

Formation ATIAM

Acoustique Traitement du Signal Informatique Appliqués à la Musique Parcours
multi-mentions du Master (M2) Sciences et Technologies de l'Université Pierre et
Marie Curie (Paris 6)

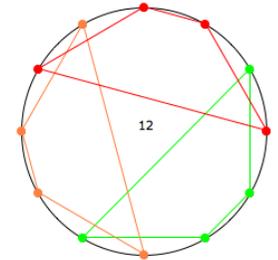
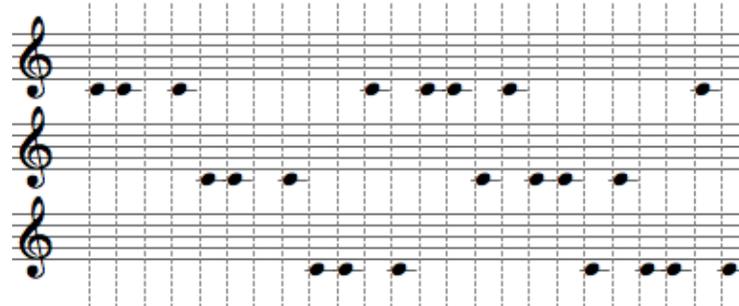
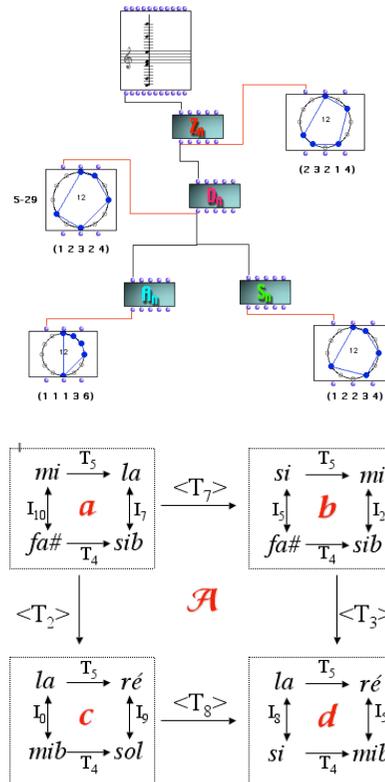
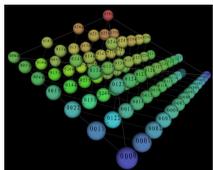
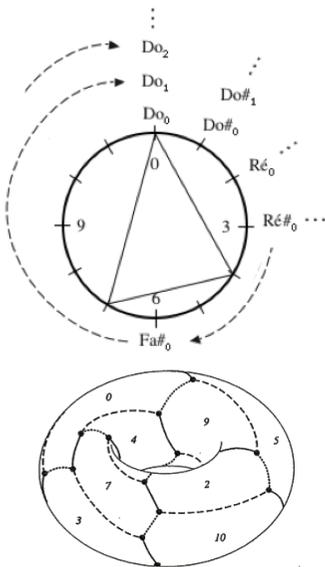
UE Méthodes mathématiques pour l'informatique musicale [MMIM] - 2013-2014

Moreno Andreatta
Equipe Représentations Musicales
IRCAM/CNRS/UPMC UMR 9912



Plan du cours : partie « structures algébriques ‘fortes’ »

- 1.) Représentation/formalisation et énumération/classification des structures musicales
- 2.) *Set Theory*, théories diatoniques, transformationnelles et néo-riemanniennes
 - Focus sur le *Tonnetz*, les orbifolds et le voice-leading (avec Mattia Bergomi, UPMC/Milan)
- 3.) Pavages en composition : la construction des canons rythmiques mosaïques
 - Focus sur les canons de Vuza et les représentations polynomiales (avec Hélianthe Caure, UPMC)



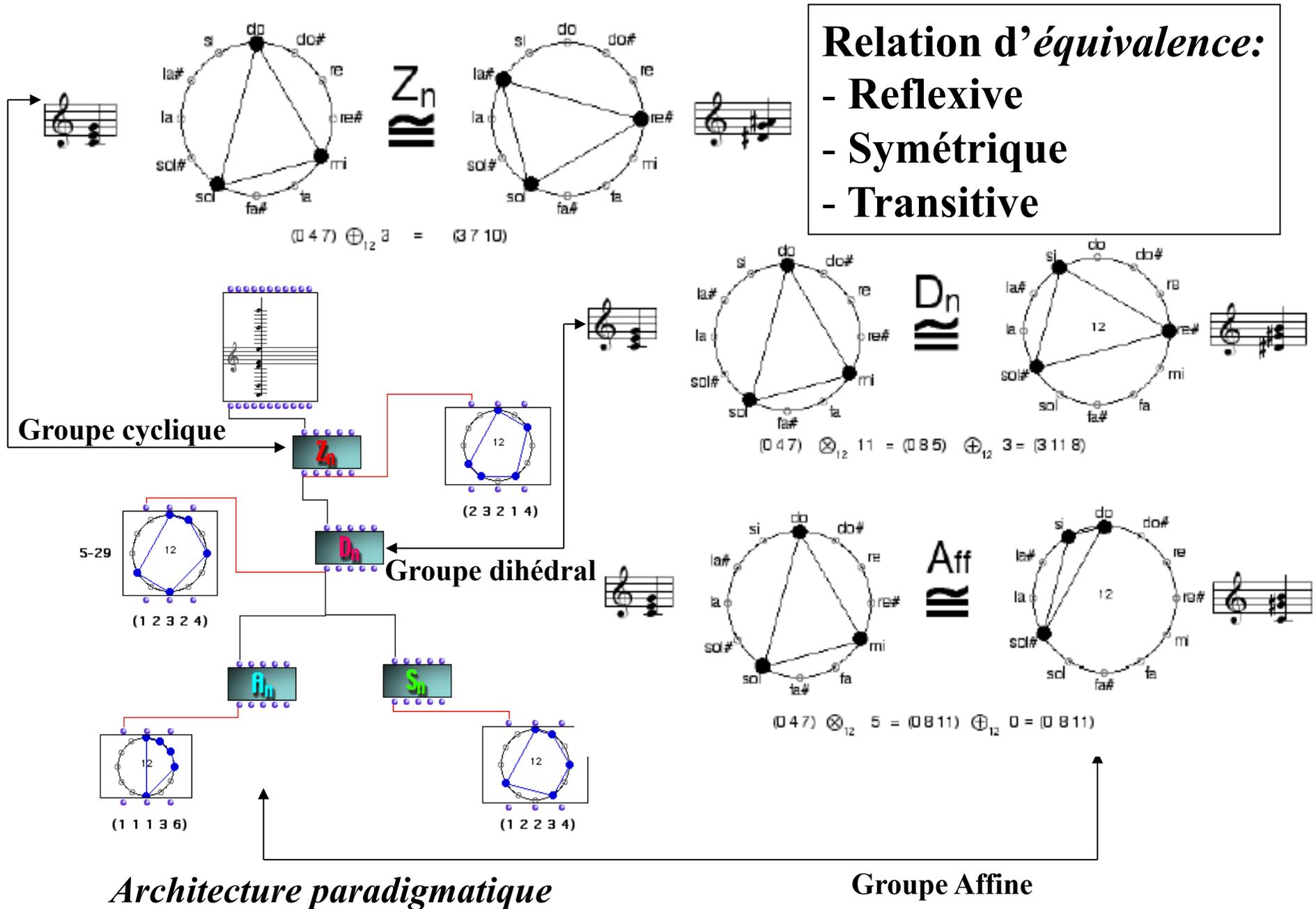
$$Df(x) = f(x) - f(x-1).$$

$$\begin{aligned}
 f &= 7 \ 11 \ 10 \ 11 \ 7 \ 2 \ 7 \ 11 \ 10 \ 11 \ 7 \ 2 \ 7 \ 11 \dots \\
 Df &= 4 \ 11 \ 1 \ 8 \ 7 \ 5 \ 4 \ 11 \ 1 \ 8 \ 7 \ 5 \ 4 \ 11 \dots \\
 D^2 f &= 7 \ 2 \ 7 \ 11 \ 10 \ 11 \ 7 \ 2 \ 7 \ 11 \ 10 \ 11 \dots \\
 D^3 f &= 7 \ 5 \ 4 \ 11 \ 1 \ 8 \ 7 \ 5 \ 4 \ 11 \ 1 \ 8 \dots \\
 D^k f &= \dots
 \end{aligned}$$

Andreatta M. et M. Chemillier (2007), « Modèles mathématiques pour l'informatique musicale (MMIM): Outils théoriques et stratégies pédagogiques », Actes des Journées d'Informatique Musicale, Lyon, avril, p. 113-12

➔ <http://articles.ircam.fr/textes/Andreatta07b/index.pdf>

Transformations géométriques et relation d'équivalence



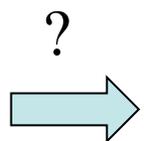
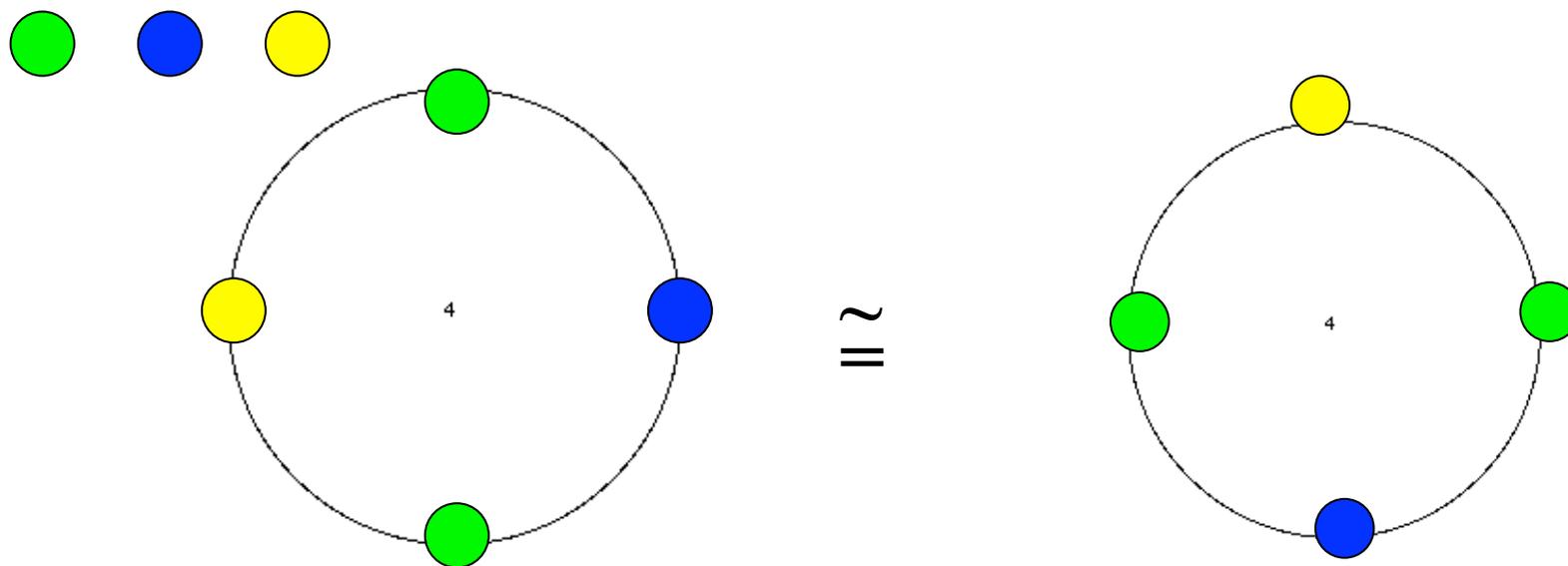
Enumeration des orbites par rapport à l'action d'un groupe



Lemme de Burnside

$$n = \frac{1}{|G|} \sum_{g \in G} |X^g|.$$

$$X^g = \{x \in X : gx = x\}$$



Trouver le nombre de configurations possibles

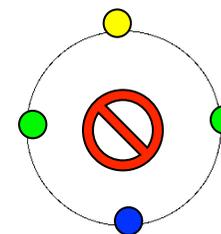
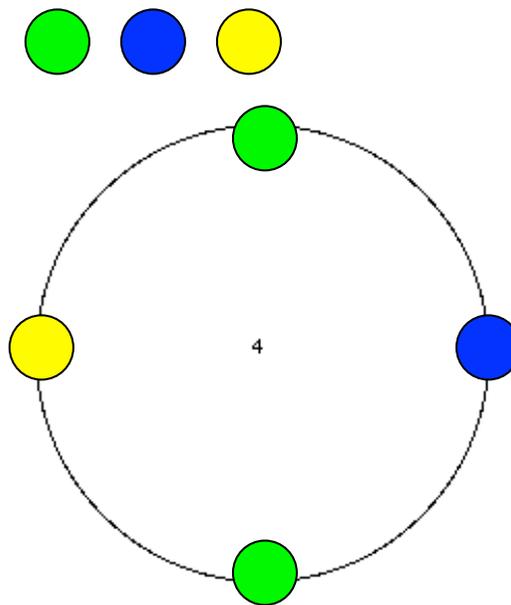
Enumération d'orbites par rapport à l'action d'un groupe



Lemme de Burnside

$$n = \frac{1}{|G|} \sum_{g \in G} |X^g|.$$

$$X^g = \{x \in X : gx = x\}$$



Action de $\mathbf{Z}/4\mathbf{Z}$

T_0 = identité

T_1 = rotation de 90°

T_2 = rotation de 180°

T_3 = rotation de 270°

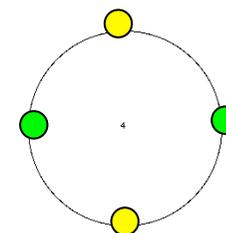
Configurations possibles = $3^4 = 81$

T_0 fixe toute configuration $\Rightarrow |X^{T_0}| = 81$

T_1 fixe toute configuration monochromes $\Rightarrow |X^{T_1}| = 3$

T_3 idem

T_2 fixe toute configuration «double-diamètre» $\Rightarrow |X^{T_2}| = 3^2 = 9$



➔ $n = 1/4 (81+3+3+9) = 24$

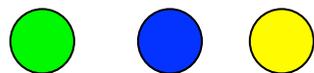
Enumération d'orbites par rapport à l'action d'un groupe



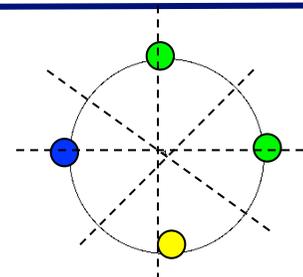
Lemme de Burnside

$$n = \frac{1}{|G|} \sum_{g \in G} |X^g|.$$

$$X^g = \{x \in X : gx = x\}$$



Action de Z_4



<i>Transformation</i>	<i>Action</i>	<i>Cycle representation</i>	<i>No. of cycles</i>	<i>Fixed configs.</i>	<i>Cycle type</i>	<i>Cycle index</i>
T_0	$0 \rightarrow 0, 1 \rightarrow 1, 2 \rightarrow 2, 3 \rightarrow 3$	$(0)(1)(2)(3)$	4	$3^4 = 81$	1^4	t_1^4
T_1	$0 \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow 0$	$(0\ 1\ 2\ 3)$	1	$3^1 = 3$	4^1	t_4^1
T_2	$0 \rightarrow 2 \rightarrow 0, 1 \rightarrow 3 \rightarrow 1$	$(0\ 2)(1\ 3)$	2	$3^2 = 9$	2^2	t_2^2
T_3	$0 \rightarrow 3 \rightarrow 2 \rightarrow 1 \rightarrow 0$	$(0\ 3\ 2\ 1)$	1	$3^1 = 3$	4^1	t_4^1

Julian Hook, « Why are there 29 Tetrachords? A Tutorial on Combinatorics and Enumeration in Music Theory », MTO, 13(4), 2007

$$n = 1/4 (81+3+3+9) = 24$$

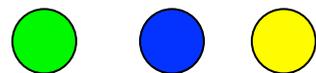
Enumération d'orbites par rapport à l'action d'un groupe



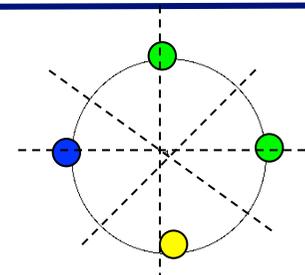
Lemme de Burnside

$$n = \frac{1}{|G|} \sum_{g \in G} |X^g|.$$

$$X^g = \{x \in X : gx = x\}$$



Action de D_4



<i>Transformation</i>	<i>Action</i>	<i>Cycle representation</i>	<i>No. of cycles</i>	<i>Fixed configs.</i>	<i>Cycle type</i>	<i>Cycle index</i>
T_0	$0 \rightarrow 0, 1 \rightarrow 1, 2 \rightarrow 2, 3 \rightarrow 3$	$(0)(1)(2)(3)$	4	$3^4 = 81$	1^4	t_1^4
T_1	$0 \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow 0$	$(0\ 1\ 2\ 3)$	1	$3^1 = 3$	4^1	t_4^1
T_2	$0 \rightarrow 2 \rightarrow 0, 1 \rightarrow 3 \rightarrow 1$	$(0\ 2)(1\ 3)$	2	$3^2 = 9$	2^2	t_2^2
T_3	$0 \rightarrow 3 \rightarrow 2 \rightarrow 1 \rightarrow 0$	$(0\ 3\ 2\ 1)$	1	$3^1 = 3$	4^1	t_4^1
I	$0 \rightarrow 0, 1 \rightarrow 3 \rightarrow 1, 2 \rightarrow 2$	$(0)(1\ 3)(2)$	3	$3^3 = 27$	$1^2 2^1$	$t_1^2 t_2^1$
$T_1 I$	$0 \rightarrow 1 \rightarrow 0, 2 \rightarrow 3 \rightarrow 2$	$(0\ 1)(2\ 3)$	2	$3^2 = 9$	2^2	t_2^2
$T_2 I$	$0 \rightarrow 2 \rightarrow 0, 1 \rightarrow 1, 3 \rightarrow 3$	$(0\ 2)(1)(3)$	3	$3^3 = 27$	$1^2 2^1$	$t_1^2 t_2^1$
$T_3 I$	$0 \rightarrow 3 \rightarrow 0, 1 \rightarrow 2 \rightarrow 1$	$(0\ 3)(1\ 2)$	2	$3^2 = 9$	2^2	t_2^2

Julian Hook, « Why are there 29 Tetrachords? A Tutorial on Combinatorics and Enumeration in Music Theory », MTO, 13(4), 2007

$$n = 1/8 (81+3+3+9+27+9+27+9) = 168/8=21$$

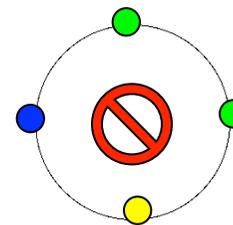
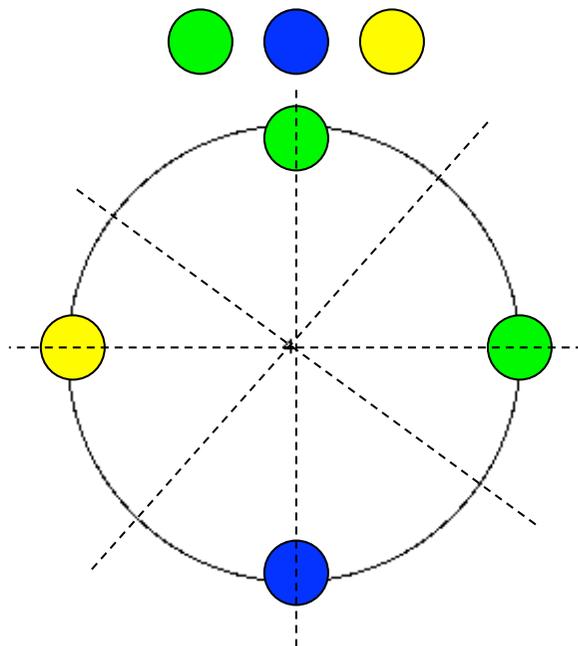
Enumération d'orbites par rapport à l'action d'un groupe



Lemme de Burnside

$$n = \frac{1}{|G|} \sum_{g \in G} |X^g|.$$

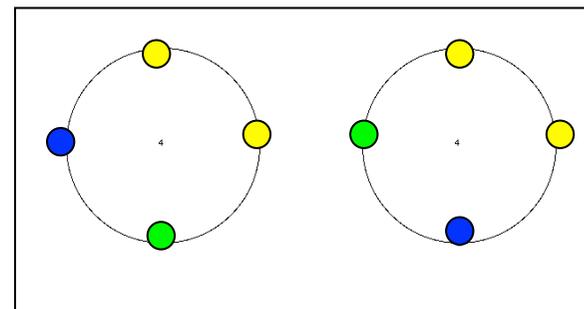
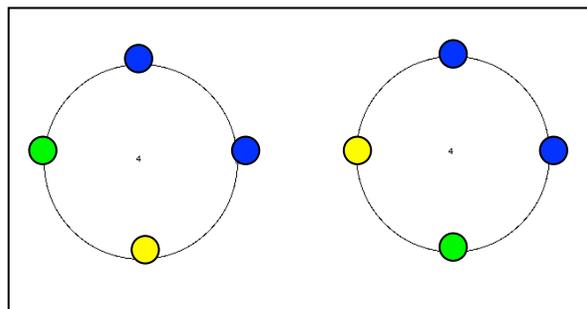
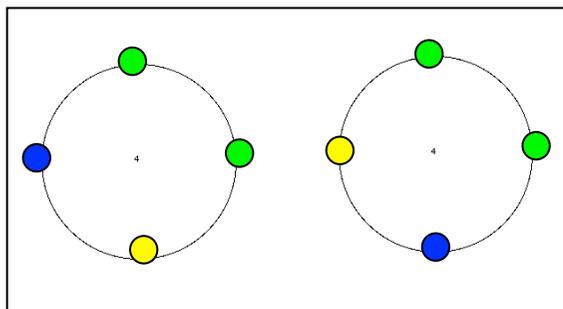
$X^g = \{x \in X : gx = x\}$



Action de D_4

- | | |
|-------------------------------|---------------------------|
| $T_0 = \text{id}$ | $T_0I = \text{inversion}$ |
| $T_1 = \text{rot } 90^\circ$ | $T_1I = \text{inv.}$ |
| $T_2 = \text{rot } 180^\circ$ | $T_2I = \text{inv.}$ |
| $T_3 = \text{rot } 270^\circ$ | $T_3I = \text{inv.}$ |

→ 21=24-3



Énumération d'accords par rapport à l'action du groupe diédrale



Transformation	Cycle representation	No. of cycles	Fixed configs.	Cycle type	Cycle index
T_0	(0)(1)(2)(3)(4)(5)(6)(7)(8)(9)(A)(B)	12	$2^{12} = 4096$	1^{12}	t_1^{12}
T_1	(0 1 2 3 4 5 6 7 8 9 A B)	1	$2^1 = 2$	12^1	t_{12}^1
T_2	(0 2 4 6 8 A)(1 3 5 7 9 B)	2	$2^2 = 4$	6^2	t_6^2
T_3	(0 3 6 9)(1 4 7 A)(2 5 8 B)	3	$2^3 = 8$	4^3	t_4^3
T_4	(0 4 8)(1 5 9)(2 6 A)(3 7 B)	4	$2^4 = 16$	3^4	t_3^4
T_5	(0 5 A 3 8 1 6 B 4 9 2 7)	1	$2^1 = 2$	12^1	t_{12}^1
T_6	(0 6)(1 7)(2 8)(3 9)(4 A)(5 B)	6	$2^6 = 64$	2^6	t_2^6
T_7	(0 7 2 9 4 B 6 1 8 3 A 5)	1	$2^1 = 2$	12^1	t_{12}^1
T_8	(0 8 4)(1 9 5)(2 A 6)(3 B 7)	4	$2^4 = 16$	3^4	t_3^4
T_9	(0 9 6 3)(1 A 7 4)(2 B 8 5)	3	$2^3 = 8$	4^3	t_4^3
T_{10}	(0 A 8 6 4 2)(1 B 9 7 5 3)	2	$2^2 = 4$	6^2	t_6^2
T_{11}	(0 B A 9 8 7 6 5 4 3 2 1)	1	$2^1 = 2$	12^1	t_{12}^1

Lemme de Burnside

$$n = \frac{1}{|G|} \sum_{g \in G} |X^g|.$$

$$X^g = \{x \in X : gx = x\}$$

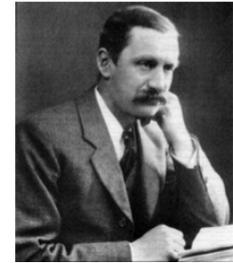
Action de Z_{12}

(Hook, MTO)



$$\# \text{ d'accords} = 1/12[4096+2+4+8+16+2+64+2+16+8+4+2]=4224/12=352$$

Énumération d'accords par rapport à l'action du groupe diédrale



Transformation	Cycle representation	No. of cycles	Fixed configs.	Cycle type	Cycle index
T_0	(0)(1)(2)(3)(4)(5)(6)(7)(8)(9)(A)(B)	12	$2^{12} = 4096$	1^{12}	t_1^{12}
T_1	(0 1 2 3 4 5 6 7 8 9 A B)	1	$2^1 = 2$	12^1	t_{12}^1
T_2	(0 2 4 6 8 A)(1 3 5 7 9 B)	2	$2^2 = 4$	6^2	t_6^2
T_3	(0 3 6 9)(1 4 7 A)(2 5 8 B)	3	$2^3 = 8$	4^3	t_4^3
T_4	(0 4 8)(1 5 9)(2 6 A)(3 7 B)	4	$2^4 = 16$	3^4	t_3^4
T_5	(0 5 A 3 8 1 6 B 4 9 2 7)	1	$2^1 = 2$	12^1	t_{12}^1
T_6	(0 6)(1 7)(2 8)(3 9)(4 A)(5 B)	6	$2^6 = 64$	2^6	t_2^6
T_7	(0 7 2 9 4 B 6 1 8 3 A 5)	1	$2^1 = 2$	12^1	t_{12}^1
T_8	(0 8 4)(1 9 5)(2 A 6)(3 B 7)	4	$2^4 = 16$	3^4	t_3^4
T_9	(0 9 6 3)(1 A 7 4)(2 B 8 5)	3	$2^3 = 8$	4^3	t_4^3
T_{10}	(0 A 8 6 4 2)(1 B 9 7 5 3)	2	$2^2 = 4$	6^2	t_6^2
T_{11}	(0 B A 9 8 7 6 5 4 3 2 1)	1	$2^1 = 2$	12^1	t_{12}^1
I	(0)(1 B)(2 A)(3 9)(4 8)(5 7)(6)	7	$2^7 = 128$	$1^2 2^5$	$t_1^2 t_2^5$
$T_1 I$	(0 1)(2 B)(3 A)(4 9)(5 8)(6 7)	6	$2^6 = 64$	2^6	t_2^6
$T_2 I$	(0 2)(1)(3 B)(4 A)(5 9)(6 8)(7)	7	$2^7 = 128$	$1^2 2^5$	$t_1^2 t_2^5$
$T_3 I$	(0 3)(1 2)(4 B)(5 A)(6 9)(7 8)	6	$2^6 = 64$	2^6	t_2^6
$T_4 I$	(0 4)(1 3)(2)(5 B)(6 A)(7 9)(8)	7	$2^7 = 128$	$1^2 2^5$	$t_1^2 t_2^5$
$T_5 I$	(0 5)(1 4)(2 3)(6 B)(7 A)(8 9)	6	$2^6 = 64$	2^6	t_2^6
$T_6 I$	(0 6)(1 5)(2 4)(3)(7 B)(8 A)(9)	7	$2^7 = 128$	$1^2 2^5$	$t_1^2 t_2^5$
$T_7 I$	(0 7)(1 6)(2 5)(3 4)(8 B)(9 A)	6	$2^6 = 64$	2^6	t_2^6
$T_8 I$	(0 8)(1 7)(2 6)(3 5)(4)(9 B)(A)	7	$2^7 = 128$	$1^2 2^5$	$t_1^2 t_2^5$
$T_9 I$	(0 9)(1 8)(2 7)(3 6)(4 5)(A B)	6	$2^6 = 64$	2^6	t_2^6
$T_{10} I$	(0 A)(1 9)(2 8)(3 7)(4 6)(5)(B)	7	$2^7 = 128$	$1^2 2^5$	$t_1^2 t_2^5$
$T_{11} I$	(0 B)(1 A)(2 9)(3 8)(4 7)(5 6)	6	$2^6 = 64$	2^6	t_2^6

Lemme de Burnside

$$n = \frac{1}{|G|} \sum_{g \in G} |X^g|.$$

$$X^g = \{x \in X : gx = x\}$$

Action de D_{12}

(Hook, MTO)



d'accords = $1/12[4096+2+4+8+16+2+64+2+16+8+4+2]=4224/12=352$



d'accords = $1/24[4224+1152] = 224$

Calcul d'orbites via le polynôme de Polya

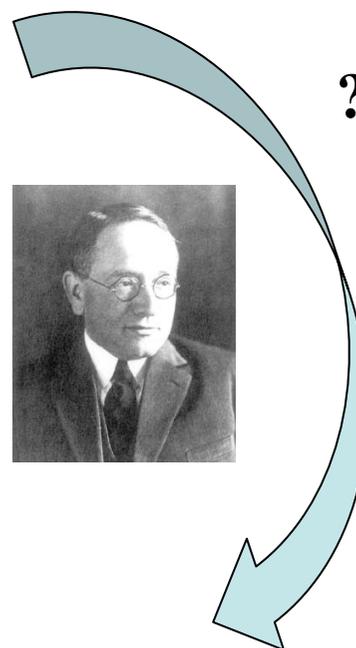
Transformation	Cycle representation	No. of cycles	Fixed configs.	Cycle type	Cycle index
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T_1	(0 1 2 3 4 5 6 7 8 9 A B)	1	$2^1 = 2$	12^1	t_{12}^1
T_2	(0 2 4 6 8 A)(1 3 5 7 9 B)	2	$2^2 = 4$	6^2	t_6^2
T_3	(0 3 6 9)(1 4 7 A)(2 5 8 B)	3	$2^3 = 8$	4^3	t_4^3
T_4	(0 4 8)(1 5 9)(2 6 A)(3 7 B)	4	$2^4 = 16$	3^4	t_3^4
T_5	(0 5 A 3 8 1 6 B 4 9 2 7)	1	$2^1 = 2$	12^1	t_{12}^1
T_6	(0 6)(1 7)(2 8)(3 9)(4 A)(5 B)	6	$2^6 = 64$	2^6	t_2^6
T_7	(0 7 2 9 4 B 6 1 8 3 A 5)	1	$2^1 = 2$	12^1	t_{12}^1
T_8	(0 8 4)(1 9 5)(2 A 6)(3 B 7)	4	$2^4 = 16$	3^4	t_3^4
T_9	(0 9 6 3)(1 A 7 4)(2 B 8 5)	3	$2^3 = 8$	4^3	t_4^3
T_{10}	(0 A 8 6 4 2)(1 B 9 7 5 3)	2	$2^2 = 4$	6^2	t_6^2
T_{11}	(0 B A 9 8 7 6 5 4 3 2 1)	1	$2^1 = 2$	12^1	t_{12}^1
I	(0)(1 B)(2 A)(3 9)(4 8)(5 7)(6)	7	$2^7 = 128$	$1^2 2^5$	$t_1^2 t_2^5$
$T_1 I$	(0 1)(2 B)(3 A)(4 9)(5 8)(6 7)	6	$2^6 = 64$	2^6	t_2^6
$T_2 I$	(0 2)(1)(3 B)(4 A)(5 9)(6 8)(7)	7	$2^7 = 128$	$1^2 2^5$	$t_1^2 t_2^5$
$T_3 I$	(0 3)(1 2)(4 B)(5 A)(6 9)(7 8)	6	$2^6 = 64$	2^6	t_2^6
$T_4 I$	(0 4)(1 3)(2)(5 B)(6 A)(7 9)(8)	7	$2^7 = 128$	$1^2 2^5$	$t_1^2 t_2^5$
$T_5 I$	(0 5)(1 4)(2 3)(6 B)(7 A)(8 9)	6	$2^6 = 64$	2^6	t_2^6
$T_6 I$	(0 6)(1 5)(2 4)(3)(7 B)(8 A)(9)	7	$2^7 = 128$	$1^2 2^5$	$t_1^2 t_2^5$
$T_7 I$	(0 7)(1 6)(2 5)(3 4)(8 B)(9 A)	6	$2^6 = 64$	2^6	t_2^6
$T_8 I$	(0 8)(1 7)(2 6)(3 5)(4)(9 B)(A)	7	$2^7 = 128$	$1^2 2^5$	$t_1^2 t_2^5$
$T_9 I$	(0 9)(1 8)(2 7)(3 6)(4 5)(A B)	6	$2^6 = 64$	2^6	t_2^6
$T_{10} I$	(0 A)(1 9)(2 8)(3 7)(4 6)(5)(B)	7	$2^7 = 128$	$1^2 2^5$	$t_1^2 t_2^5$
$T_{11} I$	(0 B)(1 A)(2 9)(3 8)(4 7)(5 6)	6	$2^6 = 64$	2^6	t_2^6



Lemme de Burnside

$$n = \frac{1}{|G|} \sum_{g \in G} |X^g|.$$

$$X^g = \{x \in X : gx = x\}$$

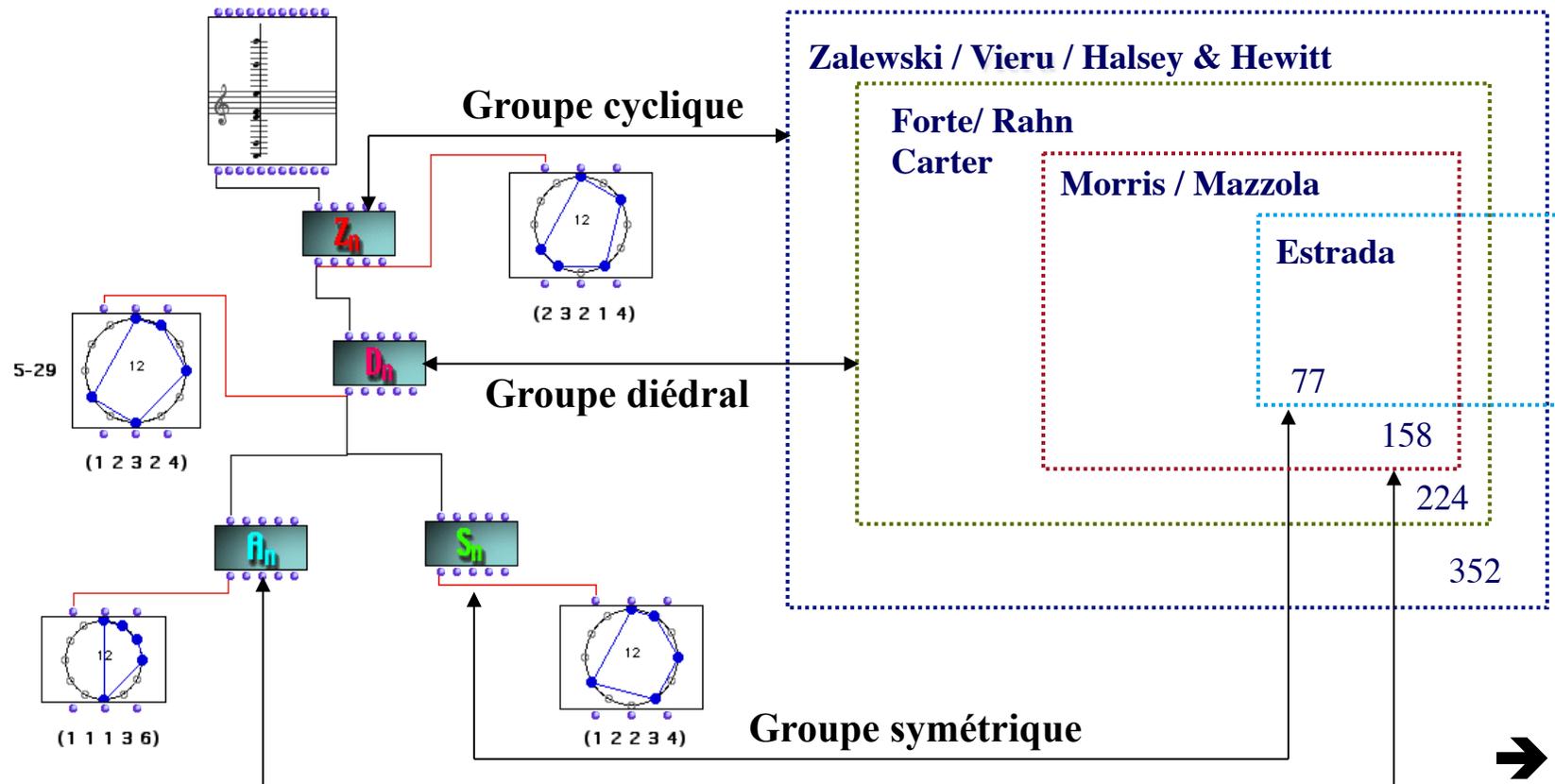


d'accords = $1/24[4224+1152] = 224$

$G \setminus k$	1	2	3	4	5	6	7	8	9	10	11	12
C_{12}	1	6	19	43	66	80	66	43	19	6	1	1
D_{12}	1	6	12	29	38	50	38	29	12	6	1	1
$\text{Aff}_1(Z_{12})$	1	5	9	21	25	34	25	21	9	5	1	1

Classification paradigmatique des structures musicales

$G \setminus k$	1	2	3	4	5	6	7	8	9	10	11	12
C_{12}	1	6	19	43	66	80	66	43	19	6	1	1
D_{12}	1	6	12	29	38	50	38	29	12	6	1	1
$\text{Aff}_1(\mathbb{Z}_{12})$	1	5	9	21	25	34	25	21	9	5	1	1



Architecture paradigmatique

Groupe affine

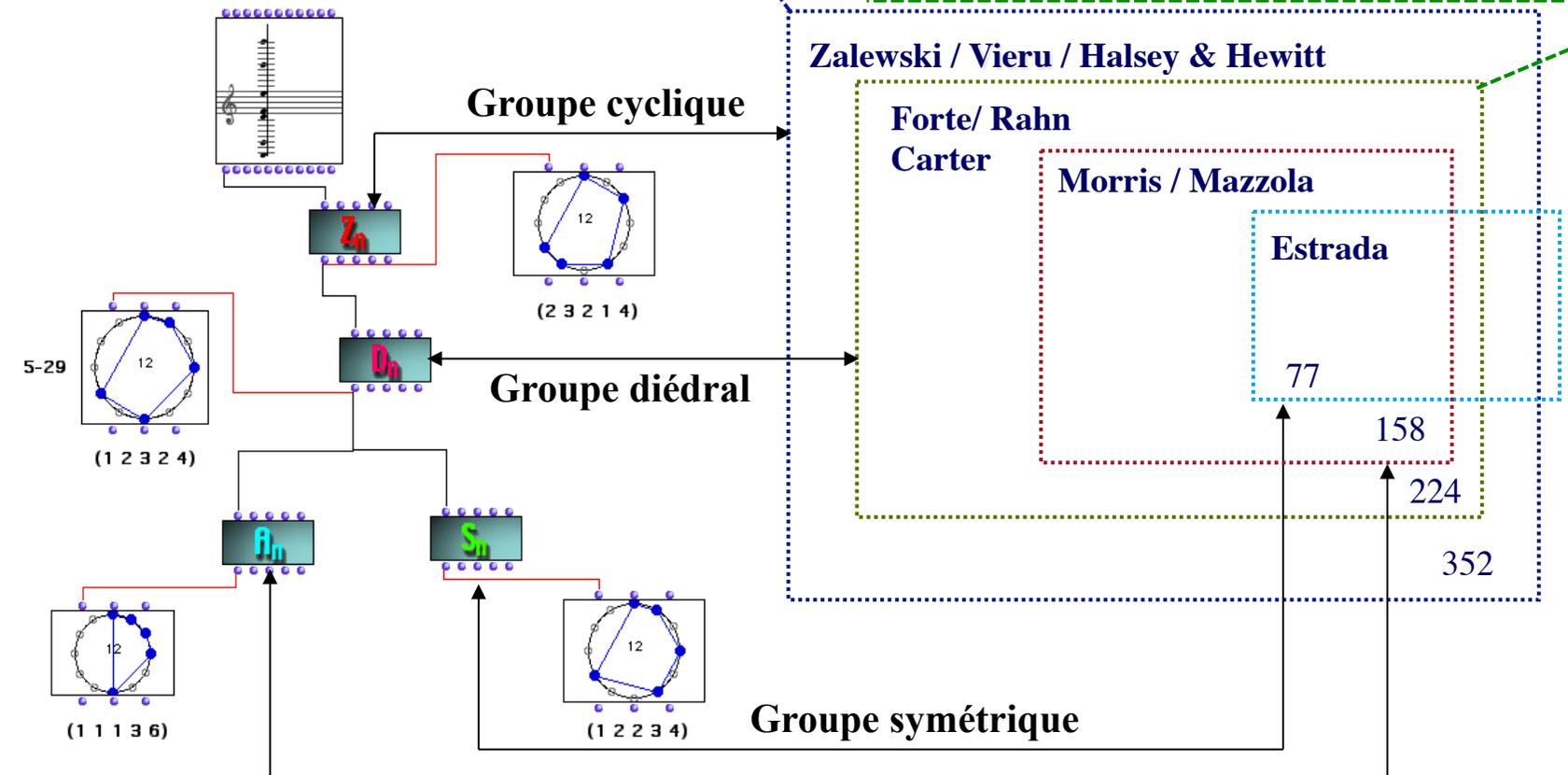


02-paradigms

Classification paradigmatic des structures musicales

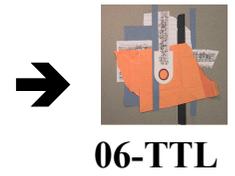
$$\# \text{ of } k\text{-chords} = \frac{1}{n} \sum_{j|k, j|(n,k)} \phi(j) \binom{n/j}{k/j} = \frac{1}{n} \Phi_n(k)$$

$$\# \text{ of } k\text{-chords} = \begin{cases} \frac{1}{2n} \left[\Phi_n(k) + n \binom{(n-1)/2}{[k/2]} \right], & \text{if } n \text{ is odd,} \\ \frac{1}{2n} \left[\Phi_n(k) + n \binom{n/2}{k/2} \right], & \text{if } n \text{ is even and } k \text{ is even,} \\ \frac{1}{2n} \left[\Phi_n(k) + n \binom{(n/2)-1}{[k/2]} \right], & \text{if } n \text{ is even and } k \text{ is odd.} \end{cases}$$



Architecture paradigmatique

- D. Halsey & E. Hewitt: « Eine gruppentheoretische Methode in der Musik-theorie », *Jahresber. Der Dt. Math.-Vereinigung*, 80, 1978.
- D. Reiner: « Enumeration in Music Theory », *Amer. Math. Month.* 92:51-54, 1985
- R.C. Read: « Combinatorial problems in the theory of music », *Discrete Math.*, 1997
- H. Friepertinger: « Enumeration of mosaics », *Discrete Math.*, 1999



Permutoèdre comme catalogue d'accords

Tabelle 1

Tabelle aller möglichen Intervallstrukturen

Diagram illustrating the permutation of intervals (1-12) and their corresponding musical staves (1-73). The diagram shows a central permutation S_n and a graph below it.

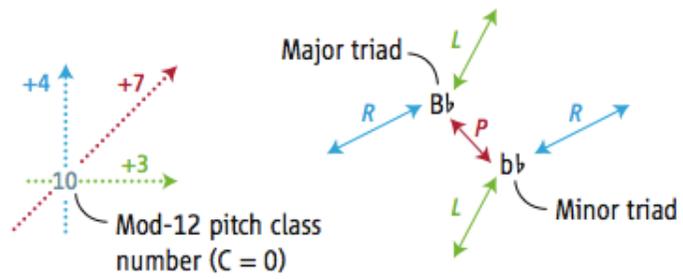
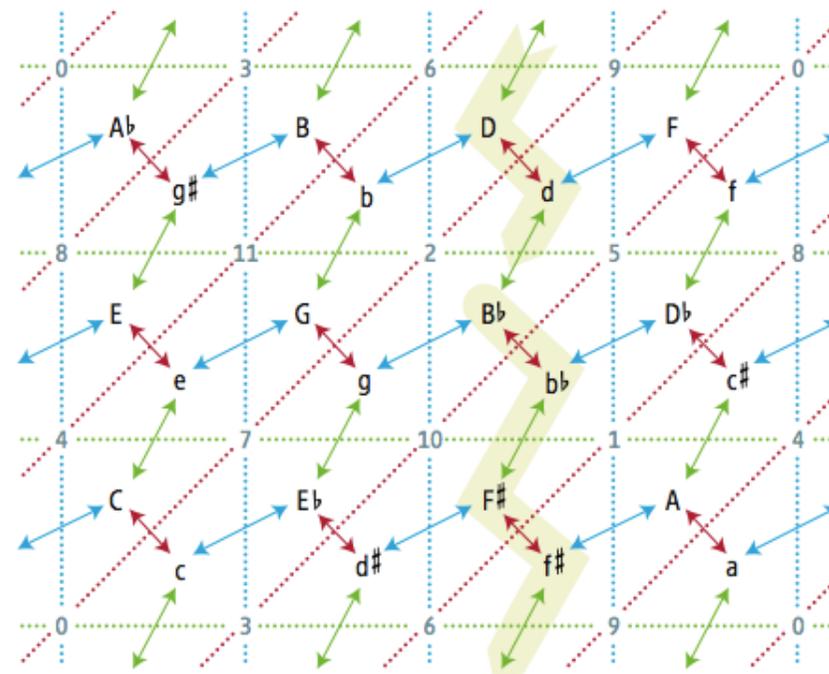
Id. Nr.	Intervalle
0	—
1	12
2	6 + 6
3	5 + 7
4	4 + 8
5	4 + 4 + 4
6	3 + 9
7	3 + 4 + 5
8	3 + 3 + 6
9	3 + 3 + 3 + 3
10	2 + 10
11	2 + 5 + 5
12	2 + 4 + 6
13	2 + 3 + 7
14	2 + 3 + 3 + 4
15	2 + 2 + 8
16	2 + 2 + 4 + 4
17	2 + 2 + 3 + 5
18	2 + 2 + 2 + 6
19	2 + 2 + 2 + 3 + 3
20	2 + 2 + 2 + 2 + 4
21	2 + 2 + 2 + 2 + 2 + 2
22	1 + 11
23	1 + 5 + 6
24	1 + 4 + 7
25	1 + 3 + 8
26	1 + 2 + 9
27	1 + 3 + 4 + 4
28	1 + 3 + 3 + 5
29	1 + 2 + 4 + 5
30	1 + 2 + 3 + 6
31	1 + 2 + 2 + 7
32	1 + 2 + 3 + 3 + 3
33	1 + 2 + 2 + 3 + 4

Studia Musicologica Academiae Scientiarum Hungaricae 9, 1967

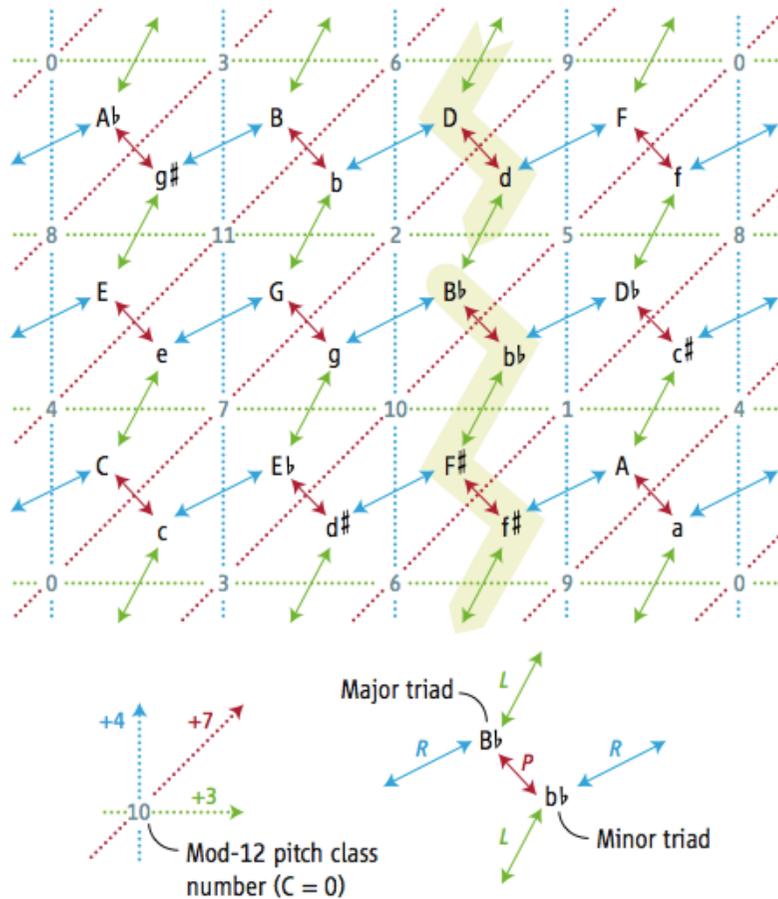
Permutoèdre et Tonnetz

73 chords with voicings and pitch class lists:

- 1: [12]
- 2: [1 11]
- 3: [2 10]
- 4: [3 9]
- 5: [4 8]
- 6: [5 7]
- 7: [6 6]
- 8: [1 1 10]
- 9: [1 2 9]
- 10: [1 3 8]
- 11: [1 4 7]
- 12: [1 5 6]
- 13: [2 3 8]
- 14: [2 3 7]
- 15: [2 4 6]
- 16: [2 5 5]
- 17: [3 5 6]
- 18: [3 4 5]
- 19: [4 4 4]
- 20: [1 1 1 9]
- 21: [1 1 2 8]
- 22: [1 1 3 7]
- 23: [1 1 4 6]
- 24: [1 1 5 5]
- 25: [1 2 2 7]
- 26: [1 2 3 6]
- 27: [1 2 4 5]
- 28: [1 3 3 5]
- 29: [1 3 4 4]
- 30: [2 2 2 6]
- 31: [2 2 3 5]
- 32: [2 2 4 4]
- 33: [2 3 3 4]
- 34: [3 3 3 3]
- 35: [1 1 1 1 8]
- 36: [1 1 1 2 7]
- 37: [1 1 1 3 6]
- 38: [1 1 1 4 5]
- 39: [1 1 2 2 6]
- 40: [1 1 2 3 5]
- 41: [1 1 2 4 4]
- 42: [1 1 3 3 4]
- 43: [1 2 2 2 5]
- 44: [1 2 2 3 4]
- 45: [1 2 2 3 3]
- 46: [1 2 2 2 2 3]
- 47: [2 2 2 3 3]
- 48: [1 1 1 1 1 7]
- 49: [1 1 1 1 2 6]
- 50: [1 1 1 1 3 5]
- 51: [1 1 1 1 4 4]
- 52: [1 1 1 2 2 5]
- 53: [1 1 1 2 3 4]
- 54: [1 1 1 3 3 3]
- 55: [1 1 2 2 2 4]
- 56: [1 1 2 2 3 3]
- 57: [1 2 2 2 2 3]
- 58: [2 2 2 2 2 2]
- 59: [1 1 1 1 1 1 6]
- 60: [1 1 1 1 1 2 5]
- 61: [1 1 1 1 1 3 4]
- 62: [1 1 1 1 2 2 4]
- 63: [1 1 1 1 2 3 3]
- 64: [1 1 1 2 2 2 3]
- 65: [1 1 2 2 2 2 2]
- 66: [1 1 1 1 1 1 1 5]
- 67: [1 1 1 1 1 1 2 4]
- 68: [1 1 1 1 1 1 3 3]
- 69: [1 1 1 1 1 2 2 3]
- 70: [1 1 1 1 2 2 2 2]
- 71: [1 1 1 1 1 1 1 1 4]
- 72: [1 1 1 1 1 1 2 2 3]
- 73: [1 1 1 1 1 1 2 2 2]

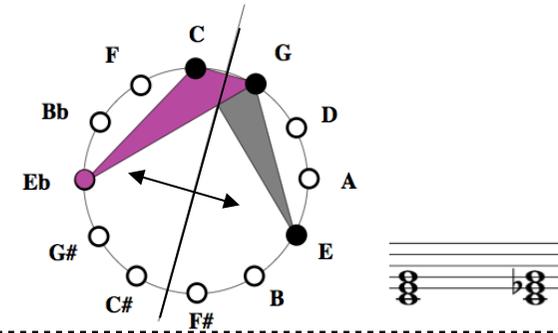


Le Tonnetz en tant que GIS

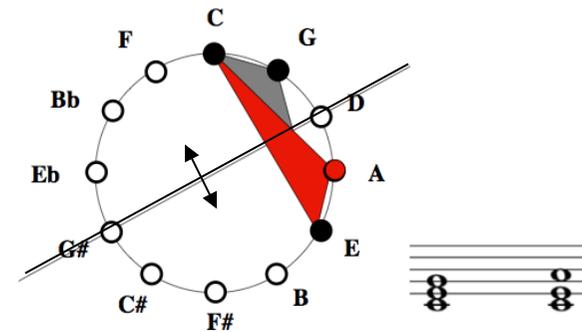


(Neo-)Riemannian Operation P = „Parallel“

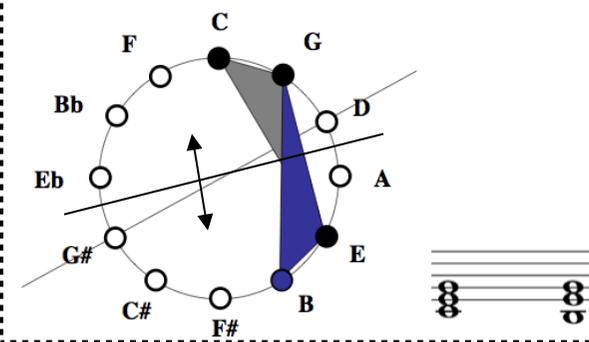
[Noll04]



(Neo-)Riemannian Operation R = „Relative“



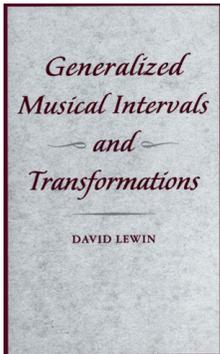
(Neo-)Riemannian Operation L = „Leading-Tone“



$$\rho = \langle L, R \mid L^2 = (LR)^{12} = 1 ; LRL = L(LR)^{-1} \rangle$$

• ρ opère de façon simplement transitive sur l'ensemble S des 24 triades consonantes

$\Rightarrow (S, \rho, \text{int})$ est un GIS



Système d'Intervalles Généralisés - Système Généralisé d'Intervalles

David Lewin's *Generalized Interval System* [GMIT, 1987]

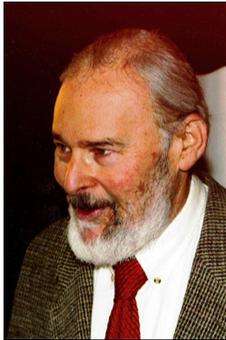
$$\text{GIS} = (S, G, \text{int})$$

S = ensemble

(G, \bullet) = groupe d'intervalles

int = fonction intervallique

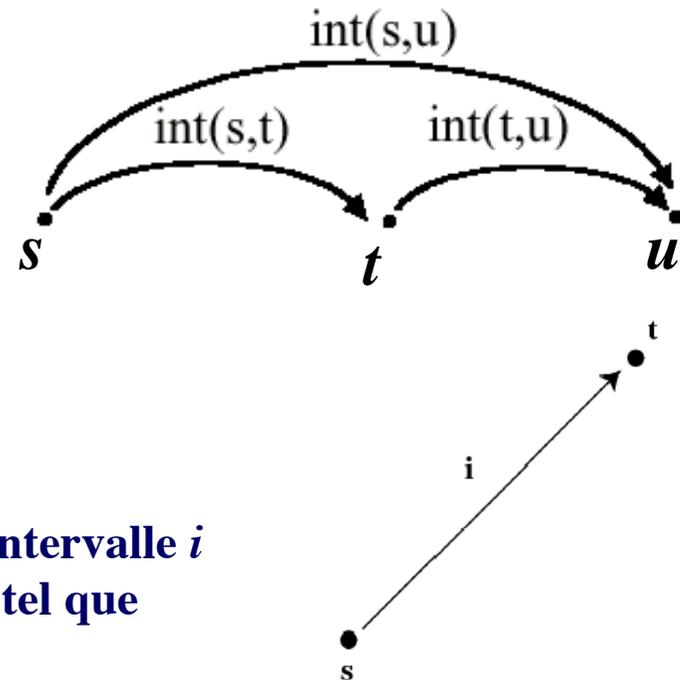
$$S \times S \xrightarrow{\text{int}} G$$



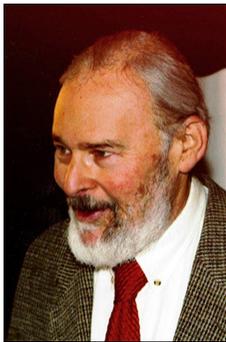
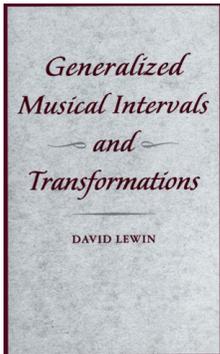
1. Pour tout objets s, t, u dans S :

$$\text{int}(s,t) \bullet \text{int}(t,u) = \text{int}(s,u)$$

2. Pour tout objet s dans S et tout intervalle i dans G il y a un seul objet t dans S tel que

$$\text{int}(s,t) = i$$


- $S = \{\dots, do, do_{\#} = ré_b, ré, \dots, si, do', \dots\}$, $G = \mathbf{Z}$, $\text{int}(do, ré) = 2$, $\text{int}(fa, do) = -5$ etc.
- $S = \{\dots, do, ré, mi, fa, sol, la, si, do', \dots\}$, $G = \mathbf{Z}$, $\text{int}(do, ré) = 1$, $\text{int}(fa, do) = -3$ etc.
- $S = G = \mathbf{Z}_{12} = \{do, do_{\#} = ré_b, ré, \dots, si\}$, $\text{int}(do, ré) = 2$, $\text{int}(fa, do) = 7$ etc.



Premières généralisations : transposition

$$\mathbf{GIS} = (S, G, \text{int})$$

S = ensemble

(G, \bullet) = groupe d'intervalles

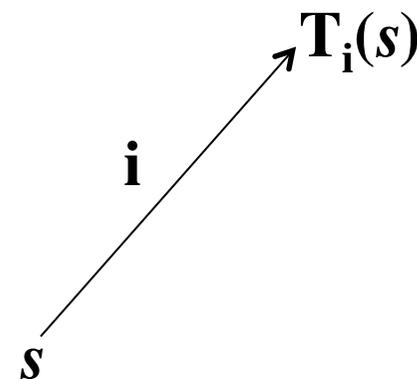
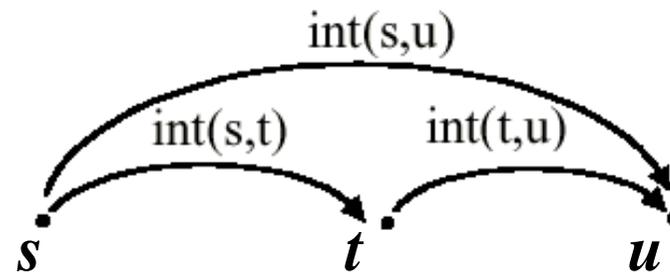
int = fonction intervallique

$$S \times S \xrightarrow{\text{int}} G$$

1. Pour tout objets s, t, u dans S :

$$\text{int}(s,t) \bullet \text{int}(t,u) = \text{int}(s,u)$$

2. Pour tout objet s dans S et tout intervalle i dans G il y a un seul objet t dans S tel que

$$\text{int}(s,t) = i$$


Généralisation de la notion de transposition (musicale)

Pour tout élément i dans G , la transposition T_i est une application

$$T_i : S \rightarrow S \quad \text{telle que} \quad \text{int}(s, T_i(s)) = i \quad \text{pour tout élément } s \text{ dans } S$$

Equivalence entre GIS et action de groupe

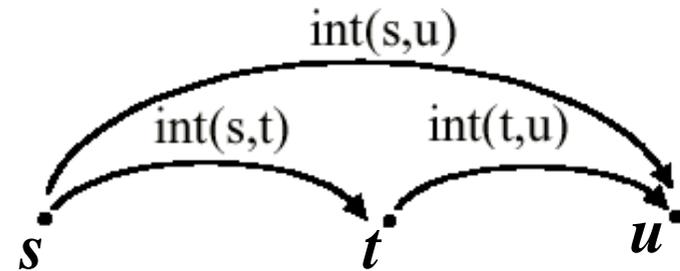
$$\text{GIS} = (S, G, \text{int})$$

S = ensemble

(G, \bullet) = groupe d'intervalles

int = fonction intervallique

$$S \times S \xrightarrow{\text{int}} G$$

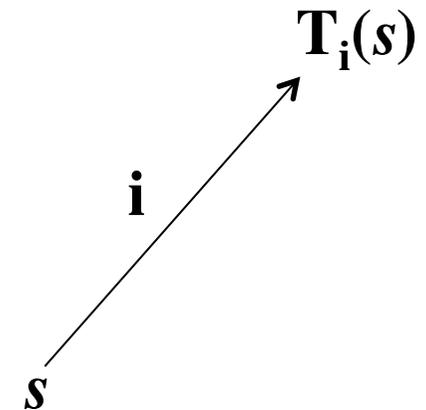


Action
simplement
transitive

1. Pour tous objets s, t, u dans S :

$$\text{int}(s,t) \bullet \text{int}(t,u) = \text{int}(s,u)$$

2. Pour tout objet s dans S et tout intervalle i dans G il y a un seul objet t dans S tel que $\text{int}(s,t) = i$



Soit $\tau = \{T_i ; i \in G\}$ le groupe des transpositions

$$\text{GIS} = (S, G, \text{int}) \Leftrightarrow \tau \times S \rightarrow S \text{ telle que } (T_i, s) \rightarrow T_i(s)$$

Terminologies équivalentes :

- Un GIS est un G -torseur à gauche
- S est un ensemble principal homogène [Bourbaki]

Une autre structure de GIS sur l'espace S

L	R	RL	I_{11}
C → e	C → a	C → G	C → e
c → A \flat	c → E \flat	c → f	c → E
D \flat → f	D \flat → B \flat	D \flat → A \flat	D \flat → e \flat
c \sharp → A	c \sharp → E	c \sharp → f \sharp	c \sharp → E \flat
D → f \sharp	D → b	D → A	D → d
d → B \flat	d → F	d → g	d → D
E \flat → g	E \flat → c	E \flat → B \flat	E \flat → c \sharp
d \sharp → B	d \sharp → F \sharp	d \sharp → g \sharp	d \sharp → C \sharp
E → g \sharp	E → c \sharp	E → B	E → c
e → C	e → G	e → a	e → C
F → a	F → d	F → C	F → b
f → D \flat	f → A \flat	f → B \flat	f → B
F \sharp → a \sharp	F \sharp → d \sharp	F \sharp → C \sharp	F \sharp → B \flat
f \sharp → D	f \sharp → A	f \sharp → b	f \sharp → B \flat
G → b	G → e	G → D	G → a
g → E \flat	g → B \flat	g → c	g → A
A \flat → c	A \flat → f	A \flat → E \flat	A \flat → a \flat
g \sharp → E	g \sharp → B	g \sharp → c \sharp	g \sharp → G \sharp
A → c \sharp	A → f \sharp	A → E	A → g
a → F	a → C	a → d	a → G
B \flat → d	B \flat → g	B \flat → F	B \flat → f \sharp
a \sharp → F \sharp	a \sharp → C \sharp	a \sharp → d \sharp	a \sharp → F \sharp
B → e \flat	B → g \sharp	B → F \sharp	B → f
b → G	b → D	b → e	b → F

[Satyendra 2004]

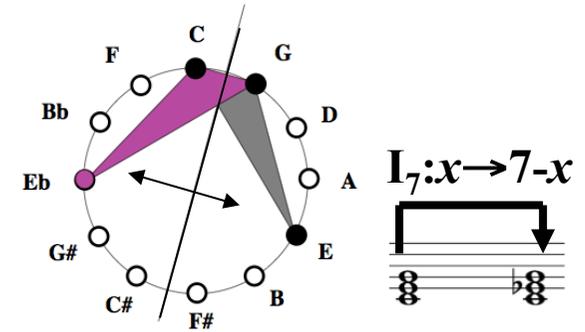
$$D_{12} = \langle I, T \mid I^2 = T^{12} = 1 ; ITI = I(IT)^{-1} \rangle$$

• D_{12} opère de façon simplement transitive sur l'ensemble S des 24 triades consonantes

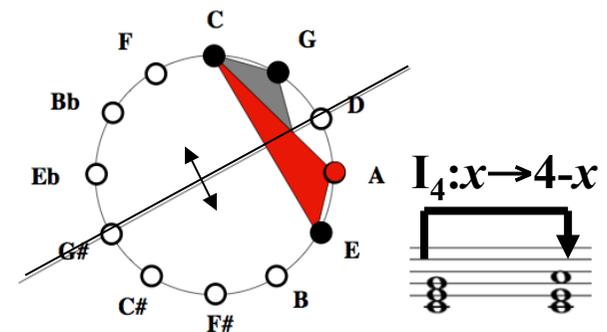
⇒ (S, D_{12}, int) est un GIS

(Neo-)Riemannian Operation P = „Parallel“

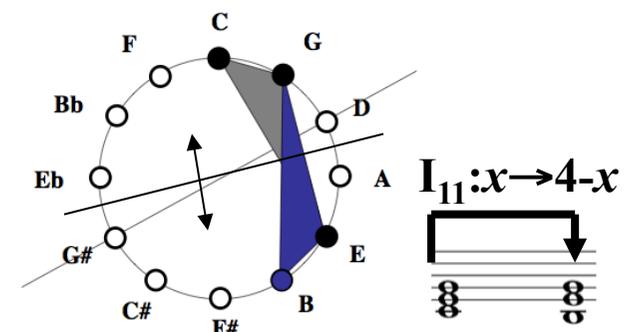
[Noll04]



(Neo-)Riemannian Operation R = „Relative“



(Neo-)Riemannian Operation L = „Leading-Tone“

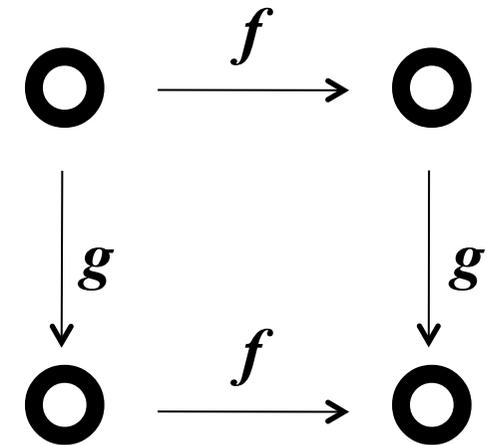
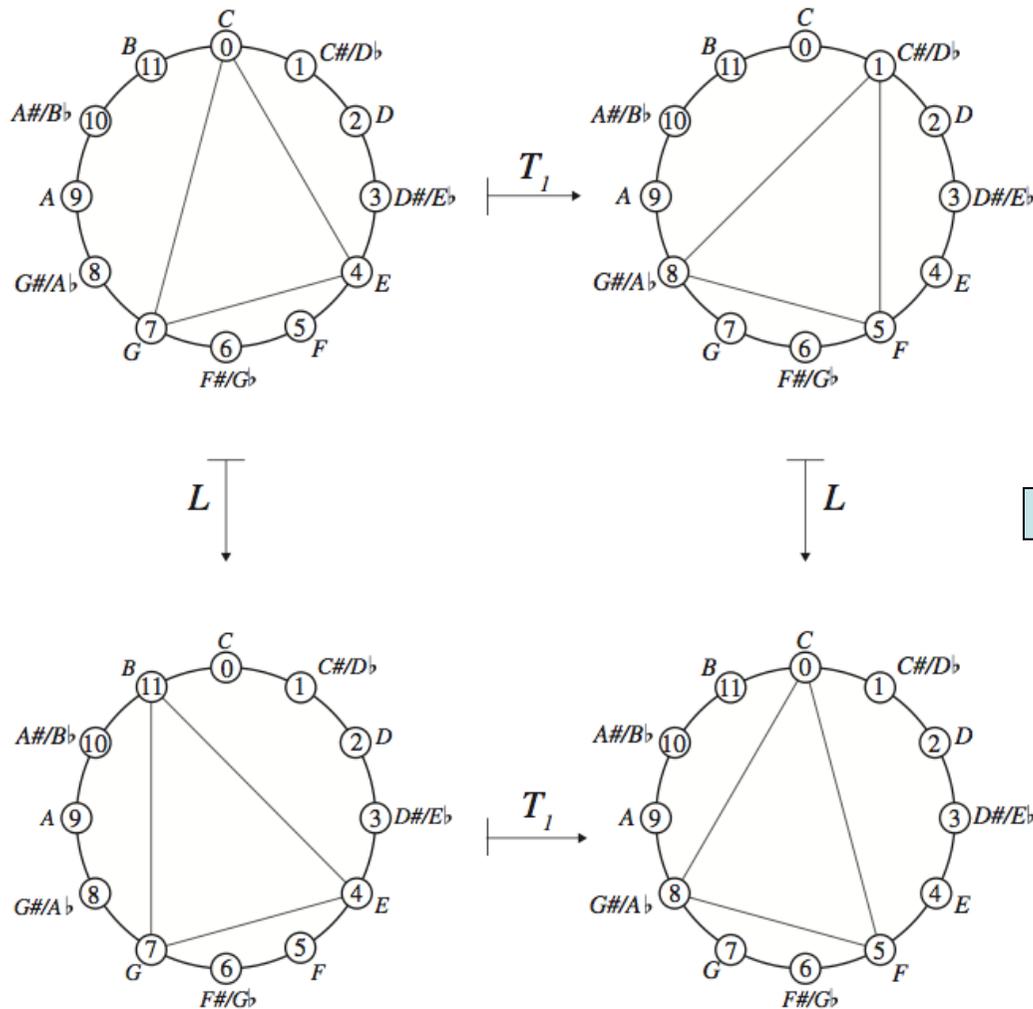


Dualité entre (S, ρ, int) et (S, D_{12}, int)

$$\rho = \langle L, R \mid L^2 = (LR)^{12} = 1 ; LRL = L(LR)^{-1} \rangle$$

$$\Leftrightarrow D_{12} = \langle I, T \mid I^2 = T^{12} = 1 ; ITI = I(IT)^{-1} \rangle$$

$\Rightarrow \rho$ et D_{12} sont l'un le *centralisateur* de l'autre (dans le groupe symétrique $\text{Sym}(S)$)



Tout diagramme commute

$$\forall f \in D_{12}$$

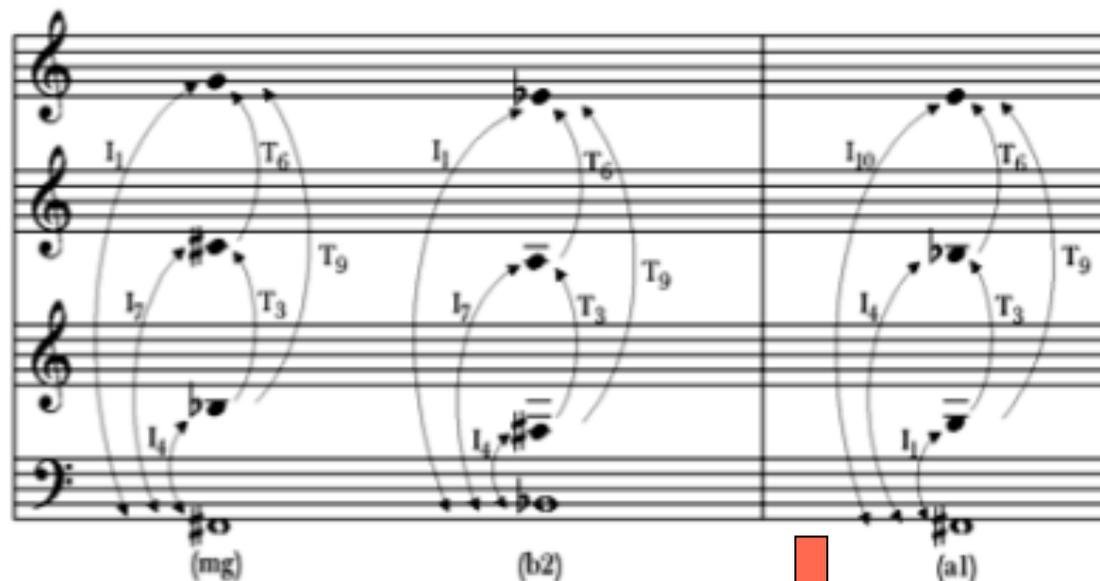
$$\forall g \in \rho$$

Klumpenhouver Networks (K-réseaux)

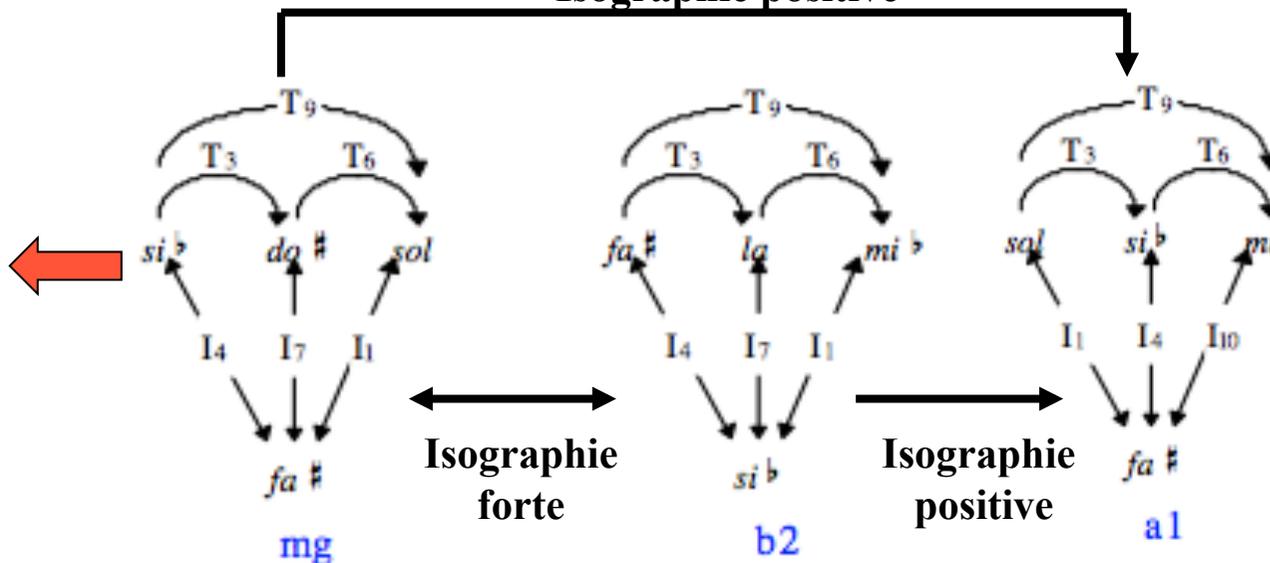
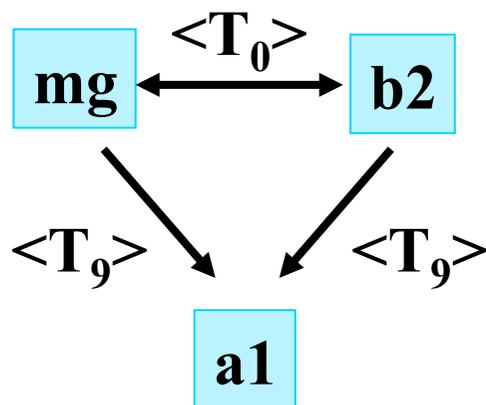
Xavier Hascher: « Liszt et les sources de la notion d'agrégat », *Analyse Musicale*, 43, 2002



Ex. 1 - « Ladislaus Teleki » (*Historische ungarische Bildnisse* n° 4), mes. 1-7
Les agrégats dans la classification de Forte



Isographie positive



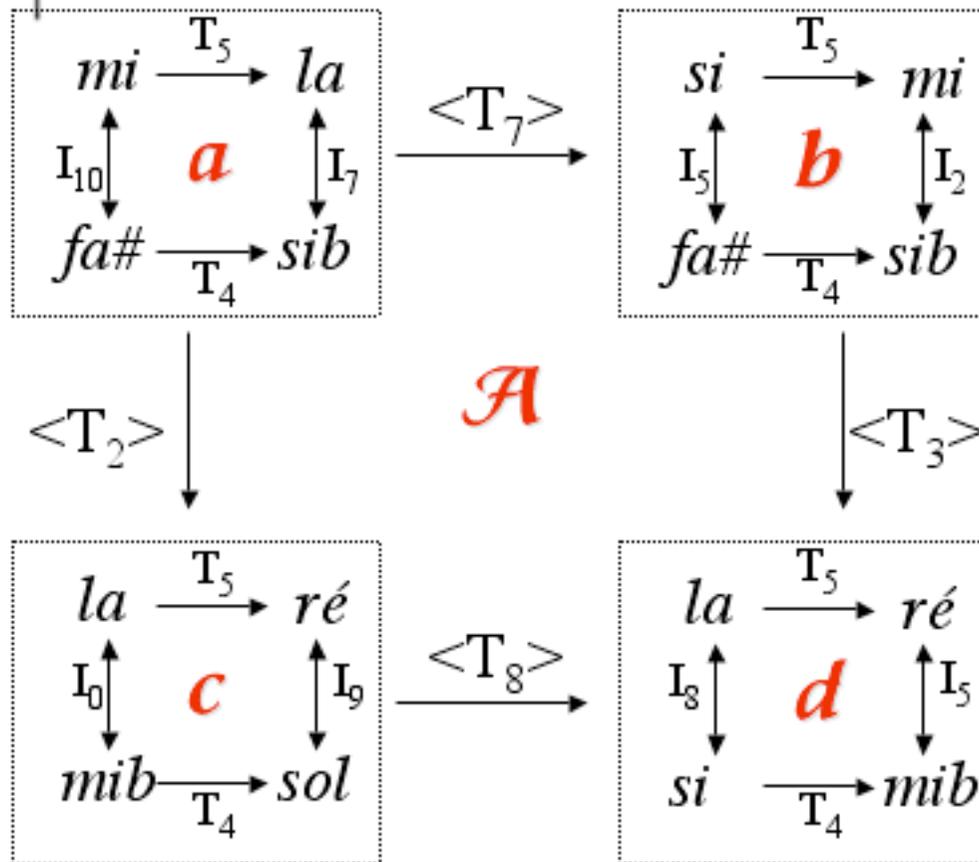
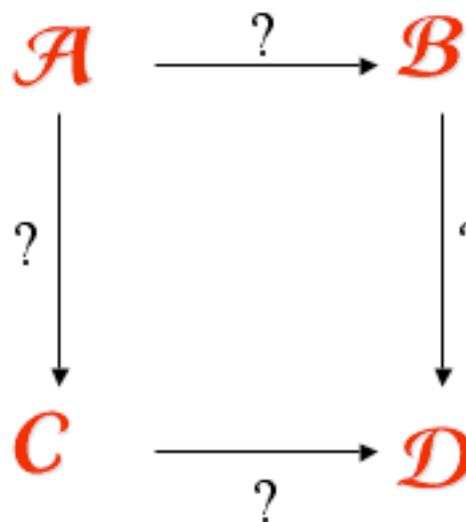
Klumpenhower Networks (K-réseaux) : isographies positives et récursivité

David Lewin: «A Tutorial on K-nets using the Chorale in Schoenberg's Op.11, N°2 », JMT, 1994



$$\langle T_k \rangle : T_m \rightarrow T_m$$

$$I_m \rightarrow I_{k+m}$$

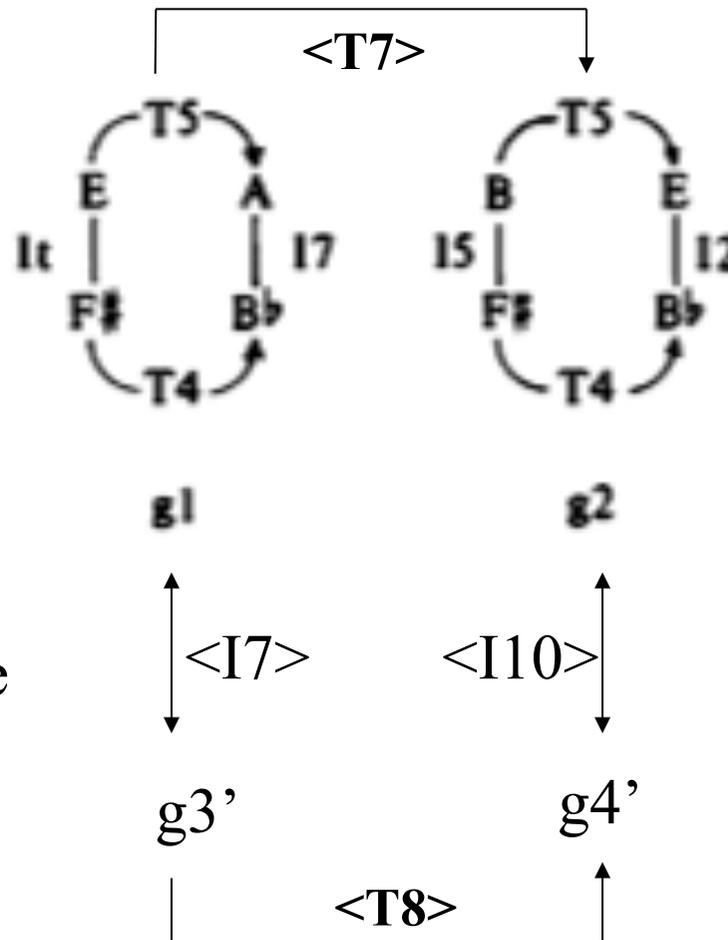
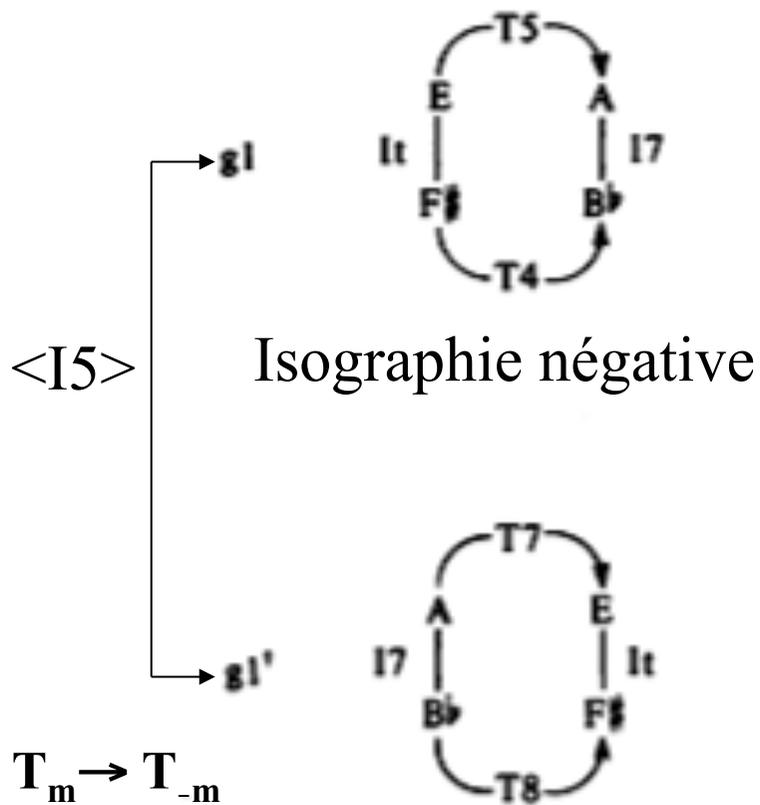


Klumpenhower Networks (K-nets) : isographies négatives

David Lewin: «A Tutorial on K-nets using the Chorale in Schoenberg's Op.11, N°2 », JMT, 1994



Example 9



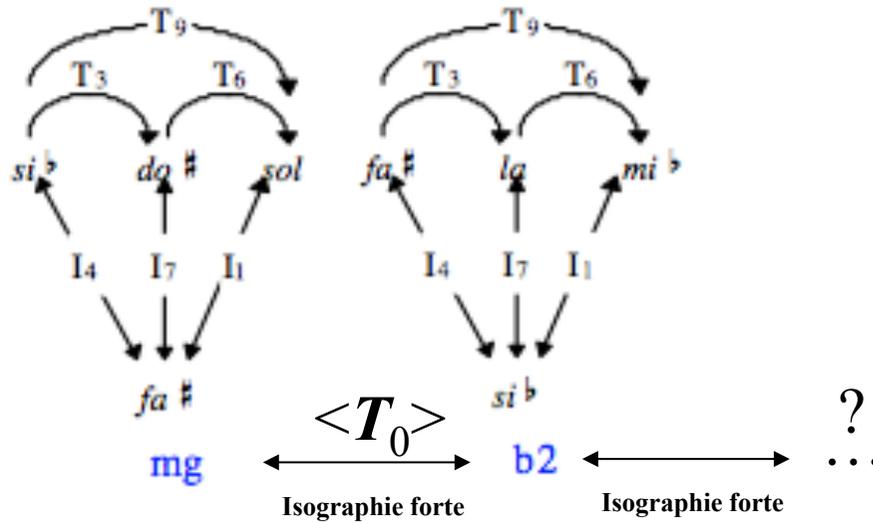
$$\langle I_k \rangle : T_m \rightarrow T_{-m}$$

$$I_m \rightarrow I_{k-m}$$

$$\langle T_k \rangle : T_m \rightarrow T_m$$

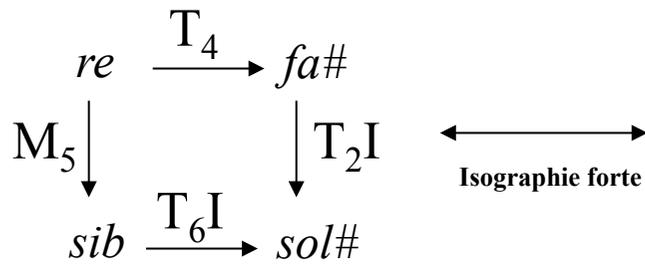
$$I_m \rightarrow I_{k+m}$$

Énumération des K-nets en relation d'isographie forte



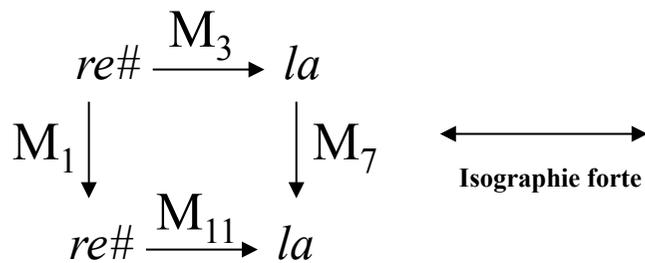
$$\begin{array}{ccccc}
 x & \xrightarrow{T_3} & x+3 & \xrightarrow{T_6} & x+9 \\
 \swarrow I_4 & & \uparrow I_7 & & \searrow I_1 \\
 4-x & = & 7-(x+3) & = & 1-(x+9)
 \end{array}$$

\Rightarrow 12 solutions



$$\begin{array}{ccc}
 x & \xrightarrow{T_4} & x+4 \\
 \downarrow M_5 & & \downarrow T_2I \\
 5x & \xrightarrow{T_6I} & 6-5x=2-(x+4) \implies 8=4x \implies x=2, 5, 8, 11
 \end{array}$$

\Rightarrow 4 solutions



$$\begin{array}{ccc}
 x & \xrightarrow{M_1} & x \\
 \downarrow M_1 & & \downarrow M_7 \\
 x & \xrightarrow{M_{11}} & 11x=7x \implies 4x=0 \implies x=0, 3, 6, 9
 \end{array}$$

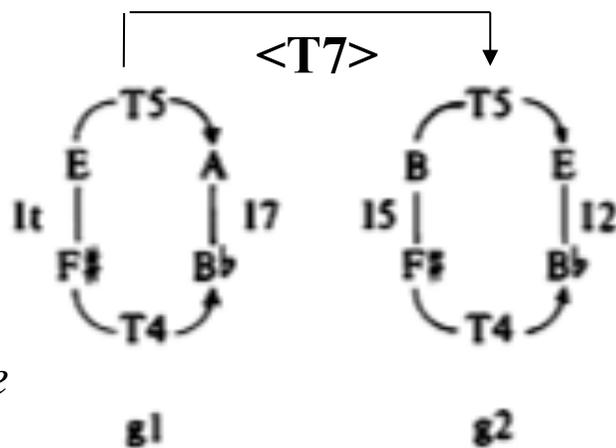
\Rightarrow 4 solutions

Isomorphismes de réseaux de Klumpenhouwer

Isographie positive

$$\langle T_k \rangle : T_m \rightarrow T_m$$

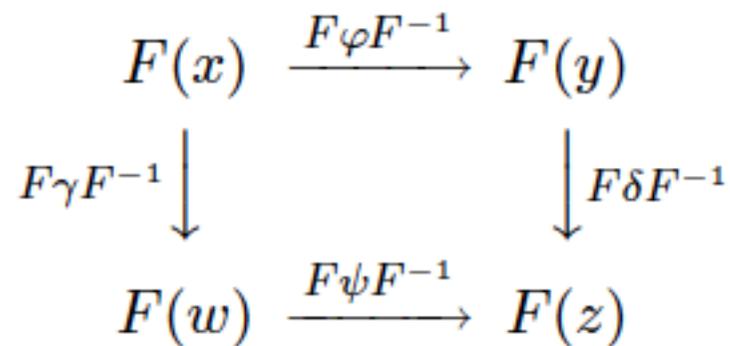
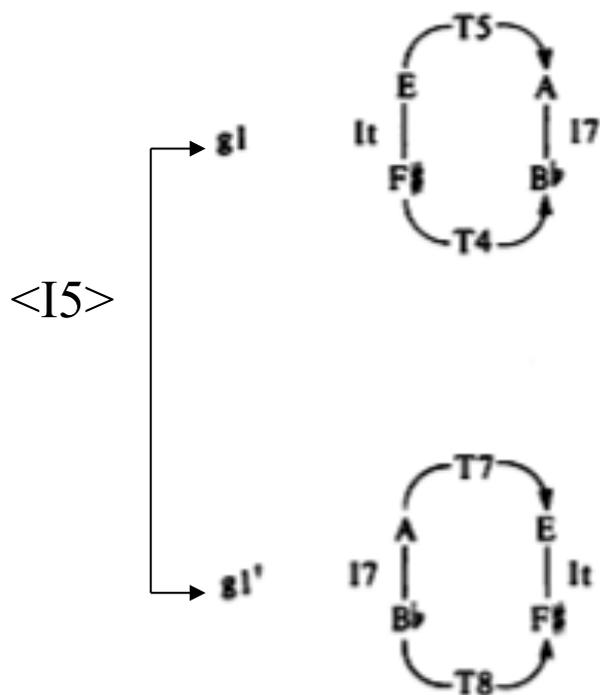
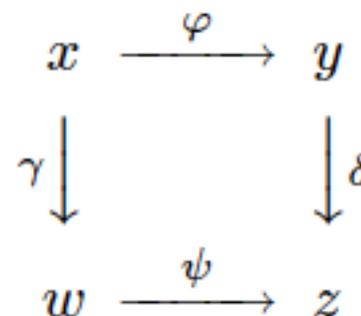
$$I_m \rightarrow I_{k+m}$$



Isographie négative

$$\langle I_k \rangle : T_m \rightarrow T_{-m}$$

$$I_m \rightarrow I_{k-m}$$

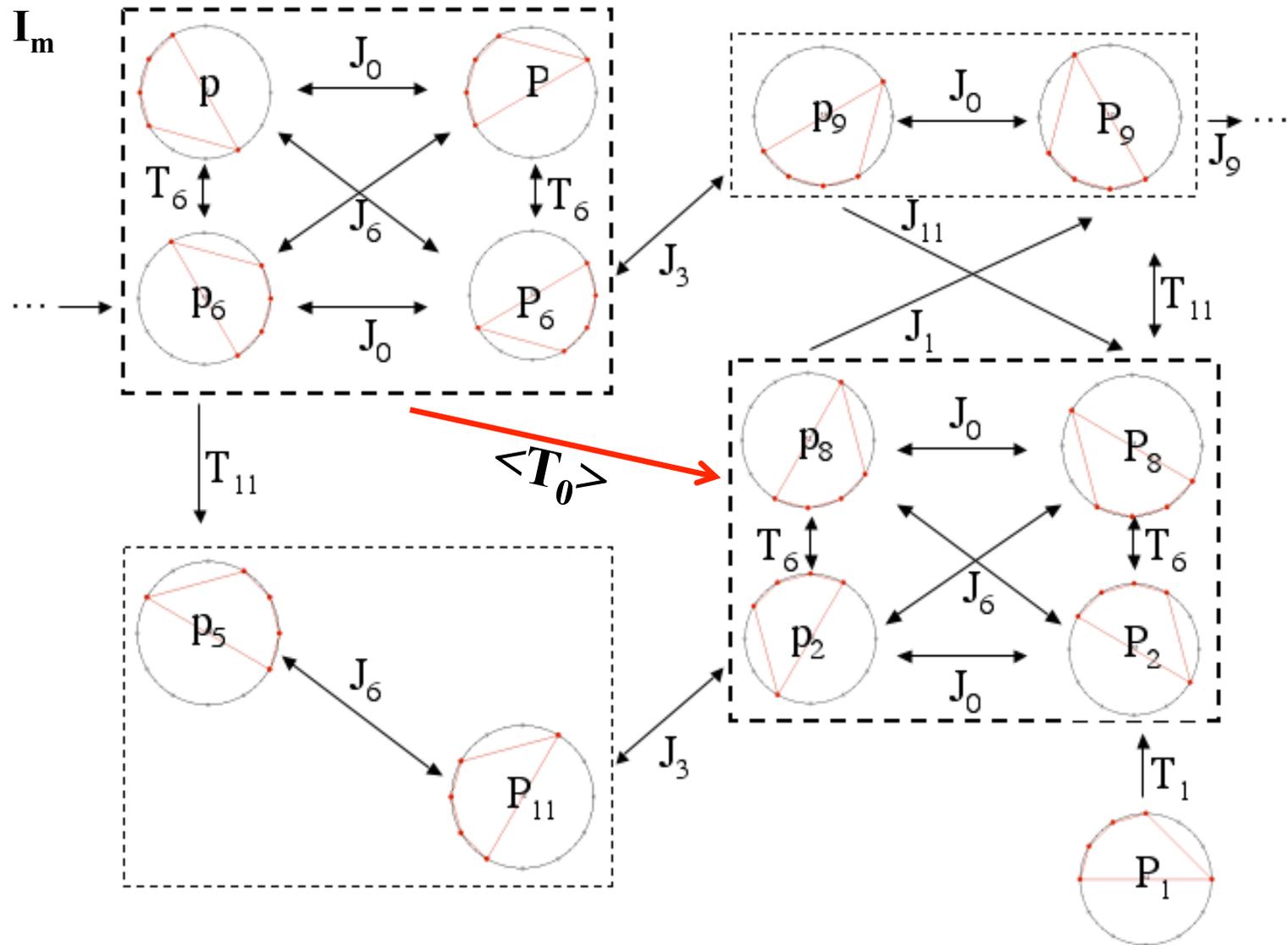


Isographies « fortes » dans le réseau transformationnel

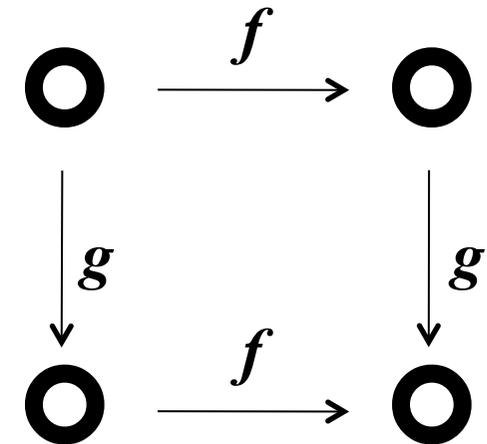
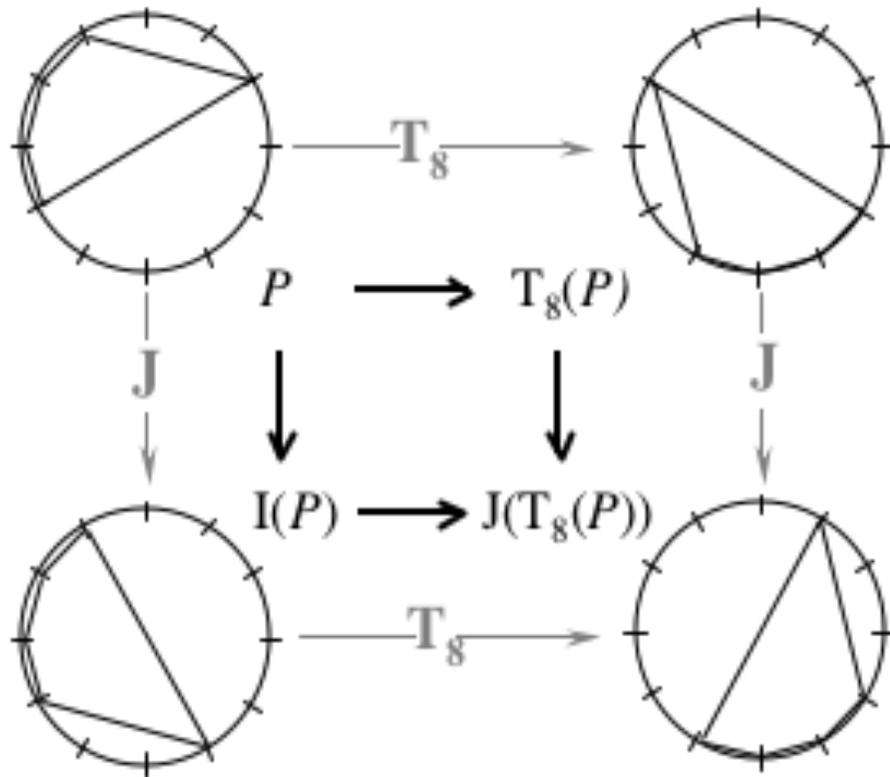
Stockhausen: *Klavierstück III* (Analyse de D. Lewin)

$$\langle T_0 \rangle : T_m \rightarrow T_m$$

$$I_m \rightarrow I_m$$



Inversions « contextuelles » et commutativité des diagrammes



Tout diagramme commute

$$\forall f, g \in \langle T, J \rangle$$

Le groupe des 24 transformations $\sigma = \{T_0, T_1, \dots, T_{11}, T_0J, T_1J, \dots, T_{11}J\}$ est commutatif et opère de manière simplement transitive sur l'espace S des 24 formes du pentacorde de base (i.e. l'ensemble de ses 12 transpositions et de ses 12 inversions)

$\Rightarrow (S, \sigma, \text{int})$ est un GIS

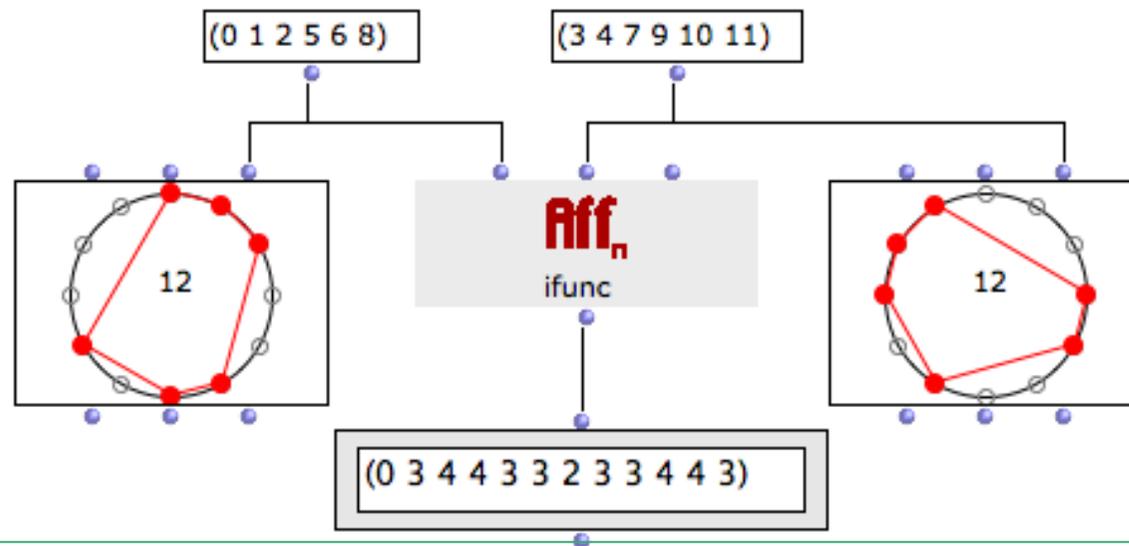
Approche transformationnelle et théorie de l'homométrie

$$\text{INJ}(A, B)(T_k) = \text{IFUNC}(A, B)(k)$$

- IFUNC peut s'écrire comme un produit de convolution de fonctions caractéristiques (Lewin, *JMT*, 1958)

$$\text{IFUNC}(A, B)(k) = \#\{(a, b) \in A \times B \mid a = b + k\}$$

$$\mathbf{1}_A * \mathbf{1}_B(k) = \sum_{x \in \mathbb{Z}_c} \mathbf{1}_A(x) \cdot \mathbf{1}_B(k - x) = \sum_{x \in \mathbb{Z}_c} \mathbf{1}_A(x) \cdot \mathbf{1}_B(x - k) = \sum_{\substack{(x, y) \in A \times B \\ y = x - k}} 1$$



$$\text{IFUNC}(A, B)(3) = \#\{(0, 9), (1, 10), (2, 11), (6, 3)\}$$

Relation Z, DFT et théorie de l'homométrie

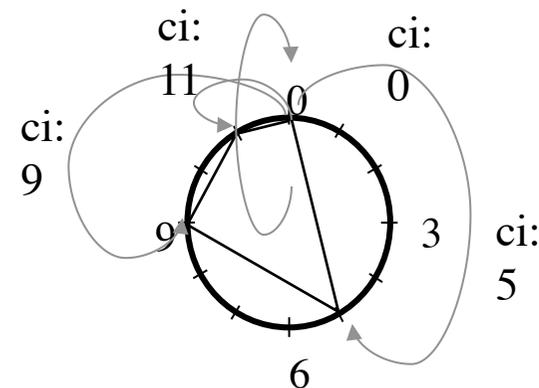
- Le contenu intervallique est équivalent à un produit de convolution de fonctions caractéristiques (Lewin, 1958)

$$\mathcal{F}_A : t \mapsto \sum_{k \in A} e^{-2i\pi kt/c}$$

$$IC_A(k) = (1_A \star 1_{-A})(k)$$

$$1_A \star \tilde{1}_B(k) = \sum_i 1_A(i) \times 1_B(i - k) = \sum_{\substack{i \in A \\ i - k \in B}} 1$$

$$\mathcal{F}(1_A \star \tilde{1}_B) = \mathcal{F}(1_A) \times \mathcal{F}(\tilde{1}_B)$$



$$A = \{0, 5, 9, 11\} \Rightarrow IC_A(k) = 1 \forall k$$

$$\mathcal{F}(IC_A) = \mathcal{F}_A \times \mathcal{F}_{-A} = |\mathcal{F}_A|^2$$

➔ **Relation Z**

$\{0, 1, 4, 6\}_{12}$

$iv = [4, 1, 1, 1, 1, 1, 2, 1, 1, 1, 1, 1]$

$\{0, 1, 3, 7\}_{12}$

- D. Ghisi, "From Z-relation to homometry: an introduction and some musical examples", Séminaire MaMuX, 10 décembre 2010
- P. Beauguitte, *Transformée de Fourier discrète et structures musicales*, Master ATIAM, 2011