Musical modeling through graphs and orbifolds

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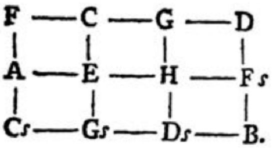
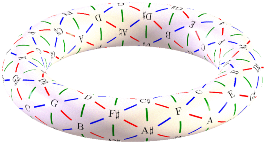
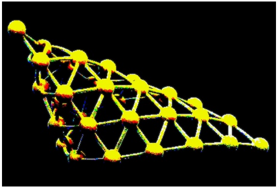
Topological & Geometrical models for Music



(a) Planar tonnetz, polygons and graphs



(b) Tonnetz on a manifold



(c) Chords as points of an orbifold

- (a) Speculum Musicum. Leonhard Euler.
- (c) Tymoczko, Dmitri. "The geometry of musical chords." Science 313.5783 (2006): 72-74.

1. Tonnetz
   - Graph Theory - Sketches
   - The tonnetz as a Graph
   - The geometrical side of the tonnetz
   - The topological side of the tonnetz
2. Orbifold
   - Geometrical intuition
   - Mathematical preliminaries
   - Space of chords - Representation and examples
   - Orbifolds in music

Mathematical preliminaries - Graph theory

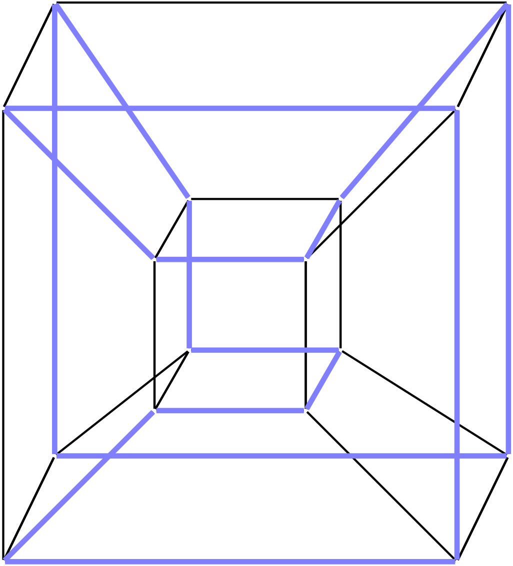


Figure : Graph examples

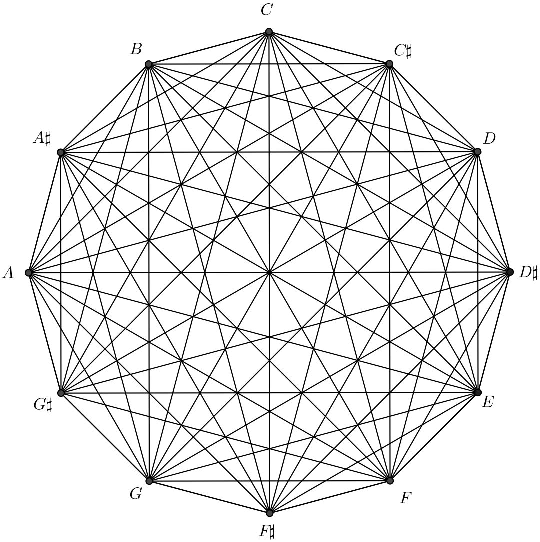


Figure : Graph examples

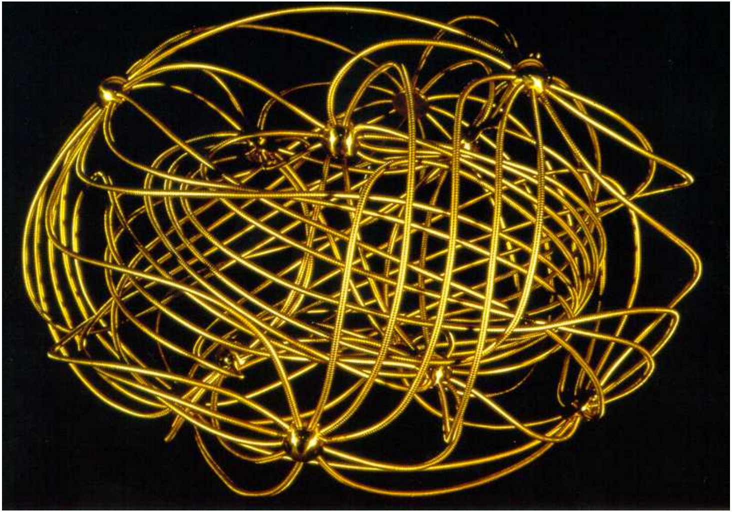


Figure : Graph examples

Definition (Graph)

An *abstract unoriented graph* is a pair (V, E) where V is a finite set and E is a set of unordered pairs of different elements of V. Thus an element of E is of the form $\{v, w\}$ where v and w belong to V and $v \neq w$. We call vertices the elements of V and edges the elements $\{v, w\}$ of E connecting v and w (or w and v).

Definition (Graph)

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Definition (Realization of a graph)

Let (V, E) be an abstract graph. A *realization* of (V, E) is a set of points in $\mathbb{R}^N$, one point for each vertex and segments joining precisely those pairs of points which correspond to edges. The points are the *vertices* and the segments are the *edges*; the realization is termed a *graph*. We require that the following two *intersection conditions* hold:

1. two edges meet either in a common end-point or at all;
2. no vertex lies on an edge except at one of its ends.

The tonnetz as a Graph

Definition (Tonnetz 1)

A tonnetz is a labeled graph, i.e. it is a sextuple $(V, E, L_V, l_V, L_E, l_E)$ such that

- $V \neq \emptyset$ is a set of vertices;
- $E \neq \emptyset$, $E \subseteq V \times V$ is the set of arrows associated to V;
- L_V and L_E are non empty set of vertices' and edges' labels repsectively;
- $l_V : V \to L_V$ is the map which allows to associate a label to a vertex. (l_E is defined in the same way on E).

The tonnetz as a Graph

Comparing two definitions

Definition (Tonnetz 1)

A tonnetz is a labeled graph, i.e. it is a sextuple $(V, E, L_V, l_V, L_E, l_E)$ such that ...[a]

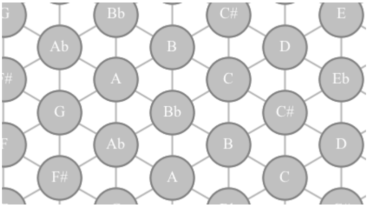
[a] *Žabka, Marek. Generalized Tonnetz and well-formed GTS: A scale theory inspired by the Neo-Riemannians. Mathematics and Computation in Music 2009. 286-298.*

Definition (Tonnetz 2)

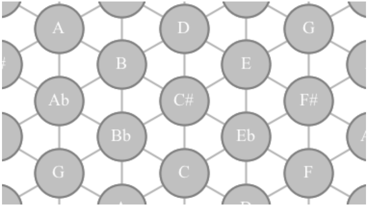
A tonnetz is a note based graph in which points represent notes and chords corresponds to extended shapes of some kind. [a]

[a] *Tymoczko, Dmitri. "The Generalized Tonnetz." Journal of Music Theory 56.1 (2012): 1-52.*

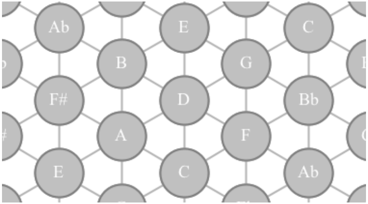
Some examples



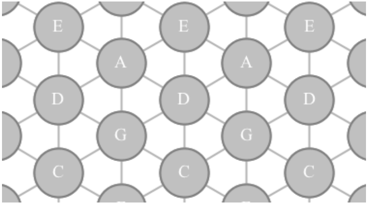
(a) C[1,1,10]



(b) C[1,2,9]



(c) C[2,3,7]

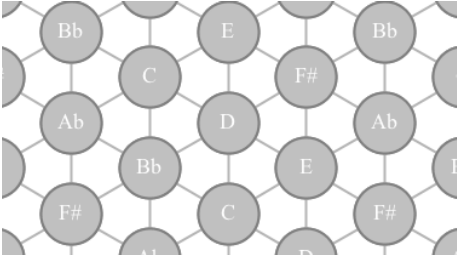


(d) C[2,5,5]

Some examples

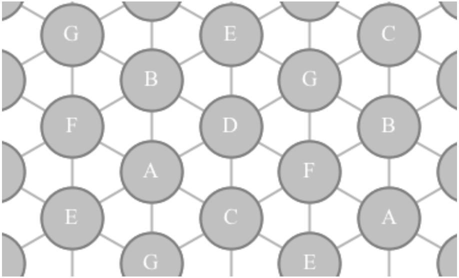
Fact

Let $C[a, b, c]$ be a tonnetz, where $\{a, b, c\} \subset \mathbb{Z}/12\mathbb{Z}$, is a representation of the whole set of pitch classes if a, b or c, is a generator of $\mathbb{Z}/12\mathbb{Z}$.



(a) C[2,2,8]

Some examples



(a) C major diatonic tonnetz

[Bigo, Louis, et al. "Computation and visualization of musical structures in chord-based simplicial complexes." Mathematics and Computation in Music. 2013. 38-51.]

The Geometrical side

From notes graphs to chords diagrams

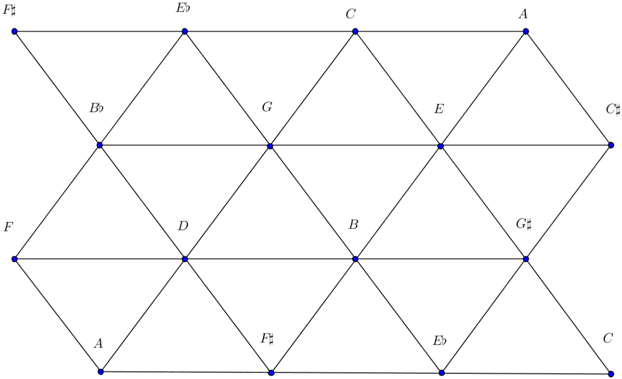


Figure : The Oettingen-Riemann Tonnetz

The Geometrical side

From notes graphs to chords diagrams

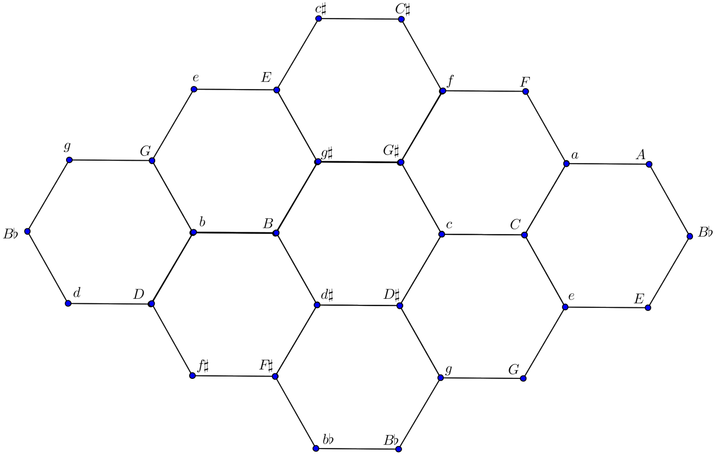


Figure : Chicken-wire torus, Douthett and Steinbach (1998).

The Geometrical side

From notes graphs to chords diagrams

Definition (Convex polytope)

A convex polytope is a compact convex set with a finite number of extreme points (i.e. vertices)[a].

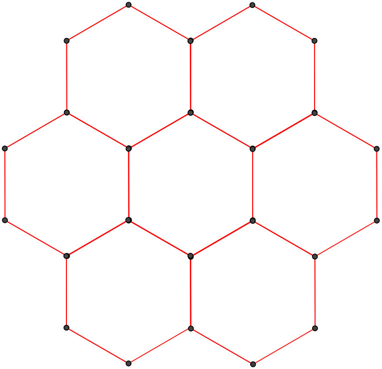
[a] [Grunbaum, Branko, et al. Convex polytopes. Vol. 2. Springer, 1967.]

Definition (Dual polytope)

Given a convex polytope P, the dual polytope associated to P is a polytope P^* whose vertices correspond to the faces of P.

The Geometrical side

From notes graphs to chords diagrams



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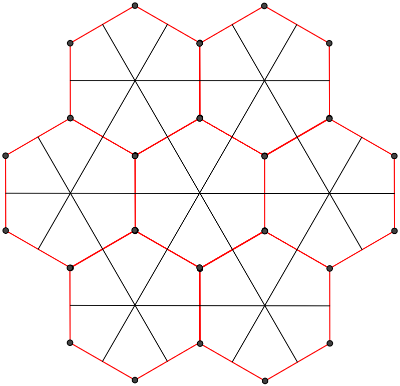
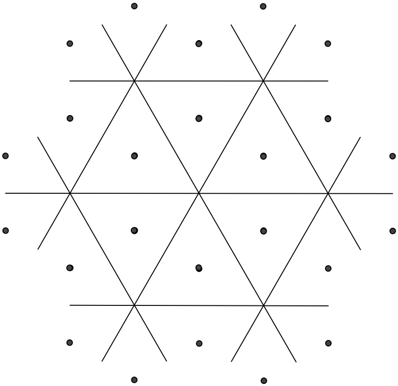
From notes graphs to chords diagrams



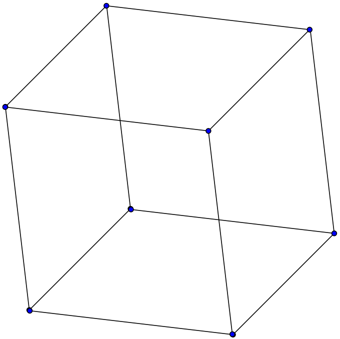
The Geometrical side

From notes graphs to chords diagrams



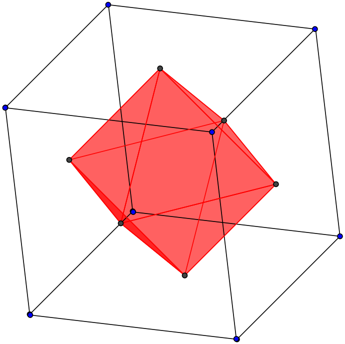
The Geometrical side

From notes graphs to chords diagrams



The Geometrical side

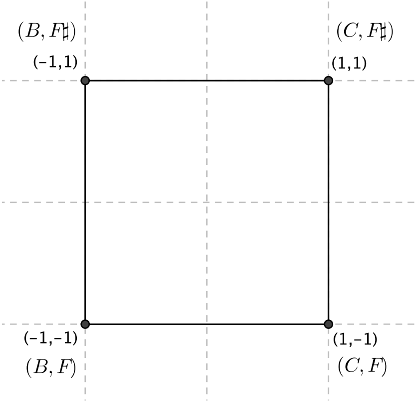
From notes graphs to chords diagrams



The Geometrical side

From notes graphs to chords diagrams

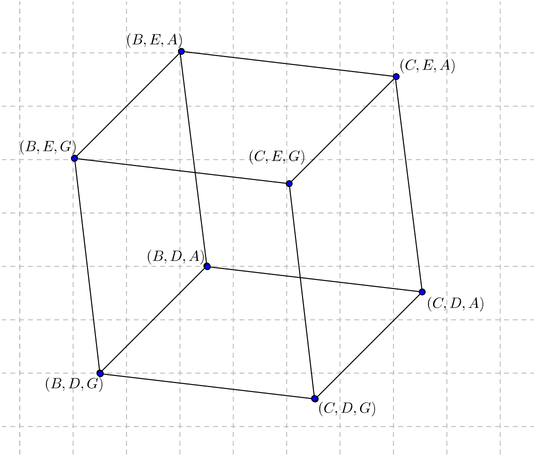
It is pretty natural to associate an interval to each vertex of a square.



The Geometrical side

From notes graphs to chords diagrams

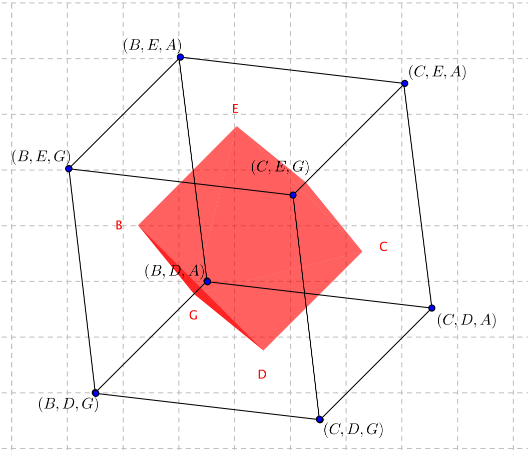
...and triads to a cube



The Geometrical side

From notes graphs to chords diagrams

The dual polytope allows to go back to the notes



Some conclusions

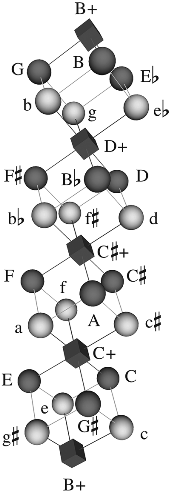
about the geometrical generalization of the tonnetz

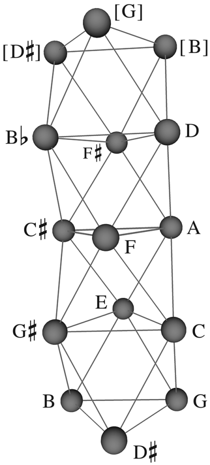
Fact (1)

The Tonnetz, while apparently a two-dimensional structure, can also be understood as a three-dimensional circle of octahedra linked by shared faces. The shared faces represent augmented triads, which do not appear on the traditional Tonnetz. The two versions of the Tonnetz are graph-theoretically identical but geometrically (and topologically) distinct.

Some conclusions

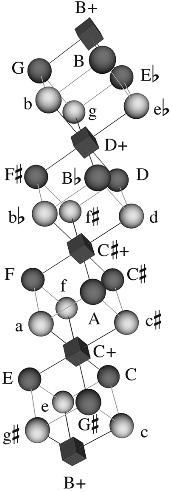
about the geometrical generalization of the tonnetz





Some conclusions

about the geometrical generalization of the tonnetz



To form the note-based graph
of nearly even three-note chromatic chords,
we start with the chord-based graph at the
center of three-note chromatic chord space

Some conclusions

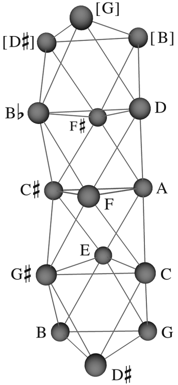
about the geometrical generalization of the tonnetz

then we replace each cube with its dual and glue the resulting octahedra together in the appropriate way.

Some conclusions

about the geometrical generalization of the tonnetz

This produces
a circle of octahedra linked by common faces. Here, triangles represent major, minor, and augmented chords, and edge-preserving flips represent single-semitone
voice leading. Note that the top face is
a $2\pi/3$ rotation of the bottom face, indicating that the structure is globally twisted.



Some conclusions

about the geometrical generalization of the tonnetz

Fact (2)

Any sufficiently large note-based graph will inevitably contain either flip restrictions or redundancies that is, the graph will either contain flips that represent nonstepwise voice leadings or multiple representations of the same chord. The traditional Tonnetz is unusual in that it lacks both features.

Some conclusions

about the geometrical generalization of the tonnetz

Fact (3)

Chord-based voice-leading graphs are associated with note-based Tonnetze by the geometrical property of duality. However, the duality relation obtains not between graphs considered as unified wholes, but rather between their cubic and octahedral components.

See *Tymoczko, Dmitri. "The Generalized Tonnetz." Journal of Music Theory 56.1 (2012): 1-52*, for further details and a deeper dissertation.

The topological side

Trajectories on the tonnetz and homology

Example

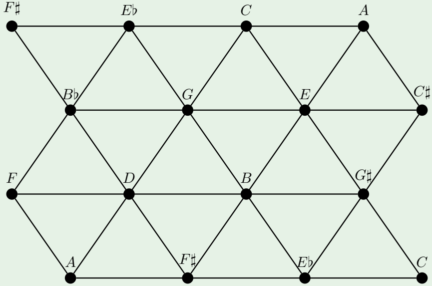


Figure : $\mathcal{S}_1\left(\left|\mathcal{K}_T\right|[3,4,5]\right)$

The topological side

Trajectories on the tonnetz and homology

Example

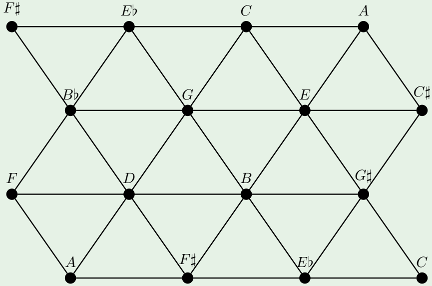


Figure : $\mathcal{S}_1\left(|\mathcal{K}_T|\,[3,4,5]\right) \leftarrow ?$

The topological side

Trajectories on the tonnetz and homology

Definition (n-simplex)

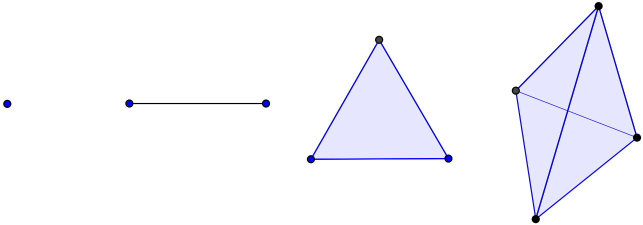
A n-simplex in $\mathbb{R}^k$ is a set of the form

$$\Delta_n = \left\{ \sum_{i=0}^{n} t_i v_i \, s.t.\, 0 \leq t_i \leq 1 \text{ end } \sum_{i=0}^{n} t_i = 1 \right\},$$

where v_i are (affine) independent points of $\mathbb{R}^k$.

The topological side

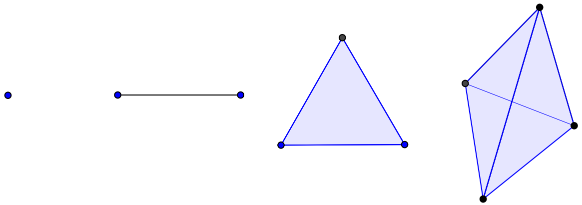
Trajectories on the tonnetz and homology



The topological side

Trajectories on the tonnetz and homology

Let σ and τ be two simplices in $\mathbb{R}^n$. τ is a face of σ if $V_\tau \subseteq V_\sigma$. If $V_\tau \subset V_\sigma$ then τ is said to be a proper face of σ.



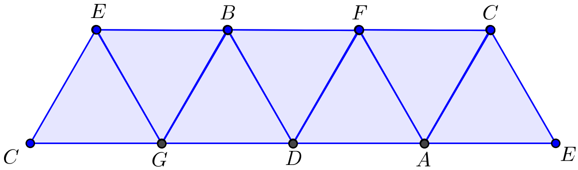
The topological side

Trajectories on the tonnetz and homology

Definition (Simplicial complex)

A finite collection K of simplices in $\mathbb{R}^n$ is said to be a simplicial complex if

- $\sigma \in K$, then every face of σ belongs to K;
- $\sigma_1 \in K$, $\sigma_2 \in K$, then either $\sigma_1 \cap \sigma_2 = \emptyset$ or else $\sigma_1 \cap \sigma_2$ is a common face of both σ_1 and σ_2.



The topological side

Trajectories on the tonnetz and homology

Remark

The dimension of a simplicial complex K is the greatest non-negative integer n with the property that K contains an n-simplex. The union of all the simplices of K is a compact subset $|K|$ of $\mathbb{R}^k$ referred to as the polyhedron of K. (The polyhedron is compact since it is both closed and bounded in $\mathbb{R}^k$.)

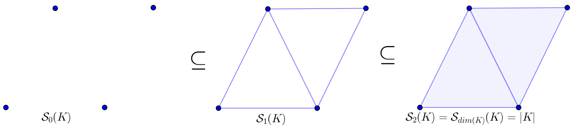
Definition (Subcomplex)

Let K be a simplicial complex in $\mathbb{R}^k$. A subcomplex of K is a collection L of simplices belonging to K with the following property:
if $\sigma \subset L$ then every face of σ belongs to L.

The topological side

Trajectories on the tonnetz and homology

A particular family of subcomplexes of a simplicial complex K is the filtration given by its n-skeletons, where $n \in \{0, \ldots, dim(K)\}$.



[Henri Cartan in 1937 and subsequently used by Bourbaki in their book Topologie Générale]

The topological side

Trajectories on the tonnetz and homology

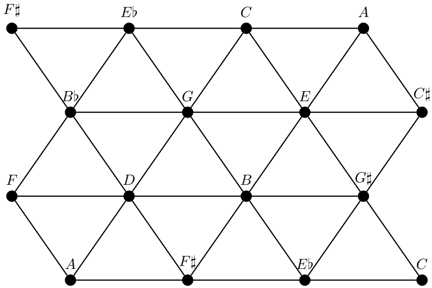
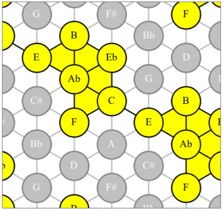


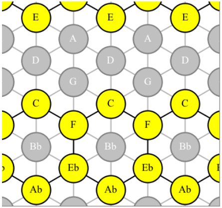
Figure : $\mathcal{S}_1\left(|\mathcal{K}_T|\,[3,4,5]\right) \leftarrow$ is the 1-skeleton of the realization of the complex $\mathcal{K}_T(3,4,5)$

Simplicial complexes on the tonnetz

Each musical piece can be seen has a simplicial complex. In the following figures an instant of Summertime is depicted



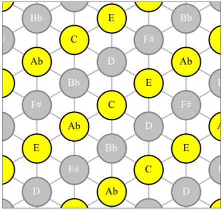
(a) C(3,4,5)



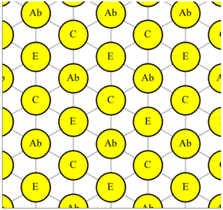
(b) C(2,2,5)

Simplicial complexes on the tonnetz

Each musical piece can be seen has a simplicial complex. In the following figures an instant of Summertime is depicted



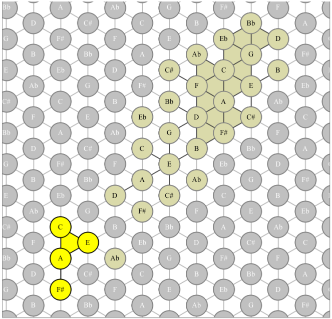
(a) C(2,2,8)



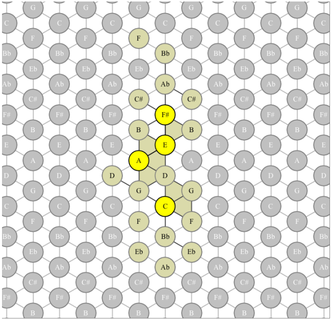
(b) C(4,4,4)

Trajectories on the tonnetz

The same holds for trajectories, this allows to relate harmony and time



(a) Trace in C(3,4,5)



(b) Trace in C(2,5,5)

The Euler characteristic and the Betti numbers

Definition

Given a polyhedron P, the Euler characteristic χ is given by

$$\chi(P) = V - E + F$$

Definition

Given a simplicial complex K, $\chi(K)$ is given by the alternate sum

$$\chi(K) = \sum_{i=0}^{n} (-1)^i k_i$$

where $k_i = |\{k - simplices \in K\}|$

The Euler characteristic is topologically invariant.

The Euler characteristic and the Betti numbers

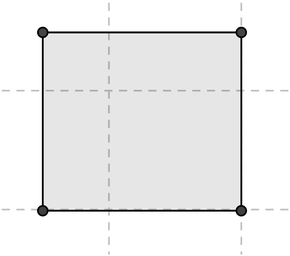
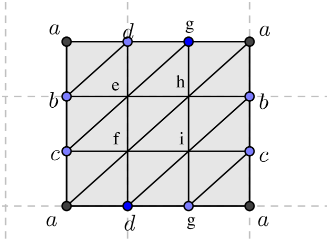


Figure : Computing $\chi(\mathbb{T}^2)$

The Euler characteristic and the Betti numbers



$$\chi(\mathbb{T}^2_{\Delta_1}) = 9 - 27 + 18 = 0$$

Figure : Computing $\chi(\mathbb{T}^2)$

The Euler characteristic and the Betti numbers

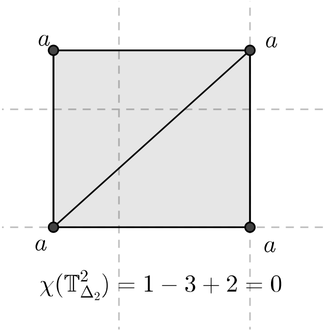


Figure : Computing $\chi(\mathbb{T}^2)$

The Euler characteristic and the Betti numbers

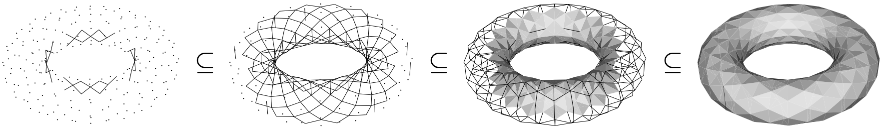
The tonnetz can be seen as a simplicial complex, in which we can compute some invariants which identifies the topology of a certain space.

Definition (Betti numbers - intuitive definition)

Given a simplicial complex, the Betti numbers count the holes of the complex in the following way:

- β_0 counts the number of connected components;
- β_1 counts the holes of dimension 1;
- β_n counts the n-dimensional holes.

The Euler characteristic and the Betti numbers



$\beta_0 = 220$	$\beta_0 = 73$	$\beta_0 = 1$	$\beta_0 = 1$
$\beta_1 = 8$	$\beta_1 = 161$	$\beta_1 = 177$	$\beta_1 = 2$
$\beta_2 = 0$	$\beta_2 = 0$	$\beta_2 = 0$	$\beta_2 = 1$

Figure : Betti numbers for a *growing* torus

The Euler characteristic and the Betti numbers

Link between χ and β_i.

Definition (Boundary homomorphism)

Let K be a simplicial complex. We define the boundary homomorphism $\partial_n : C_n(K) \to C_{n-1}(K))$, where C_{-1} is trivial, as

$$\partial_n (< v_0, \ldots, v_n >) = \sum_{i=0}^{n} (-1)^i < v_0, \ldots, \hat{v}_i, \ldots, v_n >$$

Remark

$\partial_n = 0$ *for all* $n > dim(K)$ *and for* $n \leq 0$*;*
$\partial_n \circ \partial_{n+1} = 0$

The Euler characteristic and the Betti numbers

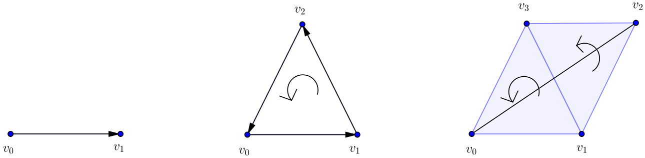


Figure : Simplicial complex homology

$$\partial[v_0, v_1] = [v_1] - [v_0]$$

$$\partial[v_0, v_1, v_2] = [v_1, v_2] - [v_0, v_2] + [v_0, v_1]$$

$$\partial[v_0, v_1, v_2, v_3] = [v_1, v_2, v_3] - [v_0, v_2, v_3] + [v_0, v_1, v_3] - [v_0, v_1, v_2]$$

The Euler characteristic and the Betti numbers

Definition

We define the i-th homology group H_i as

$$H_i = Z_i/B_i$$

where $Z_i = ker(\partial_i)$ and $B_i = im(\partial_{i+1})$.

In fact $\partial^2 = 0$ implies that $im(\partial_i) \subseteq ker(\partial_{i-1})$.

Definition

$\beta_i = dim(H_i)$ and

Fact

$\chi = \sum_{i=0}^{n}(-1)^i \beta_i = \sum_{i=0}^{n}(-1)^i dim(Z_i/B_i)$.

The Euler characteristic and the Betti numbers

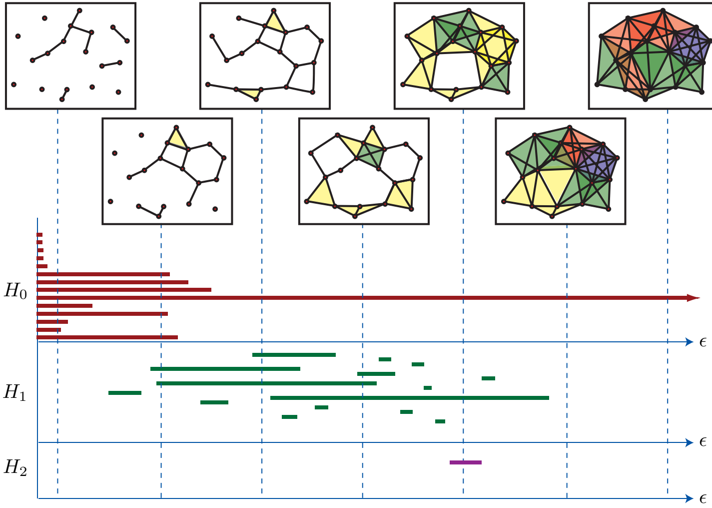


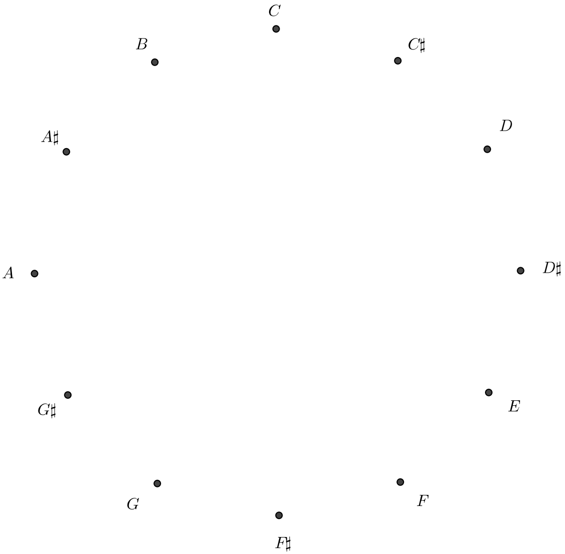
Figure : Simplicial complex homology

The Euler characteristic and the Betti numbers

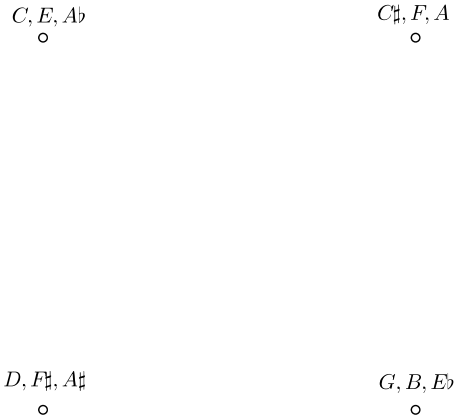
Dimension	Complex	Cardinality	Betti numbers	χ
0	$K_\emptyset$	0	0	0
0	$K_T[0]$	12	$[12]$	12
1	$K_T[5,7]$	12	$[1,1]$	0
2	$K_T[3,4,5]$	12	$[1,13,0]$	-12
2	$K_T[4,4,4]$	4	$[4,0,0]$	4

Table : Chords' complexes classification

The Euler characteristic and the Betti numbers



The Euler characteristic and the Betti numbers



The Euler characteristic and the Betti numbers

Fact

Betti numbers are not enough. One requires a means of declaring which holes are essential and which can be safely ignored. The standard topological constructs of homology and homotopy offer no such slack in their strident rigidity: a hole is a hole no matter how fragile or fine.

[Robert Ghrist Barcodes: the persistent topology of data. Bulletin of the American Mathematical Society (New Series) 45, 1 (2008), 61-75.]

The voice leading space

Intuition

Here follows a list of well known identifications in music

1. Octave
$$x \sim_{\mathbf{O}} x + 12k,\ k \in \mathbb{Z}$$
2. Transposition
$$x \sim_{\mathbf{T}} x + c(1, ..., 1),\ c \in \mathbb{R}$$
3. Permutation
$$x \sim_{\mathbf{P}} \sigma(x),\ \sigma \in \mathcal{S}_n$$
4. Inversion
$$x \sim_{\mathbf{I}} -x$$
5. Multiplicity
$$(..., x_i, x_{i+1}) \sim_{\mathbf{C}} (..., x_i, x_i, x_{i+1}, ...)$$

The voice leading space

Intuition

Each identification corresponds to a specific space

1. Octave

$$\mathbb{T}^n$$

2. Transposition

$$\mathbb{R}^n \text{ o } \mathbb{T}^n$$

3. Permutation

$$\mathbb{R}^n/\mathcal{S}_n \text{ or } \mathbb{T}^n/\mathcal{S}_n$$

4. Inversion

$$\mathbb{R}^n/\mathbb{Z}_2 \text{ or } \mathbb{T}^n/\mathbb{Z}_2$$

5. Multiplicity

$SP^n(\mathbb{R})$ or infinite dimensional *Ran space*

The voice leading space

Intuition

What we need to build an orbifold:

1. Octave

$$x \sim_{\mathbf{O}} x + 12k,\ k \in \mathbb{Z}$$

2. Permutation

$$x \sim_{\mathbf{P}} \sigma(x),\ \sigma \in \mathcal{S}_n$$

3. Multiplicity

$$(\ldots, x_i, x_{i+1}) \sim_{\mathbf{C}} (\ldots, x_i, x_i, x_{i+1}, \ldots)$$

The voice leading space

Mathematical Preliminaries

- Dmitri Tymoczko, *The Geometry of Musical Chords*, Science, 2006;
- Dmitri Tymoczko, Rachel Wells Hall, *Submajorization and the Geometry of Unordered Collections*, preprint, 2010;

The voice leading space

Mathematical Preliminaries

Definition (Multiset)

A multiset is a couple (A, m) where A is a set and $m : A \to \mathbb{N}$ is a map such that $m : a \mapsto n$ is the multiplicity of $a \in A$.

Definition (Voice leading)

A voice leading among two multisets $\{x_1, ..., x_m\}$ e $\{y_1, ..., y_n\}$ is a multiset of ordered couples (x_i, y_j) denoted by $\{x_1, ..., x_m\} \to \{y_1, ..., y_n\}$.

The lowest dimensional case: intervals

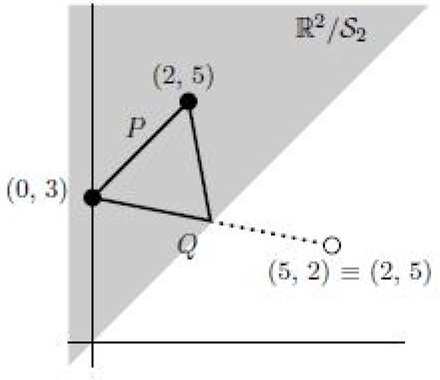


Figure : $\mathbb{R}^2/\mathcal{S}_2$

The voice leading space

Mathematical Preliminaries

The lowest dimensional case: intervals

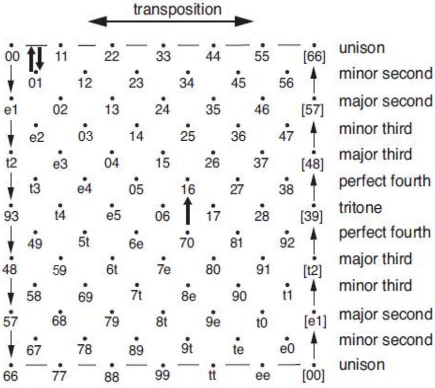
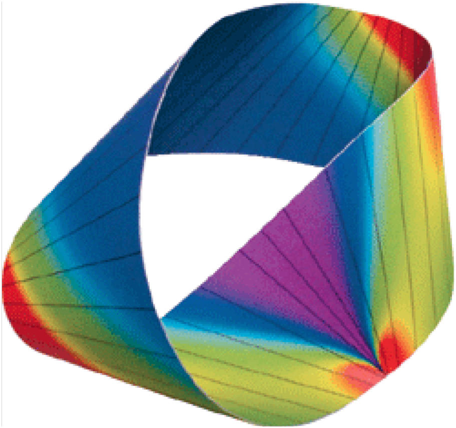


Figure : $\mathbb{T}^2/\mathcal{S}_2$

The voice leading space
Mathematical Preliminaries

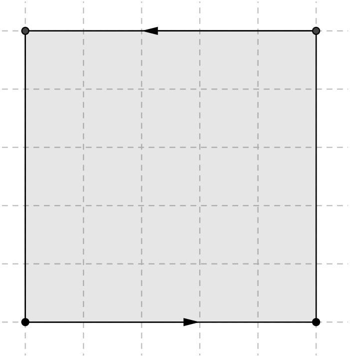
The lowest dimensional case: intervals



The voice leading space

Mathematical Preliminaries

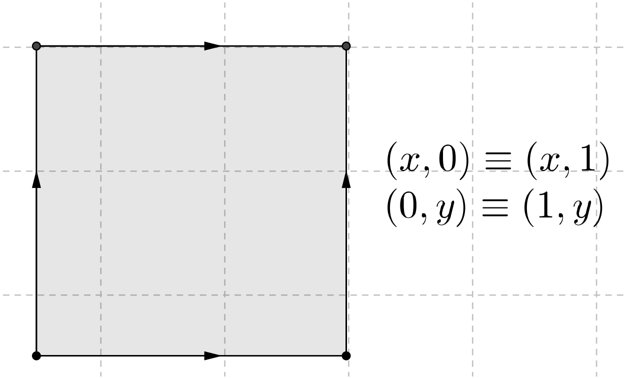
From a square to the Möbius strip.



The voice leading space

Mathematical Preliminaries

From the torus to the Möbius strip.

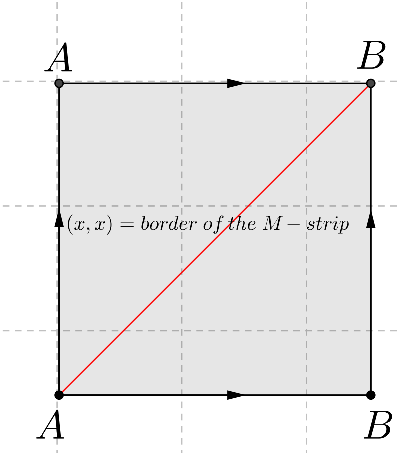


$$(x,0) \equiv (x,1)$$
$$(0,y) \equiv (1,y)$$

The voice leading space

Mathematical Preliminaries

From the torus to the Möbius strip.

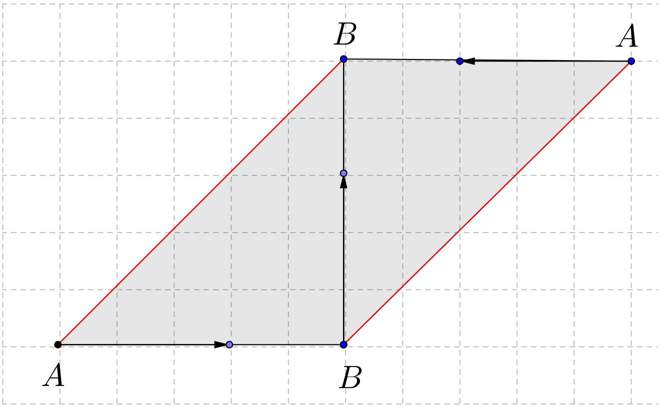


$$(x, 0) \equiv (x, 1)$$
$$(0, y) \equiv (1, y)$$
$$(x, y) \equiv (y, x)$$

The voice leading space

Mathematical Preliminaries

From the torus to the Möbius strip.



The voice leading space

Mathematical Preliminaries

Definition (Orbifold - Intuitive)

An orbifold is a space which is locally modeled on the quotient of a vector space by a finite group.

Definition (Orbifold)

Too difficult, 30' needed!

Example

A manifold is an orbifold in which each finite group is trivial.

[Vladimir G. Ivancevic, Tijana T. Ivancevic. Applied Differential Geometry: A Modern Introduction. World Scientific. 2007.]

The space of chords

$\mathbb{T}^n/\mathcal{S}_n$, $2 \leq n \leq 4$

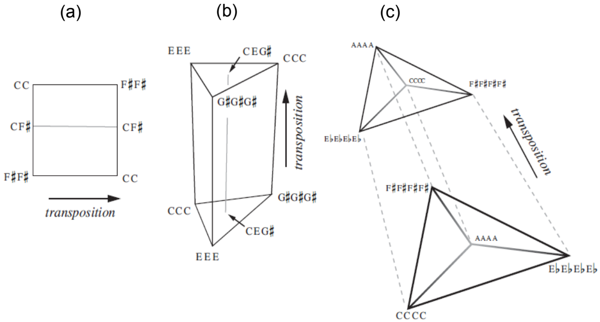


Figure : (a): The orbifold $\mathbb{T}^2/\mathcal{S}_2$, is the space of intervals. It is a two dimensional prism. The base is glued to the opposite face. Before gluing it is necessary to twist the face. The centre of the figure represent the most even interval in the octave, that is the augmented fourth.

The space of chords

$\mathbb{T}^n/\mathcal{S}_n$, $2 \leq n \leq 4$

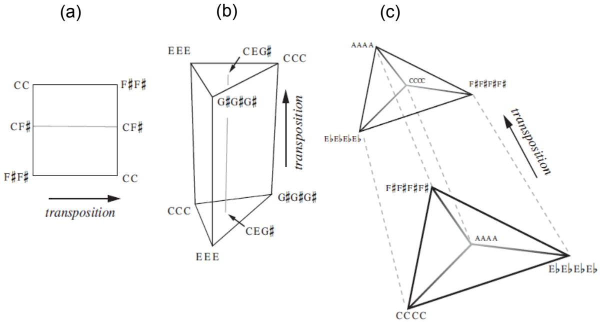


Figure : (b) The orbifold $\mathbb{T}^3/\mathcal{S}_3$, is the space of triads, it is a 3-dimensional prism, whose faces have to be identified, by a rotation of $\frac{2}{3}\pi$ such that the chords' labels match. In the middle of the orbifold we find the augmented triads.

The space of chords

$\mathbb{T}^n/\mathcal{S}_n$, $2 \leq n \leq 4$

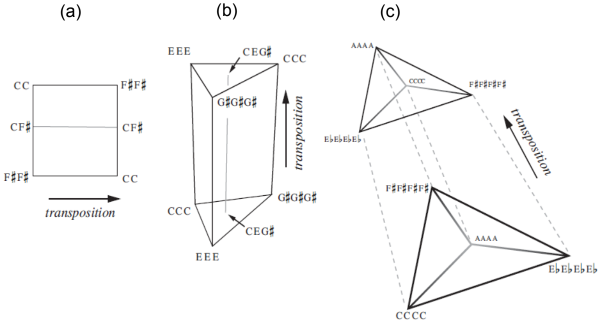
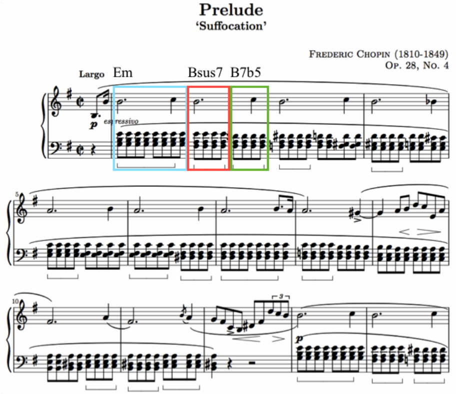


Figure : (c) The orbifold $\mathbb{T}^4/\mathcal{S}_4$, is the space of fourth chord, which is a four dimensional prism, where the two tetrahedral faces have to be indentified. The dotted line represents the fourth dimension. Before the gluing one of the face has to be twisted to make the chords labels at each vertex to correspond. The centre of this orbifold is occupied by a diminished chord.

Applications 1 - Visualization

Videos have been removed, you can find them on
http://prezi.com/tdocmoa87v47/music-math/

Chopin's prelude on $\mathbb{R}/12\mathbb{Z}$



Applications 1 - Visualization

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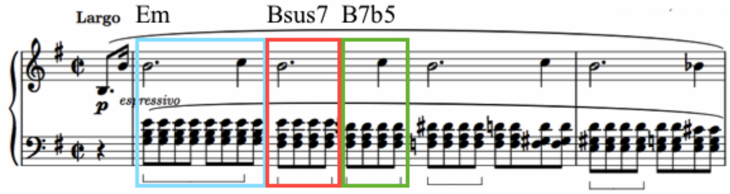

Applications 1 - Visualization

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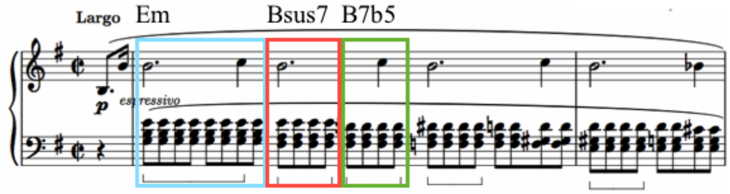
http://prezi.com/tdocmoa87v47/music-math/

Chopin's prelude on $\mathbb{T}^2/\mathcal{S}_2$



Applications 1 - Visualization

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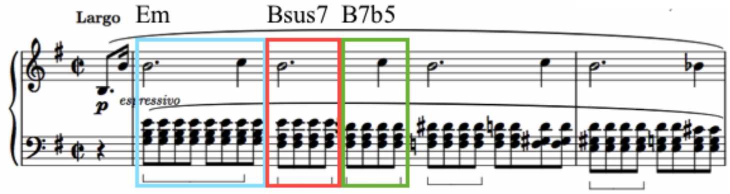
`http://prezi.com/tdocmoa87v47/music-math/`

Chopin's prelude on $\mathbb{T}^4/\mathcal{S}_4$



Application 2 - Efficient voice leadings

Voice leading metric constraints:

Definition

Let $<_{MNC}$ be the partial order induced on $\mathbb{R}^n/\mathcal{S}_n$ by:

- *The monotonicity principle.* Let X e Y be two multisets of cardinality n of non-negative real numbers, such that $y_i \geq x_i$ for all i, we have $X \leq_{MNC} Y$.
- *No-Crossings Principle*. If $A = \{a_1, ..., a_n\}$ e $B = \{b_1, ..., b_n\}$ are multisets of real numbers, then

$$\{|b_{[1]} - a_{[1]}|, ..., |b_{[n]} - a_{[n]}|\} \leq_{MNC} \{|b_1 - a_1|, ..., |b_n - a_n|\},$$

where $a_{[i]}$ denotes the ith maximum elements of A.

Application 2 - Efficient voice leadings

$\mathbb{T}^2/\mathcal{S}_2$

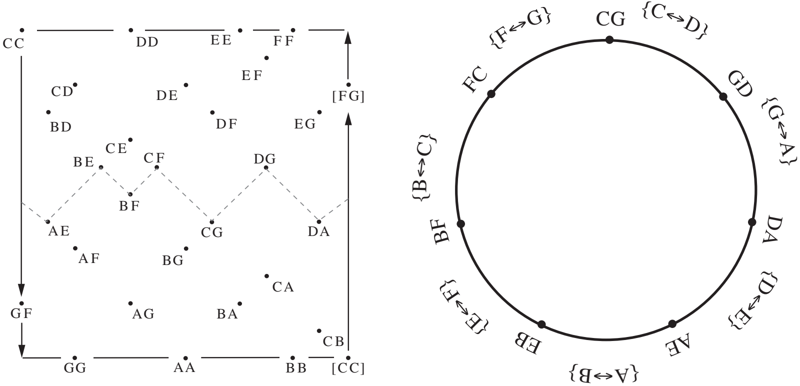
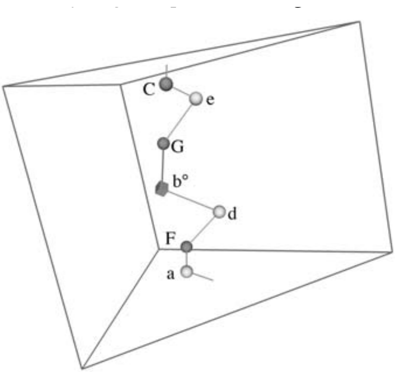


Figure : Efficient voice leadings between diatonic fifths

Application 2 - Efficient voice leadings

$\mathbb{T}^3/\mathcal{S}_3$



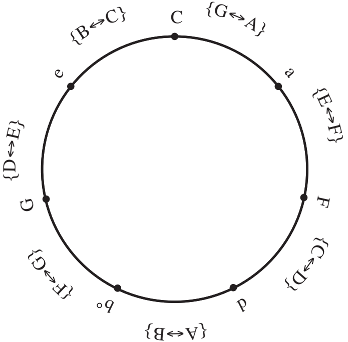


Figure : Efficient voice leadings between diatonic triads

Application 2 - Efficient voice leadings

Chords configuration in the space of triads

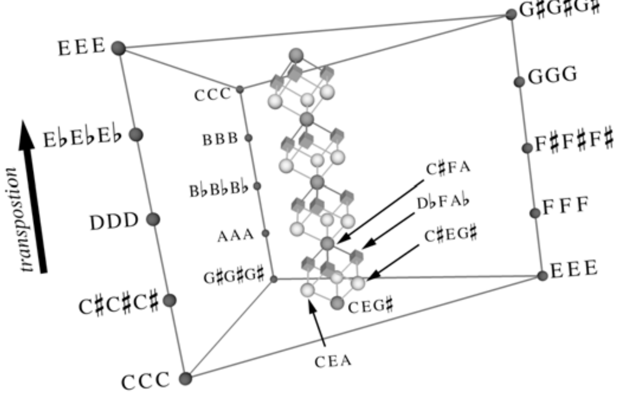


Figure : Here, triads are particularly close to their major-third transpositions

Application 2 - Efficient voice leadings

[Tymoczko, Dmitri. "Three conceptions of musical distance." Mathematics and computation in music. Springer Berlin Heidelberg, 2009. 258-272.]
[Tymoczko, Dmitri. "Scale Theory, Serial Theory and Voice Leading." Music Analysis 27.1 (2008): 1-49.]
[Hall, Rachel Wells, and Dmitri Tymoczko. "Submajorization and the geometry of unordered collections." The American Mathematical Monthly 119.4 (2012): 263-283.]

Thank you!