeats of each line should be fixed at twelve repeats per line. Since the first performer's part does not change, it is up to the second performer to vary from one line to the next. The second performer should try to keep an even distribution where it is written, i.e. on the first beat of each measure (and on the first beat of the group of three steps), so that his distribution always falls on a new beat of the first performer's pattern.

The choice of a particular clapping sound, or with cuffed or flat hands, is left up to the performer, whichever takes is chosen, both performers should try to get the same one so that their two parts will blend together and sound really pretty.



The circle and its 'canonic' rotations

CLAPPING MUSIC

FOR TWO PERFORMERS



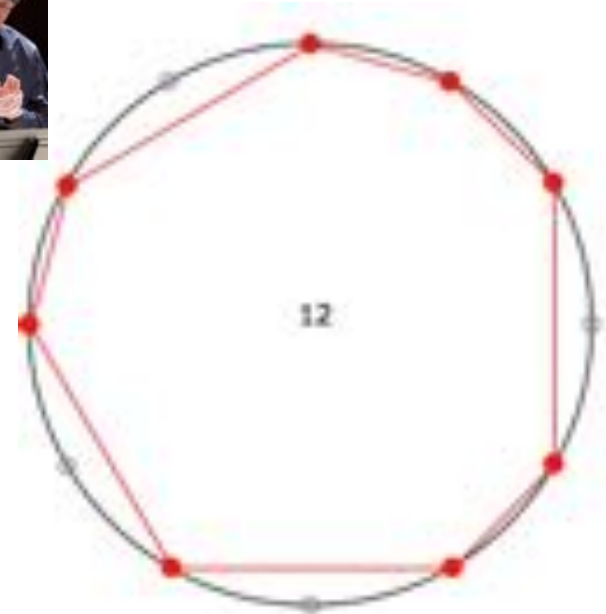
CLAP 1
CLAP 2

♩ = 1/4 = 100

Repeat back, then end

The performers begin and end with both performers in unison at the Ⓢ. The number of repeats of each line should be fixed at twelve repeats per line. Since the first performer's part does not change, it is up to the second performer to vary from one line to the next. The second performer should try to keep an even distribution where it is written, i.e. on the first beat of each measure (not on the first beat of the group of three claps), so that his distribution always falls in a new kind of dynamic relationship.

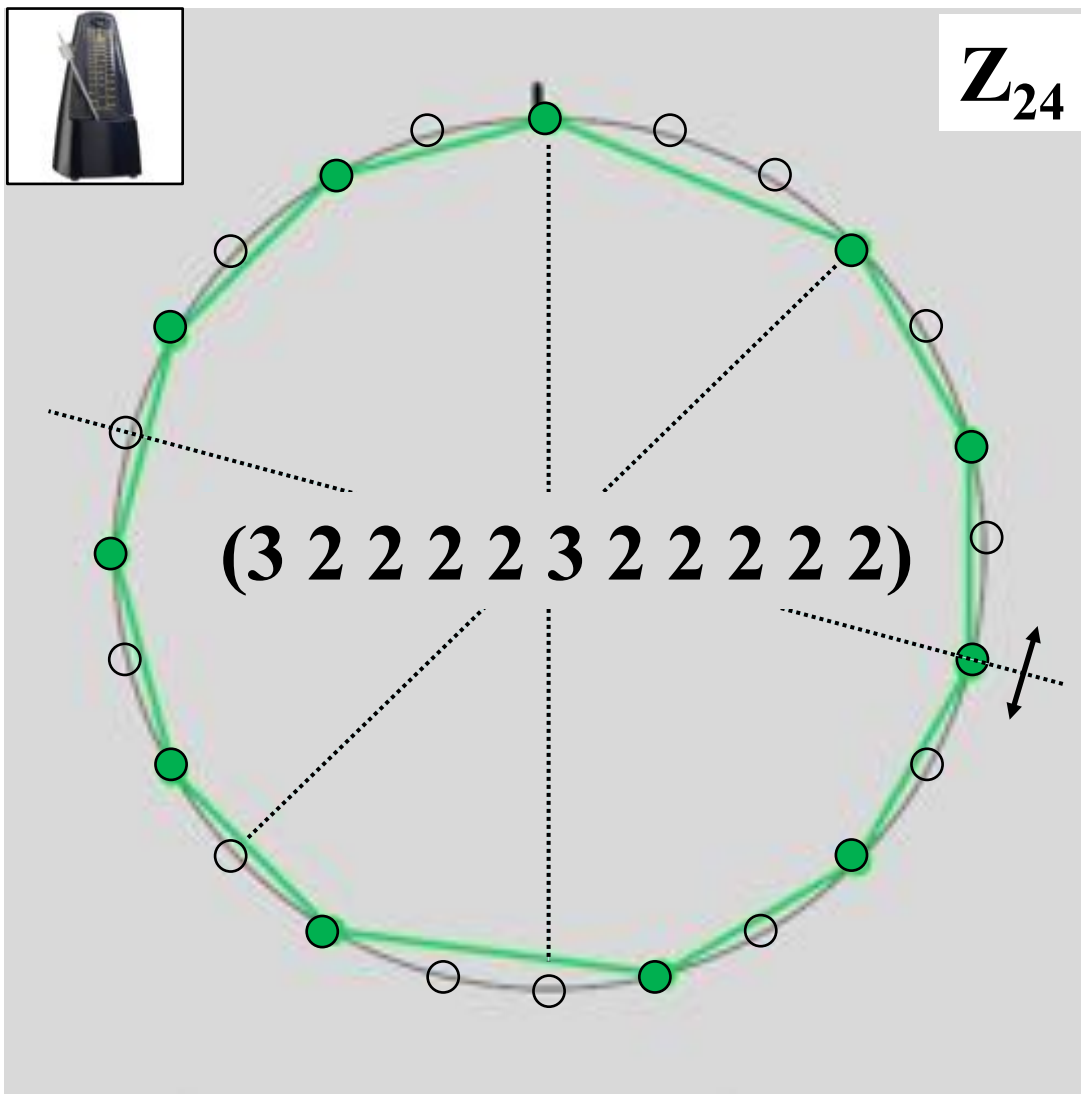
The choice of a particular clapping sound, or with cuffed or flat hands, is left up to the performer, whose task is chosen, both performers, should try to get the same one so that their two parts will blend together in a most pleasing pattern.



1 2 3 4 5 6 7 8

(SHIFT)

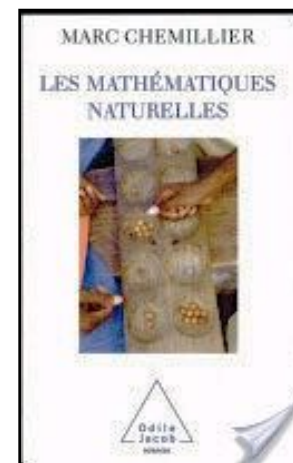
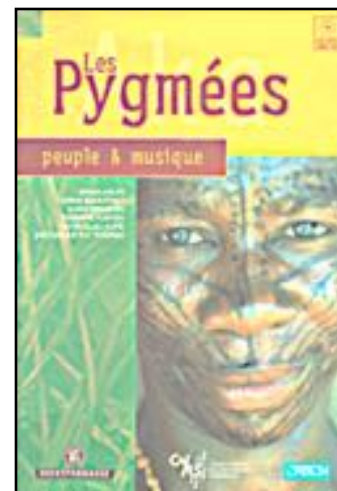
Odditive property of orally-transmitted practices



Simha Arom



Marc Chemillier

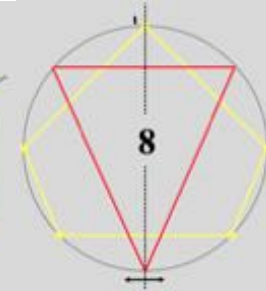
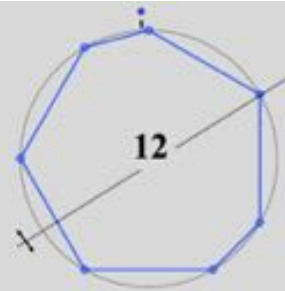
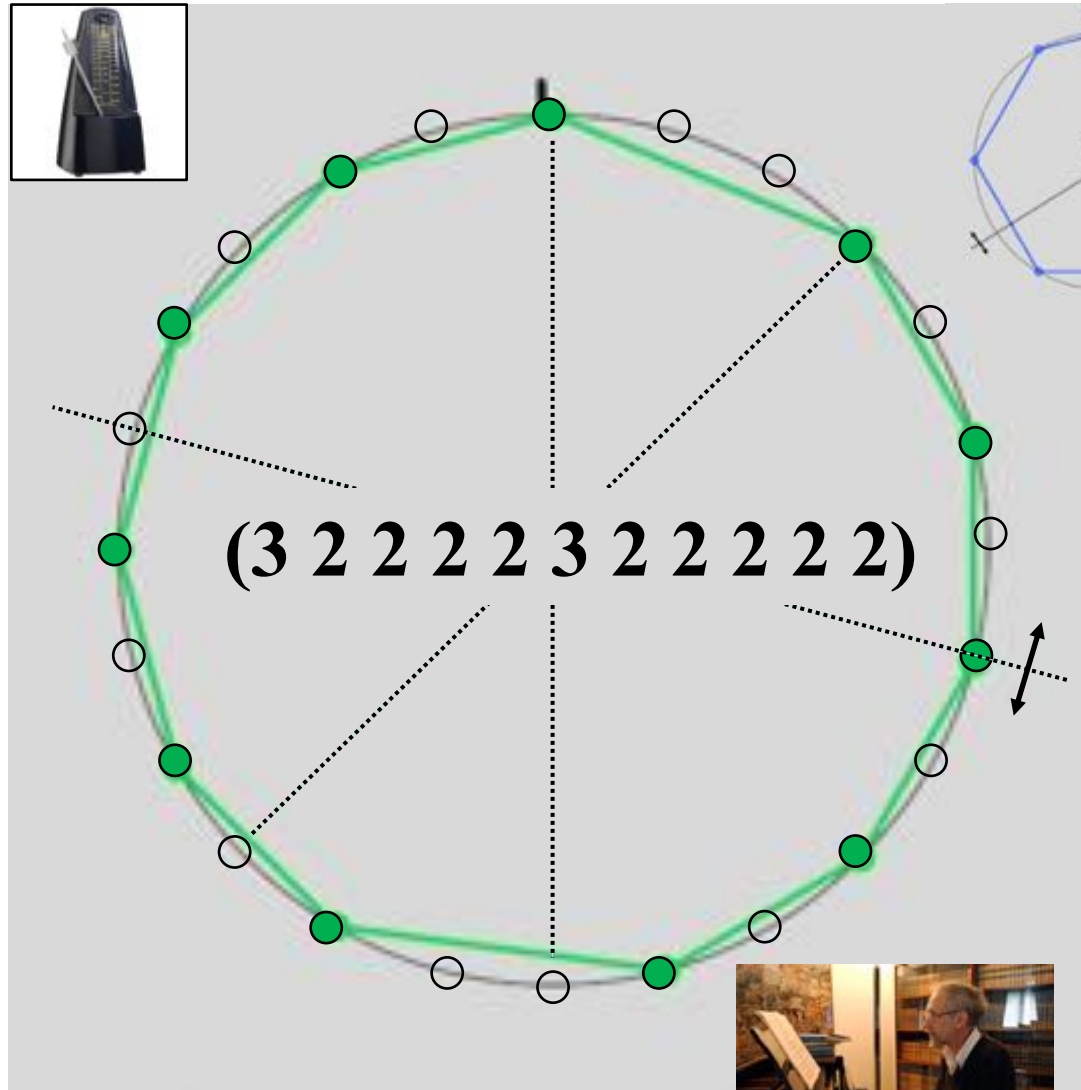


musimédiane

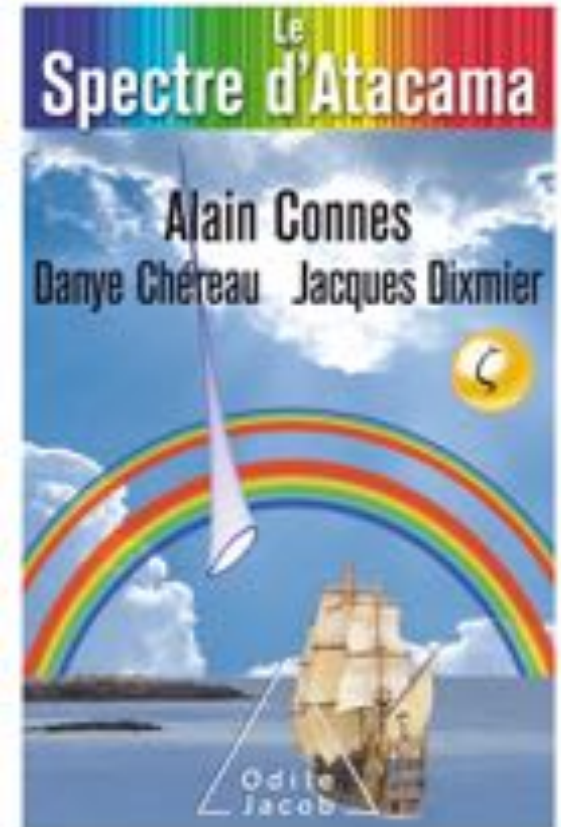
publiée avec le concours de la SFAM

revue audiovisuelle et multimédia d'analyse musicale

Olivier Messiaen's non-invertible rhythms



Olivier Messiaen

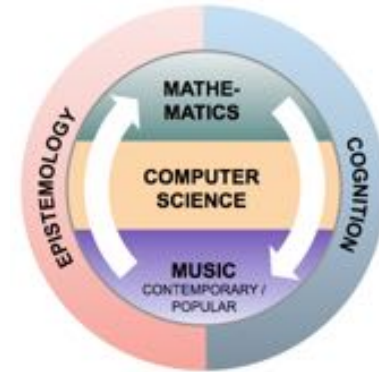
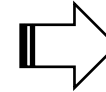
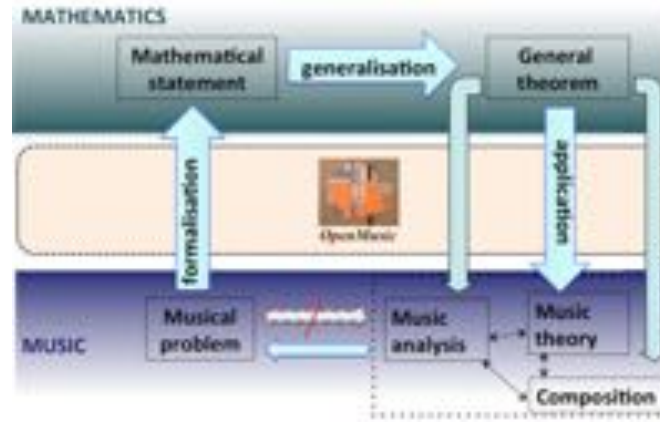


Alain Connes

A short catalogue of mathematical problems

M. Andreatta : *Mathematica est exercitium musicae*, Habilitation Thesis, IRMA University of Strasbourg, 2010

- Tiling Rhythmic Canons
- Z relation and homometry
- Transformational Theory
- Music Analysis, SC and FCA
- Diatonic Theory and ME-Sets
- Periodic sequences and FDC
- Block-designs in composition



Rhythmic Tiling Canons

Z-Relation and Homometric Sets

18 → (0 1 4 6) → [111111] → 4-Z15

23 → (0 1 3 7) → [111111] → 4-Z29

Finite Difference Calculus

$$Df(x) = f(x) - f(x-1)$$

7 11 10 11 7 2 7 11 10 11 7 2 7 11...
 4 11 1 8 7 5 4 11 1 8 7 5 4 11...
 7 2 7 11 10 11 7 2 7 11 10 11...
 7 5 4 11 1 8 7 5 4 11 18...

Set Theory, and Transformation Theory

Neo-Riemannian Theory and Spatial Computing

Diatonic Theory and ME-Sets

Block-designs

OpenMusic, a Visual Programming Language for computer-aided composition

www.repmus.ircam.fr/openmusic/home

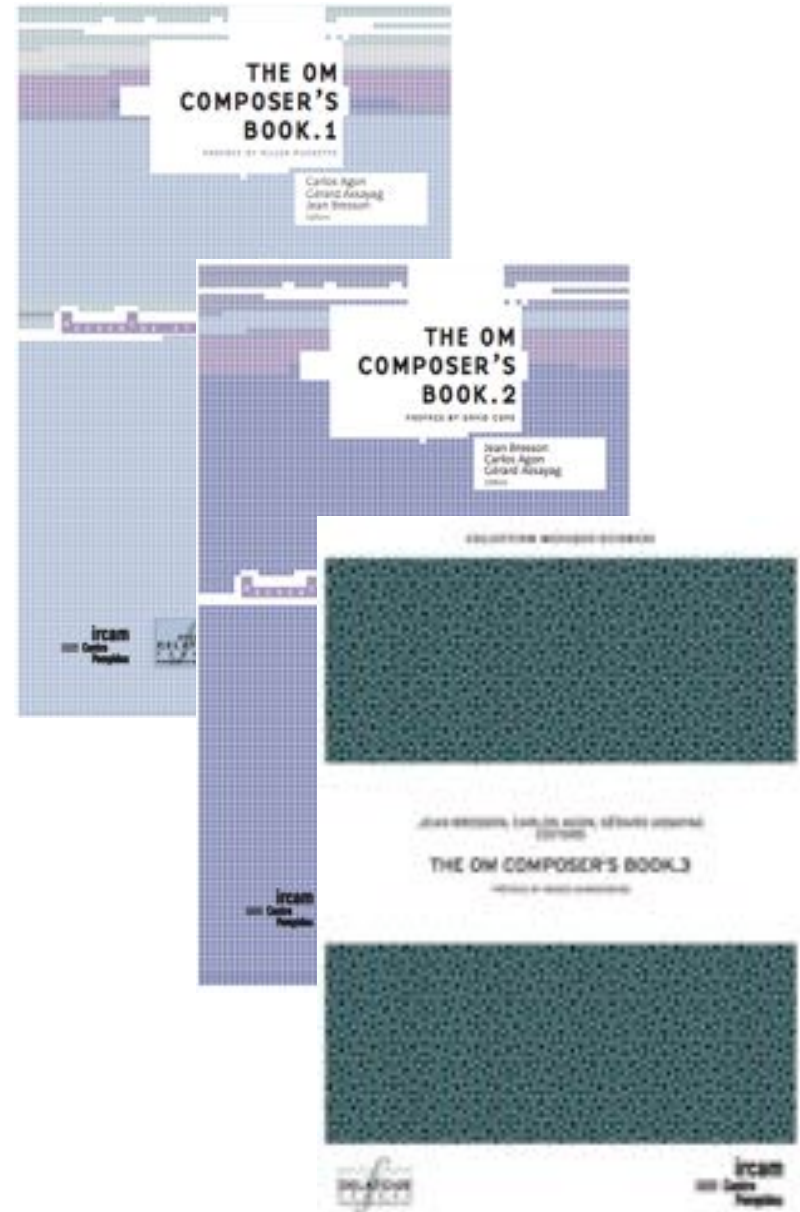
OpenMusic

(c) Ircam - Centre Pompidou



Dedicated to the memory of Gérard Grisey (French composer, 1946-1998)

Design and development : G. Assayag, A. Agon and J. Bresson
with help from C. Rueda, D. Celeriac, Ugo Malashère (Drame)
Musical expertise by : M. Andreatta, J. Babou, J. Fineberg, K. Haddad,
C. Malherbe, M. Malt, T. Mural, O. Sander, M. Stroppa, H. Tutschku
Artwork : A. Miltner



C. Agon, G. Assayag and J. Bresson, *The OM Composer's Book* (3 volumes)
“Musique/Sciences” Series, Ircam/Delatour, 2006, 2007 and 2016

Music and mathematics: « prima la musica »!



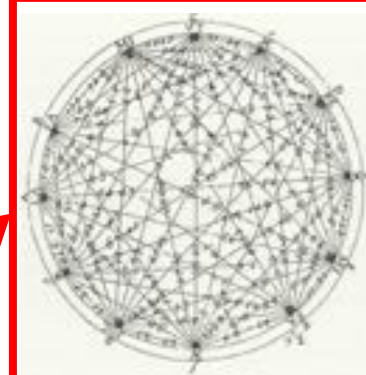
I. Xenakis



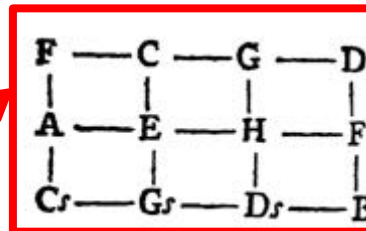
MUSIC	MATHS
500 B.C. Pitches and lengths of strings are related. Here music gives a marvelous thrust to number theory and geometry .	Discovery of the fundamental importance of natural numbers and the invention of fractions .
No correspondence in music.	Positive irrational numbers [...]
300 B.C. [...] Music theory highlights the discovery of the isomorphism between the logarithms (musical intervals) and exponentials (string lengths) more than 15 centuries before their discovery in mathematics; also a premonition of group theory is suggested by Aristoxenos.	No reaction in mathematics. [...]
1000 A.D. Invention of the two-dimensional spatial representation of pitches linked with time by means of staves and points [...] seven centuries (1635-37) before the magnificent analytical geometry of Fermat and Descartes.	No parallel in mathematics.
1500 No response or development of the preceding concepts.	Zero and negative numbers are adopted. Construction of the set of rationals.
1600 No equivalence, no reaction.	The sets of real numbers and of logarithms are invented.
1648 Invention of musical combinatorics by Marin Mersenne (<i>Harmonicorum Libri</i>)	Probability theory by Bernoulli (<i>Ars Conjectandi</i> , 1713)
1700 [...] The fugue , for example, is an abstract automaton used two centuries before the birth of the science of automata. Also, there is an unconscious manipulation of finite groups (Klein group) in the four variations of a melodic line used in counterpoint.	Number theory is ahead of but has no equivalent yet in temporal structures. [...]
1773 A first geometric and graph-theoretic representation of pitches (<i>Speculum Musicum</i>)	Invention of graph theory
1900 Liberation from the tonal yoke. First acceptance of the neutrality of chromatic totality (Loquin [1895], Hauer, Schoenberg).	The infinite and transfinite numbers (Cantor). Peano axiomatics. [...] The beautiful measure theory (Lebesgue, ...)
1920 First radical formalization of macrostructures through the serial system of Schoenberg.	No new development of the number theory.
1929 and 1937-1939 Susanne K. Langer and Ernst Krenek on the role of axioms in music	David Hilbert, <i>Die Grundlage der Geometrie</i> [1899]
1946 Milton Babbitt on group theory and integral serialism	Rudolf Carnap, <i>The Logical Syntax of Language</i> [1937]



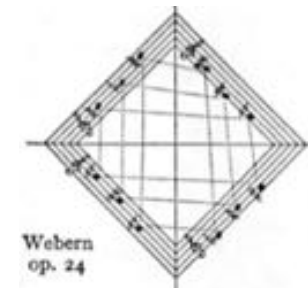
Pythagoras and the monochord, VIth-Vth Century B.C.



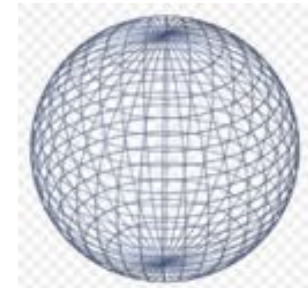
Mersenne and the 'musical clock', 1648



Euler and the *Speculum musicum*, 1773



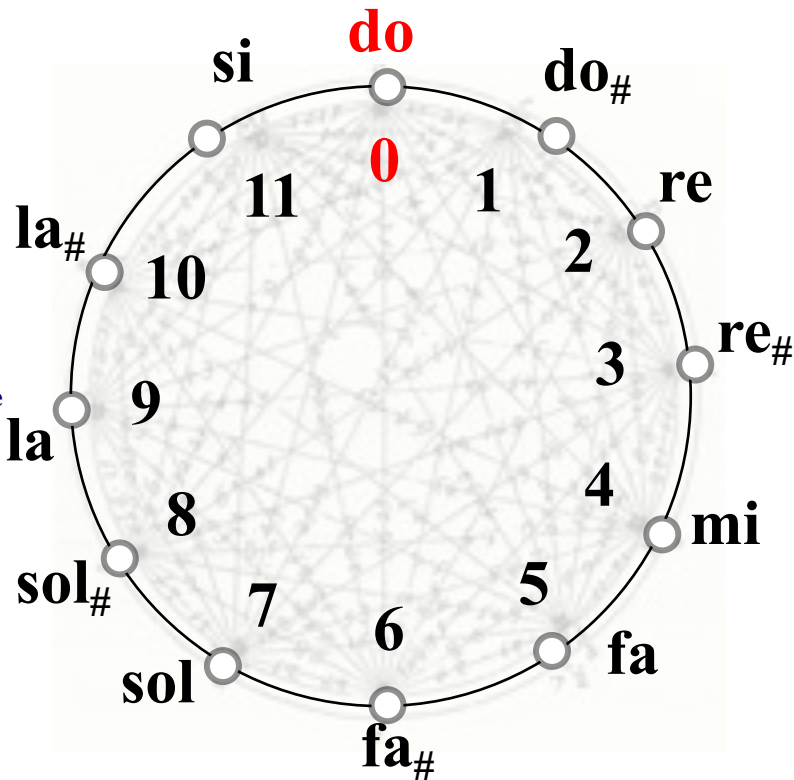
Webern op. 24



The circular representation of the pitch space



Marin Mersenne



Harmonicorum Libri XII, 1648



114

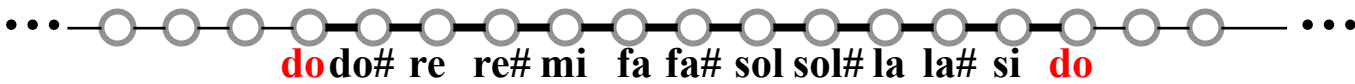
LIBER SEPTIMVS.
DE CANTIBVS, SEV CANTILENIS,
EARYMO, NYMERO, PARTIBVS, ET SPECIBVS.

Tabula Combinatiois ab 1 ad 12.

I	1
II	2
III	6
IV	24
V	110
VI	720
VII	5040
VIII	40320
IX	362880
X	3628800
XI	39916800
XII	479001600
XIII	6177018000
XIV	87178290000
XV	1307674168000
XVI	20521785880000
XVII	3126874180960000
XVIII	4402737057180000
XIX	6264800408130000
XX	8431200008766400000
XXI	110909411717094400000
XXII	144000727776076800000

Varietas, seu Combinatio quatuor notarum.

A musical score showing four staves of music. Each staff contains a sequence of notes, with numbers 1 through 24 written below them, representing the 24 possible combinations of four notes.



A musical score for the chromatic scale. It consists of two staves of music. Below the staves is a horizontal line with 12 circles, each labeled with a number from 0 to 12. The notes are: do (0), do# (1), re (2), re# (3), mi (4), fa (5), fa# (6), sol (7), sol# (8), la (9), la# (10), si (11), do (12). A large blue arrow points from the left towards the diagram.

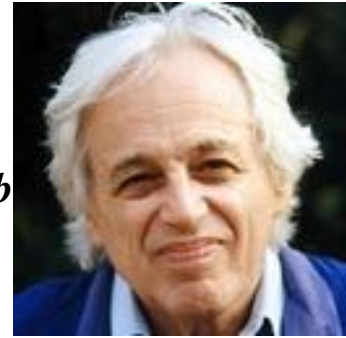
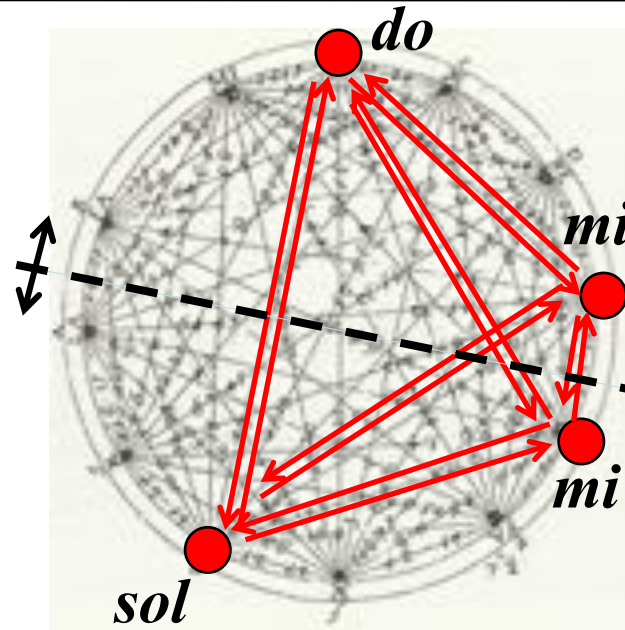
Permutational strategies in contemporary music

114. Marin Mersenne, *Harmonicorum Libri XII*, 1648

LIBER SEPTIMVS. DE CANTIBVS, SEV CANTILENIS, EARVMQ; NVMERO, PARTIBVS, ET SPECIEBVS.

Tabula Combinationis ab 1 ad 22.

I	1
II	2
III	6
IV	24
V	110
VI	710
VII	3040
VIII	40310
IX	361880
X	3618800
XI	39916800
XII	479001600
XIII	6127016800
XIV	87178120000
XV	1307674968000
XVI	20911789888000
XVII	315687418096000
XVIII	6401373705718000
XIX	121648100408810000
XX	24313010081766400000
XXI	50909418717094400000
XXII	1114000717777607480000



Six Bagatelles
(G. Ligeti, 1953)

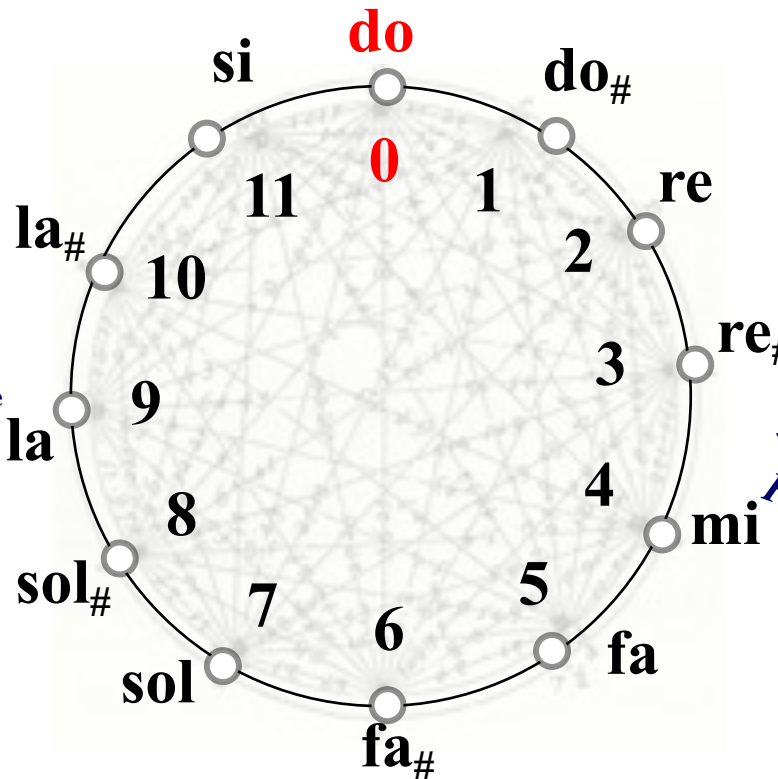
Varietas, seu Combinatio quatuor notarum.

Musical score for 'Varietas, seu Combinatio quatuor notarum' showing 24 numbered measures.

The circular representation of the pitch space



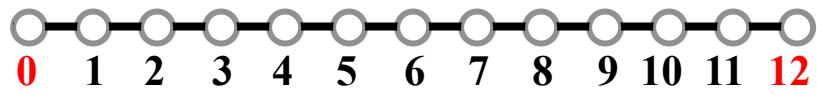
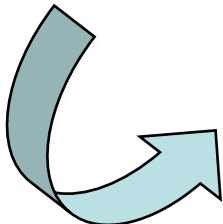
Marin Mersenne



Harmonicorum Libri XII, 1648



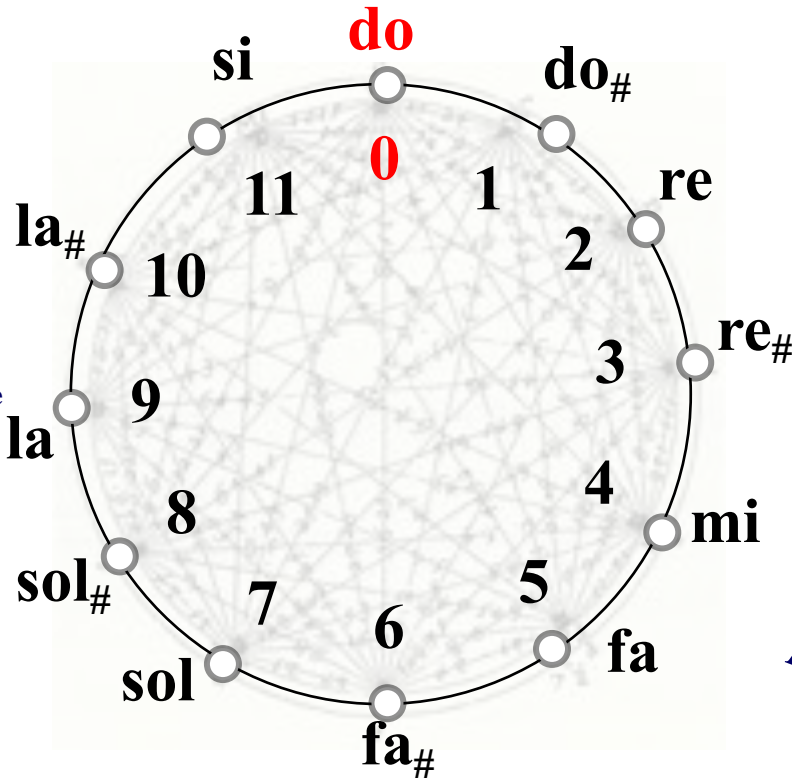
M. Andreatta, C. Agon,
«La musique mise en algèbre»,
Pour la Science, 2008



The circular representation of the pitch space



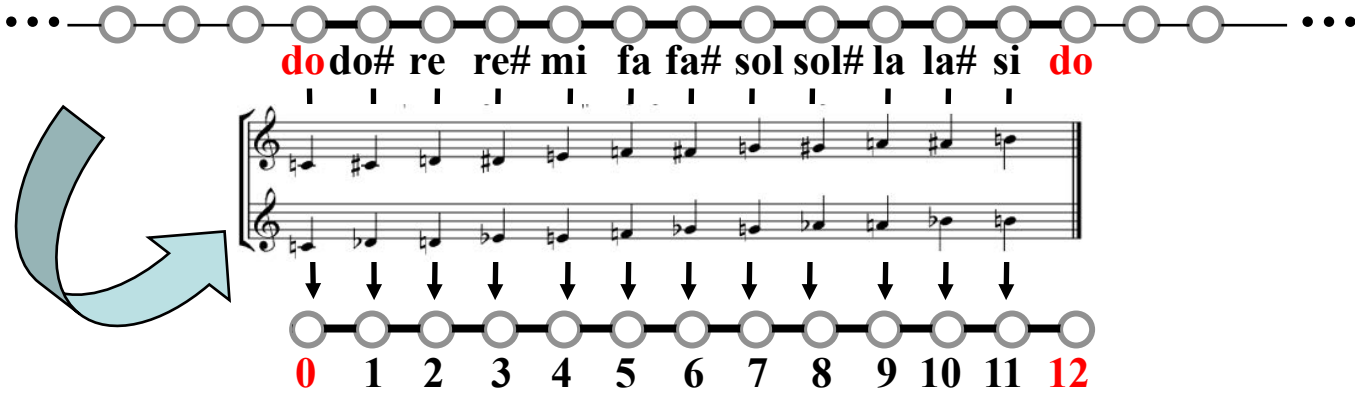
Marin Mersenne



Harmonicorum Libri XII, 1648



M. Andreatta, C. Agon,
« Algèbre et géométrie :
sont-elles inscrites dans le
cerveau ? »,
Pour la Science, 2018

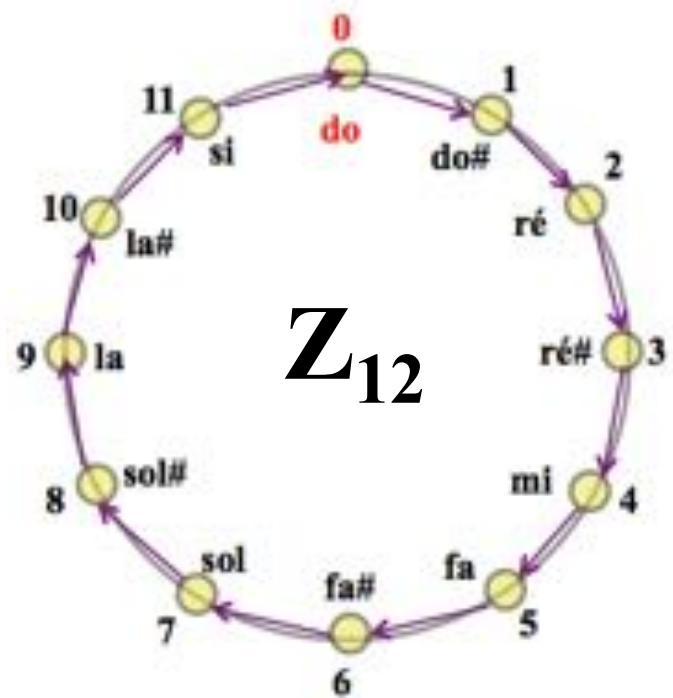
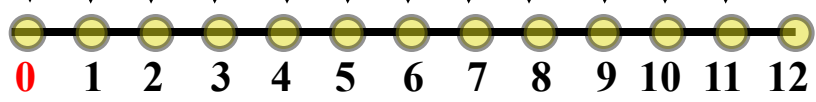
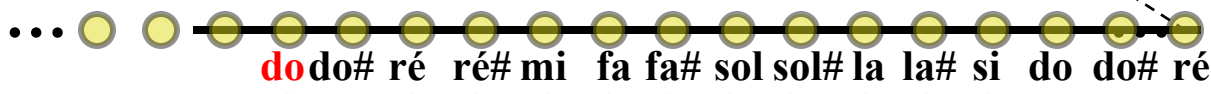


➔ DEMO

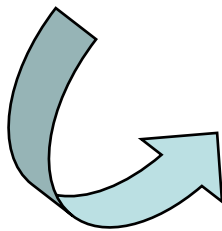
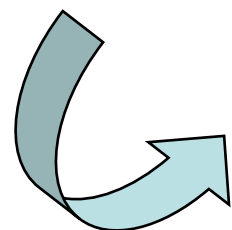
The equal tempered space is a cyclic group



The generators of the cyclic group of order 12 are the transpositions T_1 , T_5 , T_7 et T_{11} where $T_k: x \rightarrow x+k \pmod{12}$



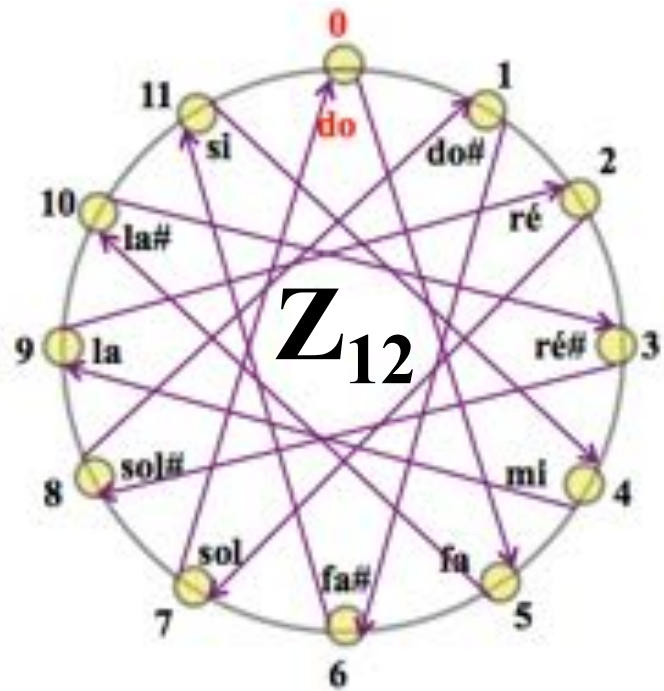
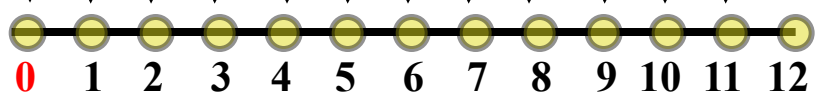
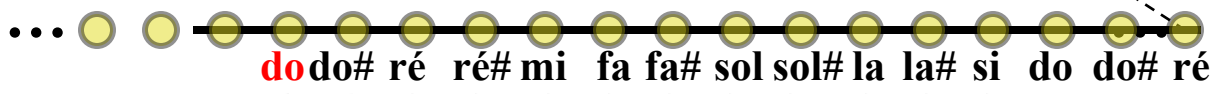
$$Z_{12} = \langle T_1 \mid (T_1)^{12} = T_0 \rangle$$



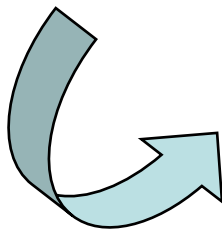
The equal tempered space is a cyclic group



The generators of the cyclic group of order 12 are the transpositions T_1 , T_5 , T_7 et T_{11} where $T_k: x \rightarrow x+k \pmod{12}$



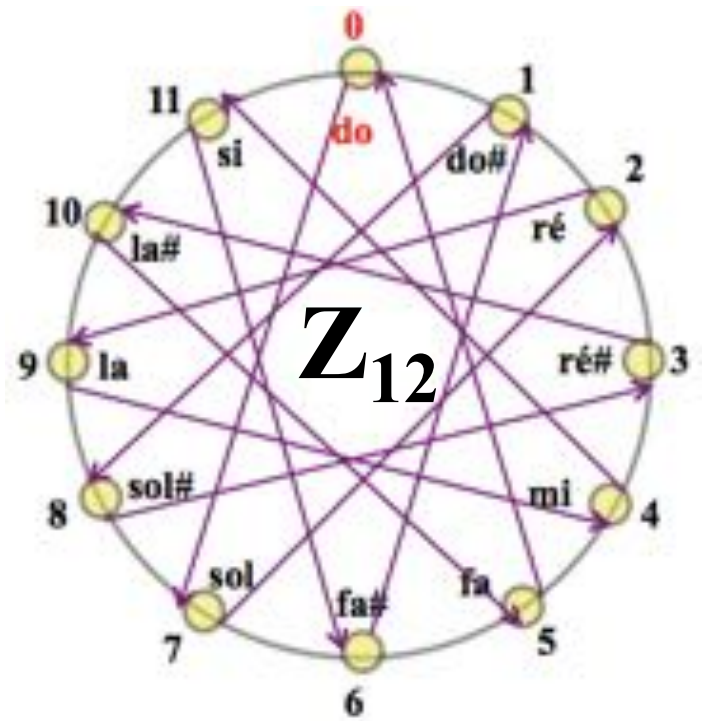
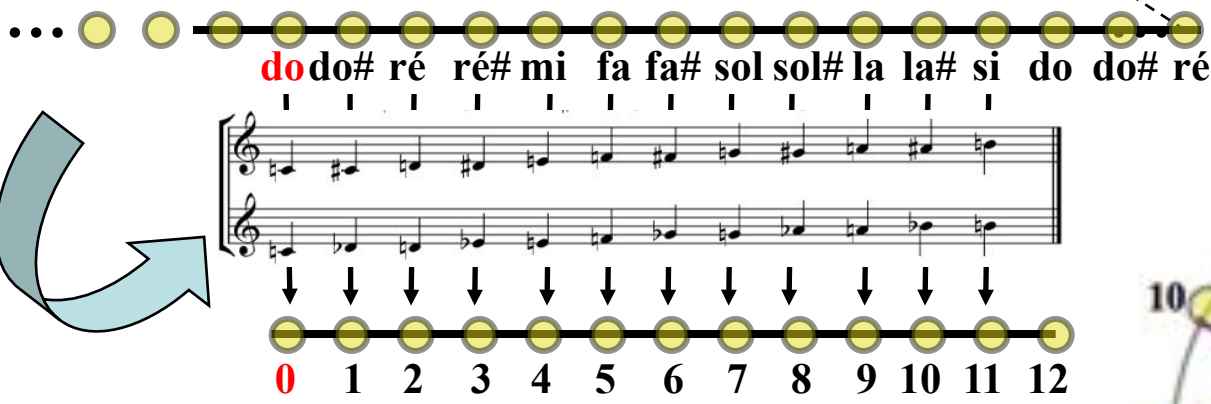
$$\begin{aligned} \mathbf{Z}_{12} &= \langle T_1 \mid (T_1)^{12} = T_0 \rangle = \\ &= \langle T_5 \mid (T_5)^{12} = T_0 \rangle \end{aligned}$$



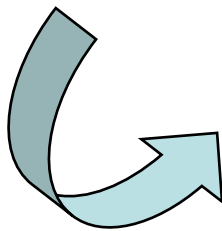
The equal tempered space is a cyclic group



The generators of the cyclic group of order 12 are the transpositions T_1 , T_5 , T_7 et T_{11} where $T_k: x \rightarrow x+k \pmod{12}$



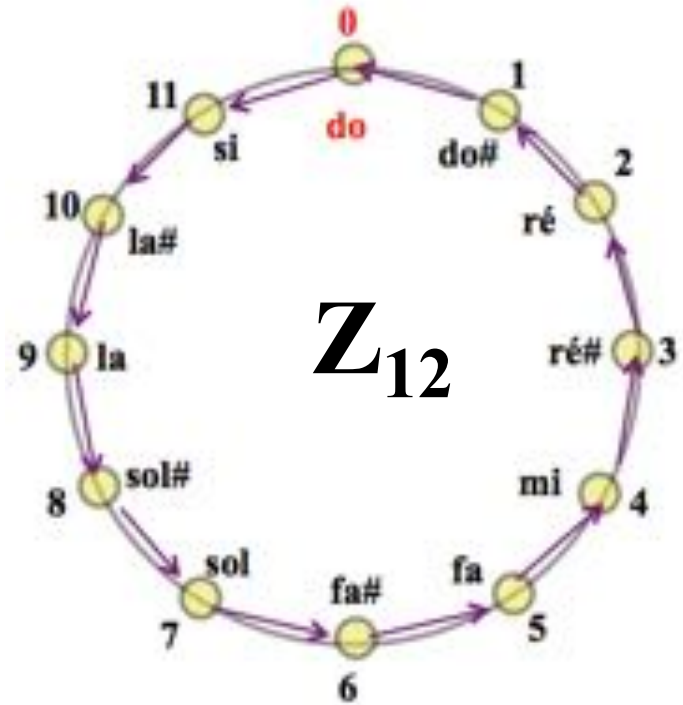
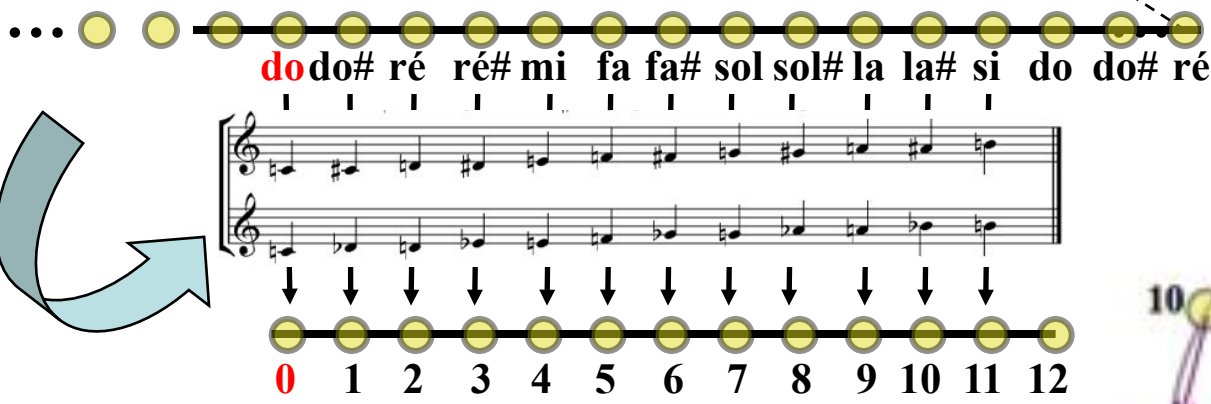
$$\begin{aligned} \mathbf{Z}_{12} &= \langle T_1 \mid (T_1)^{12} = T_0 \rangle = \\ &= \langle T_5 \mid (T_5)^{12} = T_0 \rangle = \\ &= \langle T_7 \mid (T_7)^{12} = T_0 \rangle \end{aligned}$$



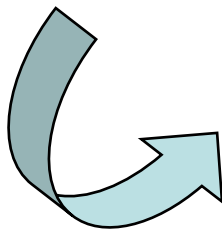
The equal tempered space is a cyclic group



The generators of the cyclic group of order 12 are the transpositions T_1 , T_5 , T_7 et T_{11} where $T_k: x \rightarrow x+k \pmod{12}$



$$\begin{aligned} \mathbf{Z}_{12} &= \langle T_1 \mid (T_1)^{12} = T_0 \rangle = \\ &= \langle T_5 \mid (T_5)^{12} = T_0 \rangle = \\ &= \langle T_7 \mid (T_7)^{12} = T_0 \rangle = \\ &= \langle T_{11} \mid (T_{11})^{12} = T_0 \rangle \end{aligned}$$



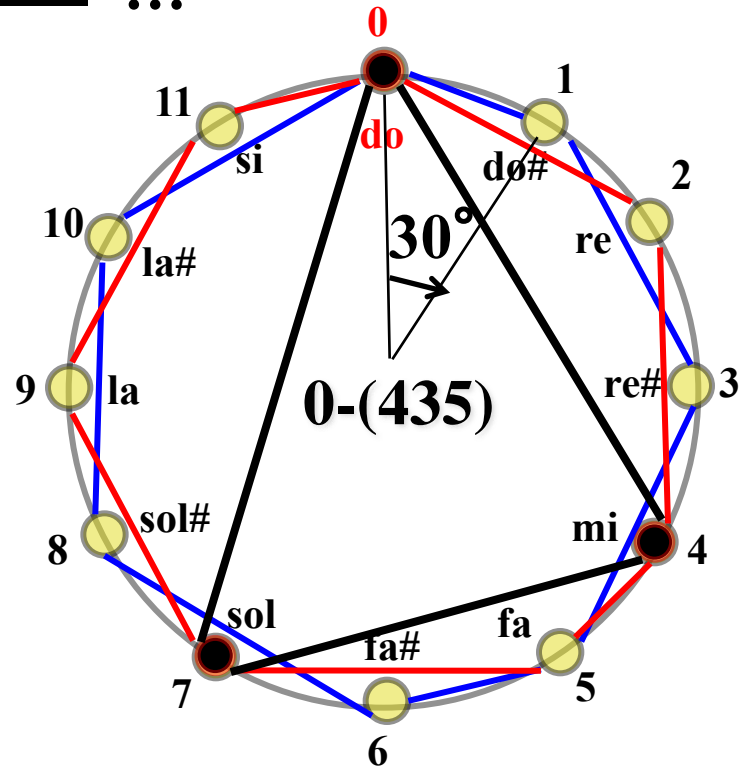
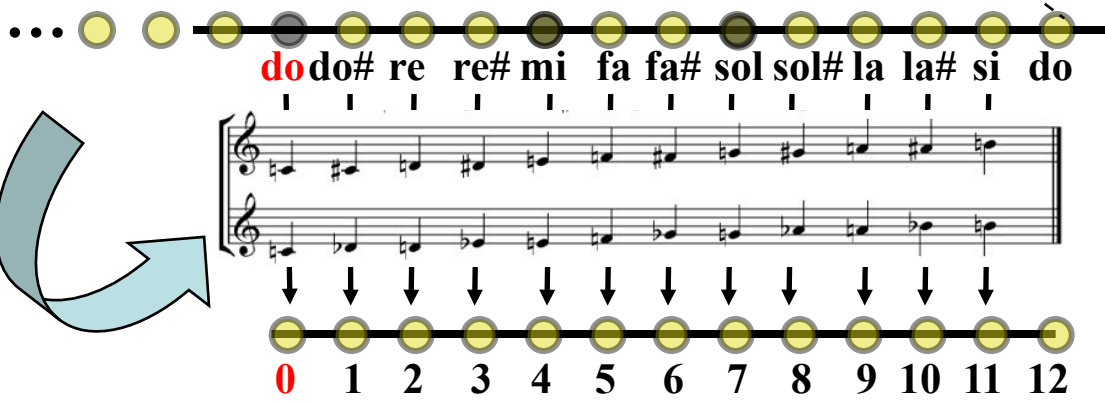
Musical transpositions are additions...

$$T_k : x \rightarrow x+k \pmod{12}$$



$$Do\ maj = \{0, 2, 4, 5, 7, 9, 11\} + 1$$

$$Do\#\ maj = \{1, 3, 5, 6, 8, 10, 0\}$$



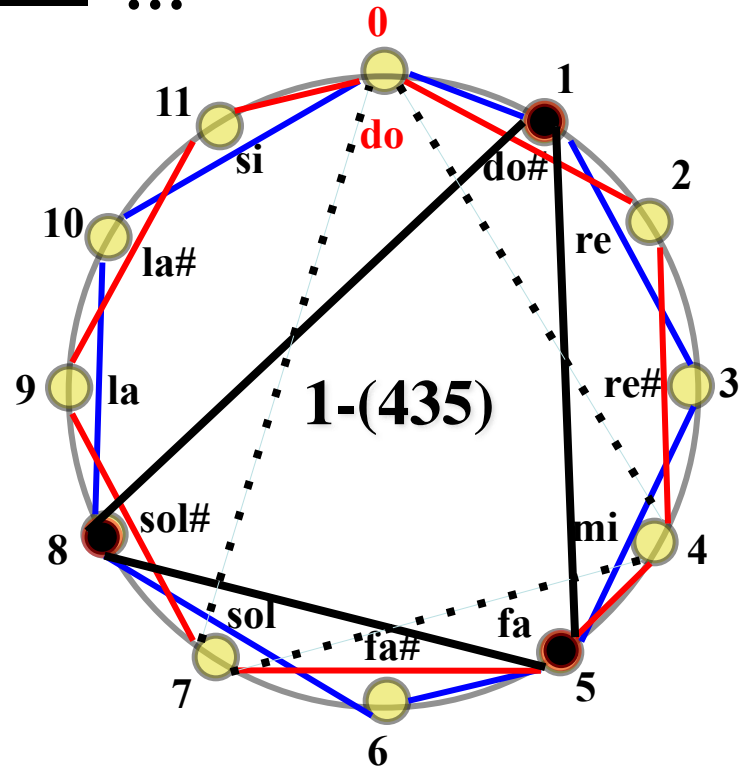
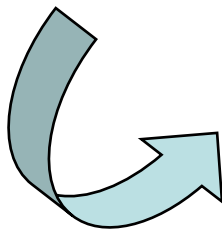
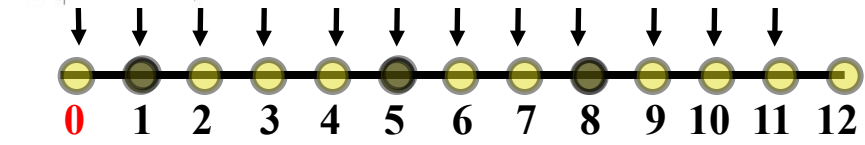
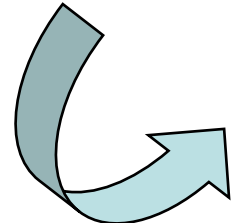
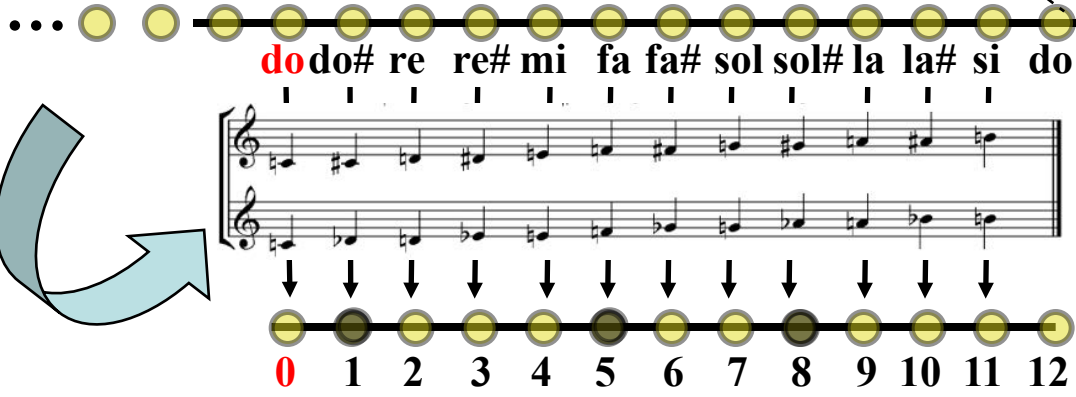
...or rotations !

Musical transpositions are additions...

$$T_k : x \rightarrow x+k \pmod{12}$$



$$\begin{aligned} \text{Do maj} &= \{0, 2, 4, 5, 7, 9, 11\} \\ \text{Do\# maj} &= \{1, 3, 5, 6, 8, 10, 0\} \end{aligned} +1$$



...or rotations !

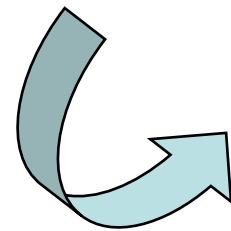
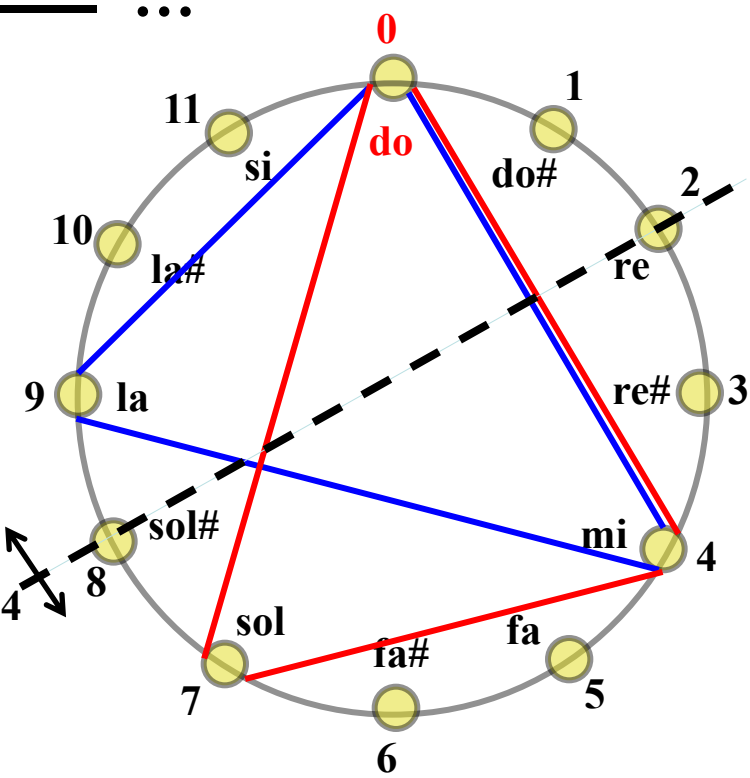
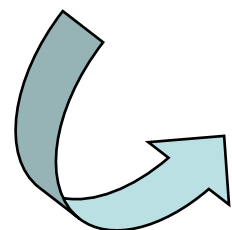
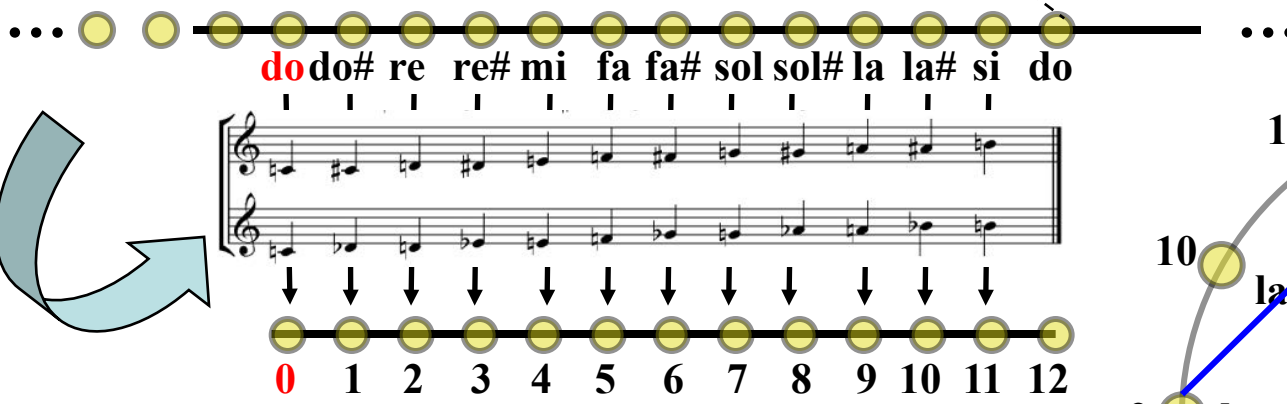
Musical inversions are differences...

$$I : x \rightarrow -x \pmod{12}$$



Do maj = {0, 4, 7}
La min = {0, 4, 9}

$I_4(x) = 4 - x$



... or axial symmetries!

Musical inversions are differences...

$$I : x \rightarrow -x \pmod{12}$$

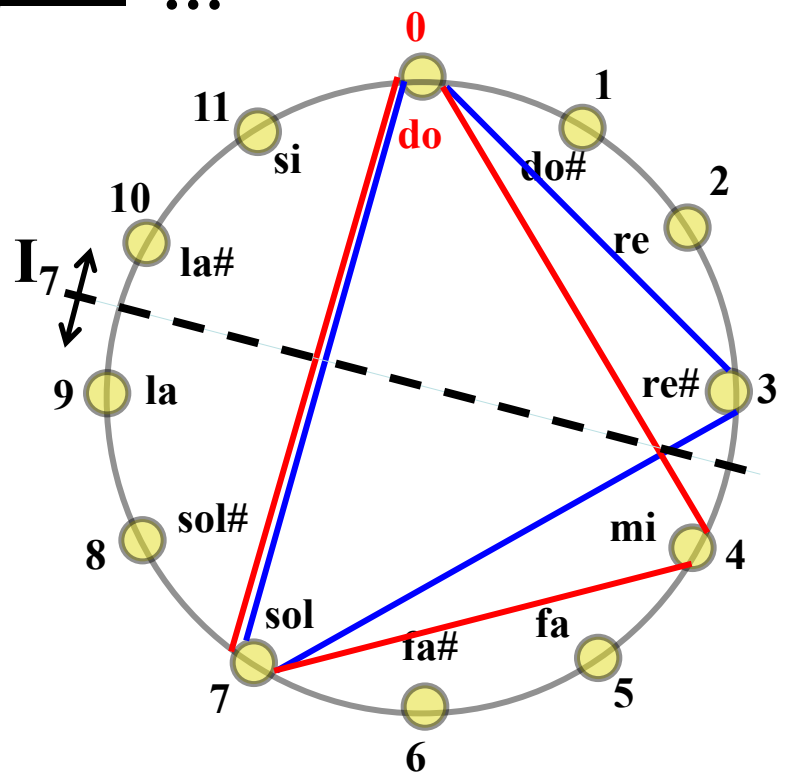
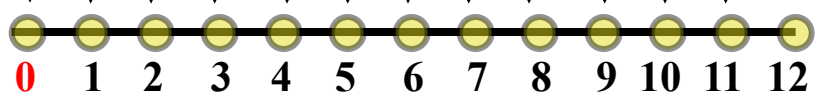


Do maj = {0,4,7}
Do min = {0,3,7}

$$I_7(x) = 7 - x$$



do do# re re# mi fa fa# sol sol# la la# si do



... or axial symmetries!

Musical inversions are differences...

$$I : x \rightarrow -x \pmod{12}$$



Do maj = {0,4,7}
Do min = {0,3,7}

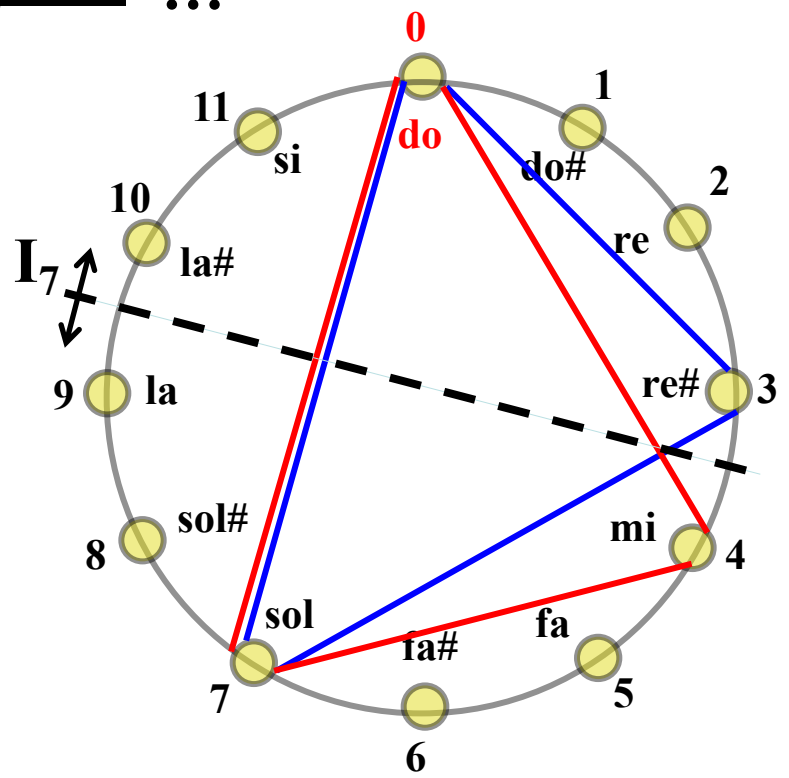
$$I_7(x) = 7 - x$$



do do# re re# mi fa fa# sol sol# la la# si do



0 1 2 3 4 5 6 7 8 9 10 11 12



... or axial symmetries!

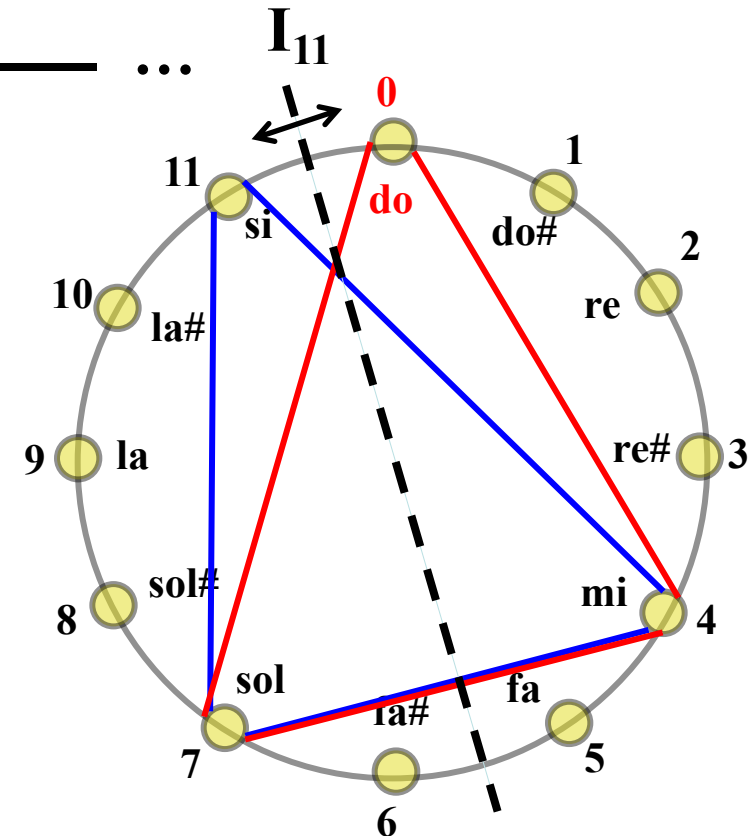
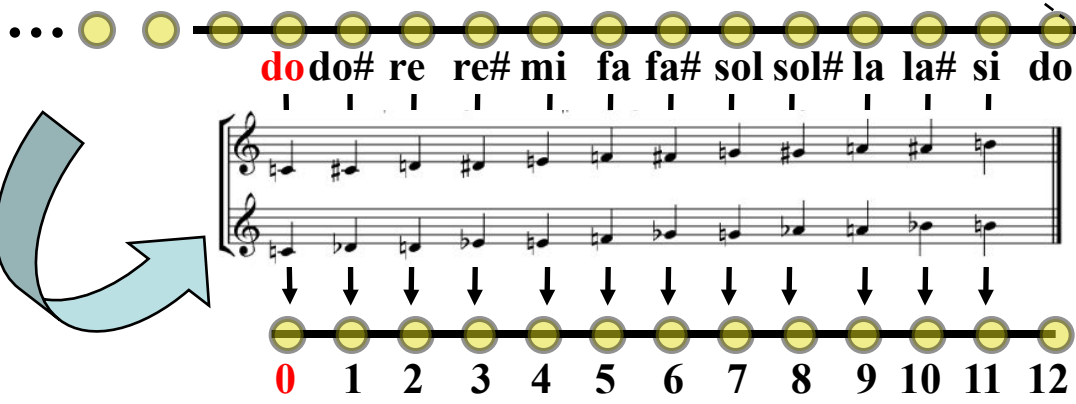
Musical inversions are differences...

$$I : x \rightarrow -x \pmod{12}$$



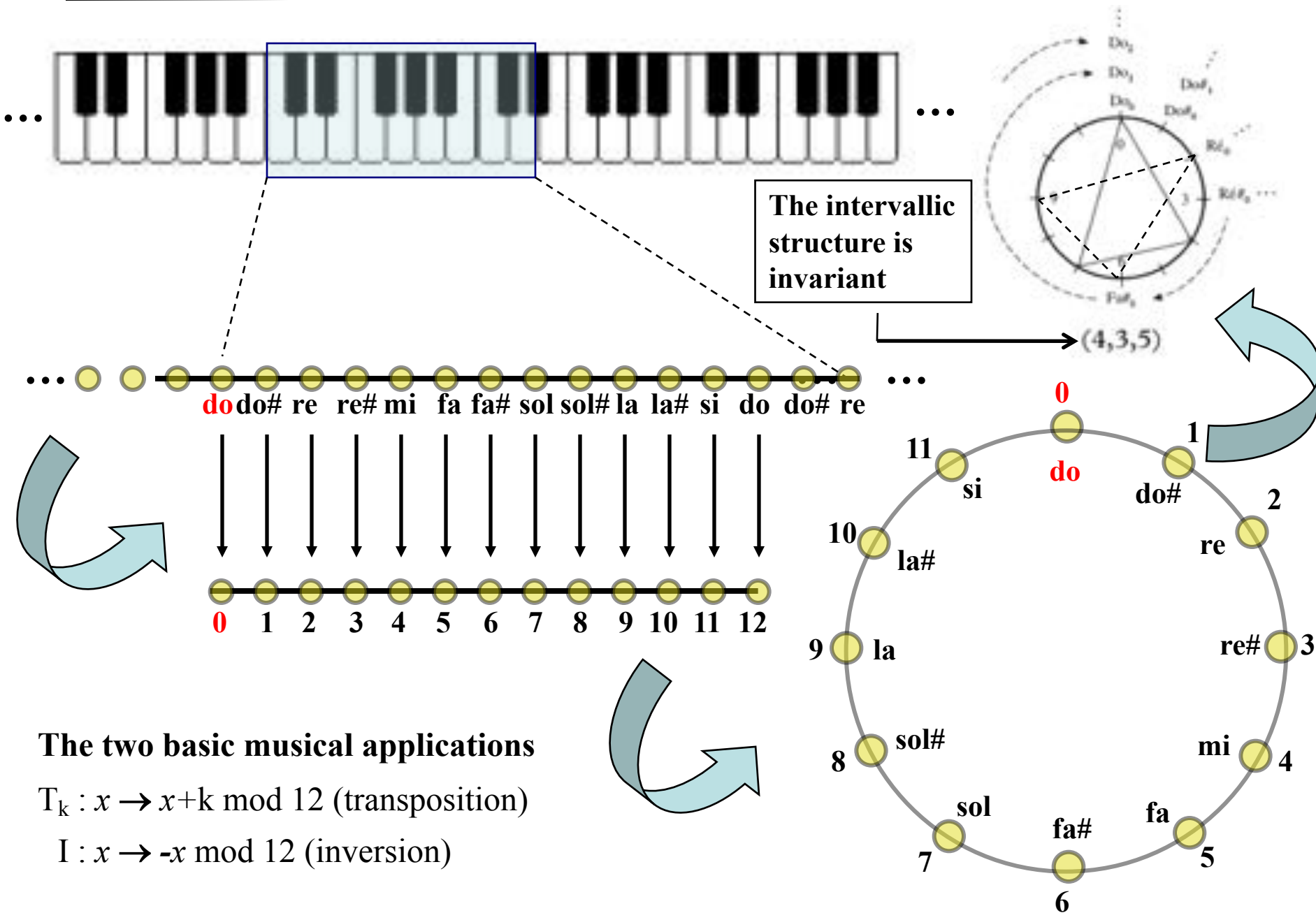
Do maj = {0,4,7}
Mi min = {4,7,11}

$$I_{11}(x) = 11 - x$$

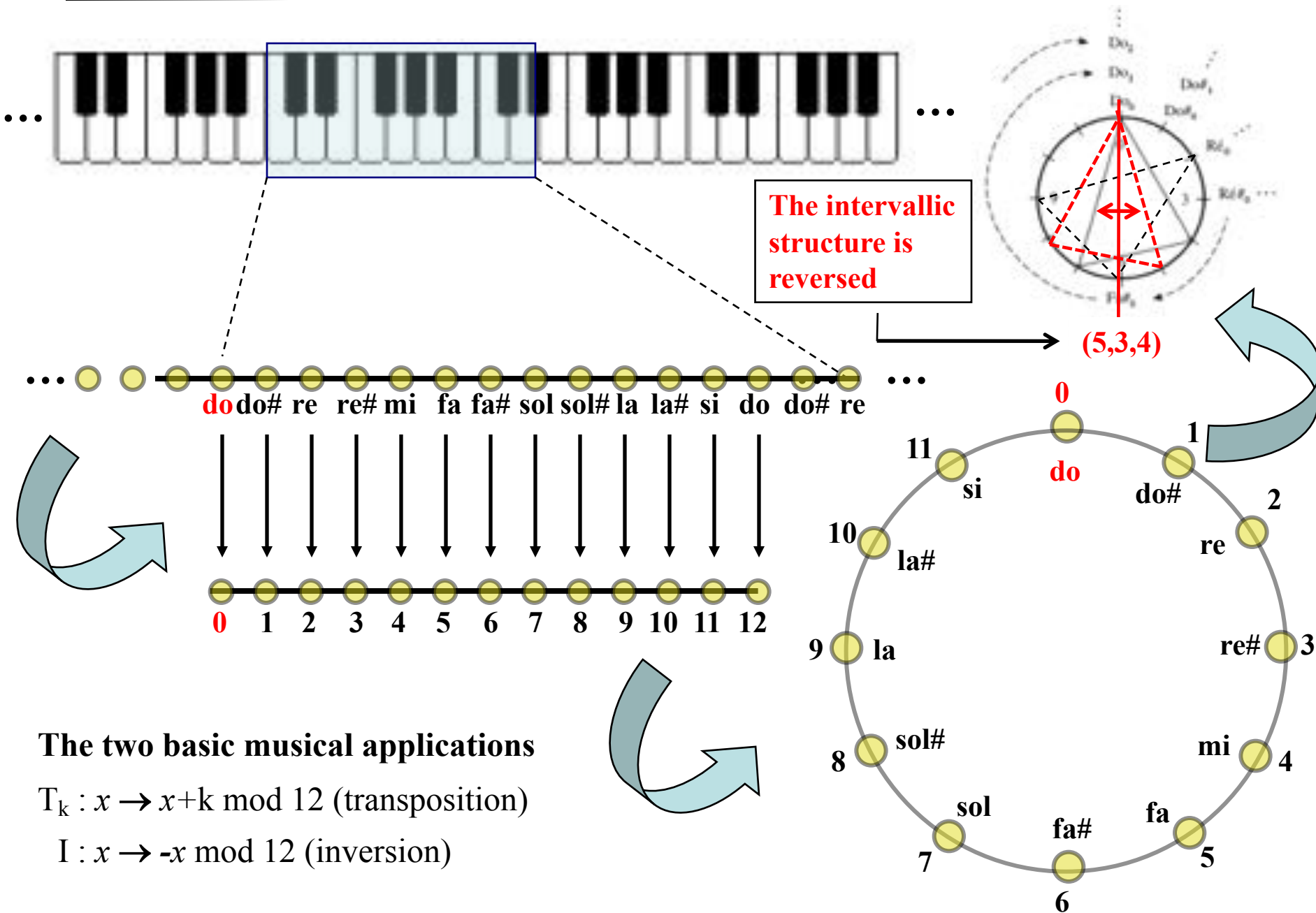


... or axial symmetries!

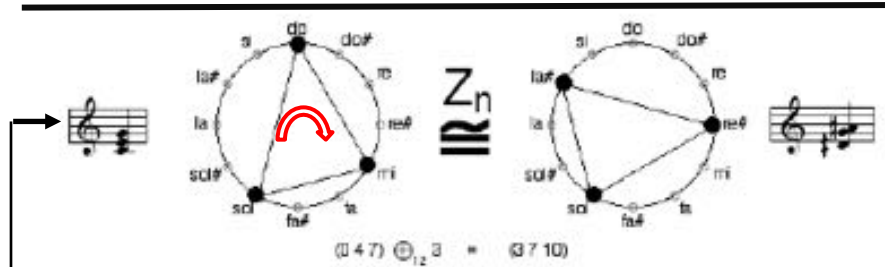
Circular representation and intervallic structure



Circular representation and intervallic structure



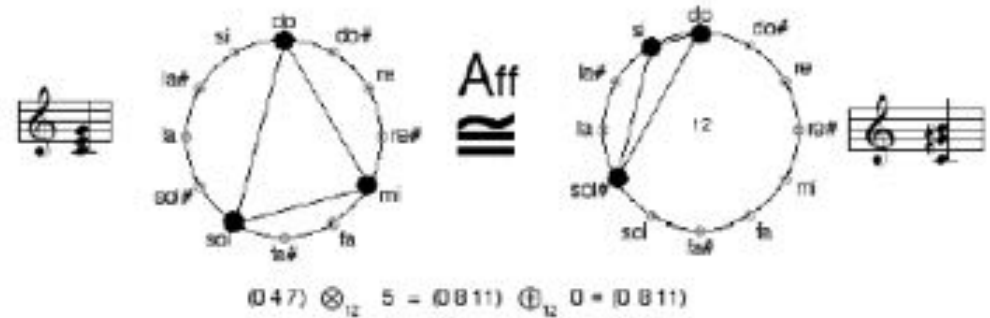
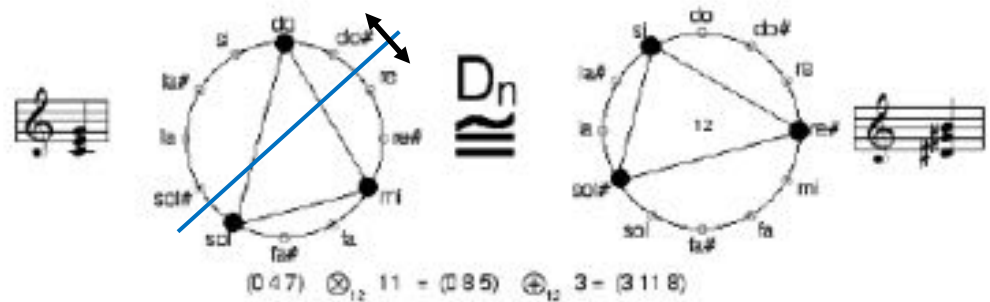
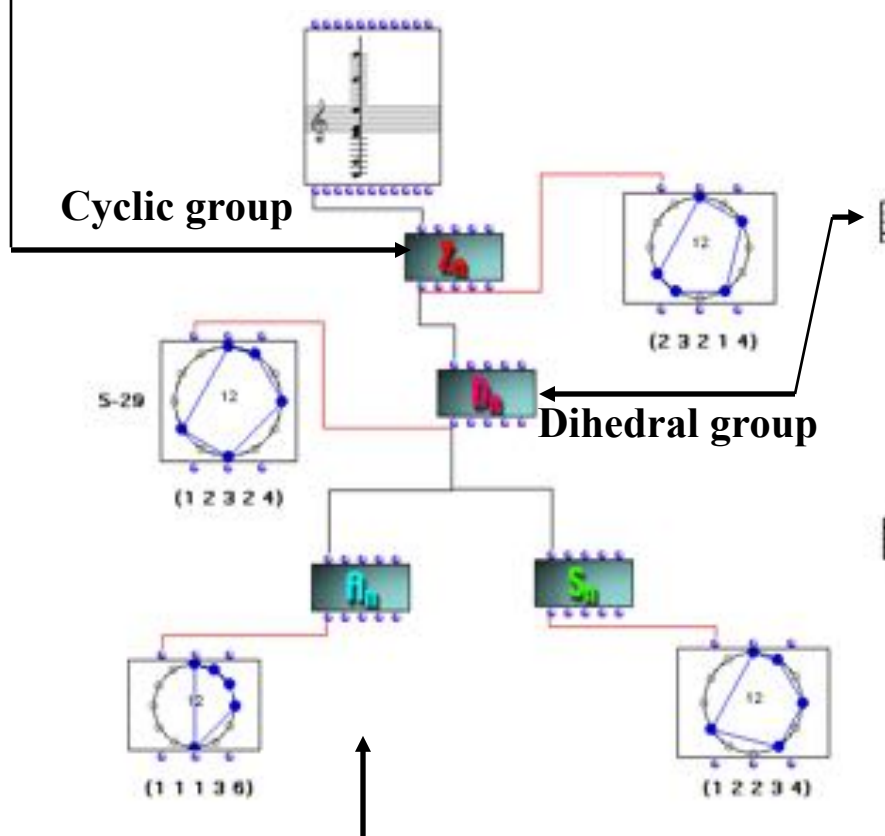
Equivalence classes of musical structures (up to a group action)



$\mathbb{Z}_{12} = \langle T_k \mid (T_k)^{12} = T_0 \rangle$ where $T_k(x) = x + k$

$\mathbb{D}_{12} = \langle T_k, I \mid (T_k)^{12} = I^2 = T_0, ITI = I(IT)^{-1} \rangle$
where $I(x) = -x$

$\text{Aff} = \{ f \mid f(x) = ax + b, a \in (\mathbb{Z}_{12})^*, b \in \mathbb{Z}_{12} \}$



Paradigmatic architecture

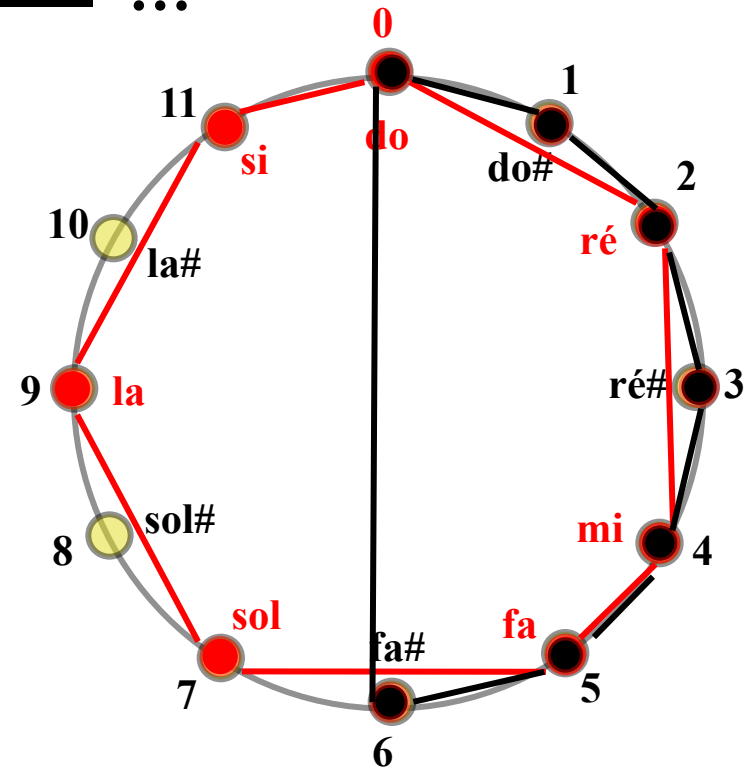
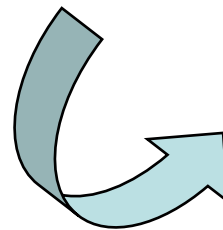
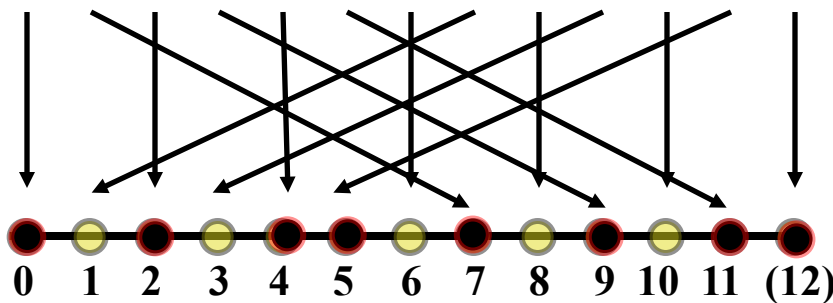
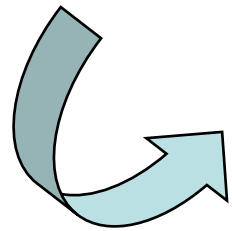
Affine group

Augmentations are multiplications...



$DIA = \{0, 2, 4, 5, 7, 9, 11\}$

$CHRO = \{0, 1, 2, 3, 4, 5, 6\}$



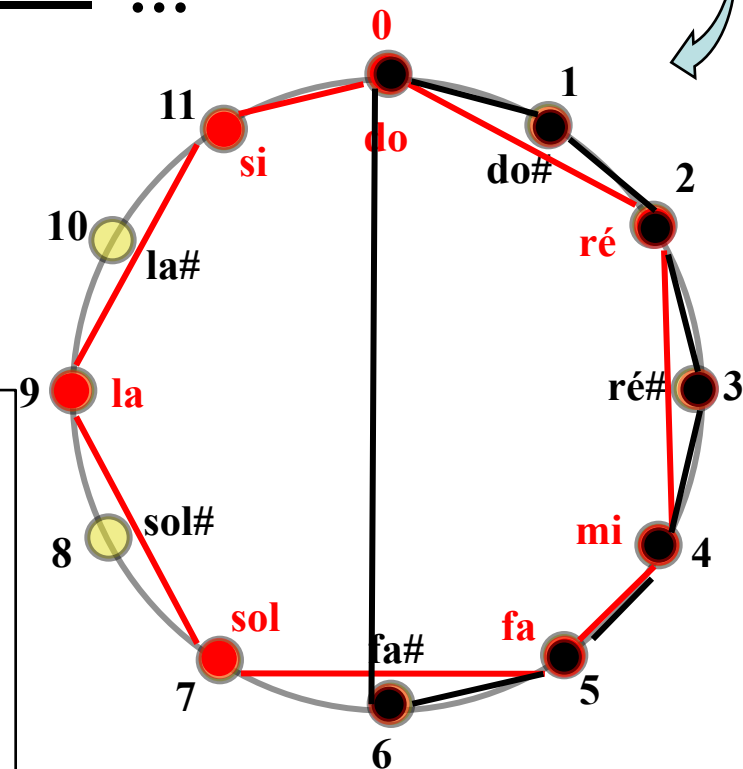
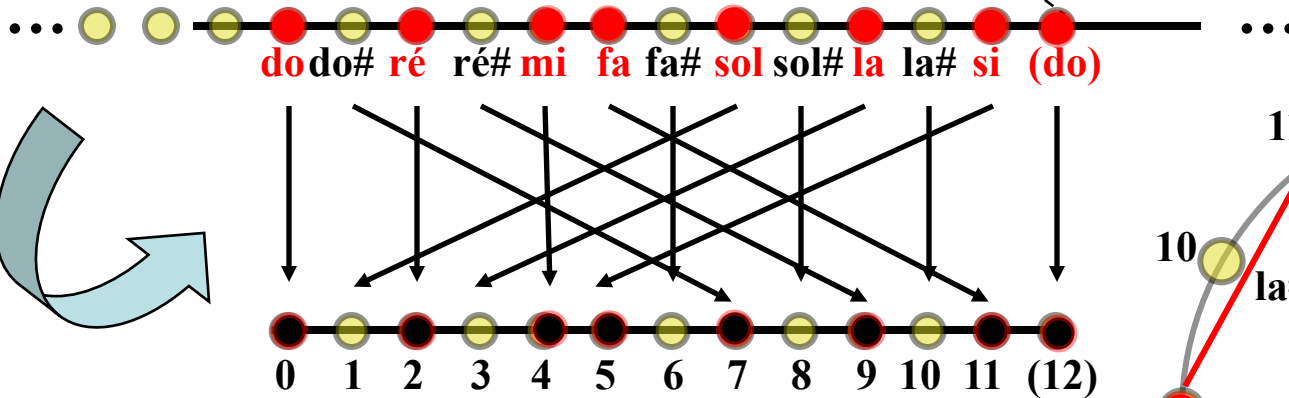
... or affine transformations!

Affine transformations and DIA/CHRO duality



$$DIA = \{0, 2, 4, 5, 7, 9, 11\}$$

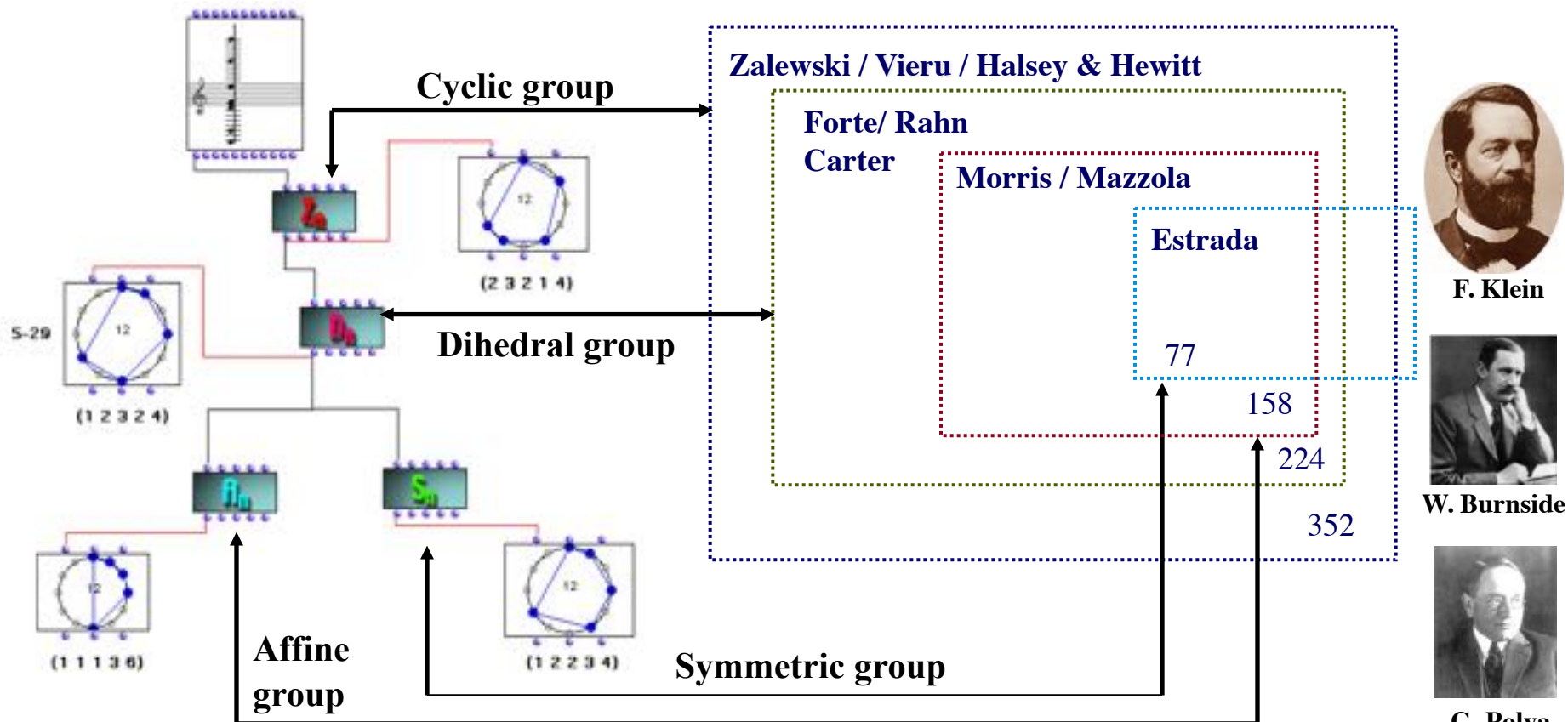
$$CHRO = \{0, 1, 2, 3, 4, 5, 6\}$$



« Le **diatonisme** et le **chromatisme** ne peuvent pas être envisagés en termes de simplicité ou de complexité, comme on le pensait jadis. Il s'agit plutôt d'une question d'**unité des contraires** dans le groupe $\mathbb{Z}/12\mathbb{Z}$ »

(A. Vieru, « The Musical Signification of Multiplication by 7. Diatonicity and Chromaticity », *Muzica*, 1995)

Group actions and the classification of musical structures



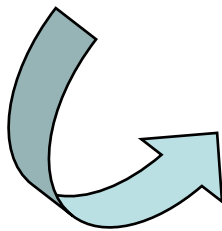
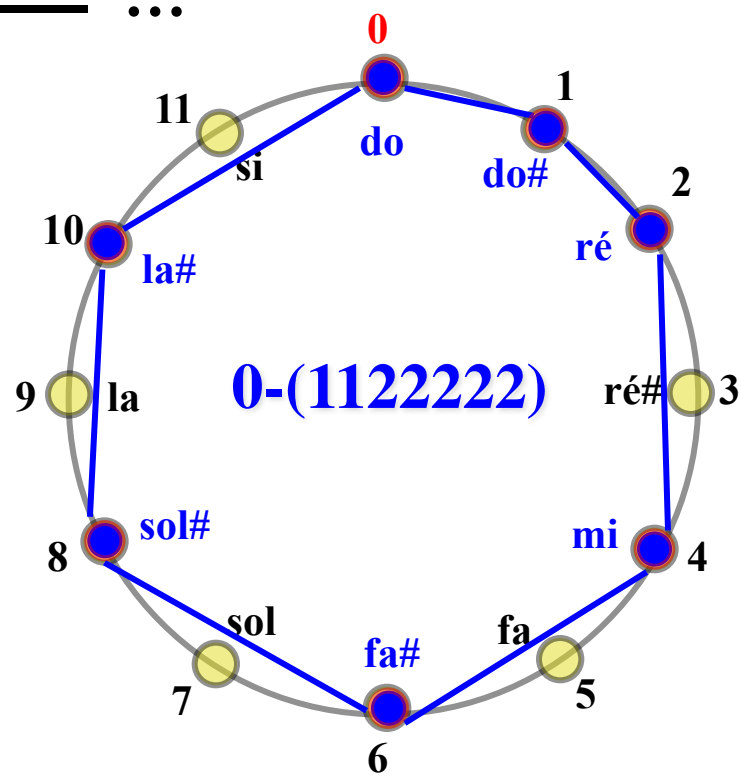
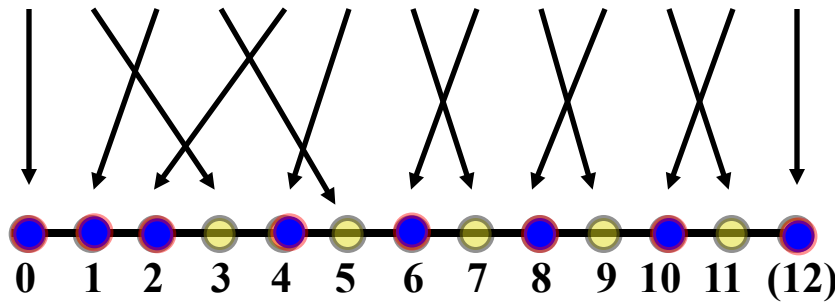
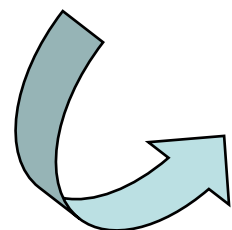
	1	2	3	4	5	6	7	8	9	10	11	12
Z_n	1	6	19	43	66	80	66	43	19	6	1	1
D_n	1	6	12	29	38	50	38	29	12	6	1	1
A_n	1	5	9	21	25	34	25	21	9	5	1	1
S_n	1	6	12	15	13	11	7	5	3	2	1	1

Classifying chords up to permutations of intervals



$$DIA = (2, 2, 1, 2, 2, 2, 1)$$

$$DIA_E = (1, 1, 2, 2, 2, 2, 2)$$



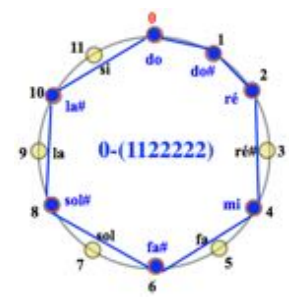
Julio Estrada, *Théorie de la composition : discontinuum – continuum*, université de Strasbourg II, 1994

The permutohedron as a combinatorial space

Julio Estrada, *Théorie de la composition : discontinuum – continuum*, université de Strasbourg II, 1994

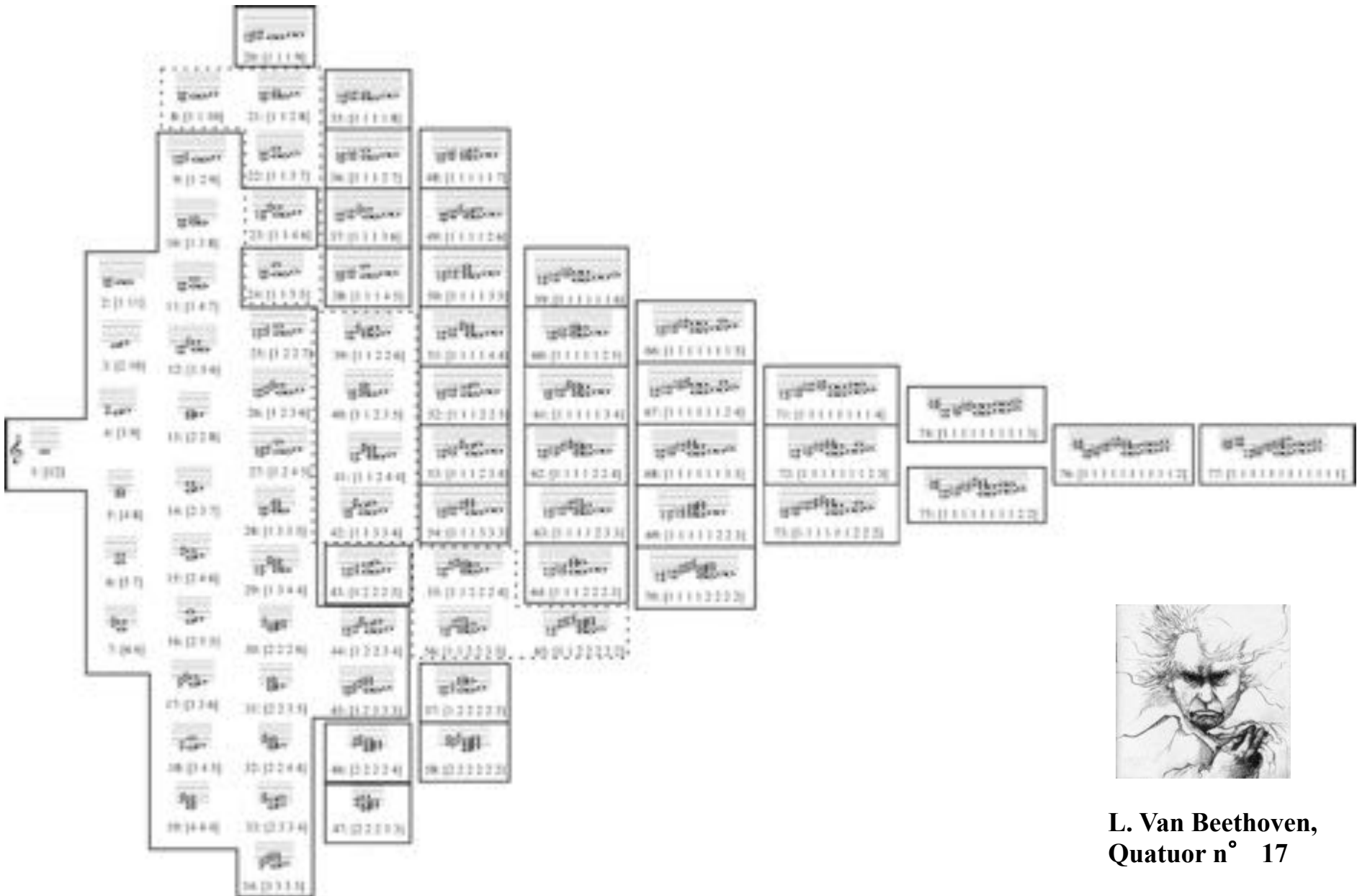
ILLUSTRATION III. REPRESENTATION EN NOTATION MUSICALE DE L'ENSEMBLE DE PARTITIONS DE L'ECHELE DE HAUTEURS D12 : 12 NIVEAUX DE DENSITE, 77 IDENTITES.

$$DIA_E = (1,1,2,2,2,2,2)$$



J. Estrada

The permutohedron as a musical conceptual space



L. Van Beethoven,
Quatuor n° 17

The permutohedron as a musical conceptual space

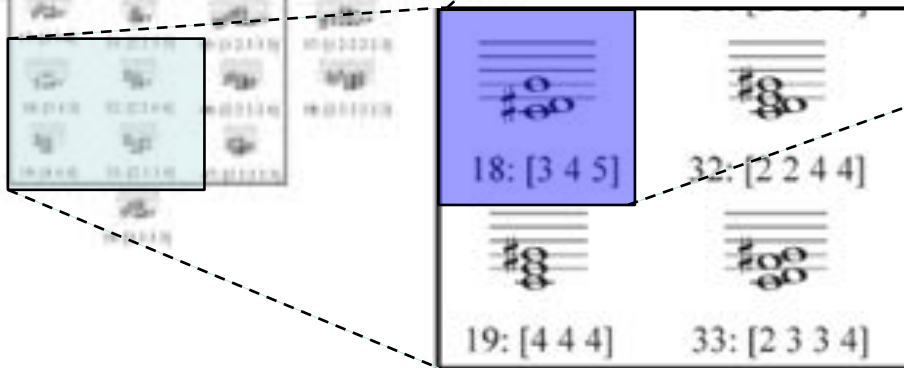
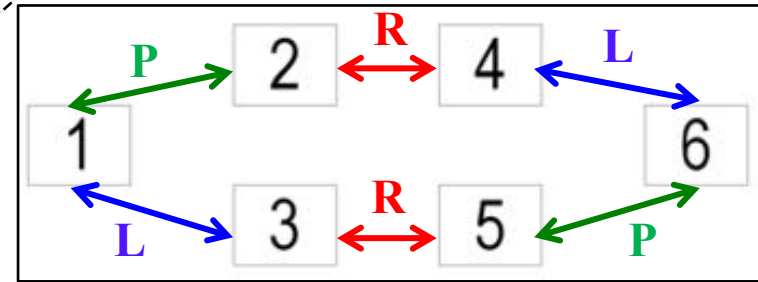
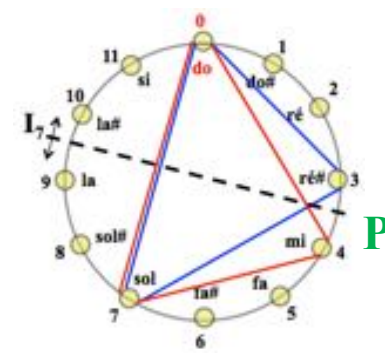
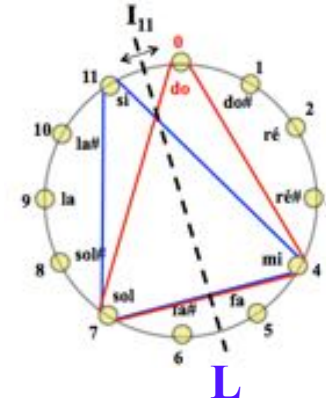
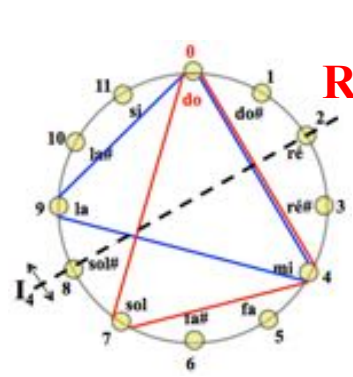
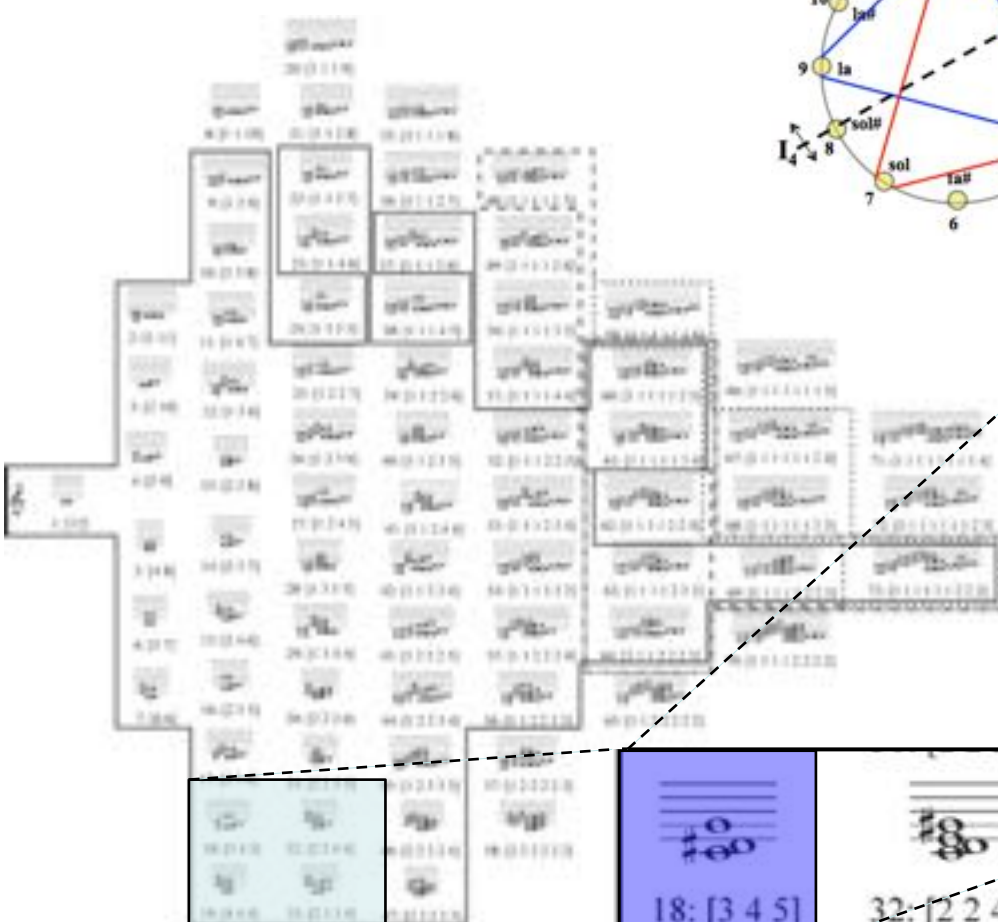
B. Bartok, Quartet n° 4
(3^d movement)



A. Schoenberg, *Six pieces* op. 19

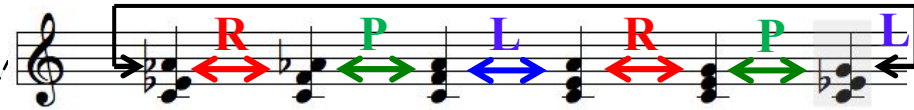


Permutohedron and *Tonnetz*: a structural inclusion

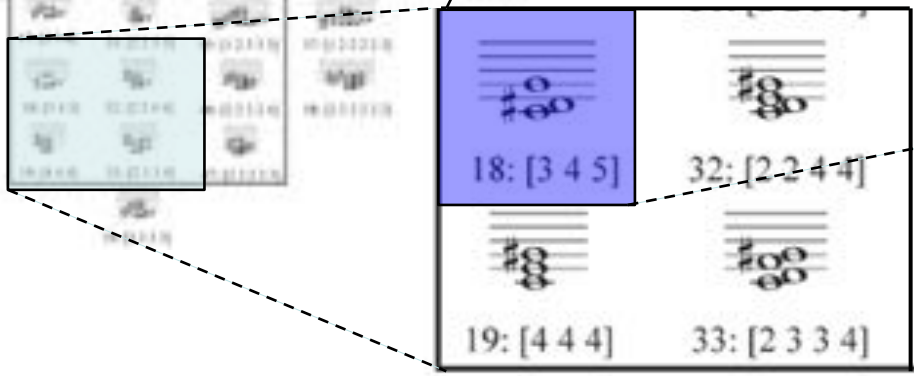
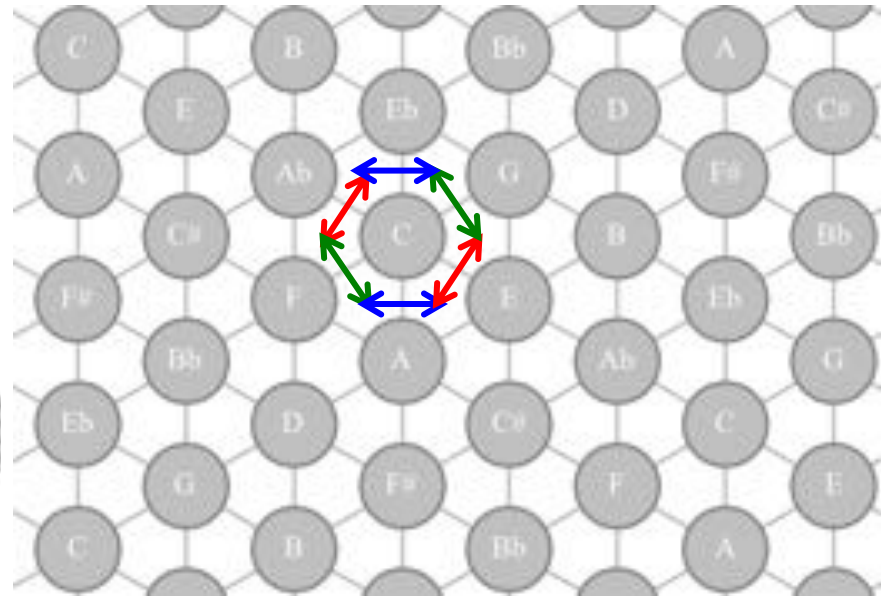


- 1 = (3 4 5)
- 2 = (4 3 5)
- 3 = (3 5 4)
- 4 = (4 5 3)
- 5 = (5 3 4)
- 6 = (5 4 3)

Permutohedron and *Tonnetz*: a structural inclusion

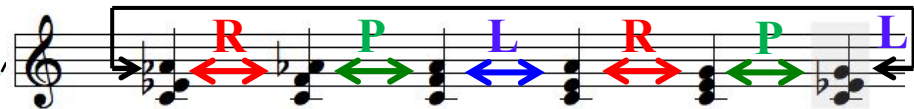


(3 5 4) (5 3 4) (5 4 3) (4 5 3) (4 3 5) (3 4 5)

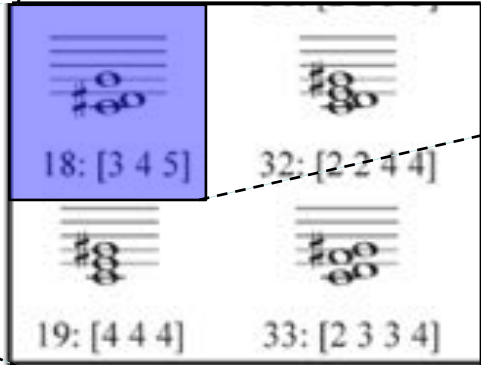
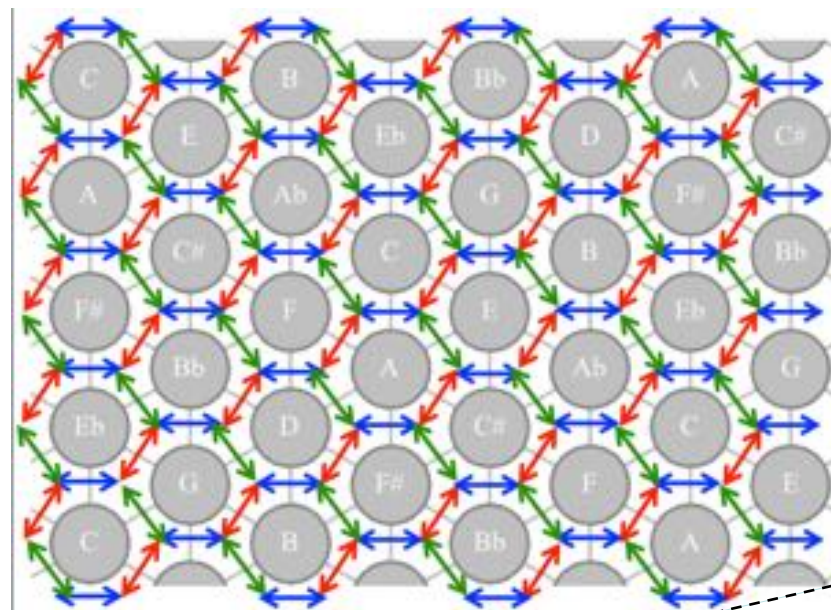


R: C_{maj} → A_{min}
L: C_{maj} → E_{min}
P: C_{maj} → C_{min}

Permutohedron and *Tonnetz*: a structural inclusion



(3 5 4) (5 3 4) (5 4 3) (4 5 3) (4 3 5) (3 4 5)



R: C_{maj} → A_{min}
L: C_{maj} → E_{min}
P: C_{maj} → C_{min}