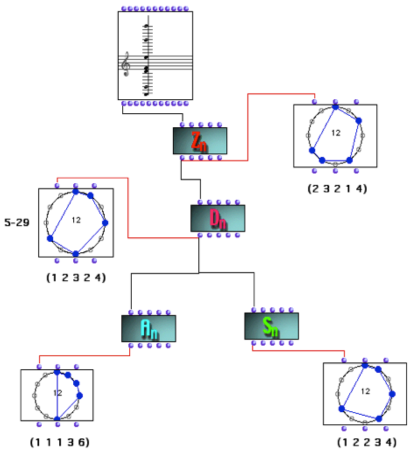
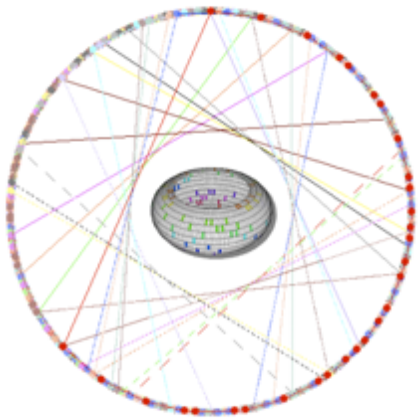
Joint International Workshop of
the National University of Singapore
Institute for Mathematical Sciences
and
Yong Siew Toh
Conservatory of Music

MATHEMUSICAL
CONVERSATIONS

# Tiling Canons as a key to approach open mathematical Conjectures?



Moreno Andreatta
Equipe Représentations Musicales
IRCAM / CNRS / UPMC

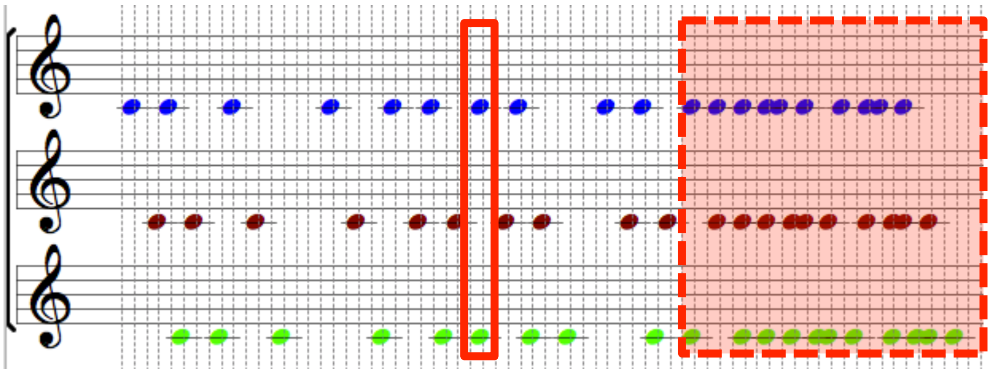
ircam Centre Pompidou

cnrs

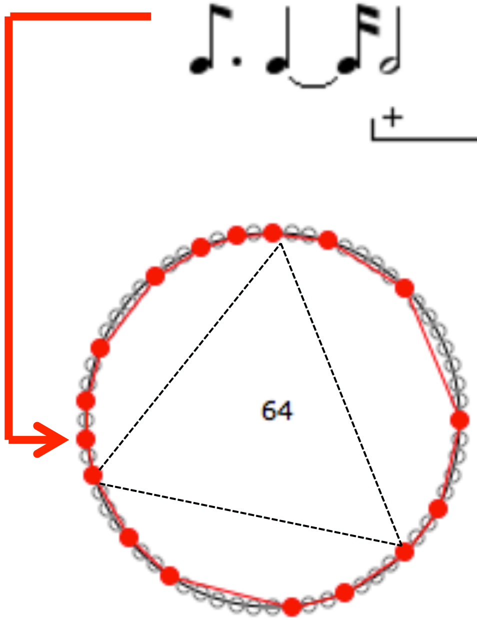
UPMC SORBONNE UNIVERSITÉS

# The 'emergence' of tiling rhythmic canons

*Harawi* (1945)



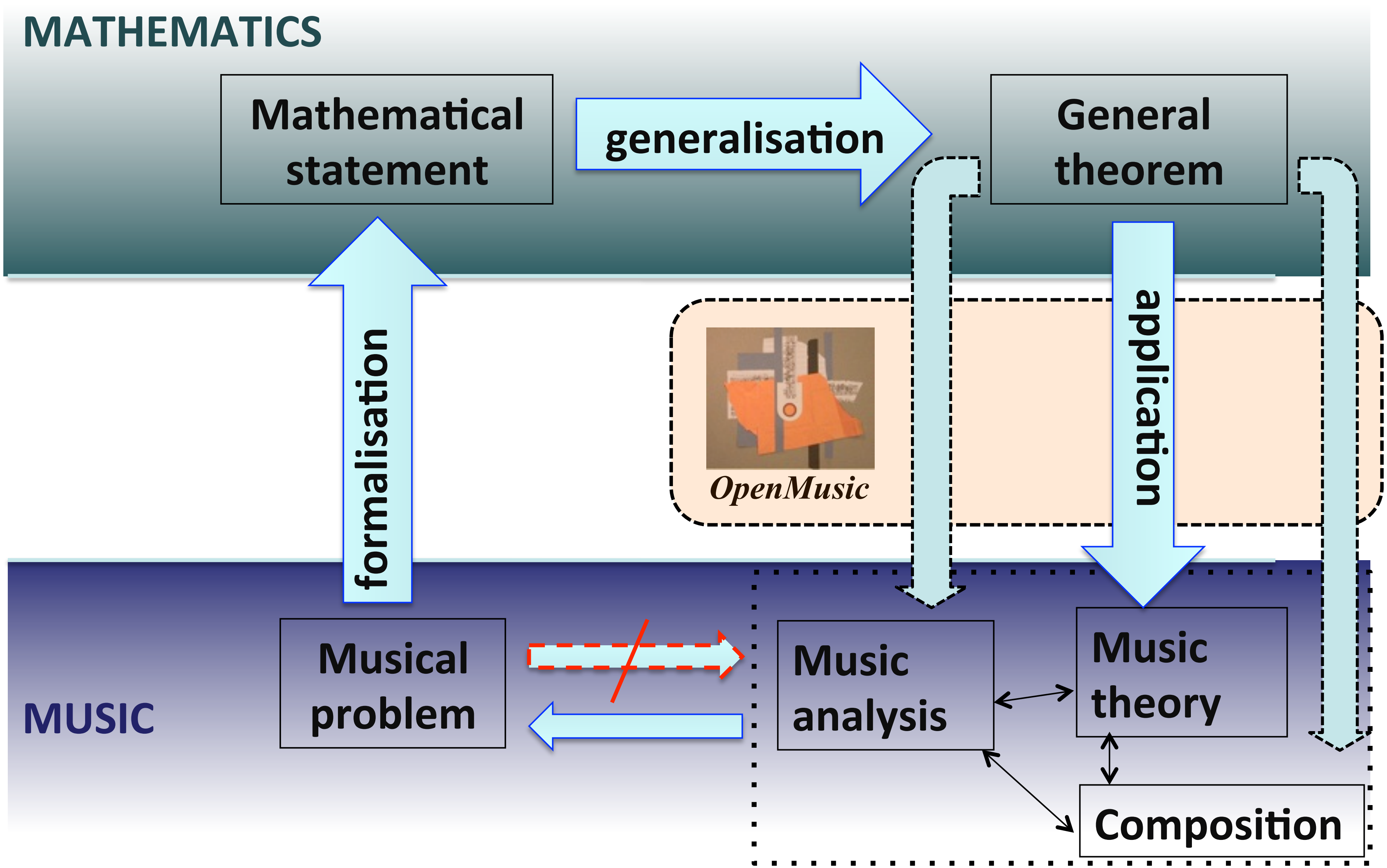
*Harawi*: rhythmic reduction


« …il résulte de tout cela que les différentes sonorités se mélangent ou s'opposent de manières très diverses, jamais au même moment ni au même endroit […]. C'est du désordre organisé »

O. Messiaen: *Traité de Rythme, de Couleur et d'Ornithologie*, tome 2, Alphonse Leduc, 1992.

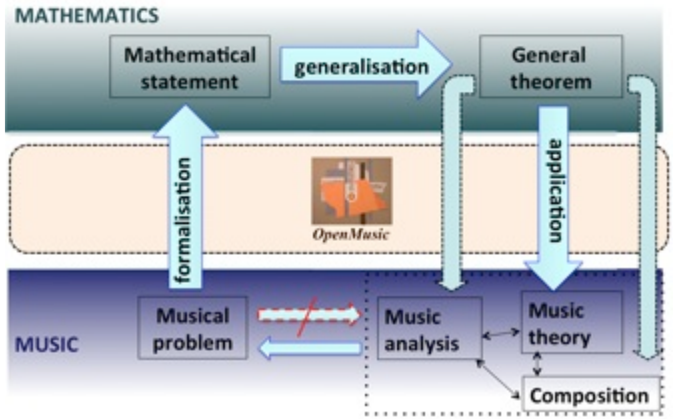
# The double movement of a 'mathemusical' activity

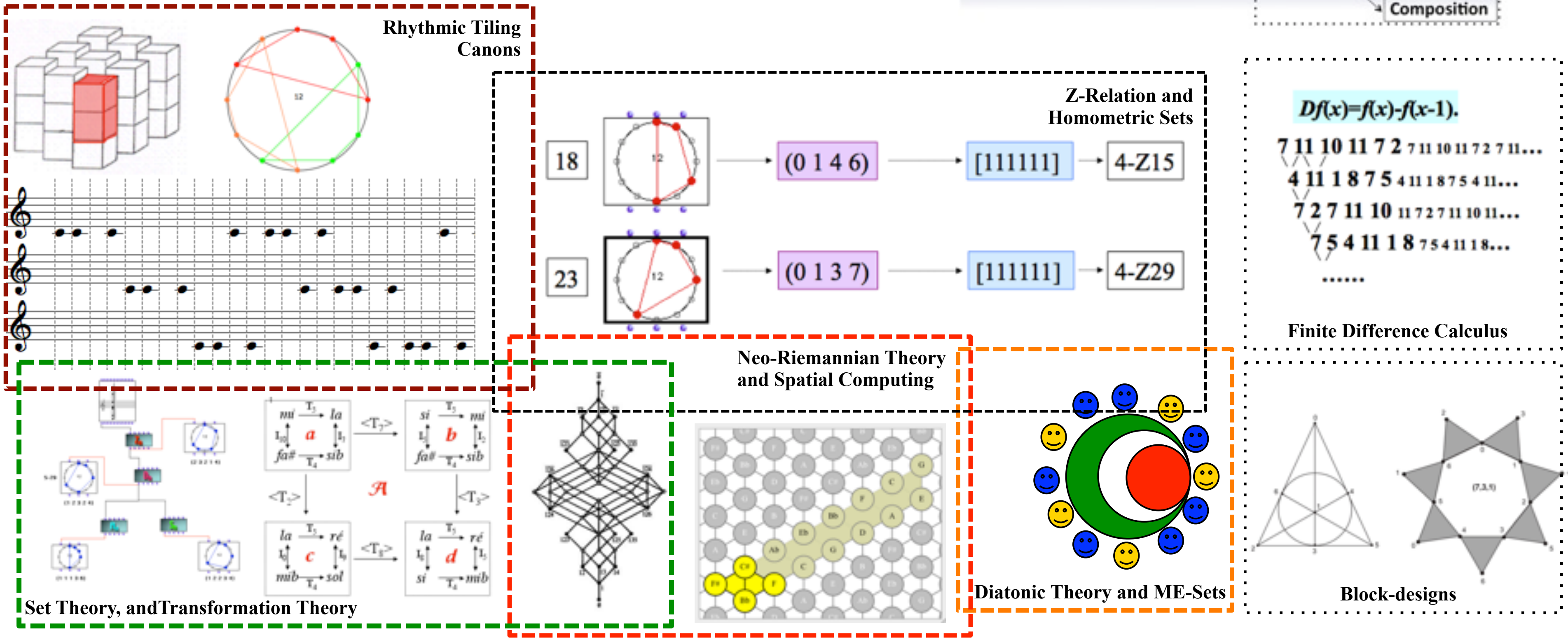


# Some examples of 'mathemusical' problems

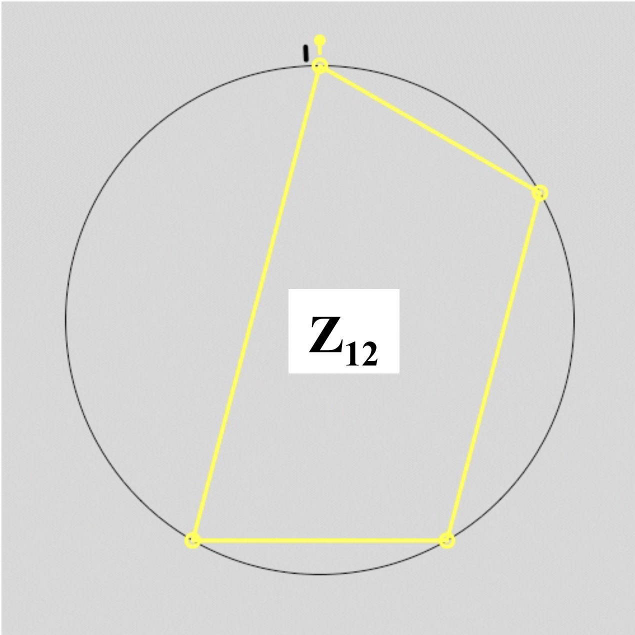
**M. Andreatta : *Mathematica est exercitium musicae*, Habilitation Thesis, IRMA University of Strasbourg, 2010**

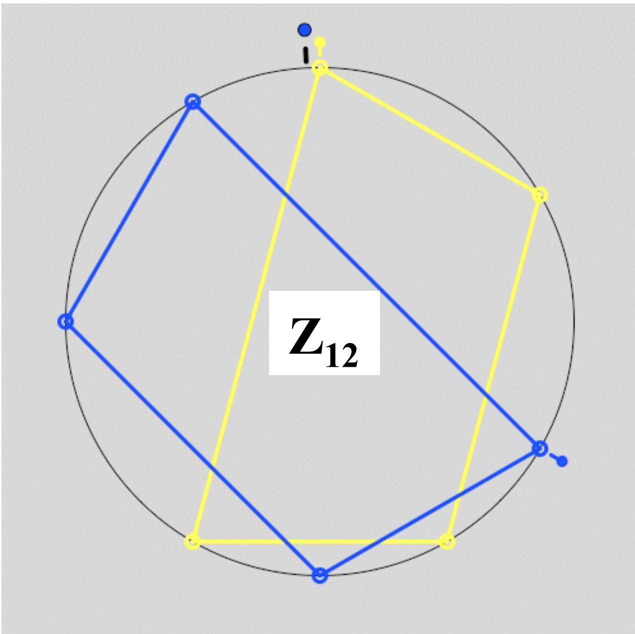
- **The construction of Tiling Rhythmic Canons**
- The *Z* relation and the theory of homometric sets
- *Set Theory* and Transformational Theory
- Neo-Riemannian Theory, Spatial Computing and FCA
- Diatonic Theory and Maximally-Even Sets
- Periodic sequences and finite difference calculus
- Block-designs and algorithmic composition

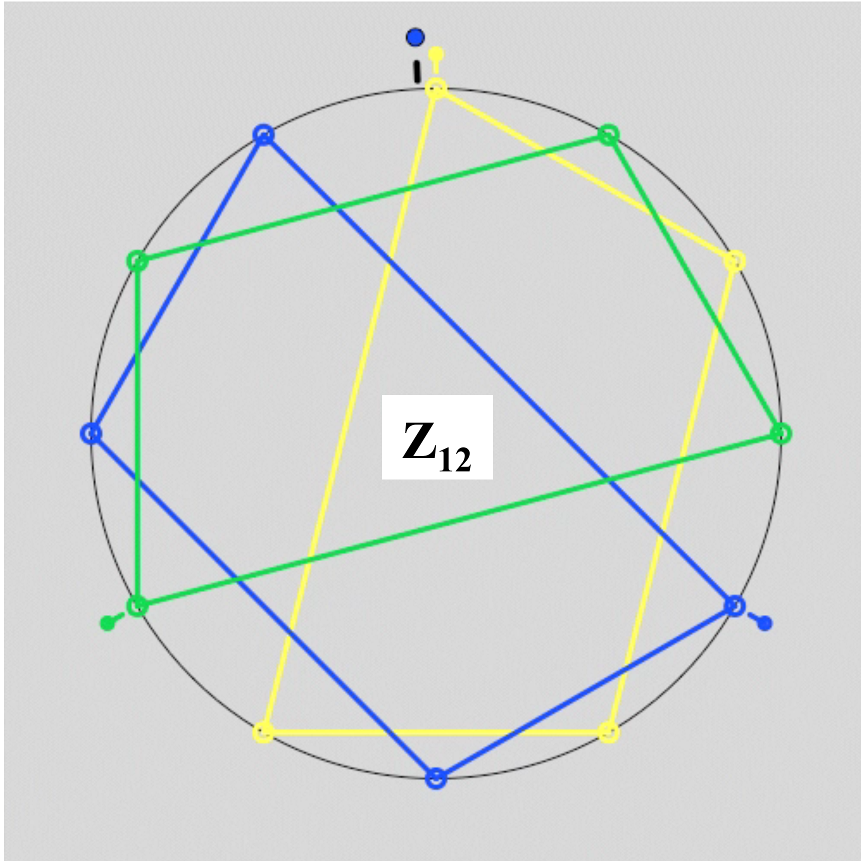


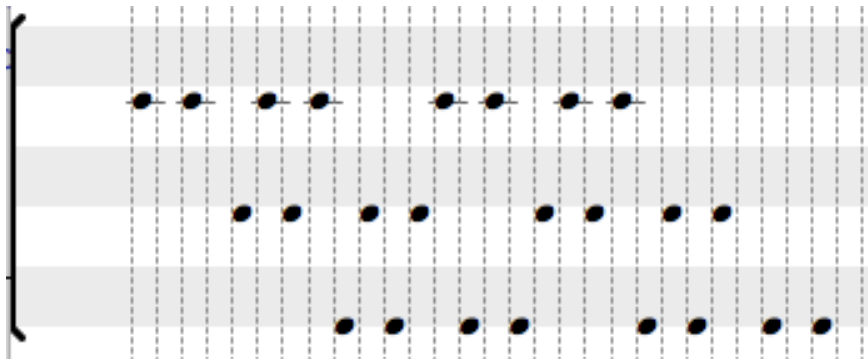



# Tiling the line with translates of one tile

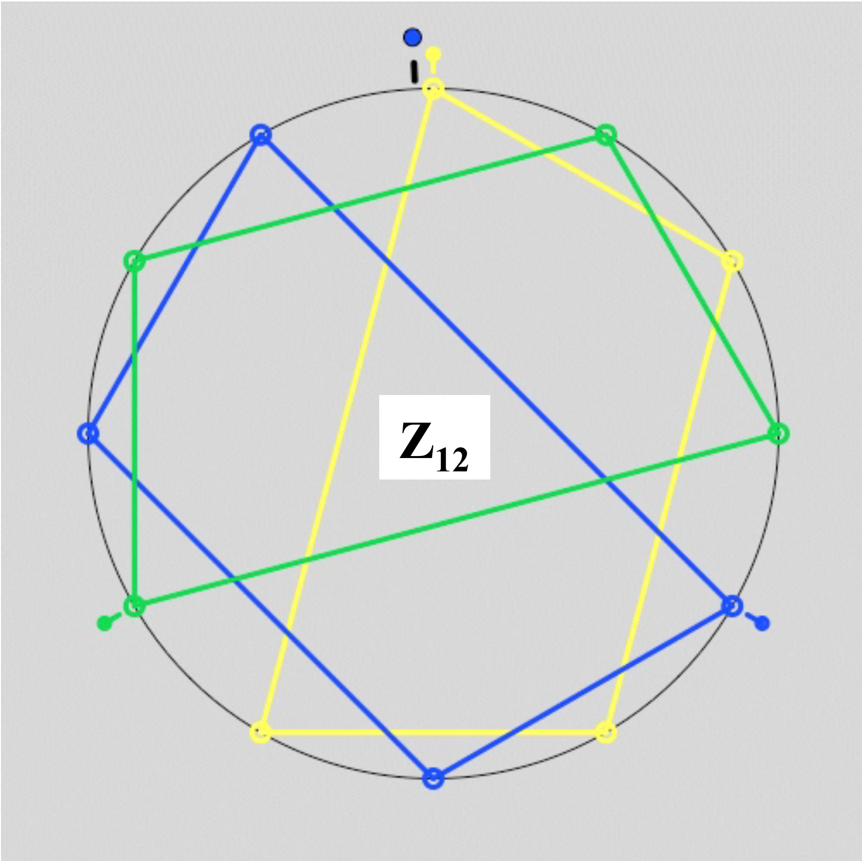


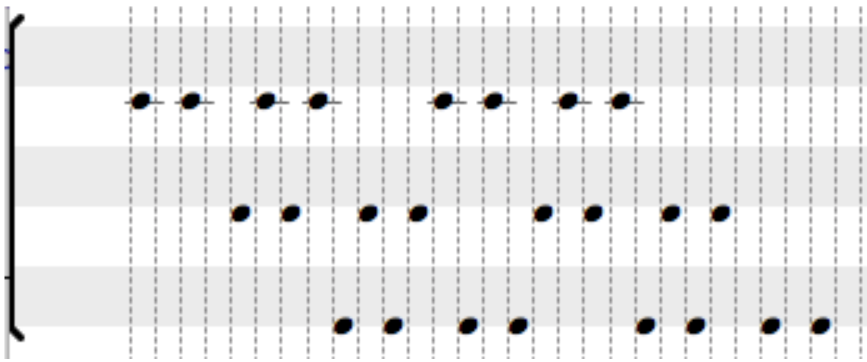



# Formalizing the tiling process as set-theoretical operations



$A_1 = \{0, 2, 5, 7\}$

$T_4$

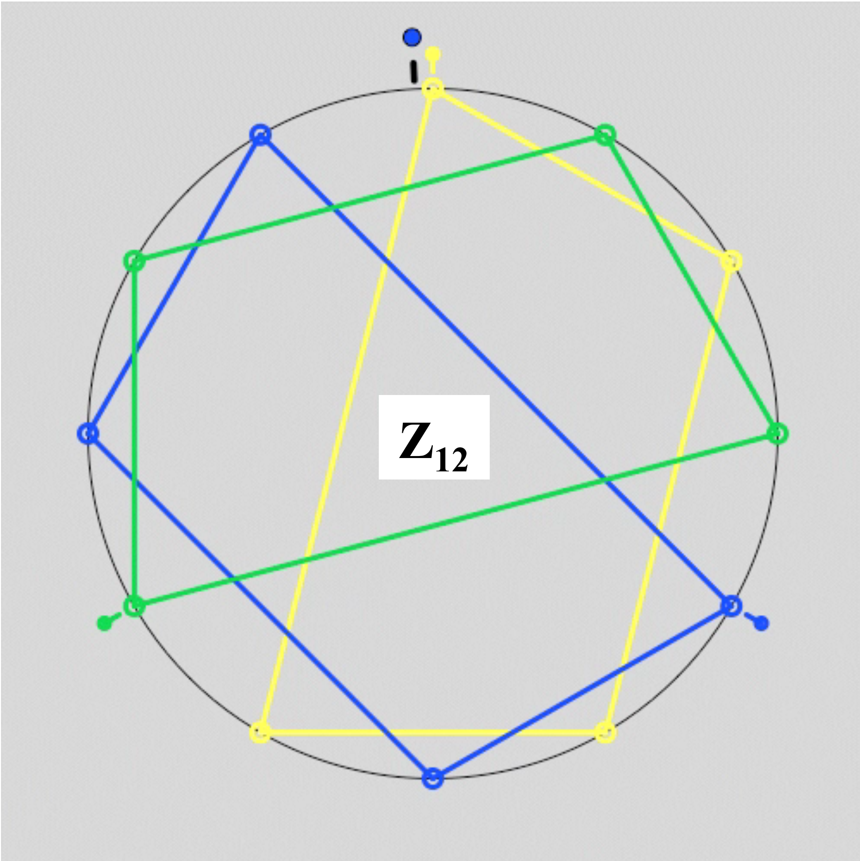
$A_2 = \{4, 6, 9, 11\}$

$T_4$

$A_3 = \{8, 10, 1, 3\}$

$Z_{12} = A_1 \cup A_2 \cup A_3$

# Formalizing the tiling process as a direct sum of subsets




$A_1 = \{0, 2, 5, 7\}$

$\downarrow T_4$

$A_2 = \{4, 6, 9, 11\}$

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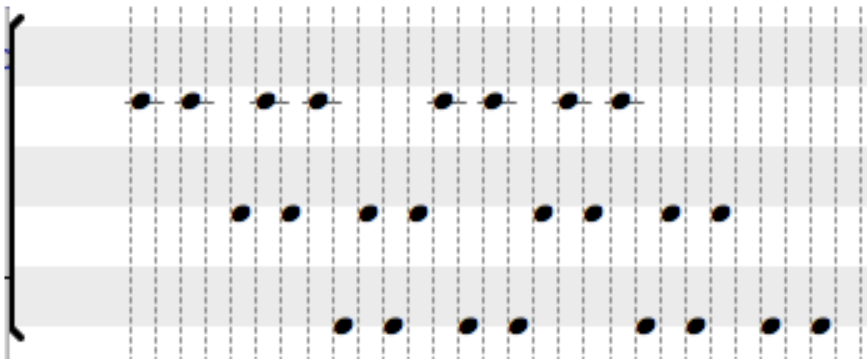
$A_3 = \{8, 10, 1, 3\}$



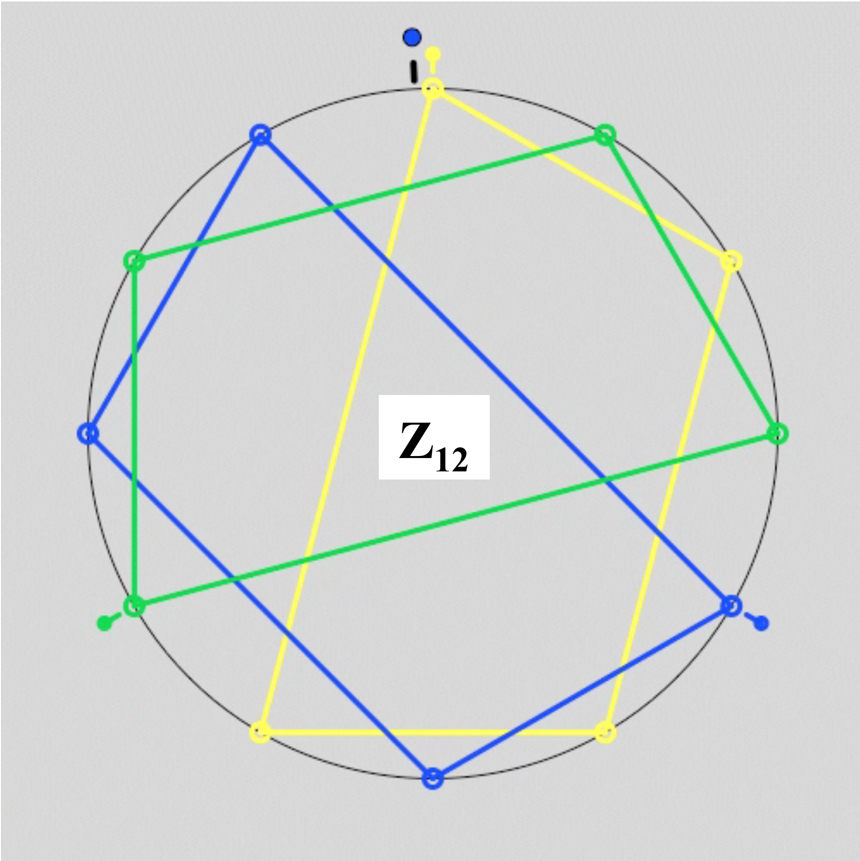
$Z_{12} = A_1 \cup A_2 \cup A_3$

$Z_{12} = A \oplus B$

A=?

B=?

# Formalizing the tiling process as a direct sum of subsets




$A_1 = \{0, 2, 5, 7\}$

$\downarrow T_4$

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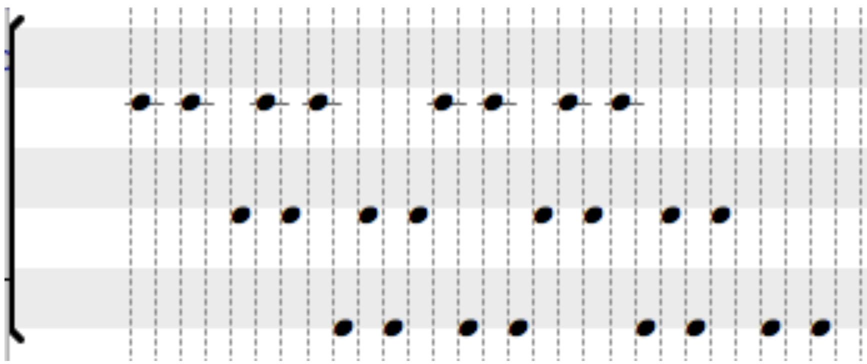
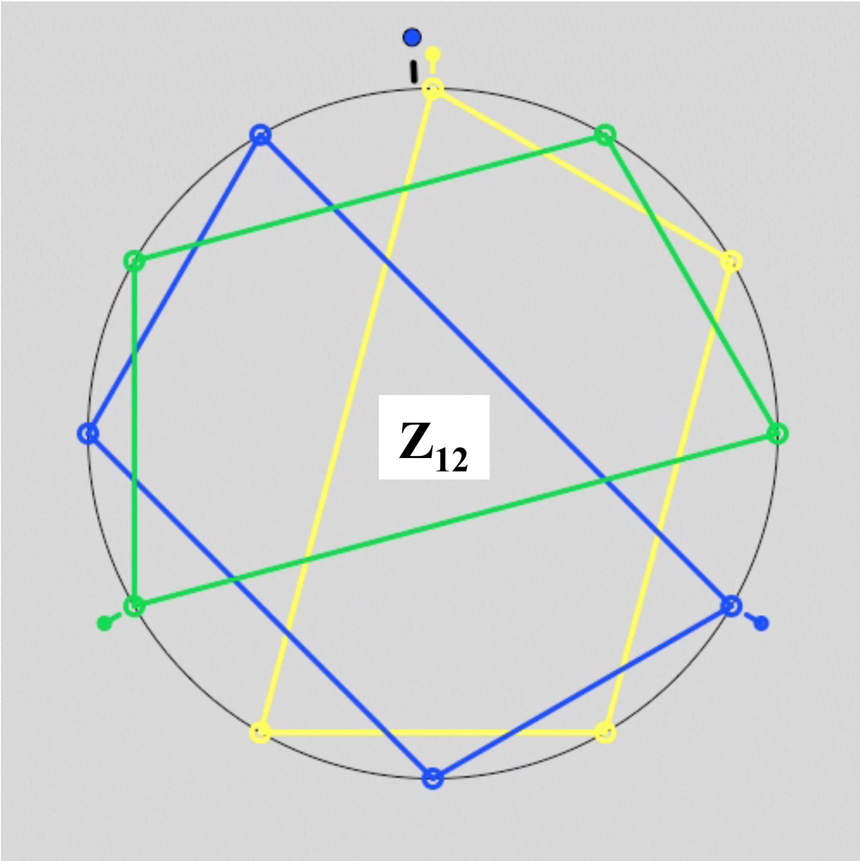


$Z_{12} = A_1 \cup A_2 \cup A_3$

$Z_{12} = A \oplus B$

$A = \{0,2,5,7\}$
$B = \{0,4,8\}$

# Formalizing the tiling process as a product of polynomials




$A_1 = \{0, 2, 5, 7\}$

$A_2 = \{4, 6, 9, 11\}$

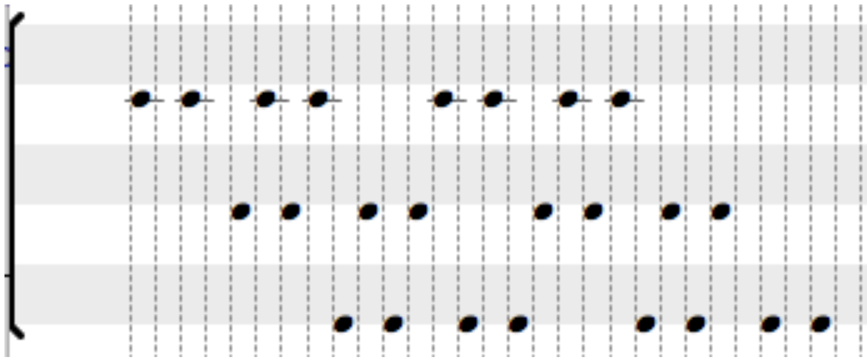
$A_3 = \{8, 10, 1, 3\}$

$T_4$ ($A_1 \to A_2$), $T_4$ ($A_2 \to A_3$)

$Z_{12} = A_1 \cup A_2 \cup A_3$

$Z_{12} = A \oplus B$

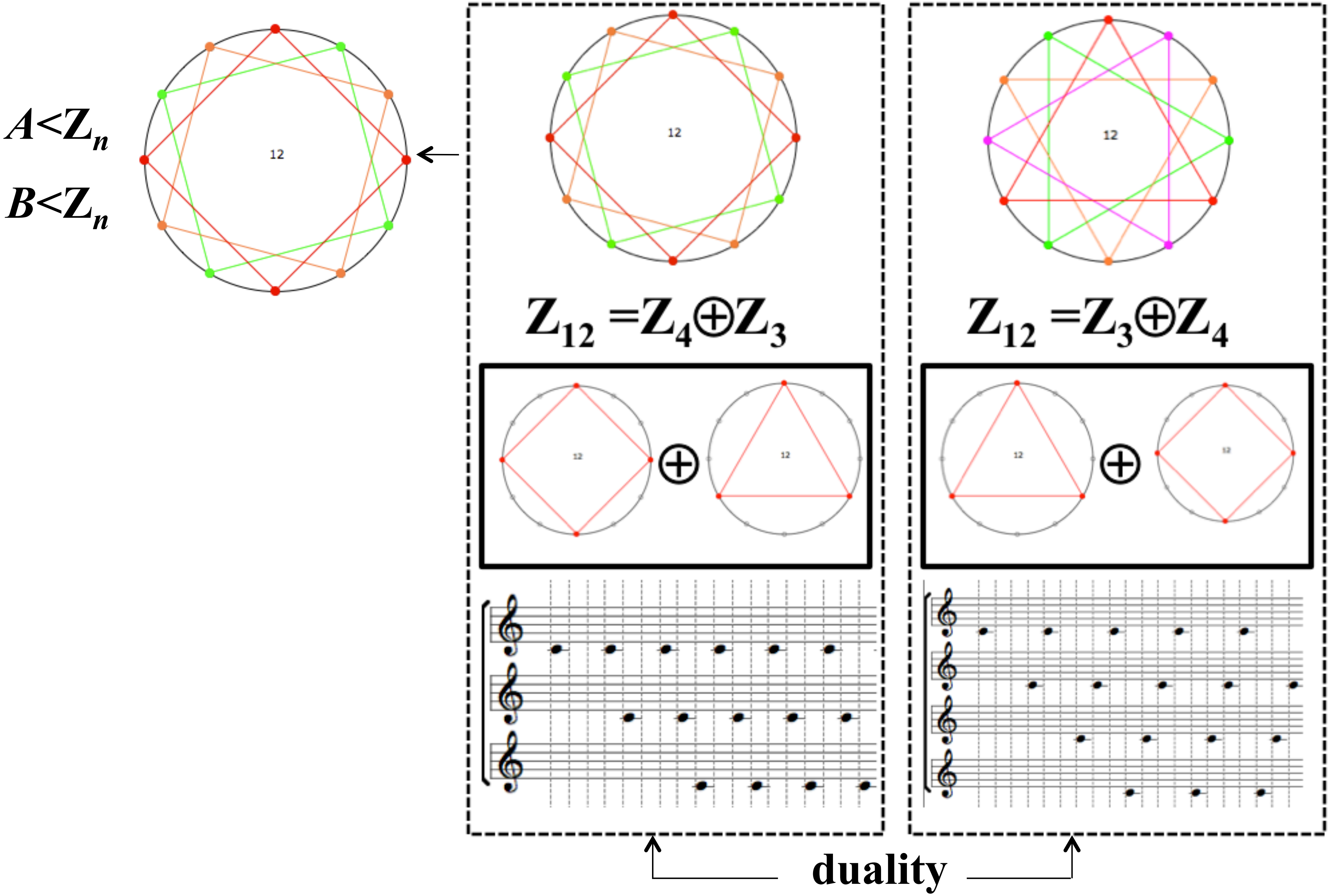
$A = \{0,2,5,7\}$
$B = \{0,4,8\}$



$1 + \ldots + x^{11} = A(x) \times B(x) \mod x^{12}-1$

$A(x) = 1 + x^2 + x^5 + x^7$
$B(x) = 1 + x^4 + x^8$

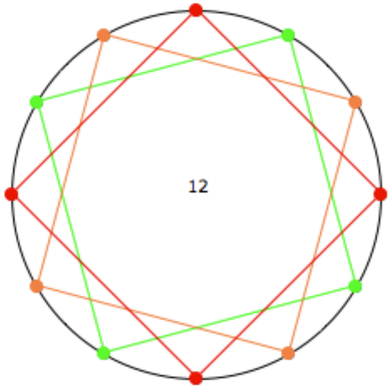
# The three ‘elementary’ types of tiling rhythmic canons

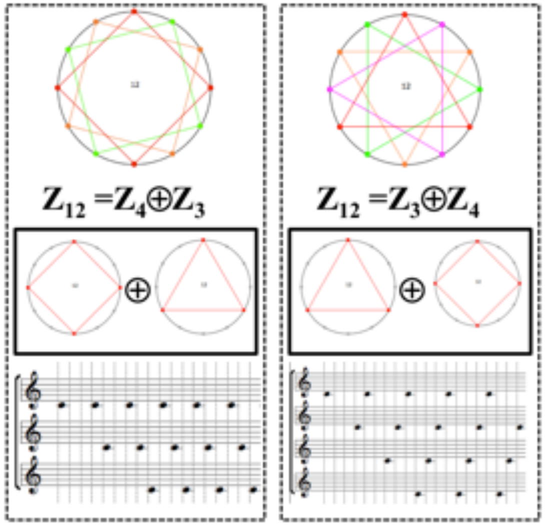


# The three ‘elementary’ types of tiling rhythmic canons

$A<Z_n$

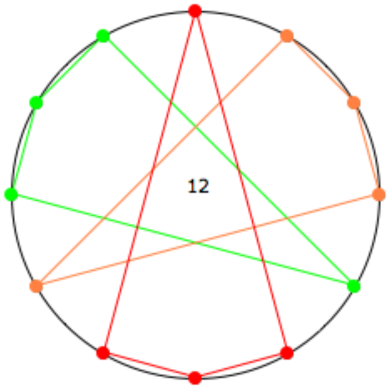
$B<Z_n$

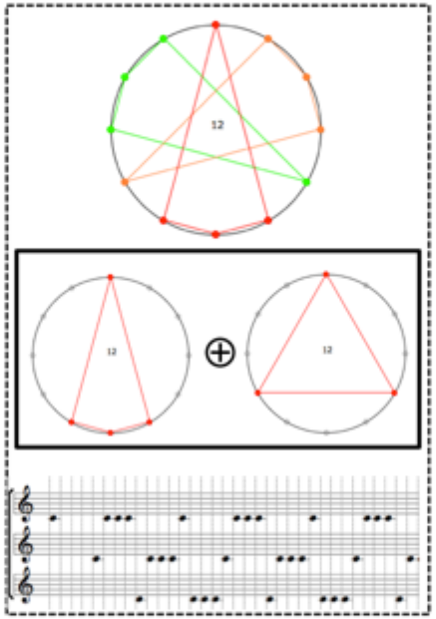



$A \subset Z_n$

$B<Z_n$

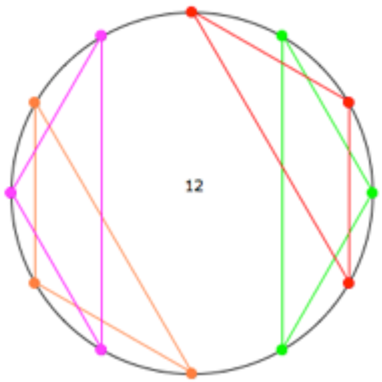


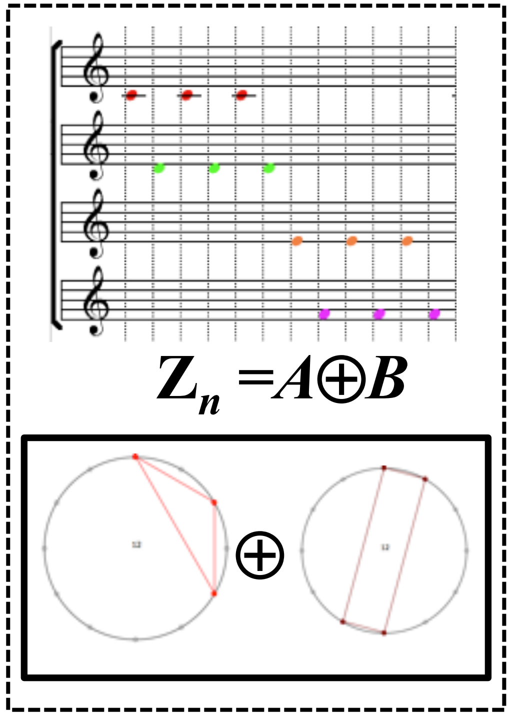



$A \subset Z_n$

$B \subset Z_n$

**$B$ periodic subset**

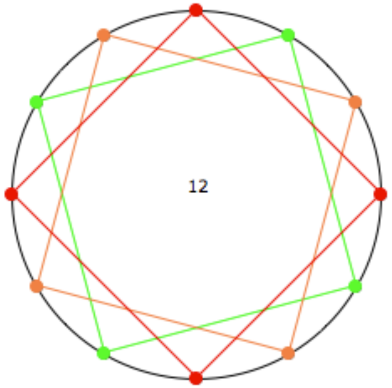


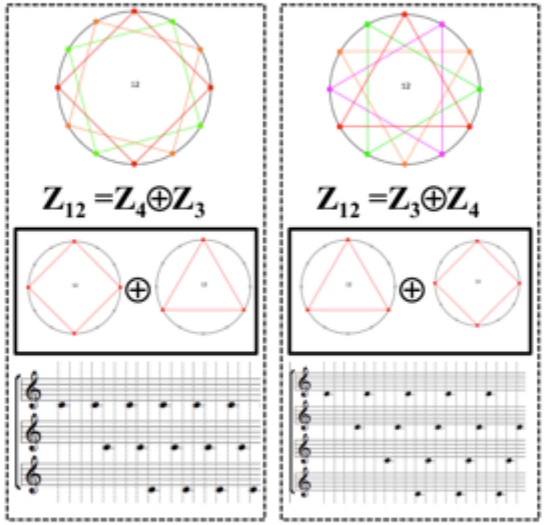



# The three ‘elementary’ types of tiling rhythmic canons

$A<Z_n$

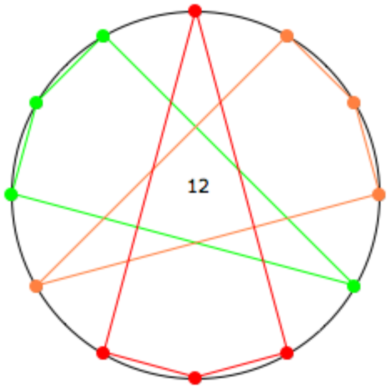
$B<Z_n$

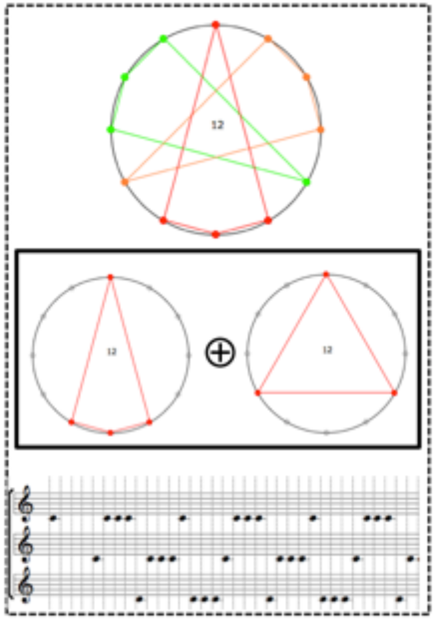



$A\subset Z_n$

$B<Z_n$

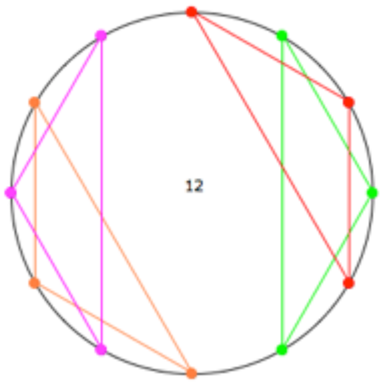


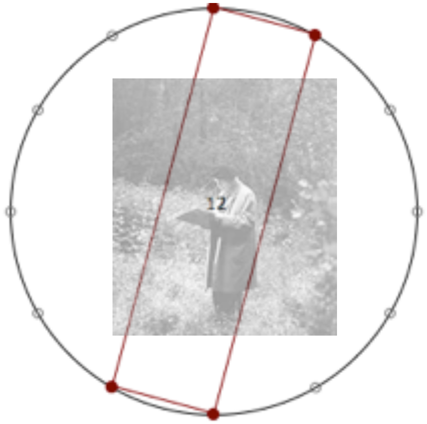



$A\subset Z_n$

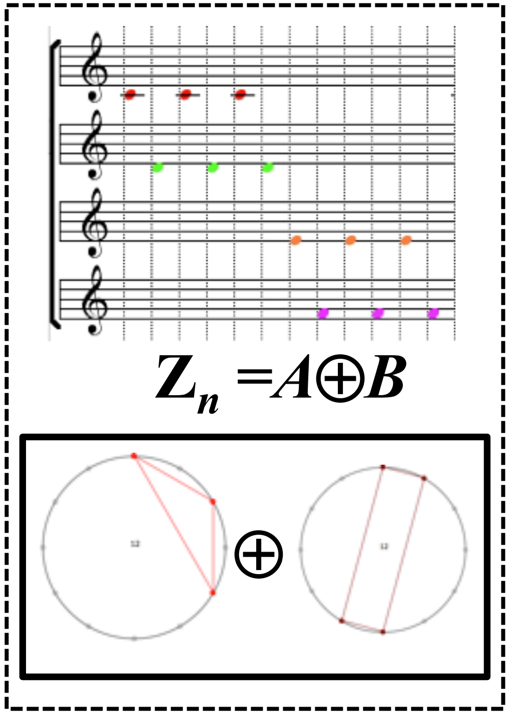
$B\subset Z_n$

***B* periodic subset**



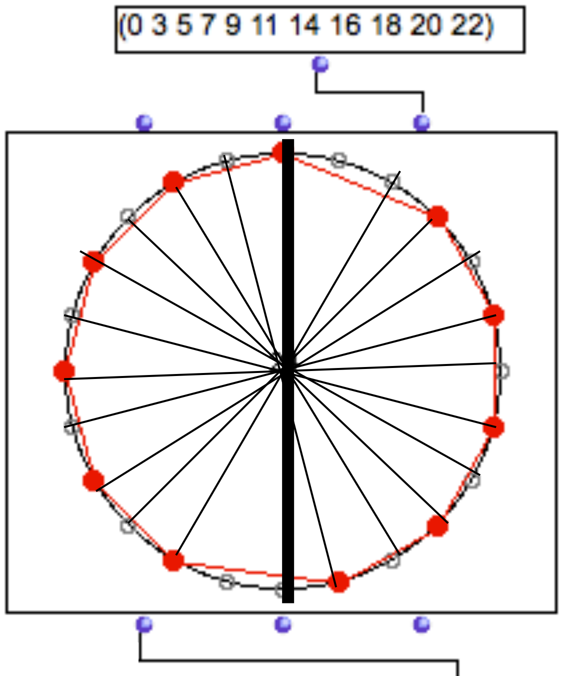

**Is Messiaen’s Property mandatory for tiling rhythmic canons?**

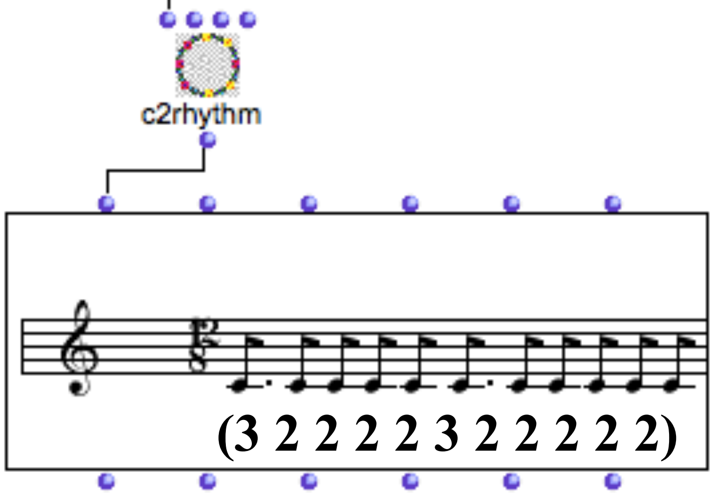


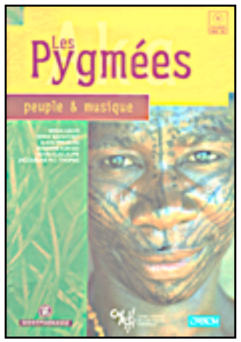
# The Oddity Property and its tiling generalization

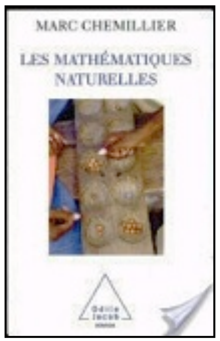
(Simha Arom & Marc Chemillier)

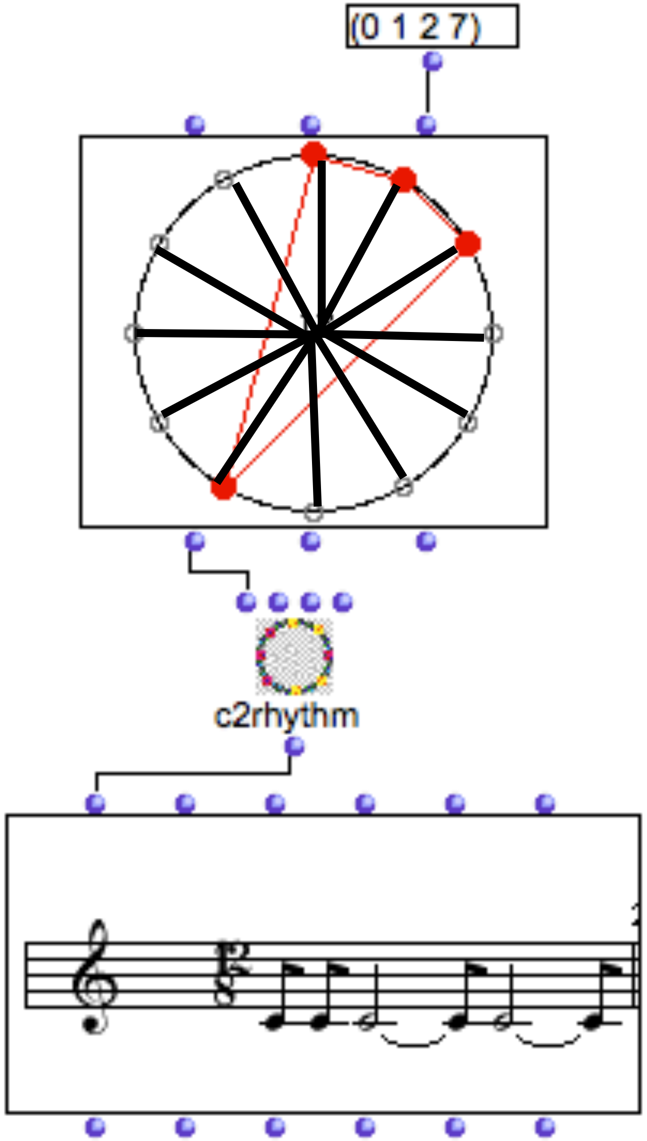
(Rachel W. Hall & P. Klingsberg)



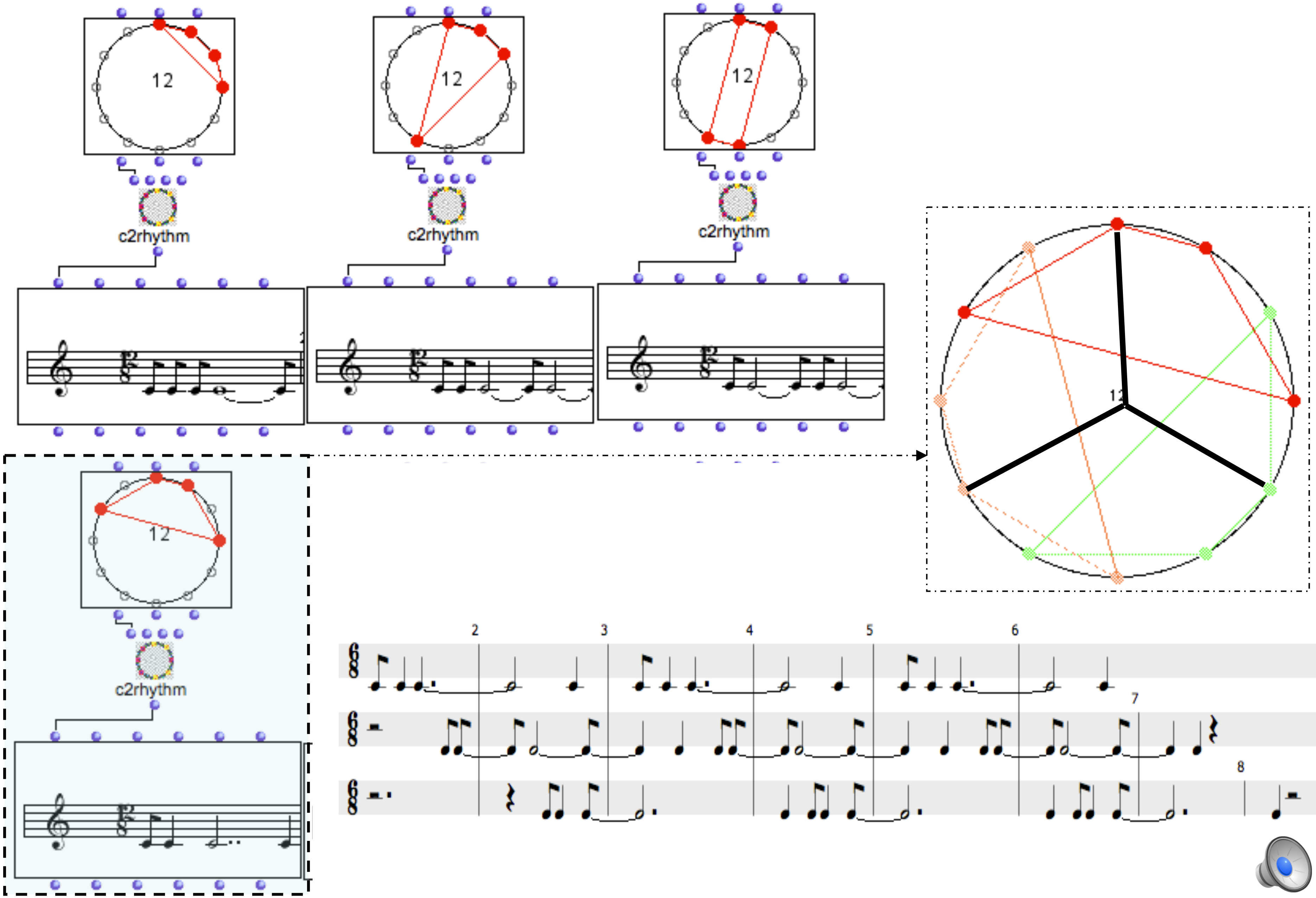



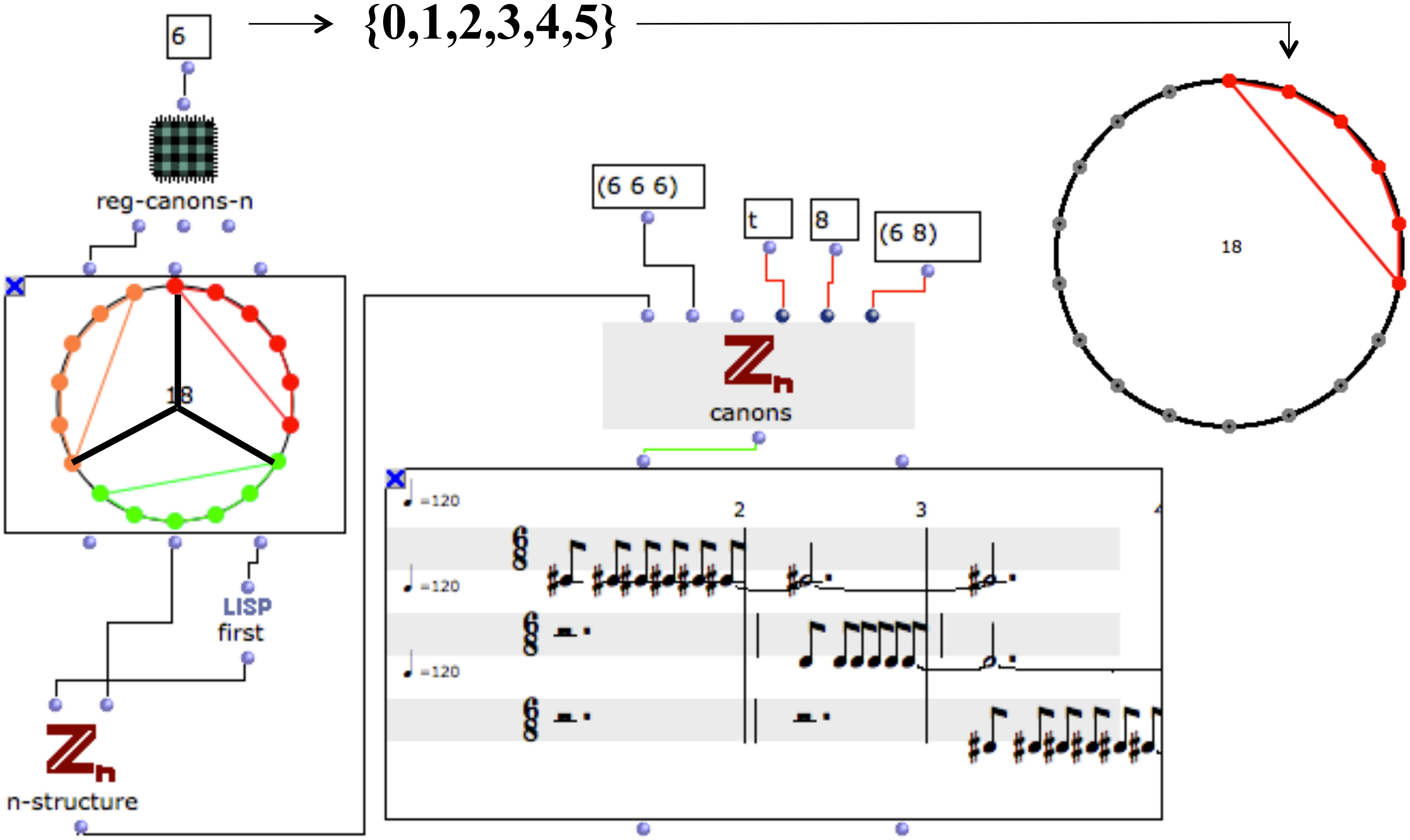



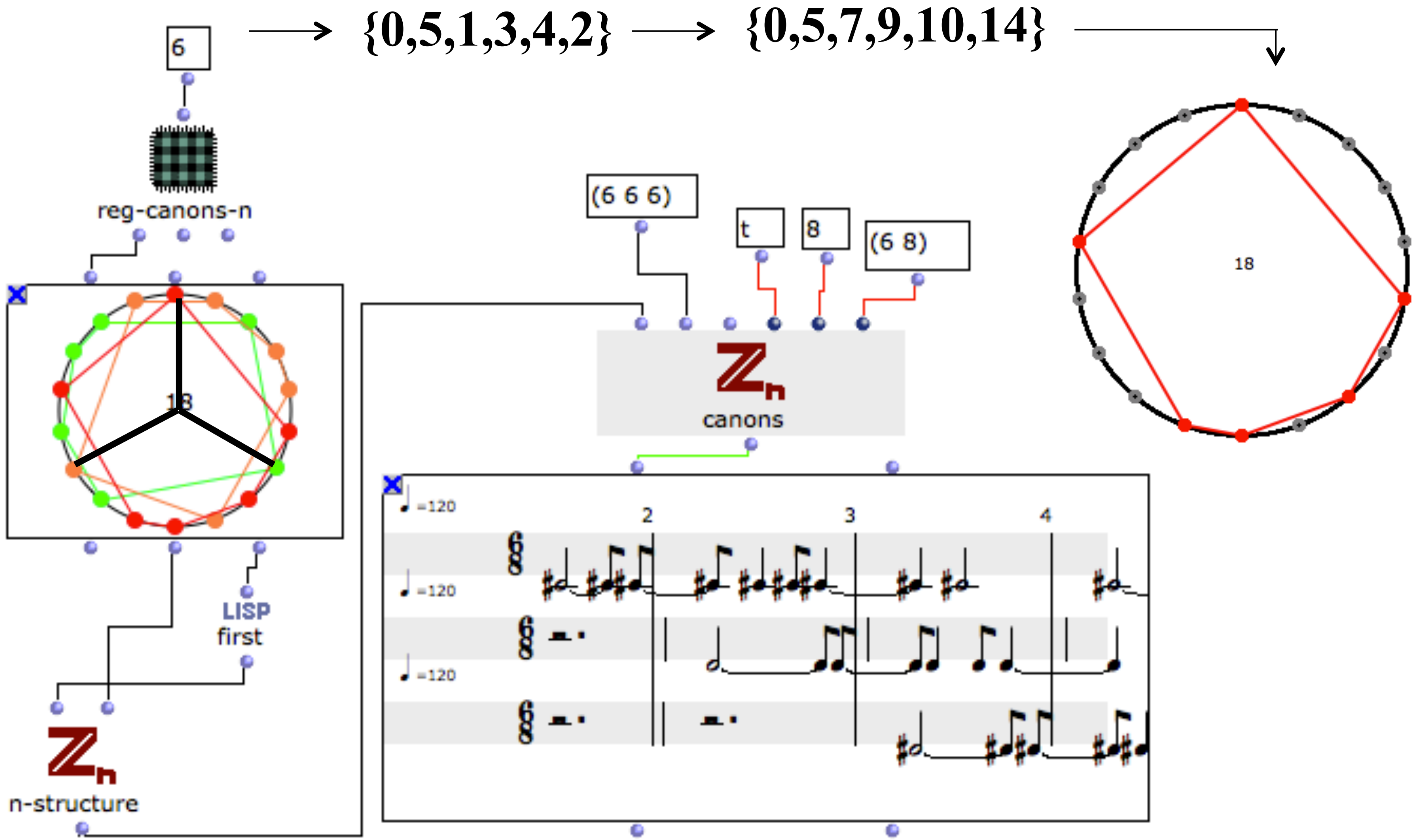
# 3-asymmetric rhythmic pattern and tiling canons



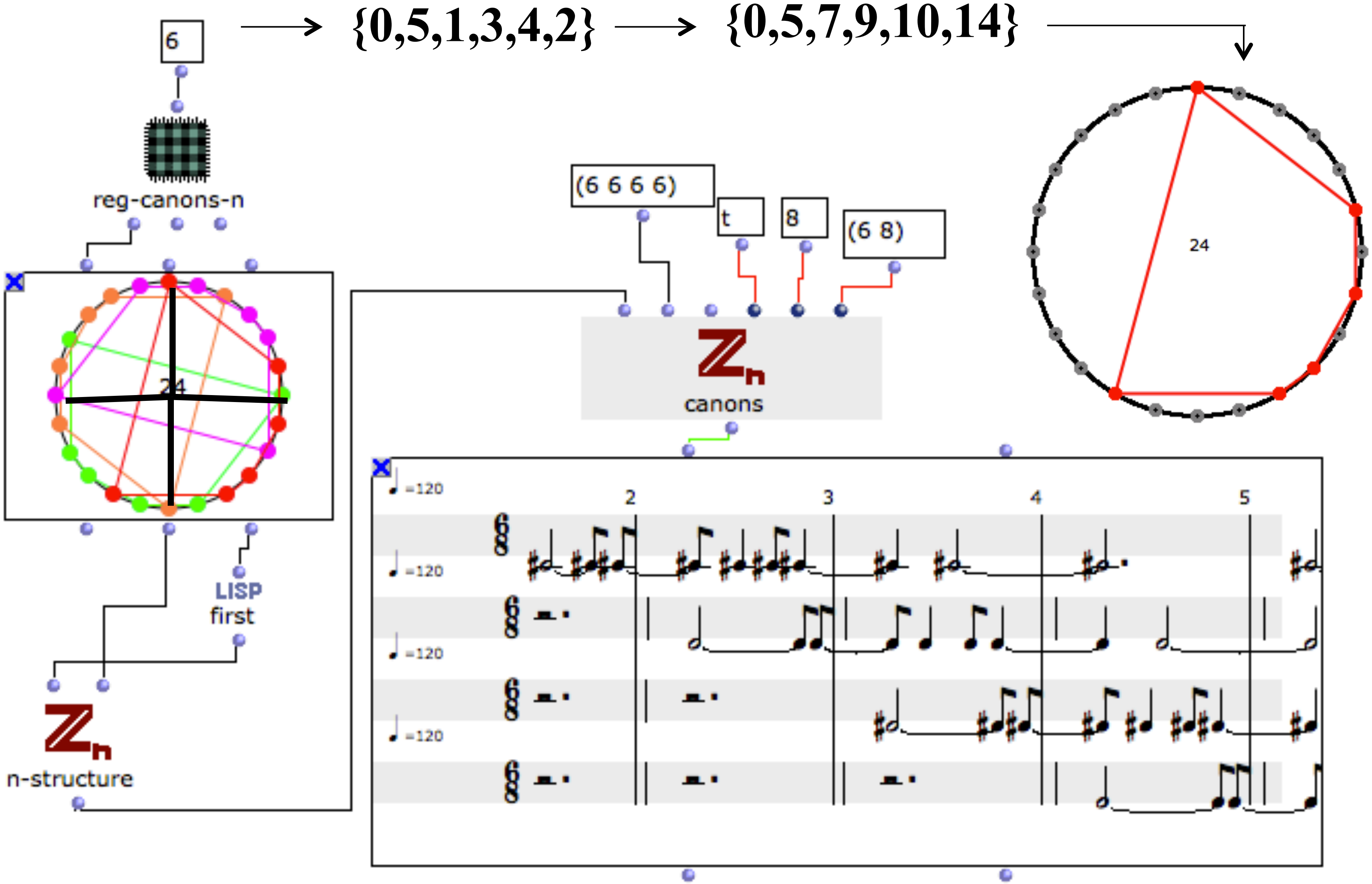
# Permutational construction of a *k*-asymmetric pattern



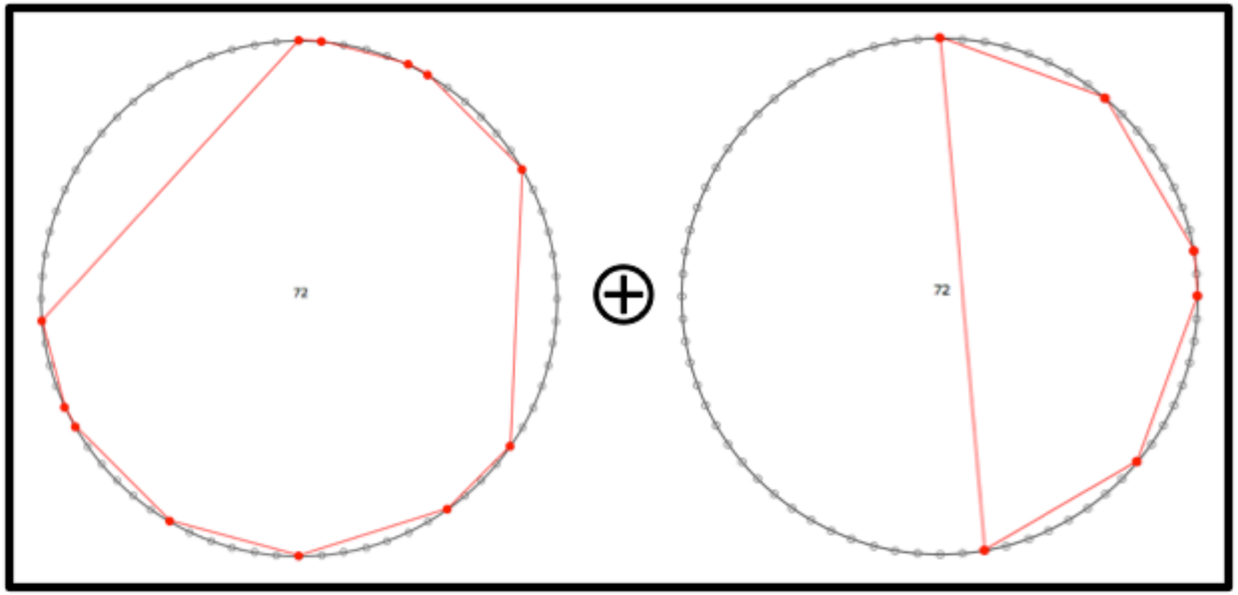
# Permutational construction of a *k*-asymmetric pattern



# Permutational construction of a *k*-asymmetric pattern

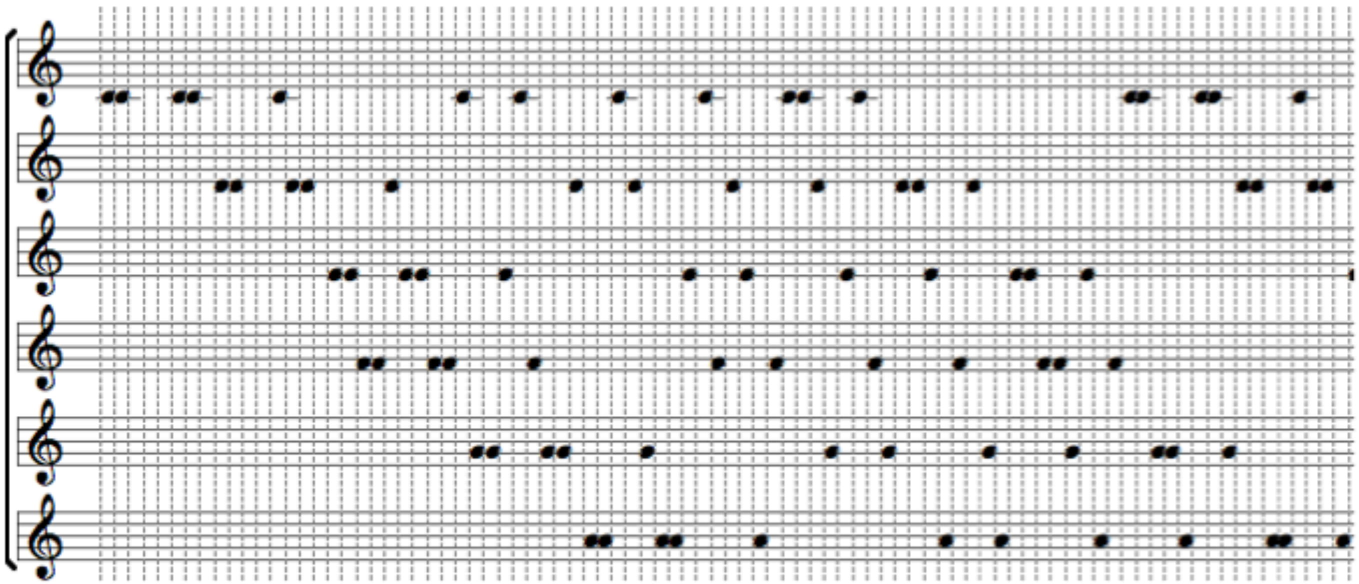


# Aperiodic Rhythmic Tiling Canons (Vuza Canons)


Dan Vuza

Anatol Vieru

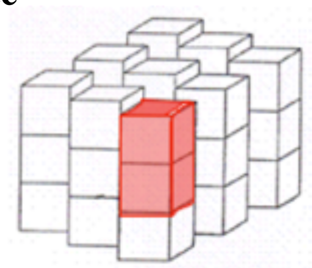


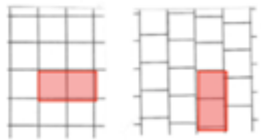
[ ]

# Tiling Rhythmic Canons as a ‘mathemusical’ problem

***Minkowski/Hajós Problem (1907-1941)***

**In any simple lattice tiling of the *n*-dimensional Euclidean space by unit cubes, some pair of cubes must share a complete (*n*-1)-dimensional face**

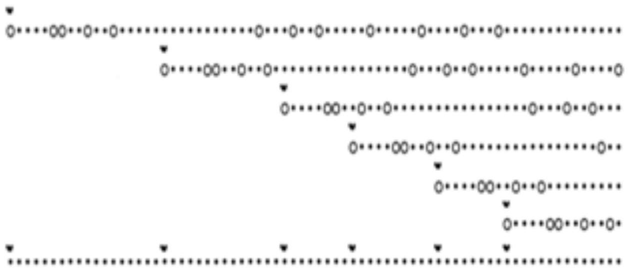




***Vieru’s problem and Vuza’s formalization (PNM, 1991)***

**A Vuza Canon is a factorization of a cyclic group in a direct sum of two non-periodic subsets**

**$Z/nZ = R \oplus S$**



***Link between Minkowski problem and Vuza Canons (Andreatta, Master diss. 1996)***

**Hajós groups (*good groups*)**

**$Z/nZ$ with $n \in \{p^{\alpha}, p^{\alpha}q, pqr, p^2q^2, p^2qr, pqrs\}$ where $p, q, r, s$, are distinct prime numbers**

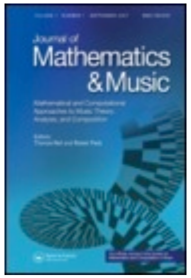
**Non-Hajós group (*bad groups*)**

**72**
**108 120 144 168 180**
**200 216 240 252 264 270 280 288**
**300 312 324 336 360 378 392 396**
**400 408 432 440 450 456 468 480**
**500 504 520 528 540 552 560 576 588 594**
**600 612 616 624 648 672 675 680 684 696**
**700 702 720 728 744 750 756 760 784 792**
**800 810 816 828 864 880 882 888…**

**(Sloane’s sequence A102562)**

***Fuglede Spectral Conjecture* (1974)**

**M. Andreatta & C. Agon (eds), « Tiling Problems in Music », Special Issue of the *Journal of Mathematics and Music*, Vol. 3, Number 2, July 2009 (with contributions by E. Amiot, F. Jedrzejewski, M. Kolountzakis and M. Matolcsi)**




**S. Stein, S. Szabó: *Algebra and Tiling*, Carus Math. Mon. 1994**

# Fuglede Spectral Conjecture

A subset of the $n$-dimensional Euclidean space tiles by translation *iff* it is *spectral*.
(*J. Func. Anal.* 16, 1974)

➔False in dim. $n \geq 3$ (Tao, Kolountzakis, Matolcsi, Farkas and Mora)

➔ Open in dim. 1 et 2

DEFINITION 6 *A subset A of some vector space (say $\mathbb{R}^n$) is spectral iff it admits a Hilbert base of exponentials, i.e. if any map $f \in L^2(A)$ can be written*

$$f(x) = \sum f_k \exp(2i\pi\lambda_k.x)$$

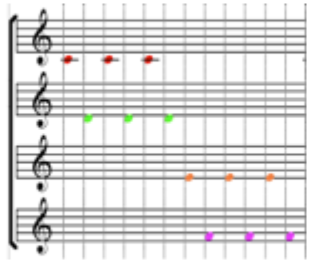
*for some fixed family of vectors $(\lambda_k)_{k\in\mathbb{Z}}$ where the maps $e_k : x \mapsto \exp(2i\pi\lambda_k.x)$ are mutually orthogonal (i.e. $\int_A \overline{e_k} e_j = 0$ whenever $k \neq j$).*

↓ $n$=1

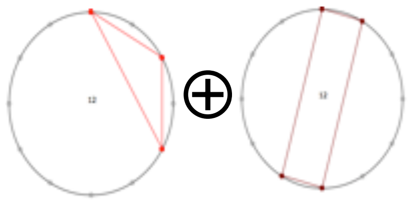
DEFINITION 8. *A subset $A \in \mathbb{Z}$ is* spectral *if there exists a spectrum $\Lambda \subset [0,1]$ (i.e., a subset with the same cardinality as $A$) such that $e^{2i\pi(\lambda_i - \lambda_j)}$ is a root of $A(X)$ for all distinct $\lambda_i, \lambda_j \in \Lambda$.*

**Example:**
$A$={0,1,6,7} ➔ Λ={0,1/12,1/2,7/12}
since exp(πi), exp(πi/6), exp(-πi/6), exp(5πi/6), exp(-5πi/6) are the roots of the associated polynomial $A(X) = 1 + X + X^6 + X^7$

→



$Z_n = B \oplus A$



# Fuglede Spectral Conjecture and Vuza Canons

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**(*J. Func. Anal.* 16, 1974)**

➔**False in dim. $n \geq 3$**
(Tao, Kolountzakis, Matolcsi, Farkas and Mora)

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↓ ***n*=1**

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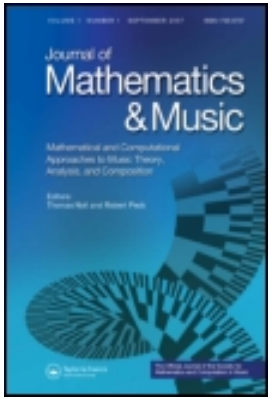
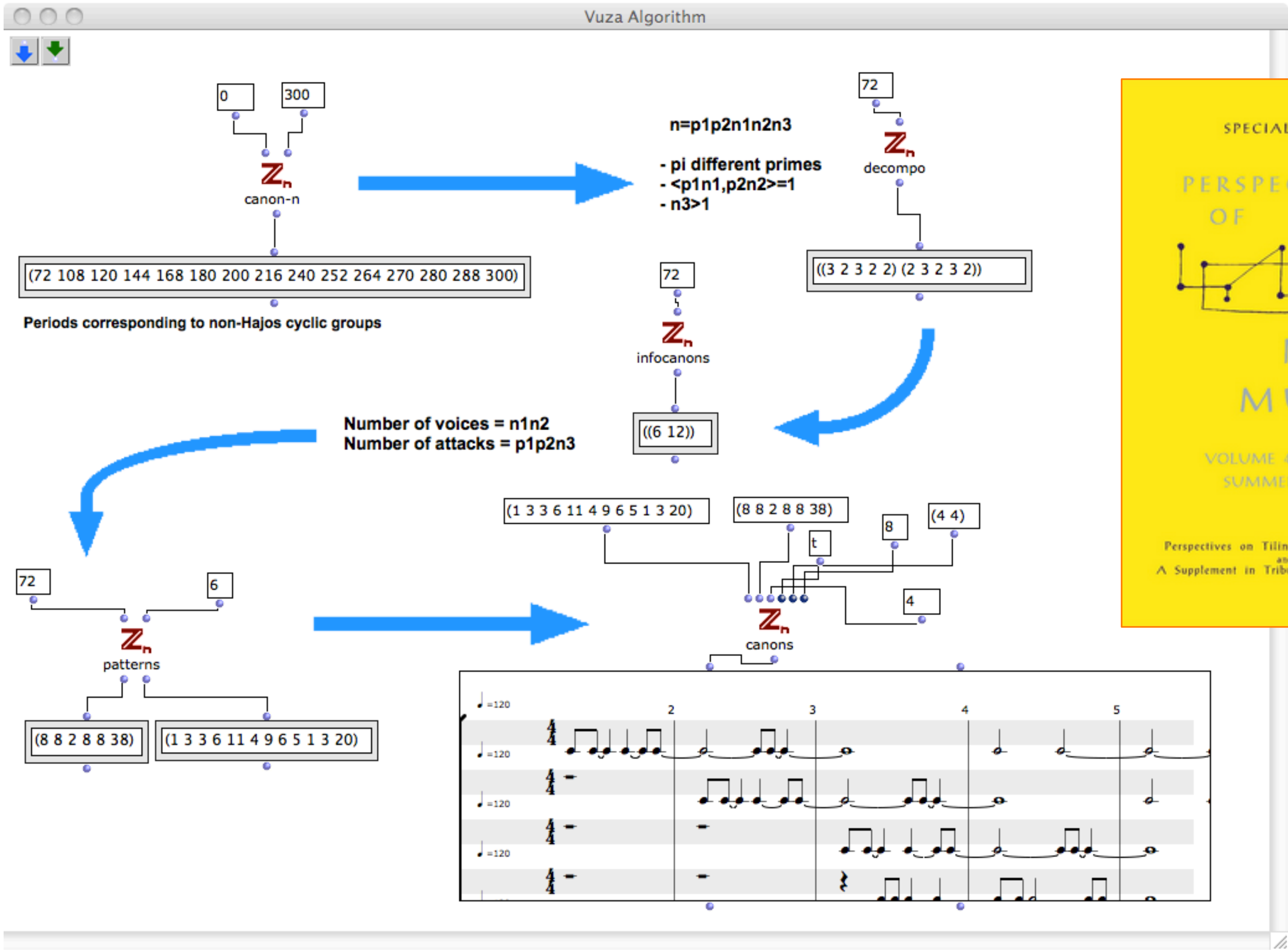
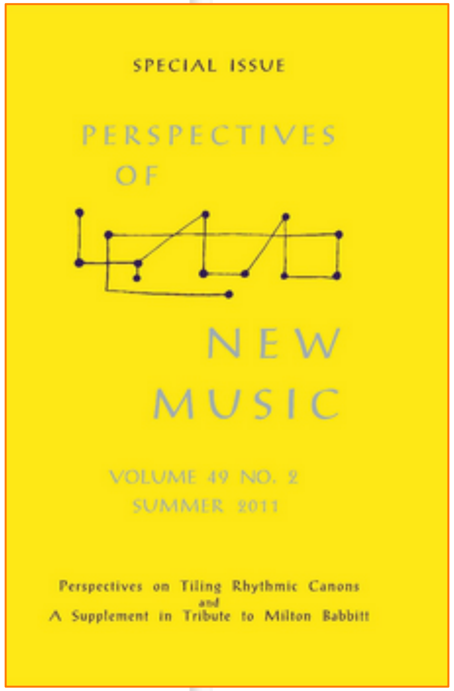
**Theorem (Amiot, 2009)**

- *All non-Vuza canons are spectral.*
- *Fuglede Conjecture is true (or false) iff it is true (or false) for Vuza Canons*



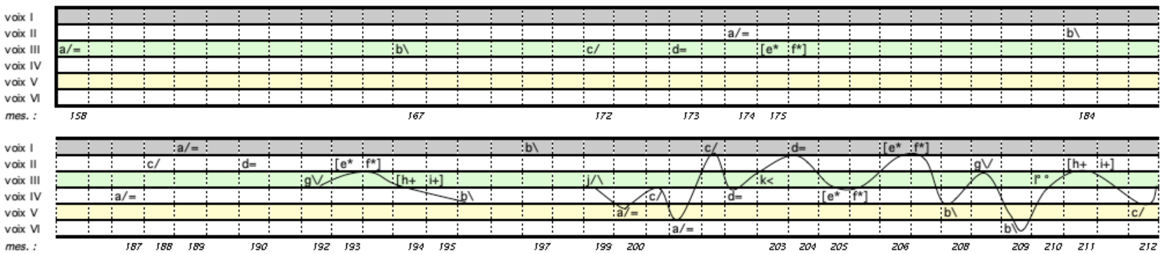

# Vuza Canons in OpenMusic 'MathTool' environment

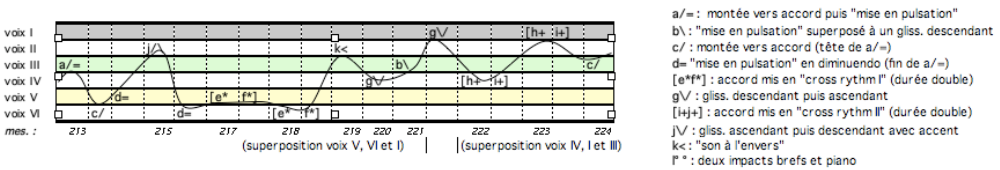



Vuza Algorithm

# Some compositional applications of the Vuza Canons model





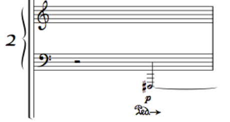
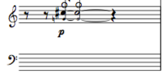

*Coïncidences* (1999)

Coïncidences - Fabien Levy : déroulement du canon (mes. 158 à 226)
(chaque impact fait 3 temps)

**F. Lévy**

*La notte poco prima della foresta*
(opéra de chambre
pour acteur, mezzo-soprano, baryton, ensemble et électronique, 2009)

**D. Ghisi**

***La bataille de caresme et de charnage***
(pour violoncelle et accompagnement, 2012)

**M. Lanza**

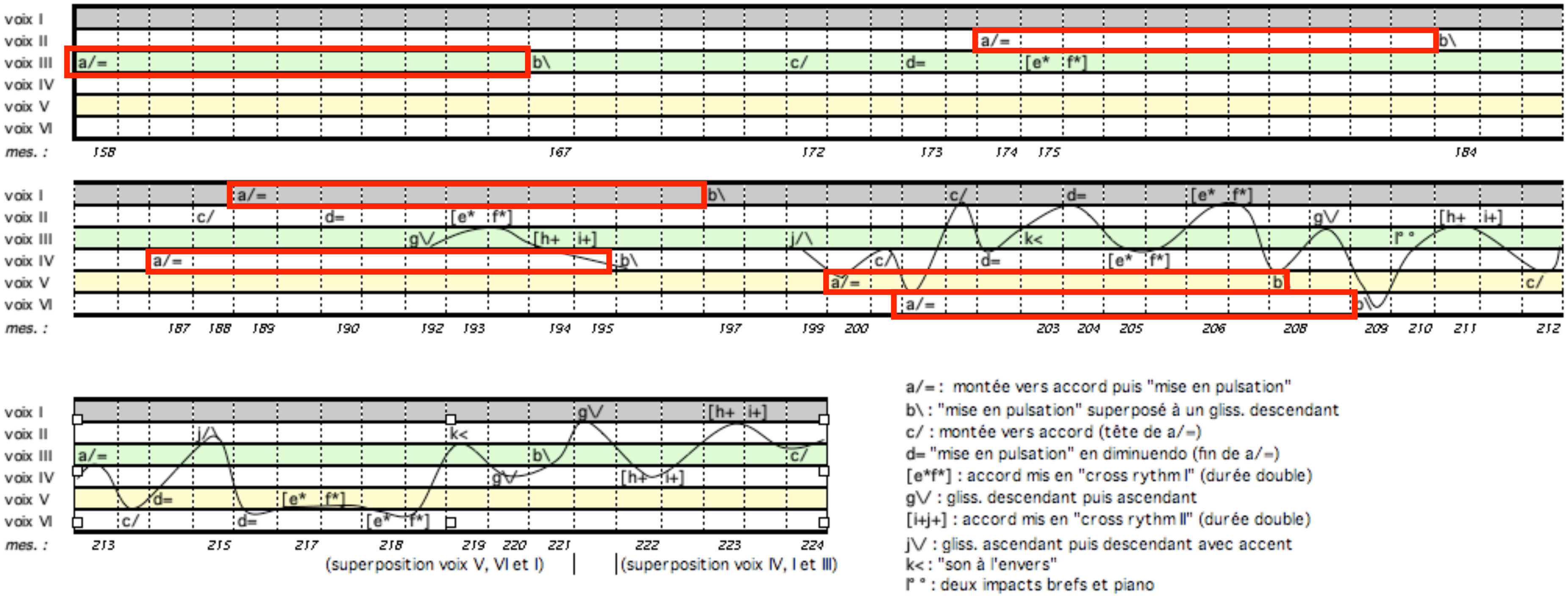
***A piece based on Monk*** **(2007)**
**(« Well You Need'nt »)**

**G. Bloch**

# Fabien Lévy

## Morphological Tiling Canons

- *Coïncidences* (pour 33 musiciens, 1999-2007)


Coïncidences - Fabien Levy : déroulement du canon (mes. 158 à 226)
(chaque impact fait 3 temps)

Tokyo Symphony Orchestra, Dir.: Kazuyoshi Akiyama, 05/09/2007, Suntory Hall, Tokyo, Japon

F. Lévy, « Three uses of Vuza Canons », *Perspectives of New Music*, 49(2), 2011, p. 23-31.

# Georges Bloch
## Several compositional strategies

- **Metrical organization of a tiling canon**
- **Self-similarity processes**
- **Metrical modulations between canons**

- ***Projet Beyeler* (2001)**
- *Projet Hitchcock*
- *Visite des tours de la cathédrale de Reims*
- *Noël des Chasseurs*
- *Canons à marcher*
- *Canon à eau*
- *Harawun* (2004)
- *L'Homme du champ* (2005)
- ***A piece based on Monk* (2007)**
- *Peking Duck Soup* (2008)



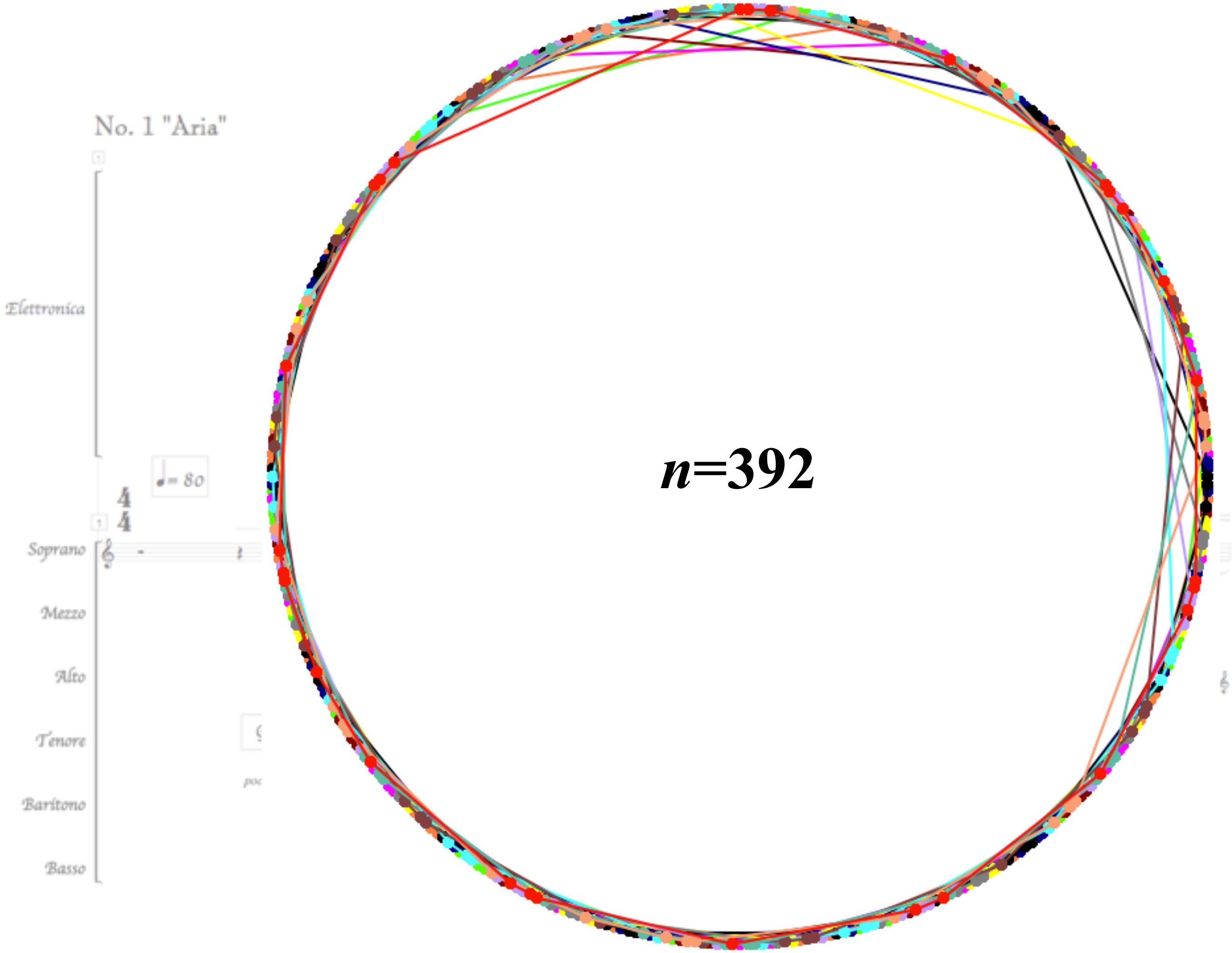
- ***A piece based on Monk*, 2007 (« Well You Need'nt »)**

**G. Bloch, « Vuza Canons into the museum », *The OM Composers' Book*, 2006**

# Mauro Lanza

## Vuza Canons and local periodicities


• ***La descrizione del diluvio*** (Ricordi, 2007-2008)




*"6 voices are live and 8 are in the electronic part. The choice of the notes and the durations is made in order to stress some* ***quasi-periodicities*** *of the underlying Vuza canon and this gives to each voice a much more "redundant" character".*

# Mauro Lanza
## Vuza Canons and local periodicities


• ***La descrizione del diluvio*** (Ricordi, 2007-2008)

*"The choice of the notes and the durations is made in order to stress some **quasi-periodicities** of the underlying Vuza canon* [...]*"*

**(1 3 25 27 1 3 11 14 27 1 3 25 27 4 25 27 1 3 25 14 13 1 3 25 27 1 3 52)**

**(*1 3 25 27 1 3* 11 14 27 1 3 25 27 4 25 27 1 3 25 14 13 *1 3 25 27 1 3* 52)**

**(*1 3 25 27* 1 3 11 14 27 *1 3 25 27* 4 25 27 1 3 25 14 13 *1 3 25 27* 1 3 52)**

**(*1 3 25* 27 1 3 11 14 27 *1 3 25* 27 4 25 27 *1 3 25* 14 13 *1 3 25* 27 1 3 52)**

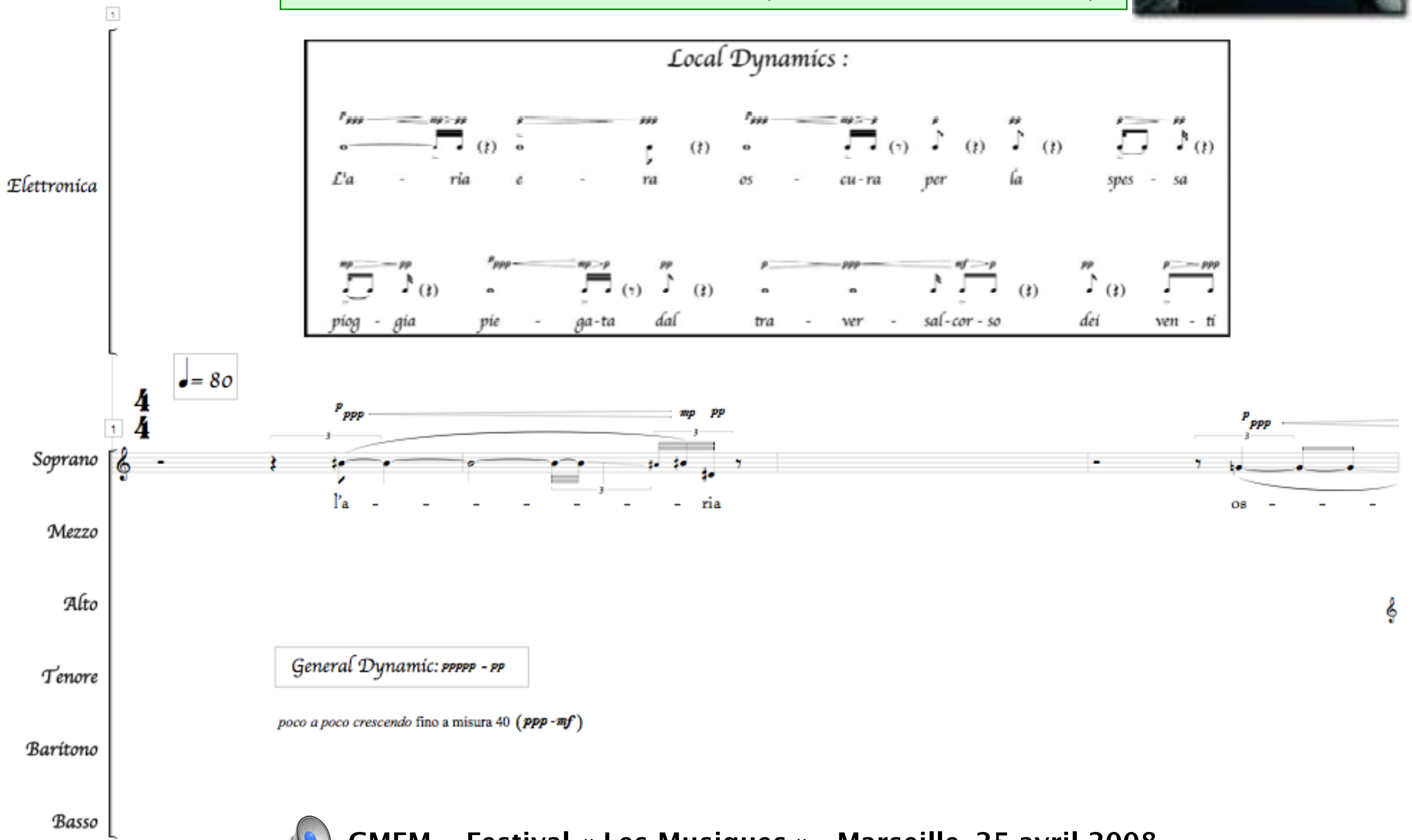
**(*1 3* 25 27 *1 3* 11 14 27 *1 3* 25 27 4 25 27 *1 3* 25 14 13 *1 3* 25 27 *1 3* 52)**

# Mauro Lanza

## Vuza Canons and local periodicities

No. 1 "Aria"

• ***La descrizione del diluvio*** (Ricordi, 2007-2008)

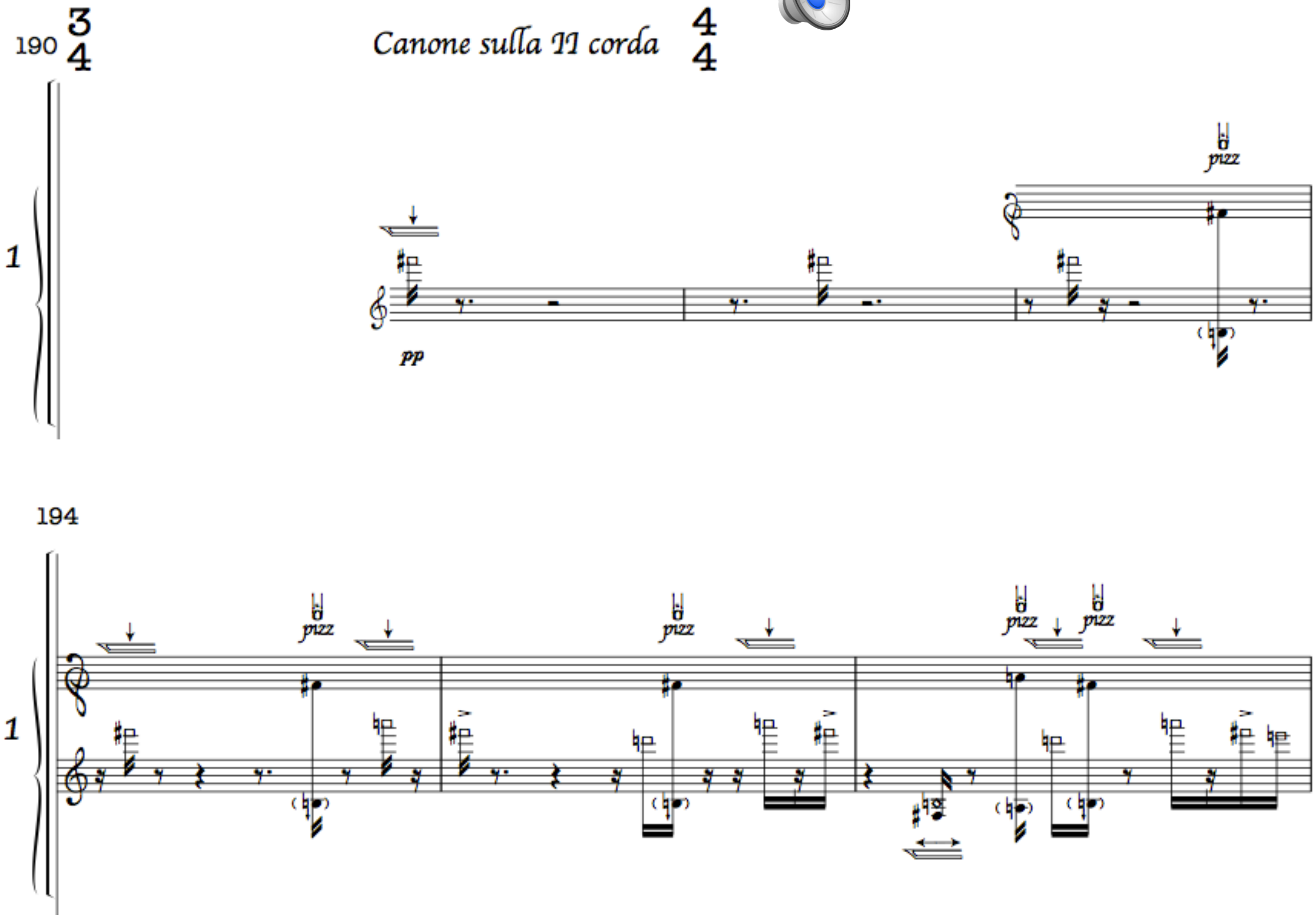


Mauro Lanza

# LA BATAILLE DE CARESME ET DE CHARNAGE

**per violoncello e accompagnamento (2012)**

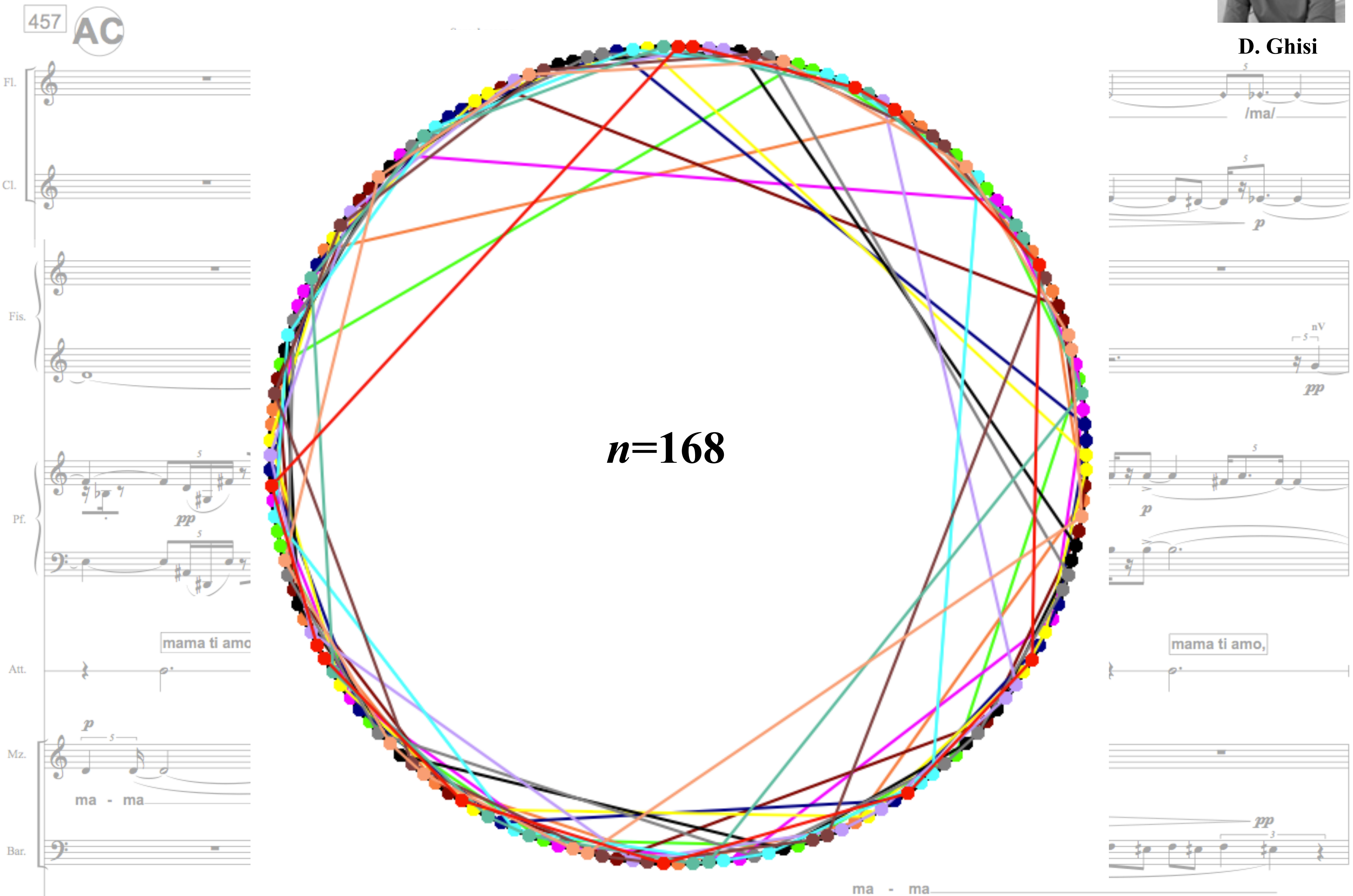

**(min. 10’15”)**



## *La notte poco prima della foresta* (2009)

(opéra de chambre pour acteur, mezzo-soprano, baryton, ensemble et électronique)

D. Ghisi

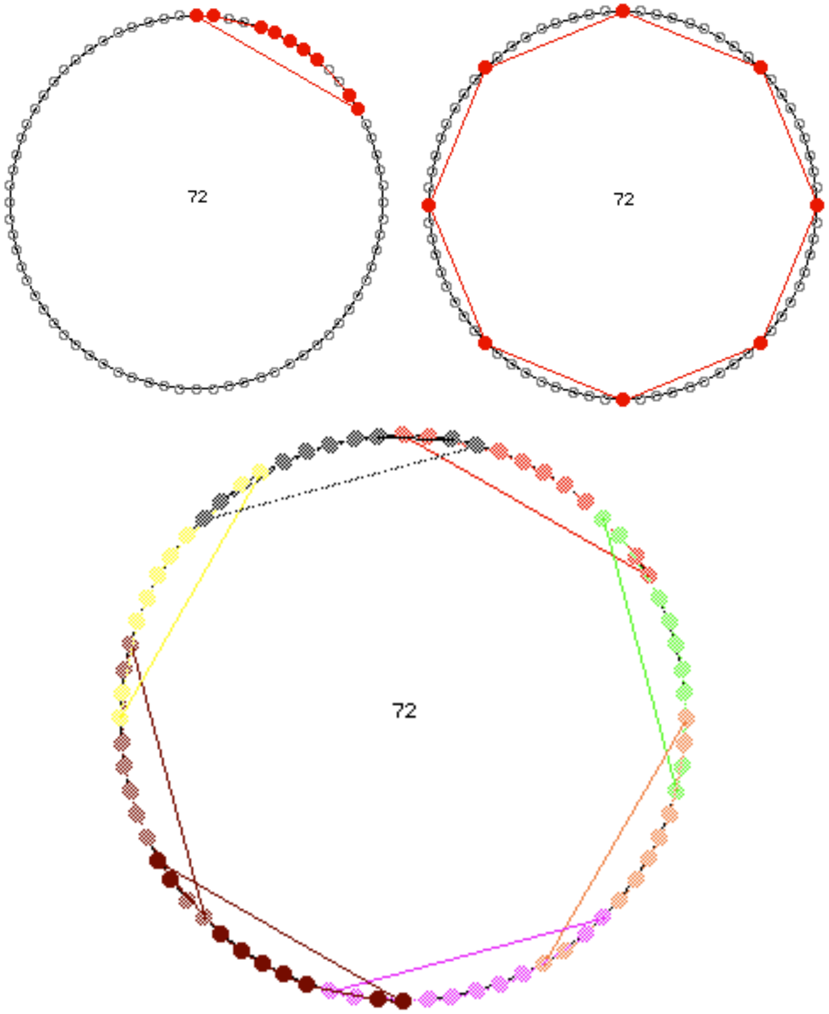


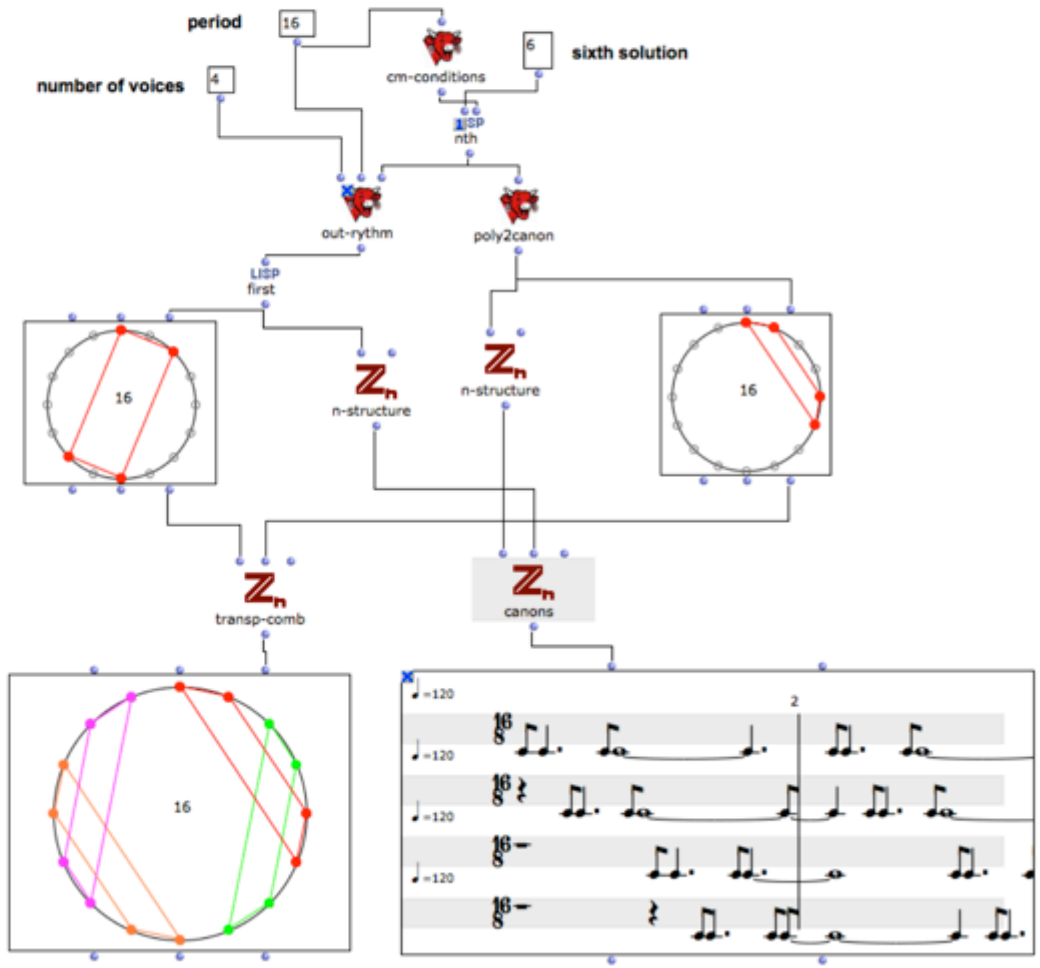
# *La notte poco prima della foresta* (2009)

(opéra de chambre pour acteur, mezzo-soprano, baryton, ensemble et électronique)

D. Ghisi

# Using (cyclotomic) polynomials to tile the line

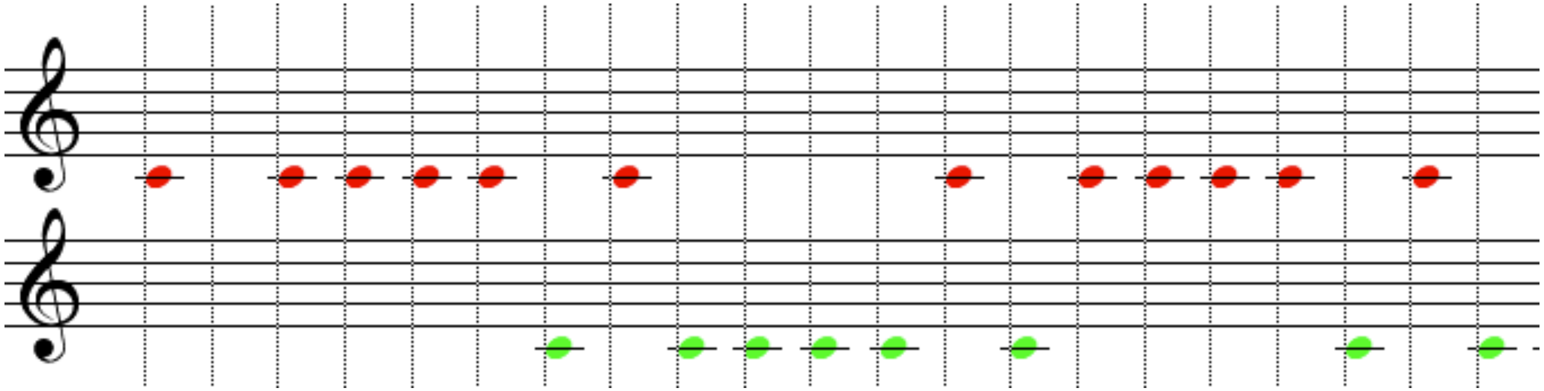


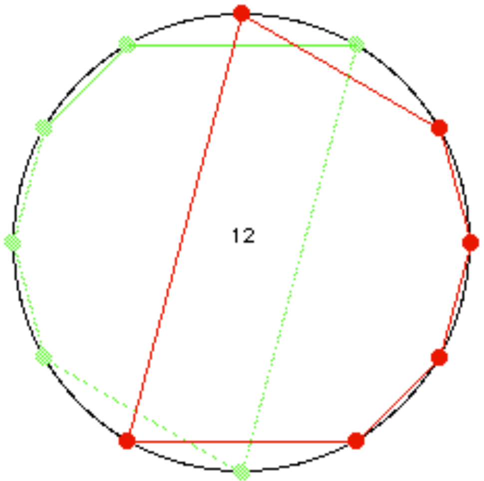

• E. Amiot, M. Andreatta, C. Agon, « Tiling the (musical) line with polynomial: some theoretical and implementational aspects », *ICMC*, Barcelona, 2005, 227-230

• C. Agon, M. Andreatta, « Modelling and Implementing Tiling Rhythmic Canons in OpenMusic Visual Programming Language », *Perspectives of New Music*, Special Issue, vol. 1-2, n° 49.

# Tiling canons by translation and augmentation

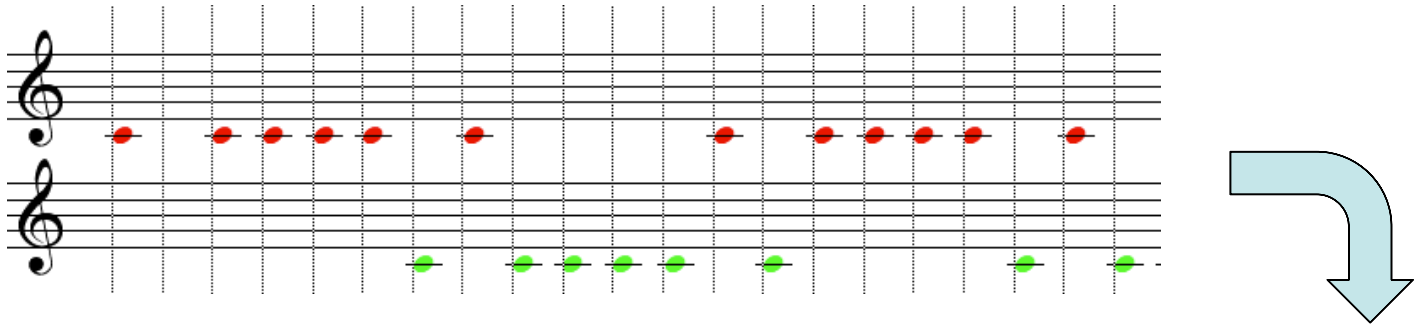


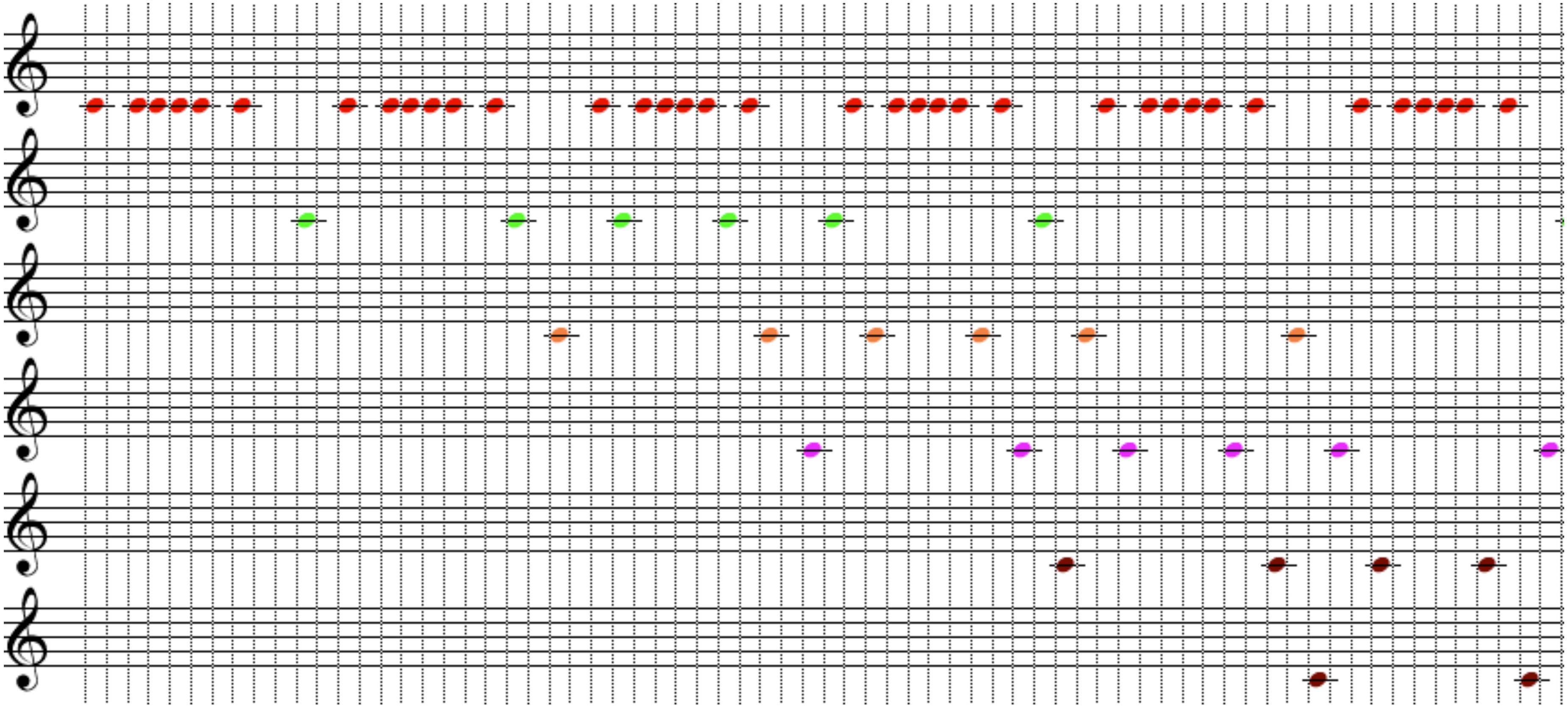
Thomas Noll




```
(((0 1 2 3 4 6) ((1 11)))
((0 1 2 3 4 5) ((1 11) (1 1)))
((0 1 2 3 5 7) ((1 11) (1 7)))
((0 1 3 4 7 8) ((1 5)))
((0 1 2 3 6 7) ((1 11)))
((0 1 3 4 6 9) ((1 11) (1 5)))
((0 1 3 6 7 9) ((1 11) (1 5)))
((0 1 2 6 7 8) ((1 11) (1 7) (1 5) (1 1)))
((0 1 4 5 8 9) ((1 11) (1 7) (1 5) (1 1)))
((0 1 2 5 6 7) ((1 7) (1 5)))
((0 2 3 4 5 7) ((1 11) (1 7) (1 5) (1 1)))
((0 1 4 5 6 8) ((1 11) (1 7)))
((0 1 2 4 5 7) ((1 5)))
((0 1 3 4 5 8) ((1 5) (1 1)))
((0 1 2 4 5 8) ((1 11)))
((0 1 2 4 6 8) ((1 11) (1 7)))
((0 2 3 4 6 8) ((1 11)))
((0 2 4 6 8 10) ((1 11) (1 7) (1 5) (1 1))))
```

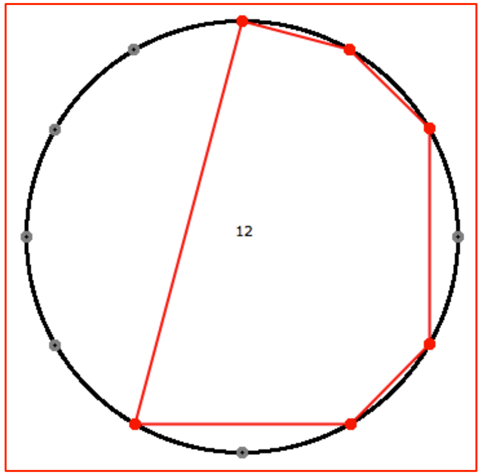
# Augmented Tiling Canons (Noll Canons)

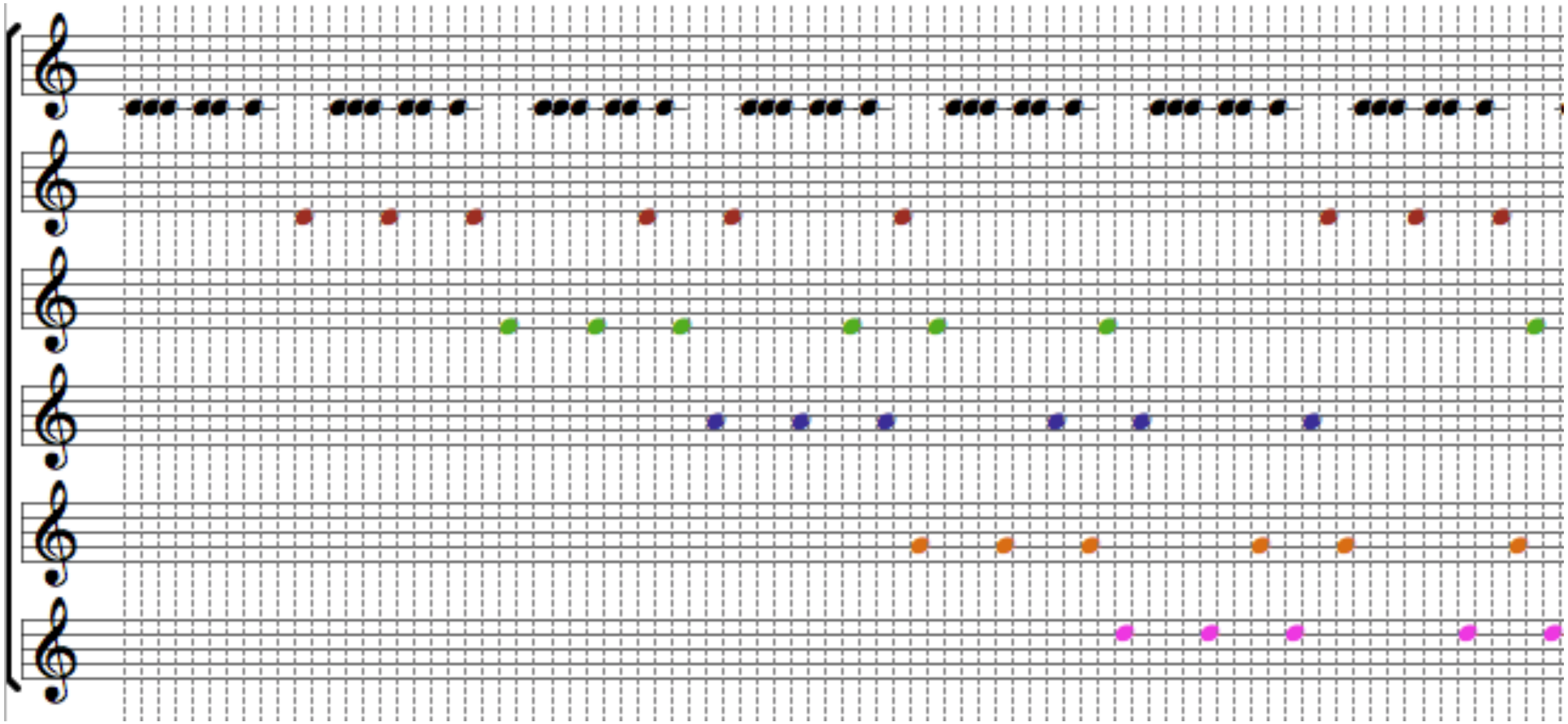




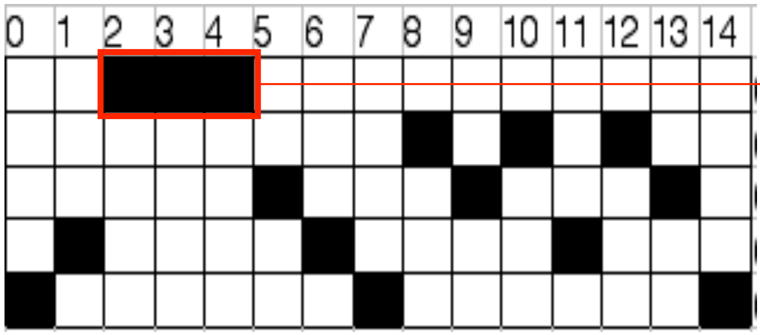
# Augmented Tiling Canons (Noll Canons)

```
(((0 1 2 3 4 6) ((1 11)))
((0 1 2 3 4 5) ((1 11) (1 1)))
((0 1 2 3 5 7) ((1 11) (1 7)))
((0 1 3 4 7 8) ((1 5)))
((0 1 2 3 6 7) ((1 11)))
((0 1 3 4 6 9) ((1 11) (1 5)))
((0 1 3 6 7 9) ((1 11) (1 5)))
((0 1 2 6 7 8) ((1 11) (1 7) (1 5) (1 1)))
((0 1 4 5 8 9) ((1 11) (1 7) (1 5) (1 1))) ((0 1 2 5 6 7) ((1 7) (1 5)))
((0 2 3 4 5 7) ((1 11) (1 7) (1 5) (1 1))) ((0 1 4 5 6 8) ((1 11) (1 7)))
((0 1 2 4 5 7) ((1 5)))
((0 1 3 4 5 8) ((1 5) (1 1)))
((0 1 2 4 5 8) ((1 11)))
((0 1 2 4 6 8) ((1 11) (1 7)))
((0 2 3 4 6 8) ((1 11)))
((0 2 4 6 8 10) ((1 11) (1 7) (1 5) (1 1))))
```

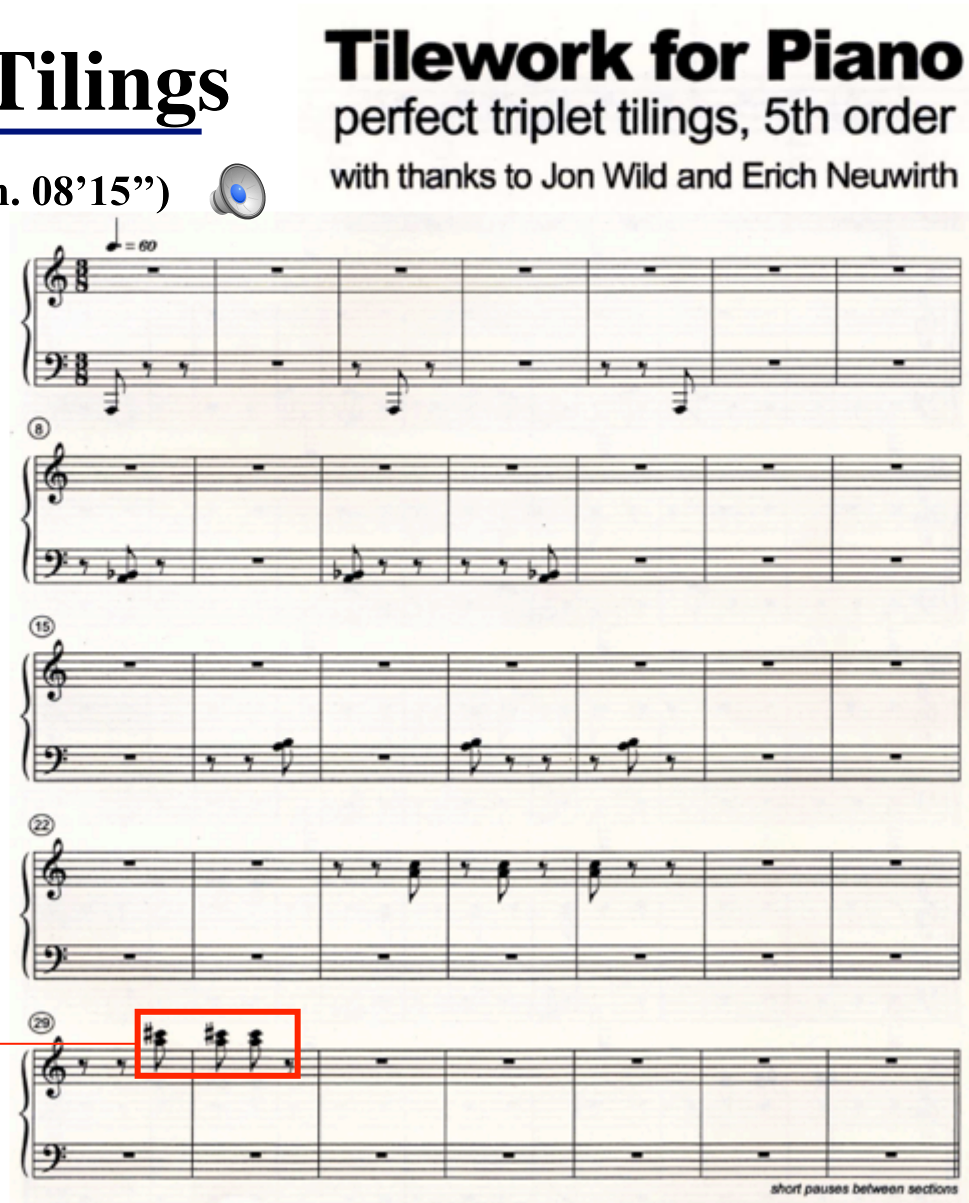


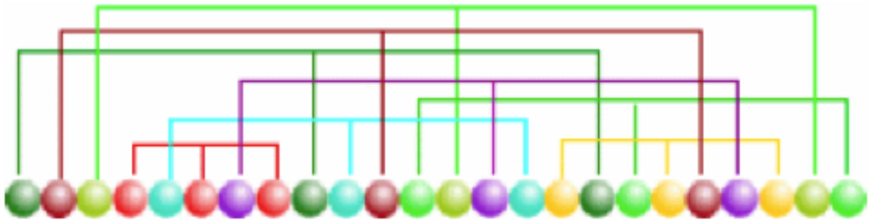



# Tom Johnson's Perfect Tilings



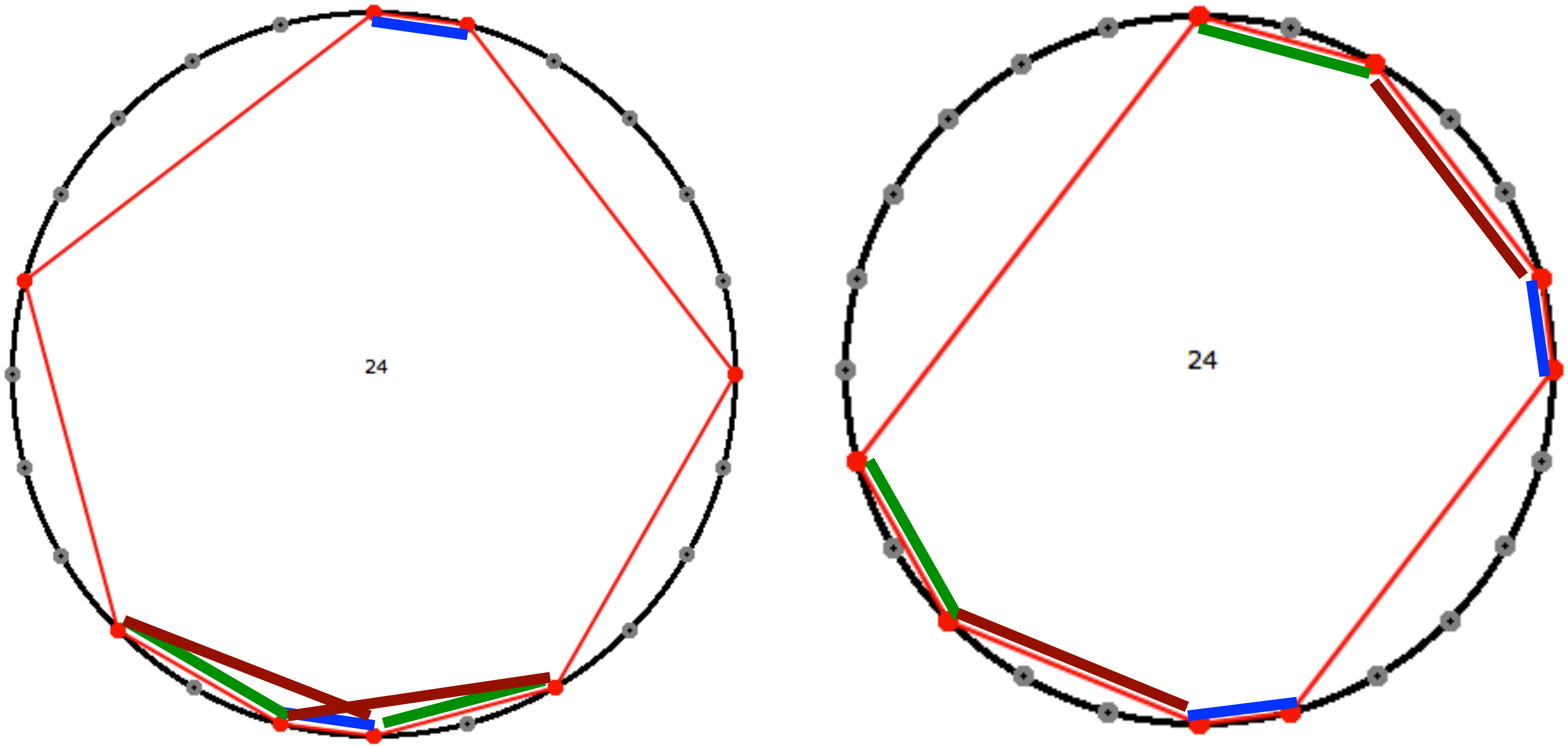

(min. 08'15")



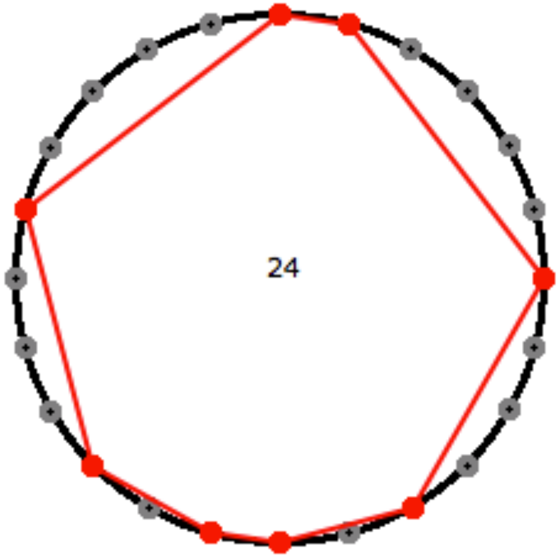
Jean-Paul Davalan, « Perfect Rhythmic Tilings », *Perspectives of New Music*, 49(2), 2011, p. 144-197

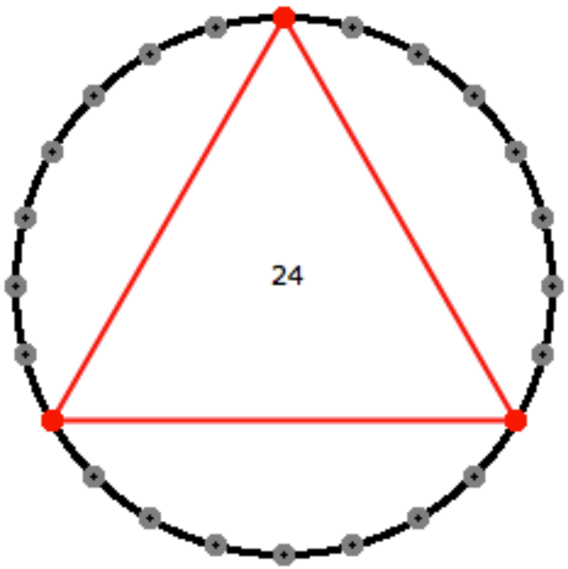
# Tiling Rhythmic Canons and Homometry

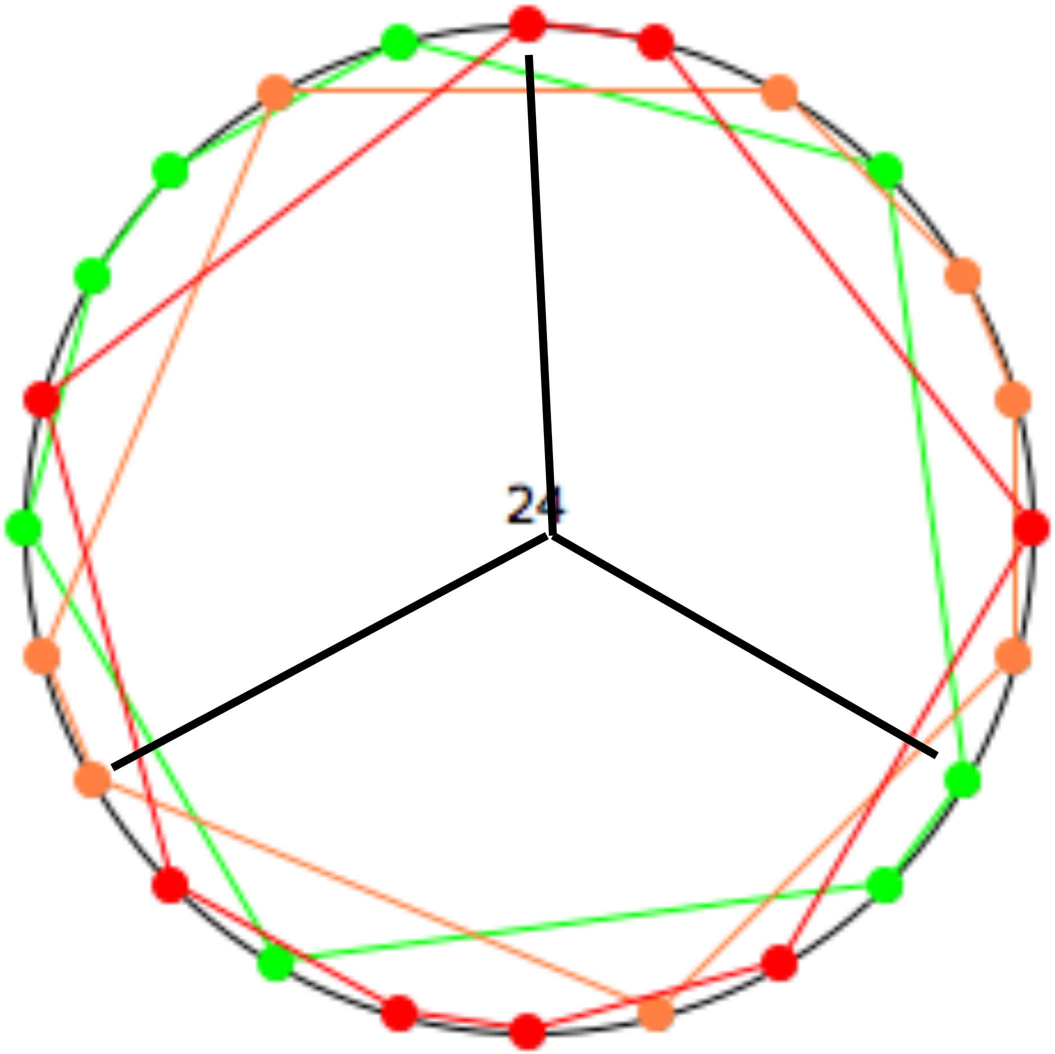



# Can you hear the shape of a canon?

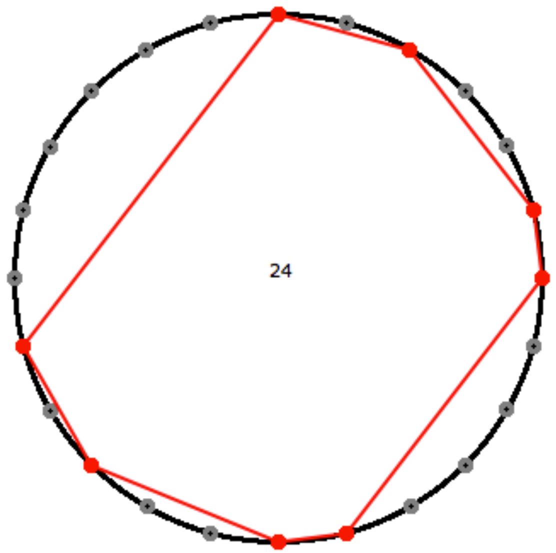
# Tiling Rhythmic Canons and Homometry

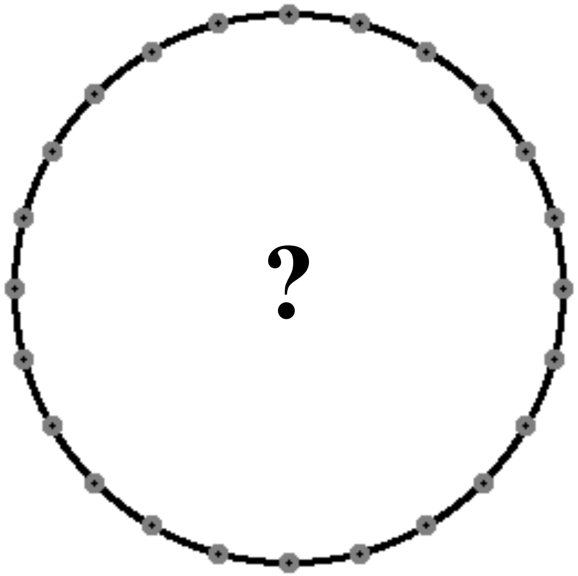


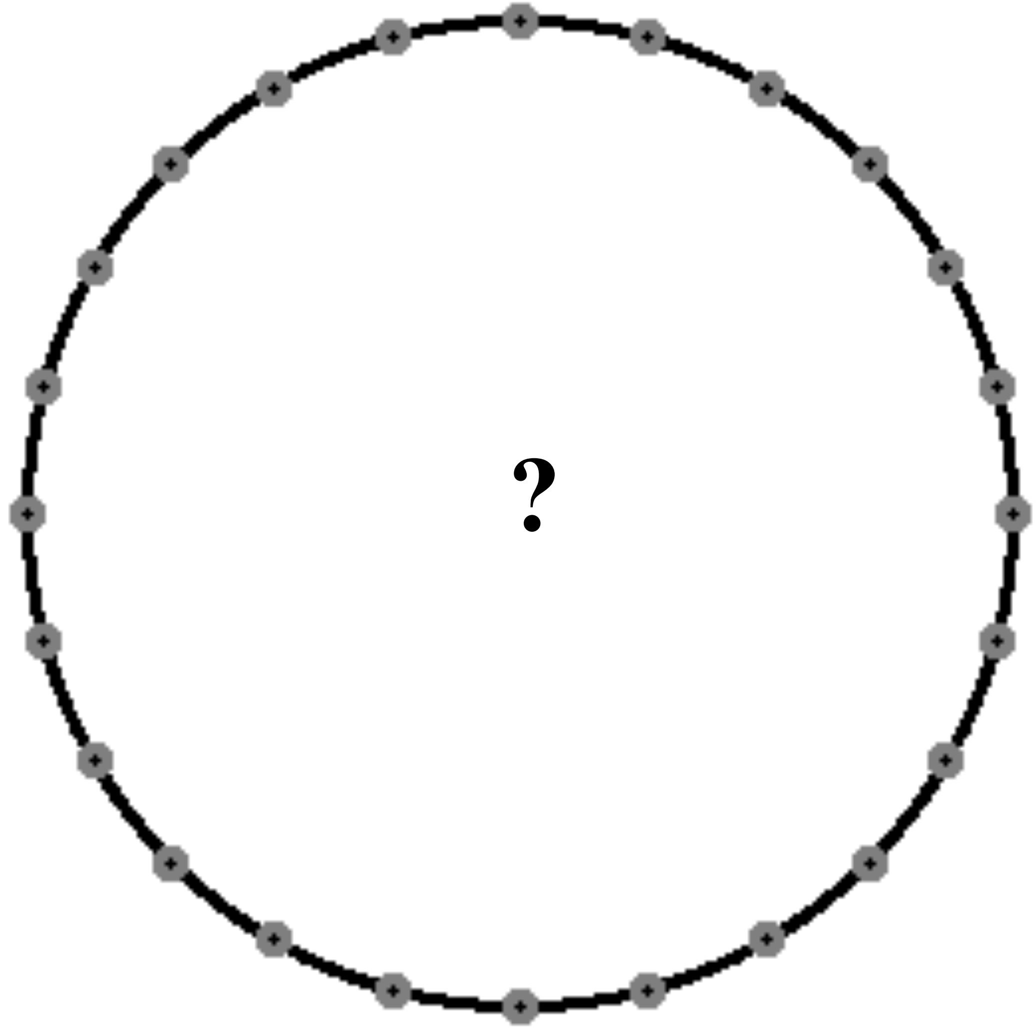



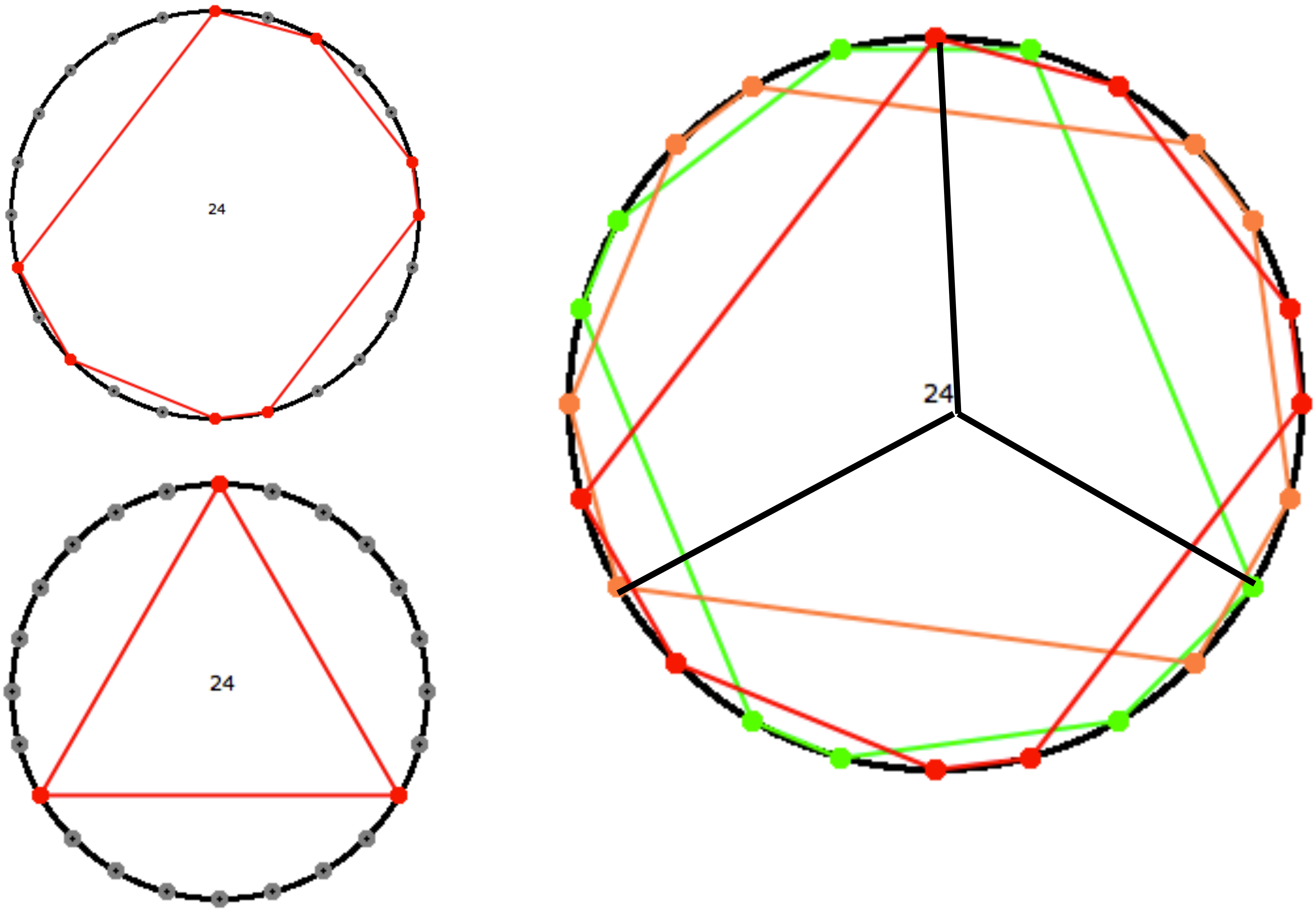
# Tiling Rhythmic Canons and Homometry



# Tiling Rhythmic Canons and Homometry



# Homometry and Tiling Rhythmic Canons

## Tiling

Let $Z_A = \{ t \in \mathbb{Z}_c, F_A(t)=0 \}$

A tiles $\mathbb{Z}_c$ when equivalently:

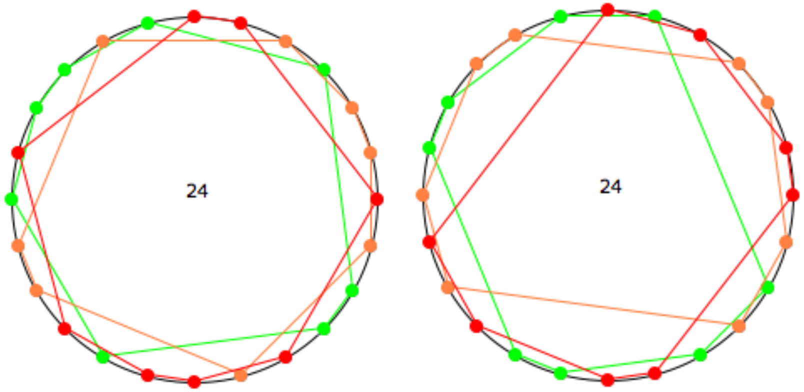
- There exists B, $A \oplus B = \mathbb{Z}_c$
- $1_A \star 1_B = 1$
- $F_A \times F_B (t) = 1 + e^{-2i\pi t/c} + \dots e^{-2i\pi t(c-1)/c}$ (0 unless t=0)
- $Z_A \cup Z_B = \{1,2 \dots c-1\}$ AND Card A × Card B = c
- $IC_A \star IC_B = IC(\mathbb{Z}_c)=c$ and Card A × Card B = c

$$\mathcal{F}_A : t \mapsto \sum_{k \in A} e^{-2i\pi kt/c}$$

$$IC_A(k) = (1_A \star 1_{-A})(k)$$

*A musical offering:*

- ***Theorem***:
  If A tiles with B and A' has the same IC, then A' tiles with B, too.




E. Amiot, « New Perspectives on Rhythmic Canons and the Spectral Conjecture », *JMM*, 2009.

# Thank you for your attention!