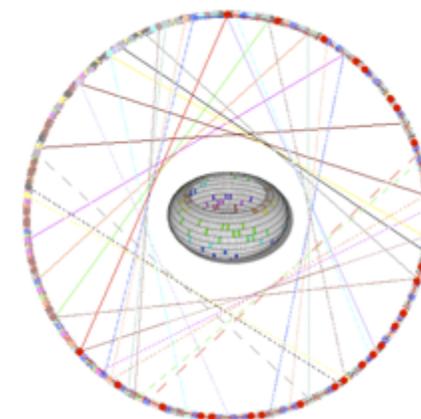
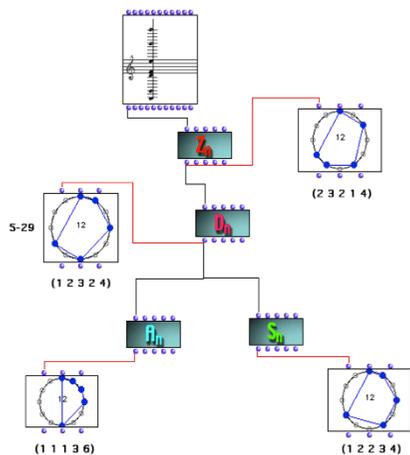


# Tiling Canons as a key to approach open mathematical Conjectures?

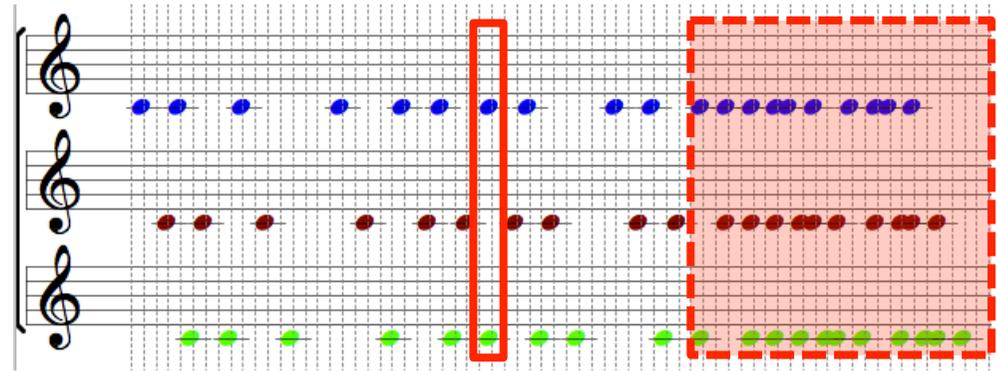


Moreno Andreatta  
Equipe Représentations Musicales  
IRCAM / CNRS / UPMC

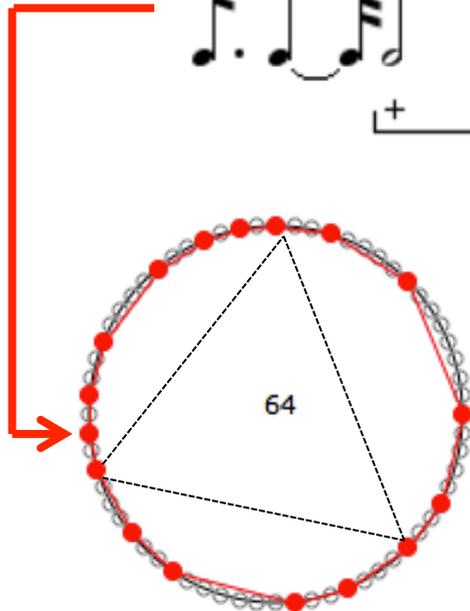
# The 'emergence' of tiling rhythmic canons



 *Harawi (1945)*



*Harawi: rhythmic reduction* 

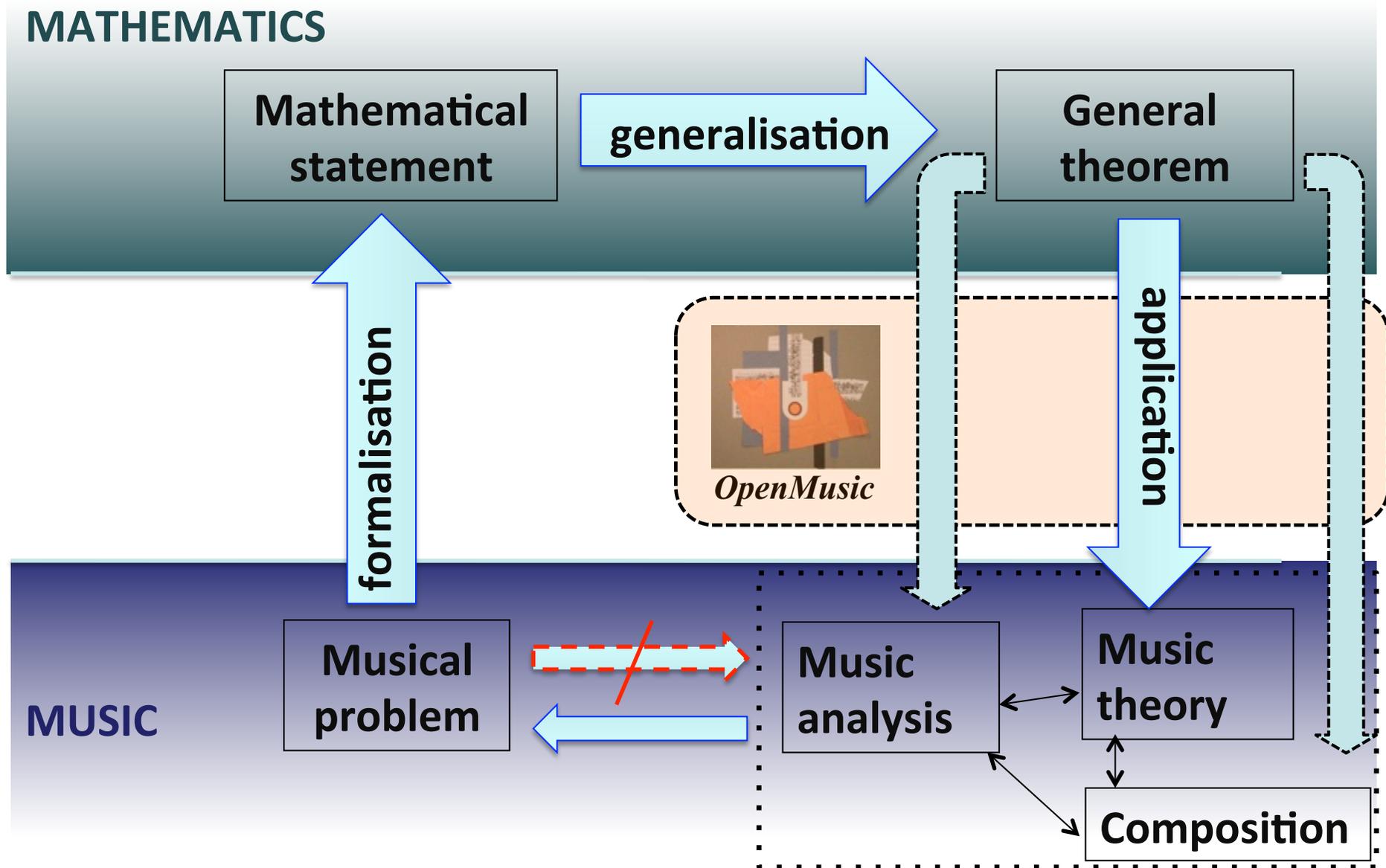


« ...il résulte de tout cela que les différentes sonorités se mélangent ou s'opposent de manières très diverses, **jamais au même moment ni au même endroit** [...]. C'est du désordre organisé »

O. Messiaen: *Traité de Rythme, de Couleur et d'Ornithologie*, tome 2, Alphonse Leduc, 1992.



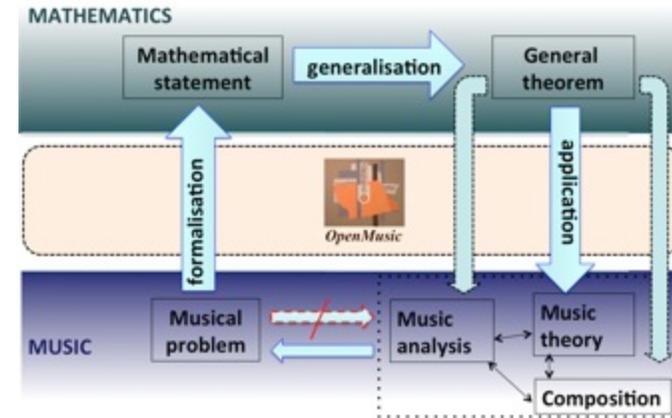
# The double movement of a 'mathemusical' activity



# Some examples of 'mathemusical' problems

M. Andreatta : *Mathematica est exercitium musicae*, Habilitation Thesis, IRMA University of Strasbourg, 2010

- The construction of Tiling Rhythmic Canons
- The Z relation and the theory of homometric sets
- Set Theory and Transformational Theory
- Neo-Riemannian Theory, Spatial Computing and FCA
- Diatonic Theory and Maximally-Even Sets
- Periodic sequences and finite difference calculus
- Block-designs and algorithmic composition



**Rhythmic Tiling Canons**

**Z-Relation and Homometric Sets**

**Finite Difference Calculus**

$$Df(x) = f(x) - f(x-1)$$

```

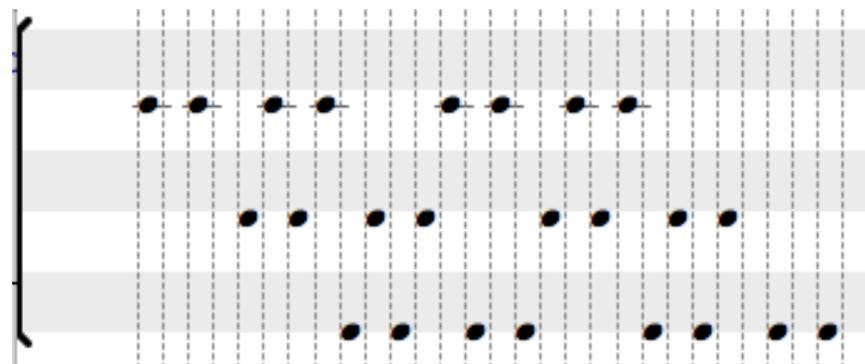
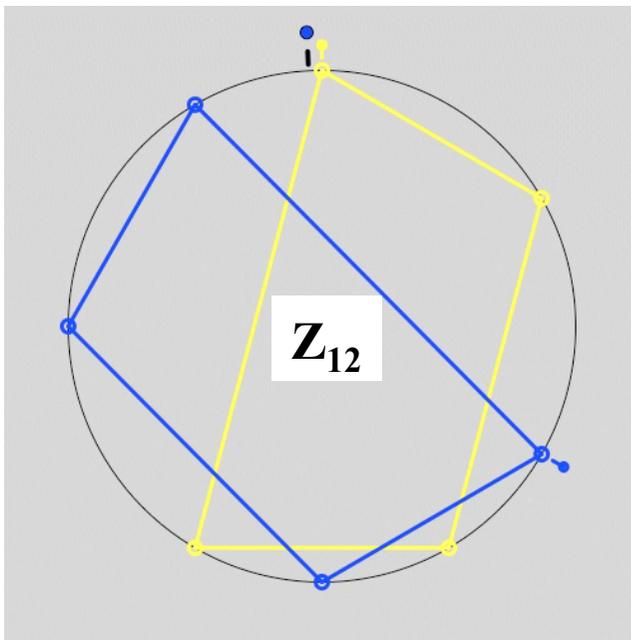
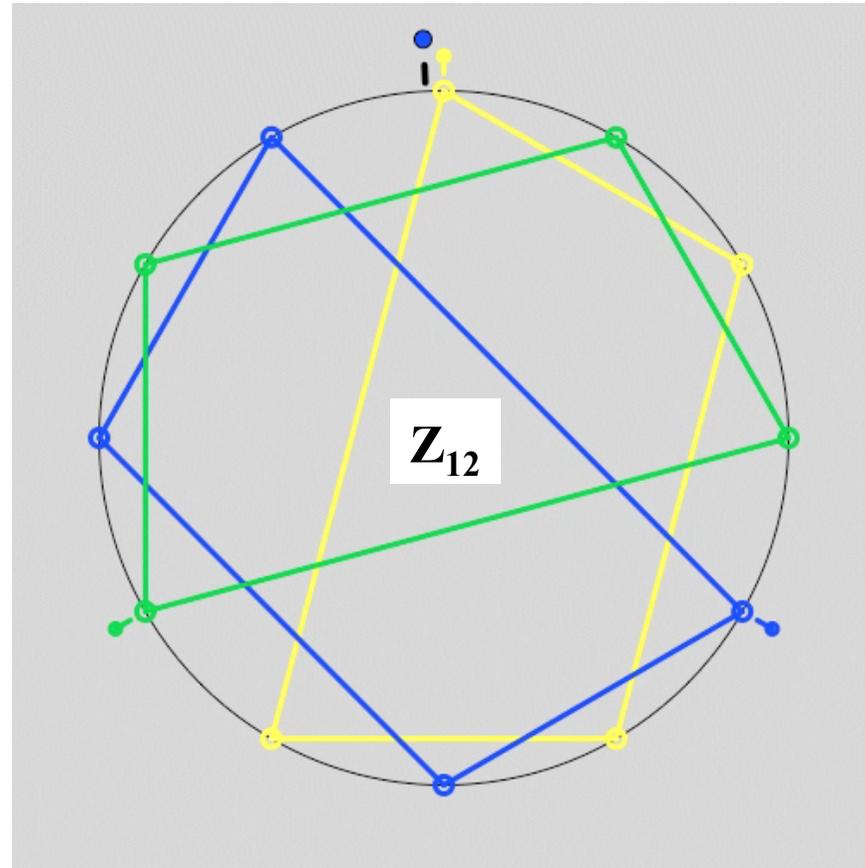
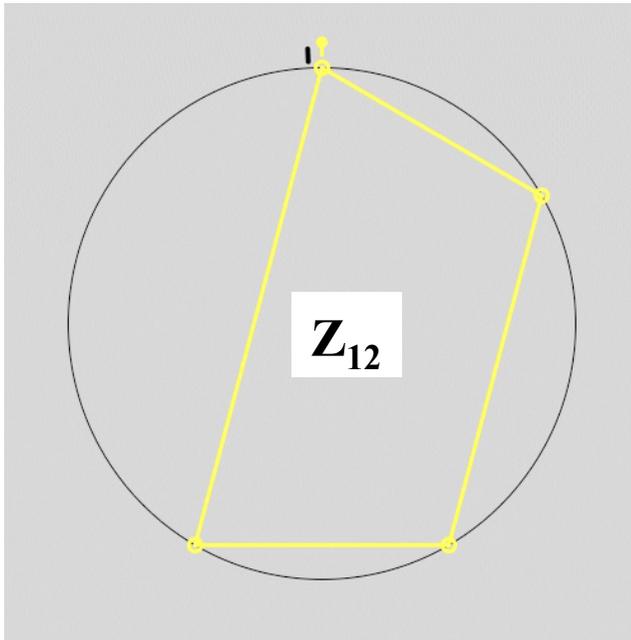
7 11 10 11 7 2 7 11 10 11 7 2 7 11...
4 11 1 8 7 5 4 11 1 8 7 5 4 11...
7 2 7 11 10 11 7 2 7 11 10 11...
7 5 4 11 1 8 7 5 4 11 18...
.....
    
```

**Set Theory, and Transformation Theory**

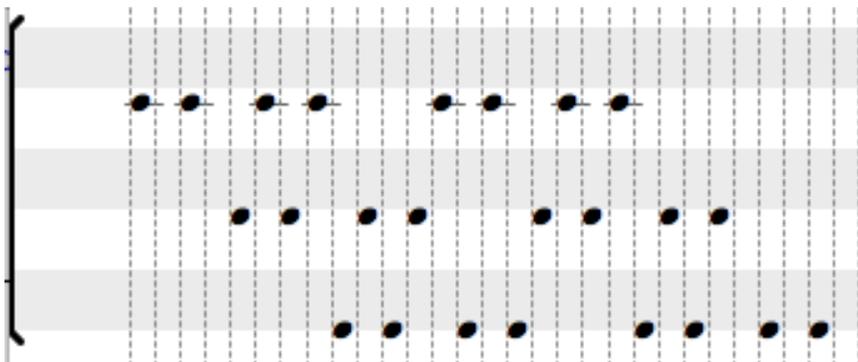
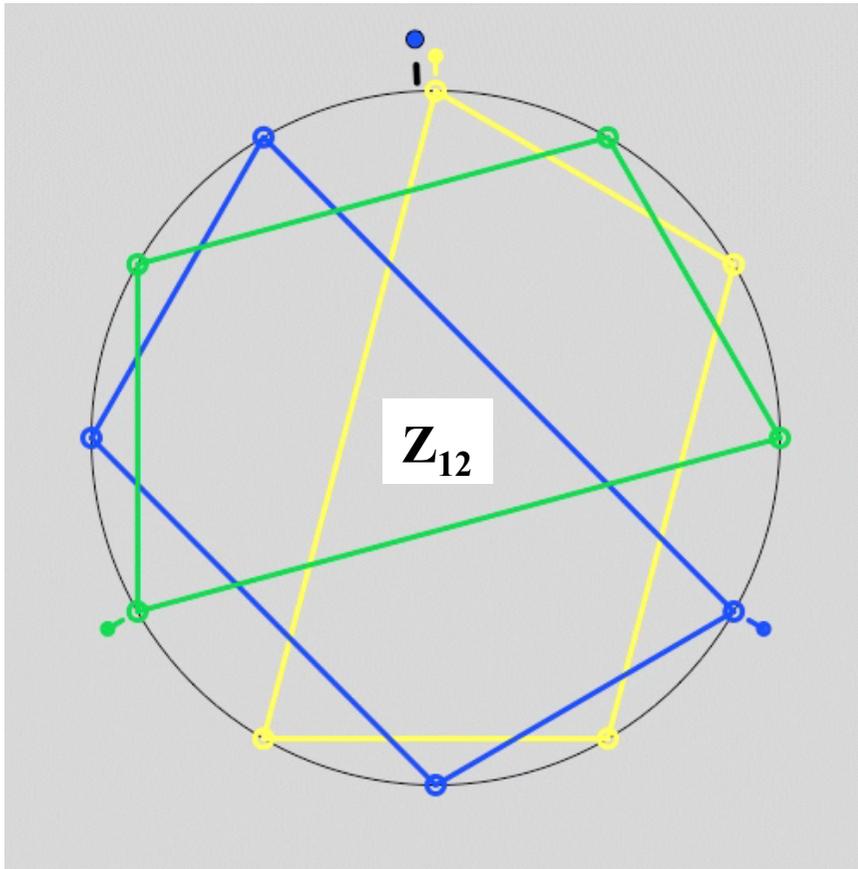
**Neo-Riemannian Theory and Spatial Computing**

**Block-designs**

# Tiling the line with translates of one tile



# Formalizing the tiling process as set-theoretical operations



$$A_1 = \{0, 2, 5, 7\}$$

$T_4 \downarrow$

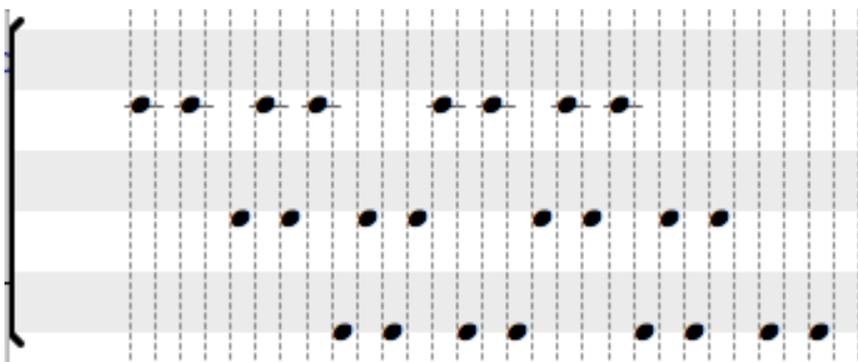
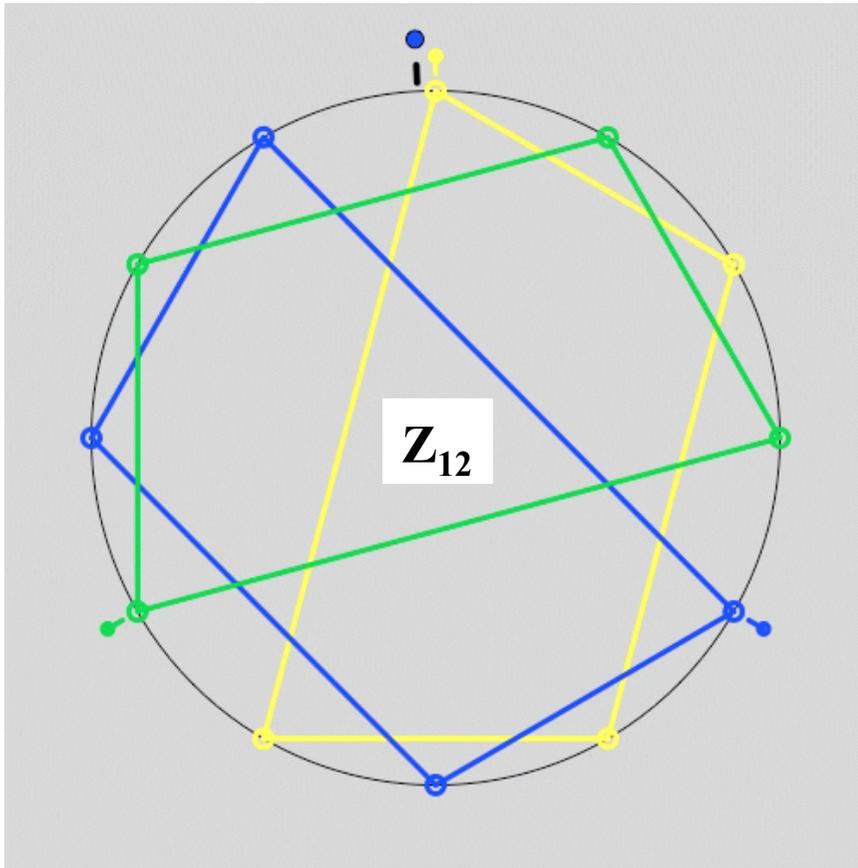
$$A_2 = \{4, 6, 9, 11\}$$

$T_4 \downarrow$

$$A_3 = \{8, 10, 1, 3\}$$

$$Z_{12} = A_1 \cup A_2 \cup A_3$$

# Formalizing the tiling process as a direct sum of subsets



$$A_1 = \{0, 2, 5, 7\}$$

$T_4 \downarrow$

$$A_2 = \{4, 6, 9, 11\}$$

$T_4 \downarrow$

$$A_3 = \{8, 10, 1, 3\}$$

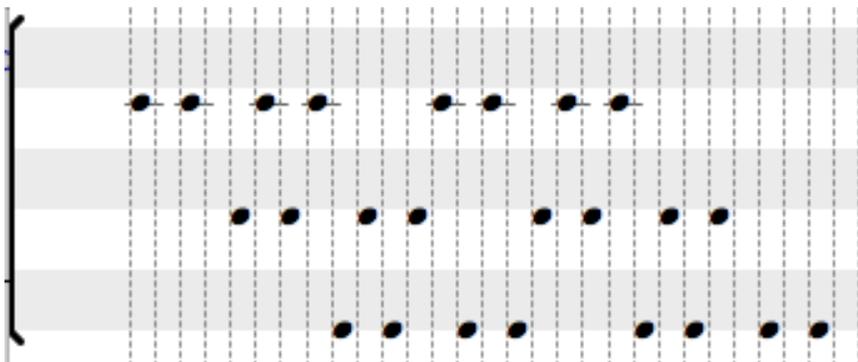
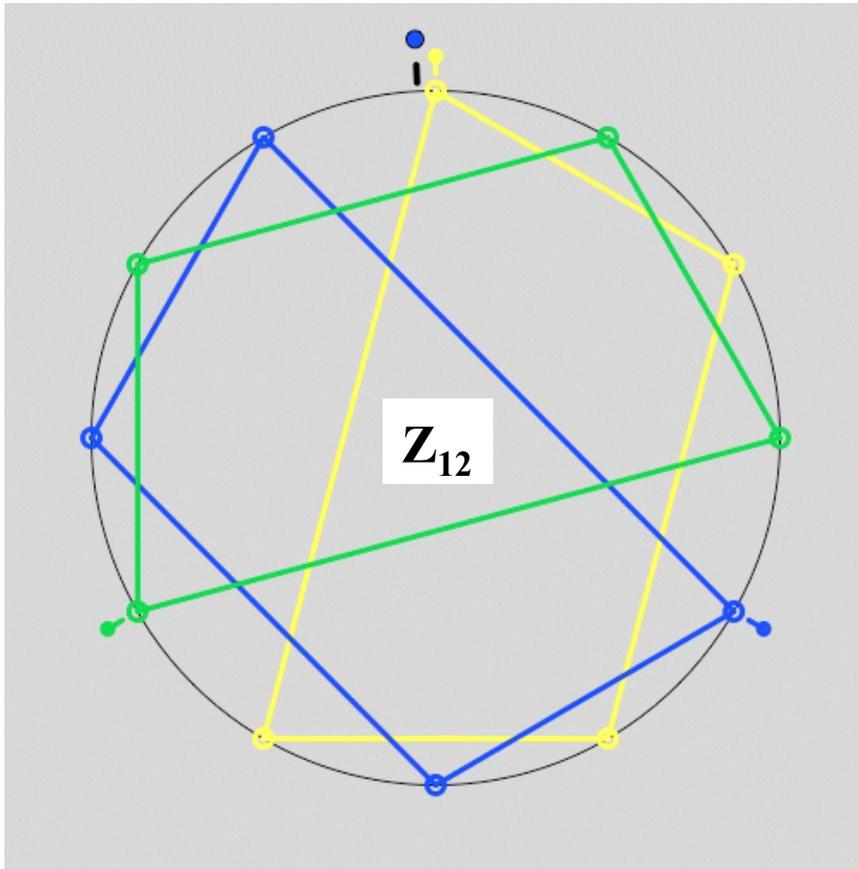
$$Z_{12} = A_1 \cup A_2 \cup A_3$$

$$Z_{12} = A \oplus B$$

$$A = ?$$

$$B = ?$$

# Formalizing the tiling process as a direct sum of subsets



$$A_1 = \{0, 2, 5, 7\}$$

$T_4 \downarrow$

$$A_2 = \{4, 6, 9, 11\}$$

$T_4 \downarrow$

$$A_3 = \{8, 10, 1, 3\}$$

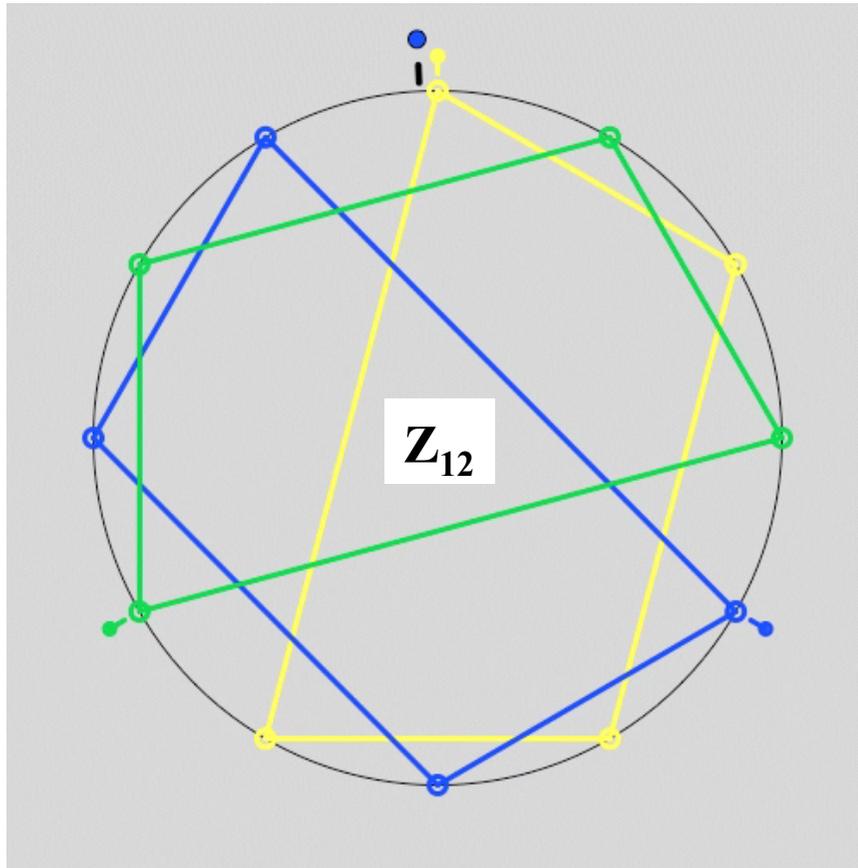
$$Z_{12} = A_1 \cup A_2 \cup A_3$$

$$Z_{12} = A \oplus B$$

$$A = \{0, 2, 5, 7\}$$

$$B = \{0, 4, 8\}$$

# Formalizing the tiling process as a product of polynomials



$$A_1 = \{0, 2, 5, 7\}$$

$$A_2 = \{4, 6, 9, 11\}$$

$$A_3 = \{8, 10, 1, 3\}$$

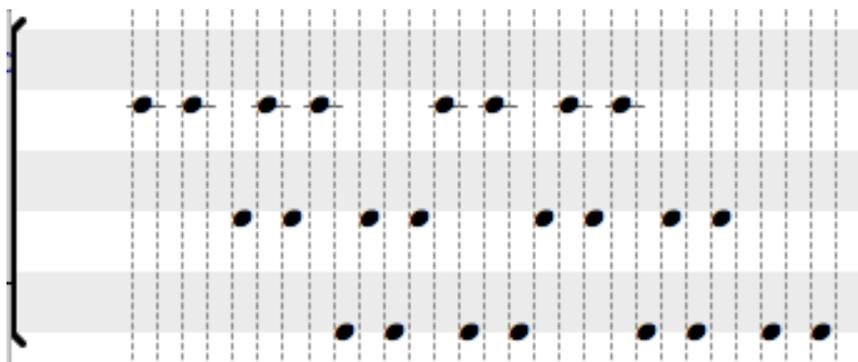
 $T_4$ 
 $T_4$ 

$$Z_{12} = A_1 \cup A_2 \cup A_3$$

$$Z_{12} = A \oplus B$$

$$A = \{0, 2, 5, 7\}$$

$$B = \{0, 4, 8\}$$



$$1 + \dots + x^{11} = A(x) \times B(x) \pmod{x^{12}-1}$$

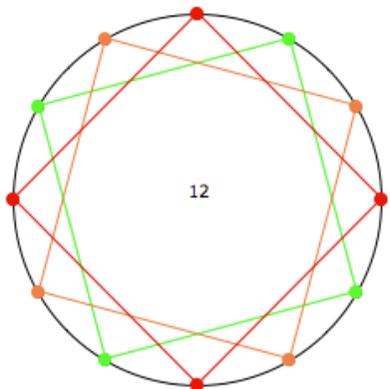
$$A(x) = 1 + x^2 + x^5 + x^7$$

$$B(x) = 1 + x^4 + x^8$$

# The three 'elementary' types of tiling rhythmic canons

$A < \mathbb{Z}_n$

$B < \mathbb{Z}_n$

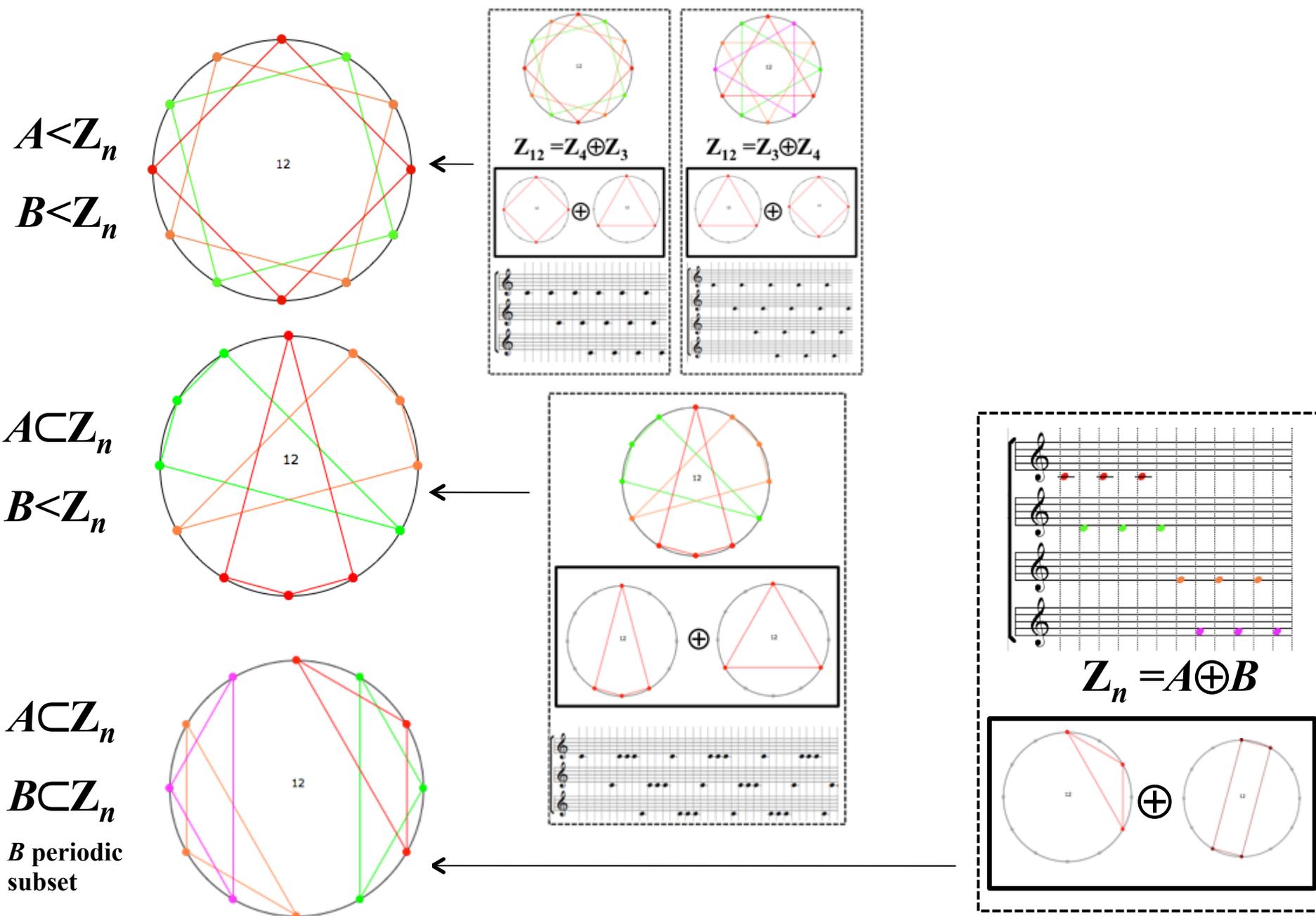


$\mathbb{Z}_{12} = \mathbb{Z}_4 \oplus \mathbb{Z}_3$

$\mathbb{Z}_{12} = \mathbb{Z}_3 \oplus \mathbb{Z}_4$

↑ duality ↑

# The three 'elementary' types of tiling rhythmic canons



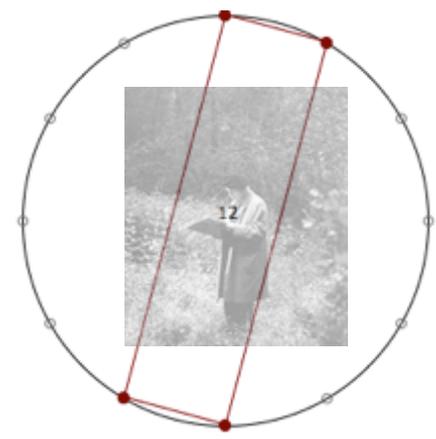
# The three 'elementary' types of tiling rhythmic canons

$A < \mathbb{Z}_n$   
 $B < \mathbb{Z}_n$

$\mathbb{Z}_{12} = \mathbb{Z}_4 \oplus \mathbb{Z}_3$

$\mathbb{Z}_{12} = \mathbb{Z}_3 \oplus \mathbb{Z}_4$

Diagram illustrating the construction of a tiling rhythmic canon with two subsets  $A$  and  $B$  of  $\mathbb{Z}_n$ . The main diagram shows a circle with 12 points and two overlapping polygons (one red, one green) representing the subsets. Two smaller diagrams show the decomposition of  $\mathbb{Z}_{12}$  into  $\mathbb{Z}_4 \oplus \mathbb{Z}_3$  and  $\mathbb{Z}_3 \oplus \mathbb{Z}_4$ , with corresponding musical notation below.



Is Messiaen's Property mandatory for tiling rhythmic canons?

$A \subset \mathbb{Z}_n$   
 $B < \mathbb{Z}_n$

$\mathbb{Z}_n = \mathbb{Z}_3 \oplus \mathbb{Z}_3$

Diagram illustrating a tiling rhythmic canon where  $A$  is a subset of  $\mathbb{Z}_n$  and  $B$  is a periodic subset. The main diagram shows a circle with 12 points and two overlapping polygons (one red, one green) representing the subsets. A smaller diagram shows the decomposition of  $\mathbb{Z}_n$  into  $\mathbb{Z}_3 \oplus \mathbb{Z}_3$ , with corresponding musical notation below.

$\mathbb{Z}_n = A \oplus B$

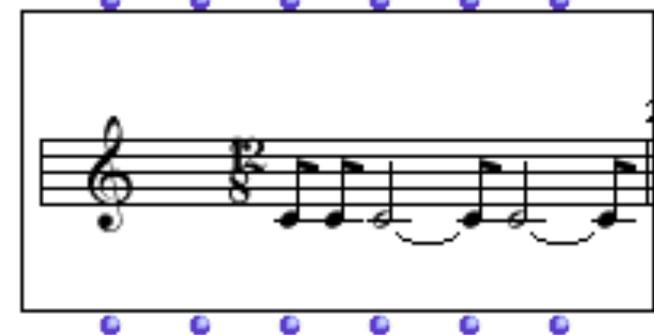
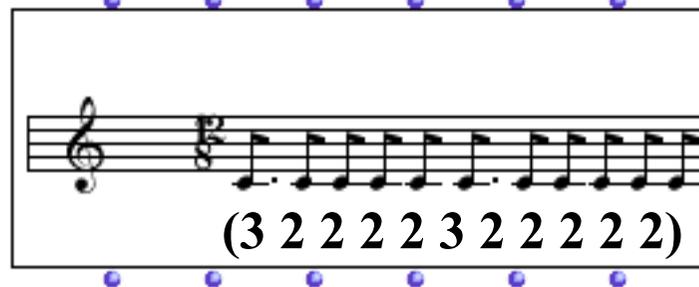
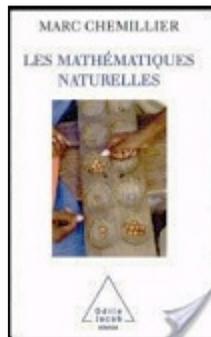
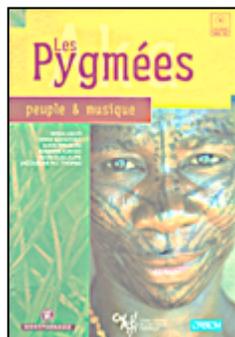
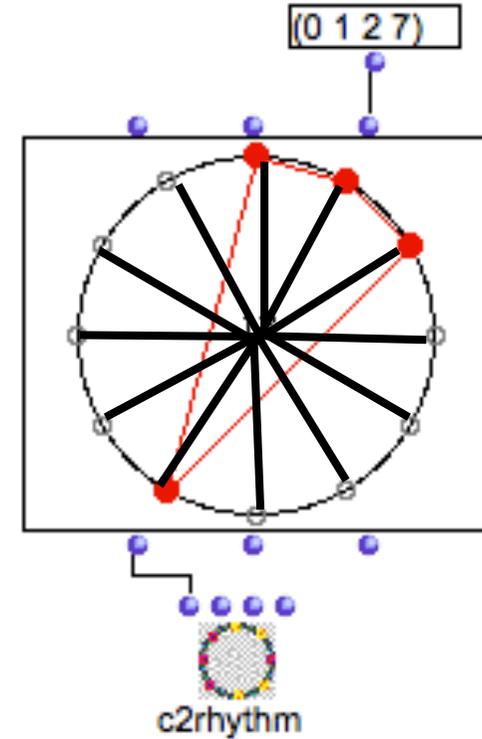
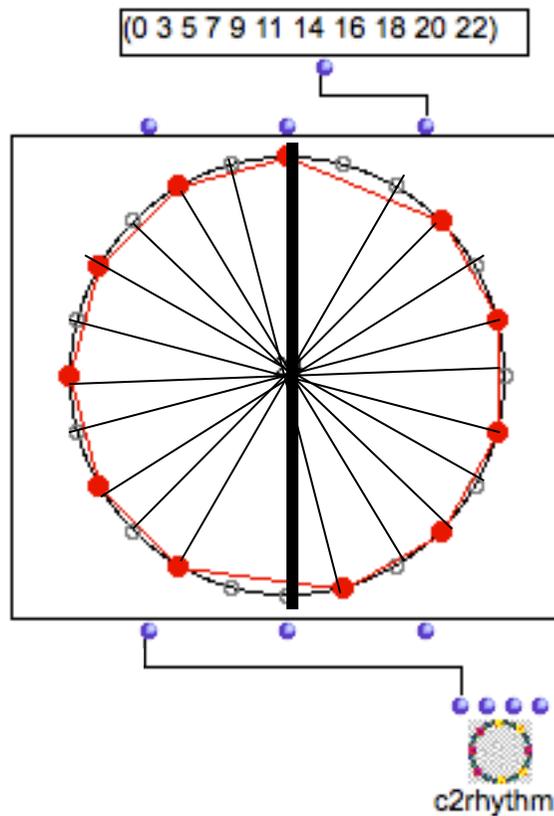
Musical notation for a tiling rhythmic canon with  $A$  subset and  $B$  periodic subset. The notation shows three staves with notes colored red, green, and purple. Below the notation is a diagram showing the decomposition of  $\mathbb{Z}_n$  into  $\mathbb{Z}_3 \oplus \mathbb{Z}_3$ .

$A \subset \mathbb{Z}_n$   
 $B \subset \mathbb{Z}_n$   
 $B$  periodic subset

# The Oddity Property and its tiling generalization

(Simha Arom & Marc Chemillier)

(Rachel W. Hall & P. Klingsberg)



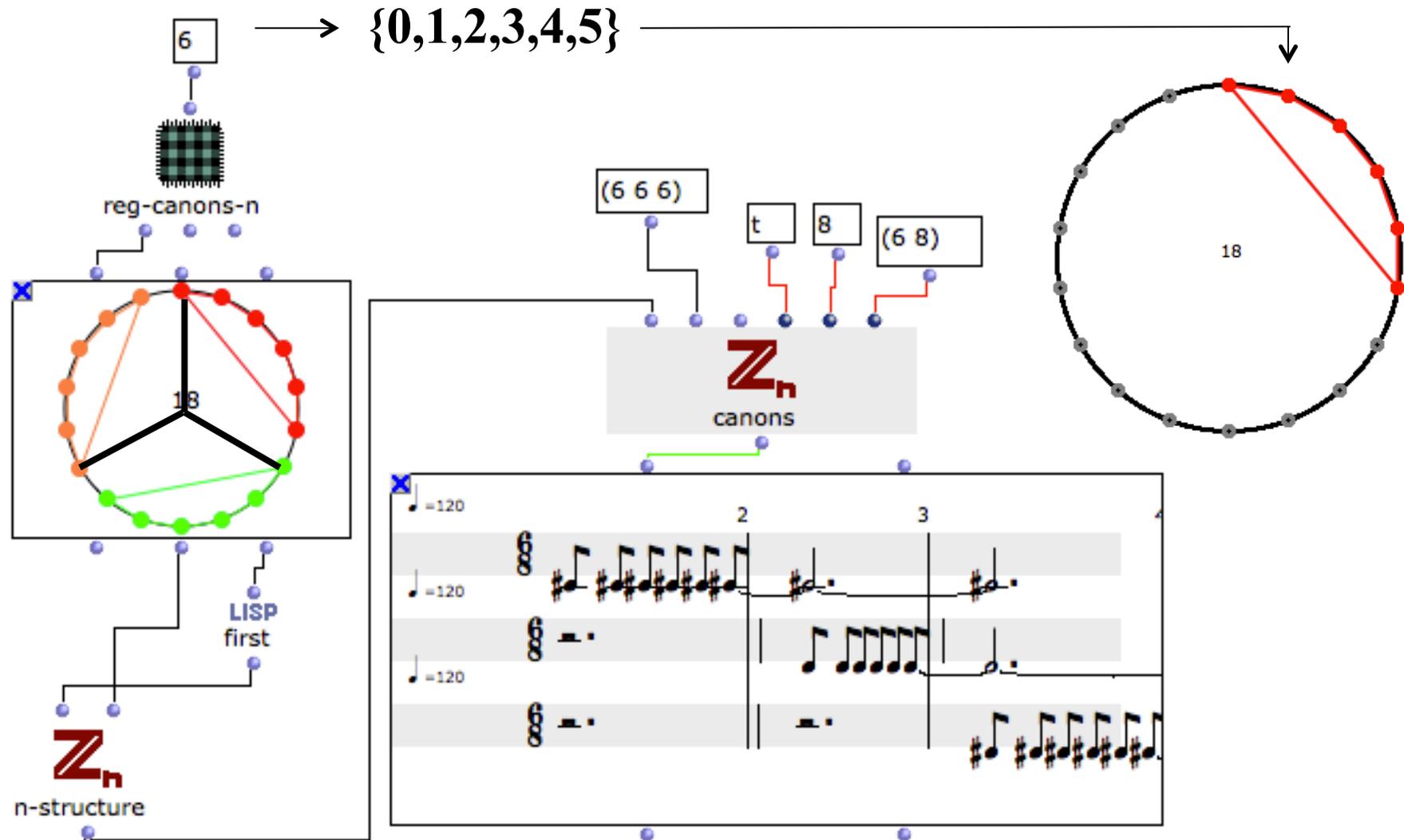
# 3-asymmetric rhythmic pattern and tiling canons

The diagram illustrates the concept of 3-asymmetric rhythmic patterns and tiling canons. It is divided into several sections:

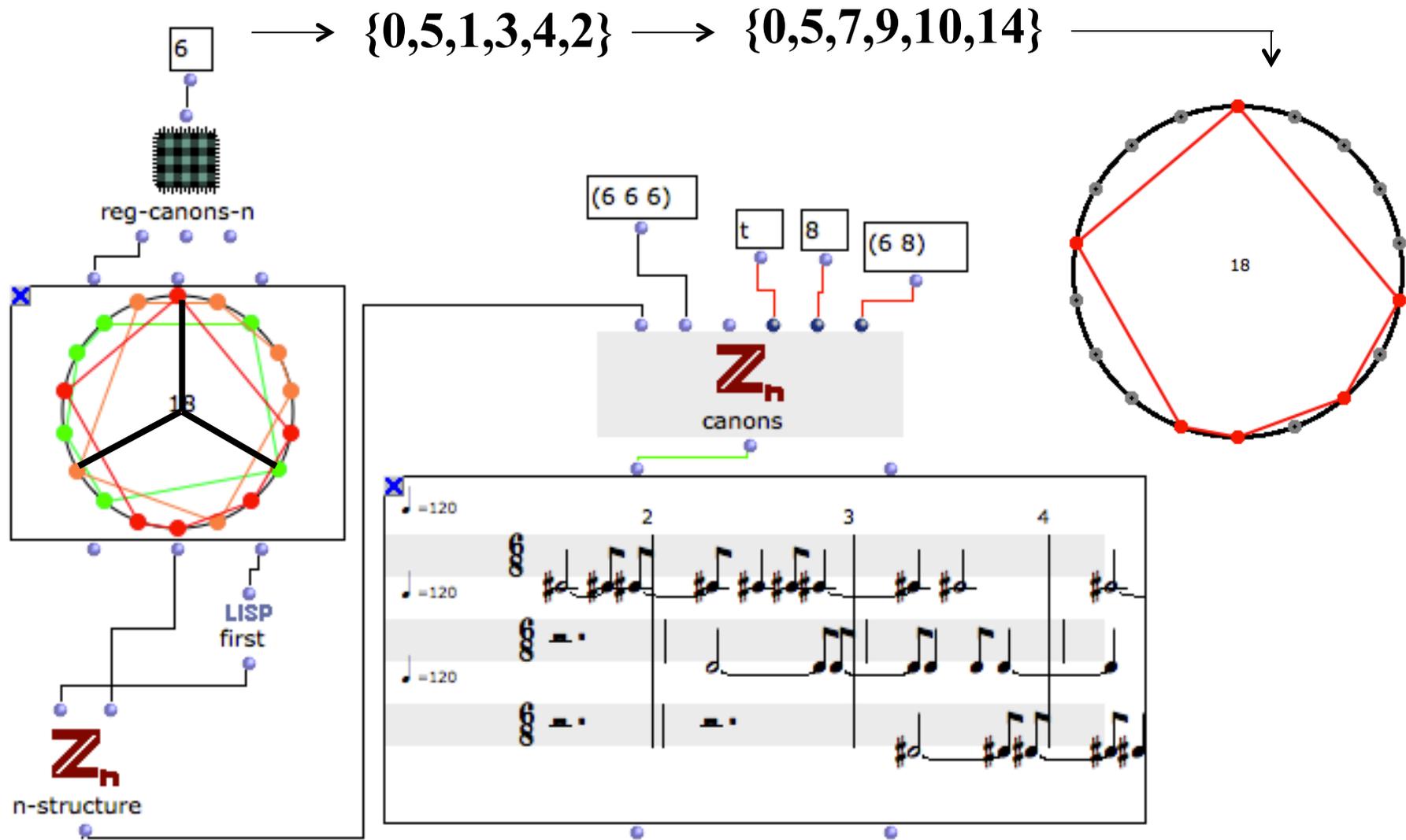
- Top Row:** Three circular diagrams, each labeled "12" and "c2rhythm". Each diagram shows a circle with 12 points. Red lines connect three points to form a triangle. Below each diagram is a small circle with 12 points and a larger circle with 12 points.
- Middle Row:** Three musical staves, each corresponding to one of the circular diagrams above. Each staff shows a rhythmic pattern of notes.
- Right Side:** A large circular diagram labeled "12" showing a tiling pattern. It features a central point and three thick black lines radiating from it to the circumference, dividing the circle into three sectors. Red lines connect points on the circumference to form a complex pattern. Green lines connect points on the circumference to form a smaller pattern. Dashed lines indicate the tiling structure.
- Bottom Left:** A dashed box containing a circular diagram labeled "12" and "c2rhythm" and a musical staff, similar to the top row but enclosed in a dashed border.
- Bottom Right:** A musical score with three staves. The top staff is labeled "2", the middle "3", and the bottom "4". The score shows a sequence of notes across 8 measures, with a treble clef and a common time signature.



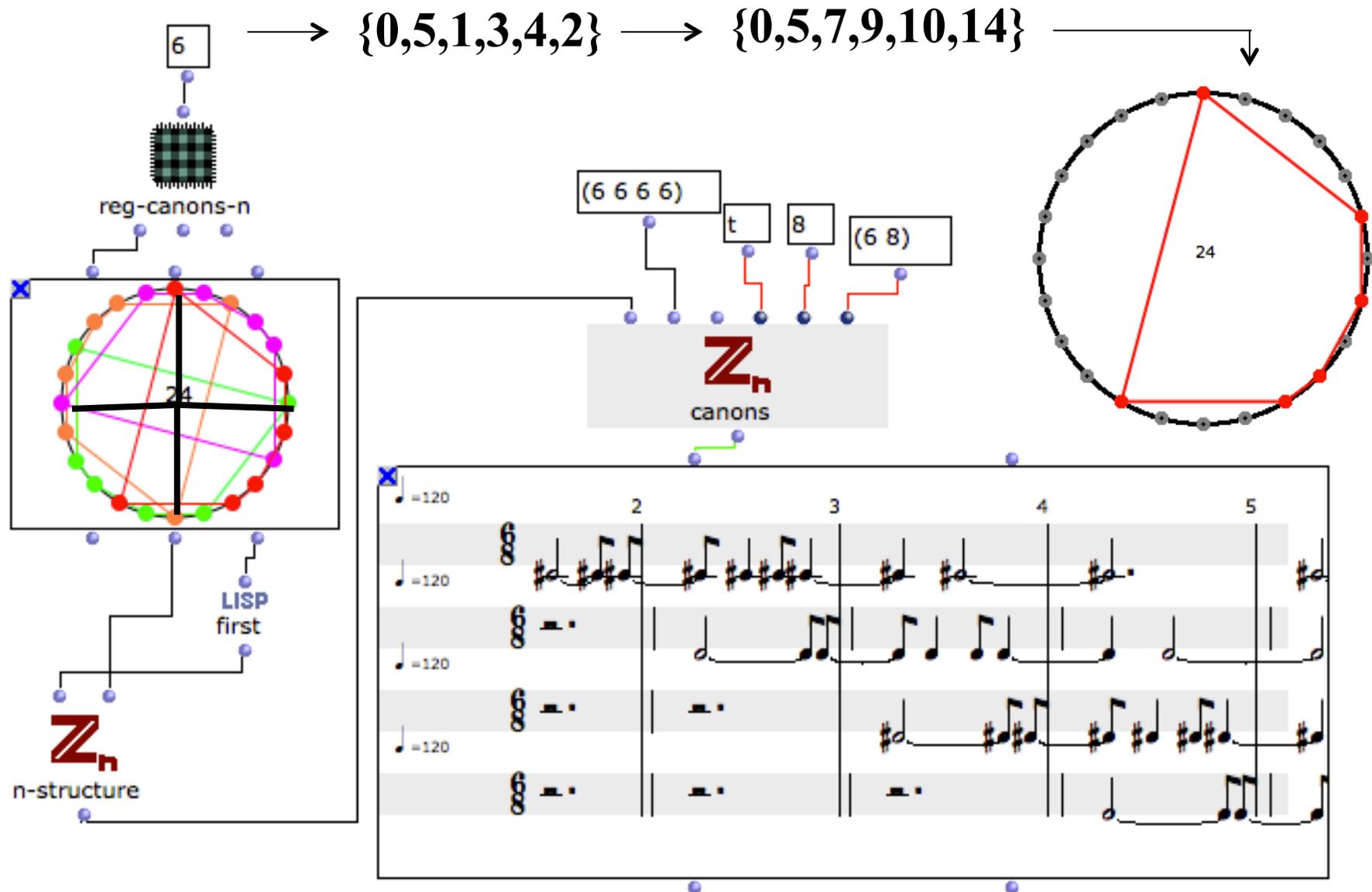
# Permutational construction of a $k$ -asymmetric pattern



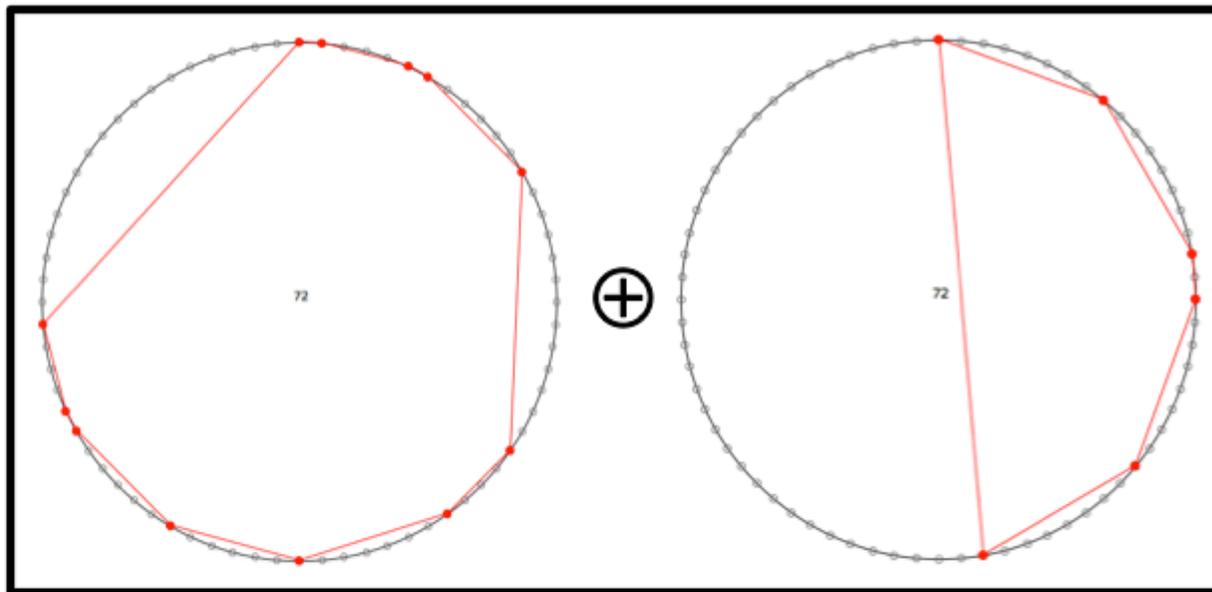
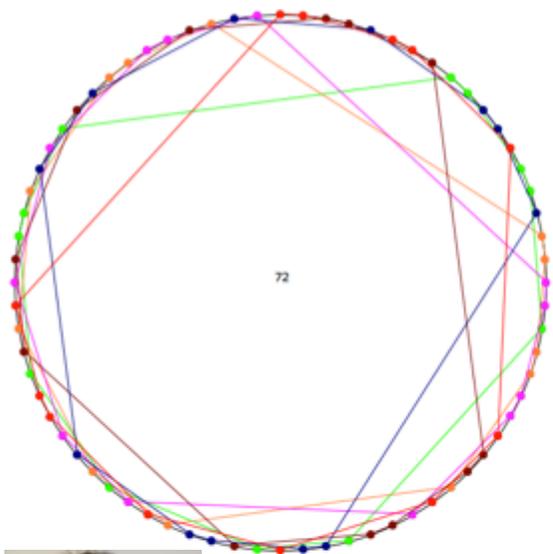
# Permutational construction of a $k$ -asymmetric pattern



# Permutational construction of a $k$ -asymmetric pattern



# Aperiodic Rhythmic Tiling Canons (Vuza Canons)



Dan Vuza

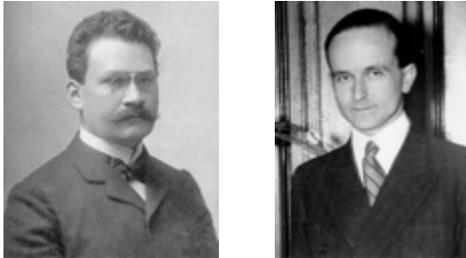


Anatol Vieru

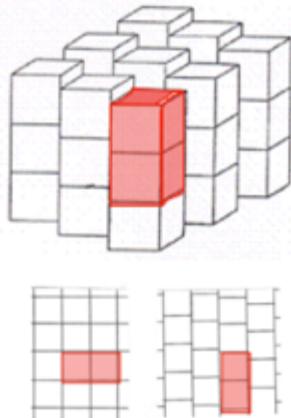


# Tiling Rhythmic Canons as a 'mathemusical' problem

## Minkowski/Hajós Problem (1907-1941)



In any simple lattice tiling of the  $n$ -dimensional Euclidean space by unit cubes, some pair of cubes must share a complete  $(n-1)$ -dimensional face

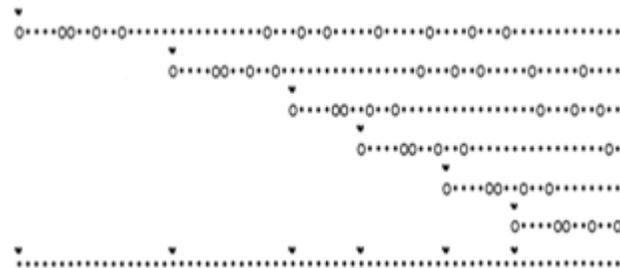


## Vieru's problem and Vuza's formalization (PNM, 1991)



A Vuza Canon is a factorization of a cyclic group in a direct sum of two non-periodic subsets

$$\mathbb{Z}/n\mathbb{Z} = R \oplus S$$



## Link between Minkowski problem and Vuza Canons (Andreatta, Master diss. 1996)

### Hajós groups (*good groups*)

$\mathbb{Z}/n\mathbb{Z}$  with  $n \in \{p^\alpha, p^\alpha q, pqr, p^2q^2, p^2qr, pqr^2\}$  where  $p, q, r, s$ , are distinct prime numbers



### Non-Hajós group (*bad groups*)

72  
 108 120 144 168 180  
 200 216 240 252 264 270 280 288  
 300 312 324 336 360 378 392 396  
 400 408 432 440 450 456 468 480  
 500 504 520 528 540 552 560 576 588 594  
 600 612 616 624 648 672 675 680 684 696  
 700 702 720 728 744 750 756 760 784 792  
 800 810 816 828 864 880 882 888...

(Sloane's sequence A102562)

S. Stein, S. Szabó:  
*Algebra and Tiling*,  
 Carus Math. Mon. 1994

M. Andreatta & C. Agon (eds), « Tiling Problems in Music », Special Issue of the *Journal of Mathematics and Music*, Vol. 3, Number 2, July 2009 (with contributions by E. Amiot, F. Jedrzejewski, M. Kolountzakis and M. Matolcsi)



*Fuglede Spectral Conjecture*  
 (1974)



# Fuglede Spectral Conjecture



A subset of the  $n$ -dimensional Euclidean space tiles by translation iff it is spectral.

(*J. Func. Anal.* 16, 1974)

→ False in dim.  $n \geq 3$   
(Tao, Kolountzakis, Matolcsi, Farkas and Mora)

→ Open in dim. 1 et 2

**DEFINITION 6** A subset  $A$  of some vector space (say  $\mathbb{R}^n$ ) is spectral iff it admits a Hilbert base of exponentials, i.e. if any map  $f \in L^2(A)$  can be written

$$f(x) = \sum f_k \exp(2i\pi \lambda_k \cdot x)$$

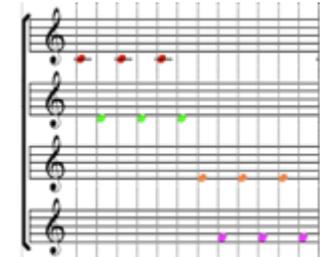
for some fixed family of vectors  $(\lambda_k)_{k \in \mathbb{Z}}$  where the maps  $e_k : x \mapsto \exp(2i\pi \lambda_k \cdot x)$  are mutually orthogonal (i.e.  $\int_A \bar{e}_k e_j = 0$  whenever  $k \neq j$ ).

↓  $n=1$

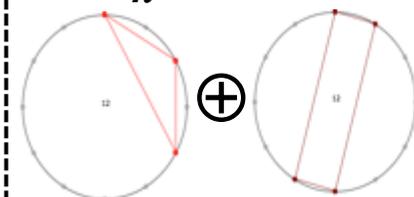
**DEFINITION 8.** A subset  $A \in \mathbb{Z}$  is spectral if there exists a spectrum  $\Lambda \subset [0, 1]$  (i.e., a subset with the same cardinality as  $A$ ) such that  $e^{2i\pi(\lambda_i - \lambda_j)}$  is a root of  $A(X)$  for all distinct  $\lambda_i, \lambda_j \in \Lambda$ .

**Example:**

$A = \{0, 1, 6, 7\} \rightarrow \Lambda = \{0, 1/12, 1/2, 7/12\}$   
since  $\exp(\pi i)$ ,  $\exp(\pi i/6)$ ,  $\exp(-\pi i/6)$ ,  $\exp(5\pi i/6)$ ,  $\exp(-5\pi i/6)$  are the roots of the associated polynomial  
 $A(X) = 1 + X + X^6 + X^7$



$$\mathbb{Z}_n = B \oplus A$$



# Fuglede Spectral Conjecture and Vuza Canons



A subset of the  $n$ -dimensional Euclidean space tiles by translation iff it is spectral.

(*J. Func. Anal.* 16, 1974)

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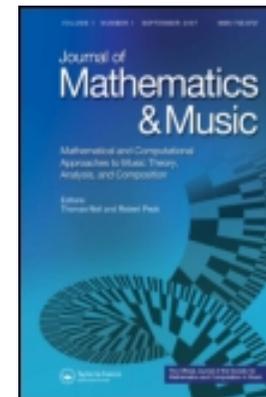
for some fixed family of vectors  $(\lambda_k)_{k \in \mathbb{Z}}$  where the maps  $e_k : x \mapsto \exp(2i\pi \lambda_k \cdot x)$  are mutually orthogonal (i.e.  $\int_A \bar{e}_k e_j = 0$  whenever  $k \neq j$ ).

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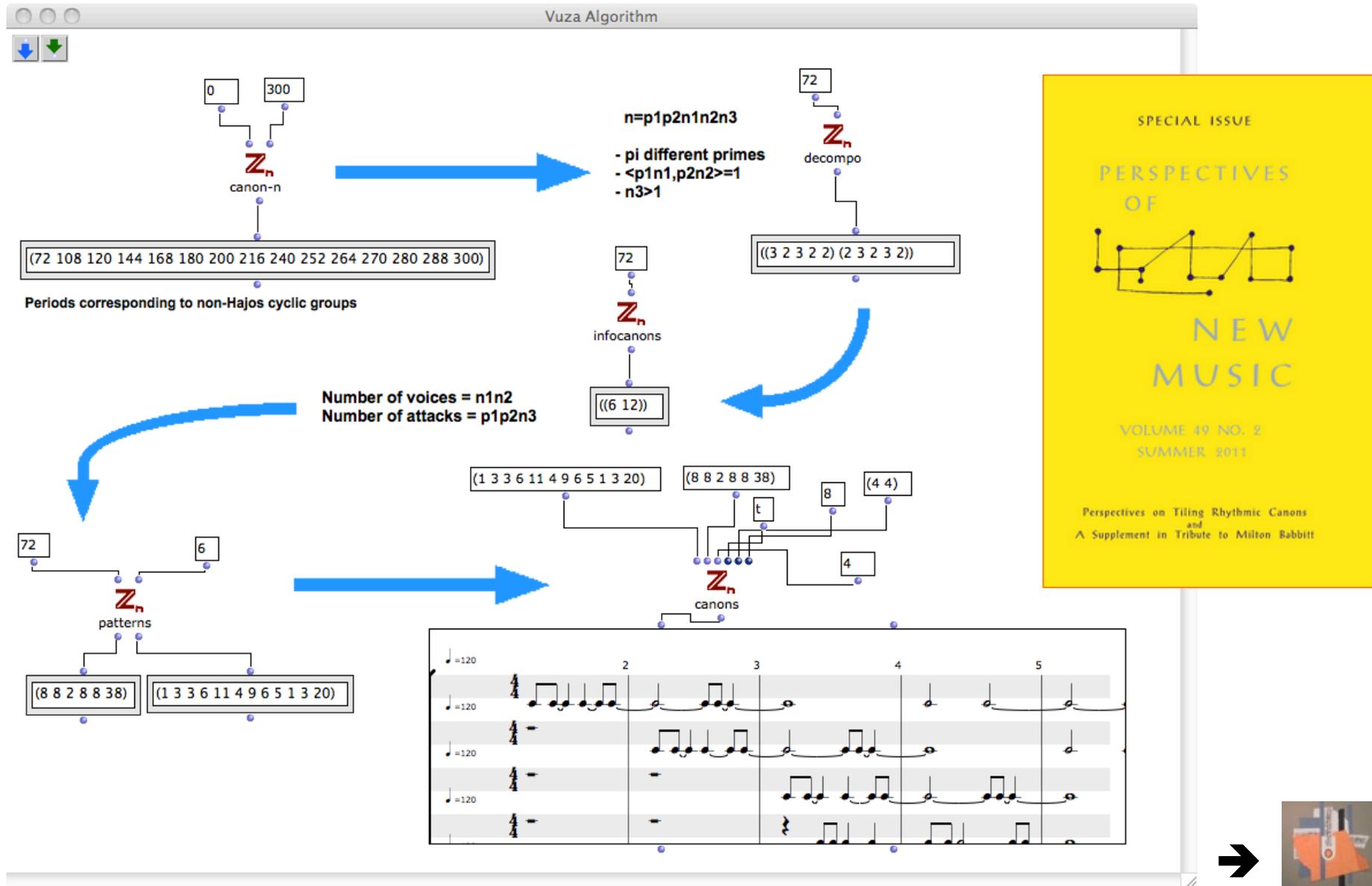
**DEFINITION 8.** A subset  $A \in \mathbb{Z}$  is spectral if there exists a spectrum  $\Lambda \subset [0, 1]$  (i.e., a subset with the same cardinality as  $A$ ) such that  $e^{2i\pi(\lambda_i - \lambda_j)}$  is a root of  $A(X)$  for all distinct  $\lambda_i, \lambda_j \in \Lambda$ .

## Theorem (Amiot, 2009)

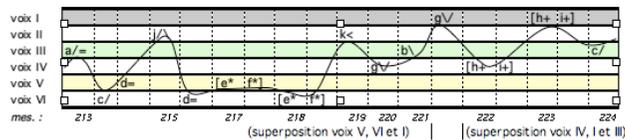
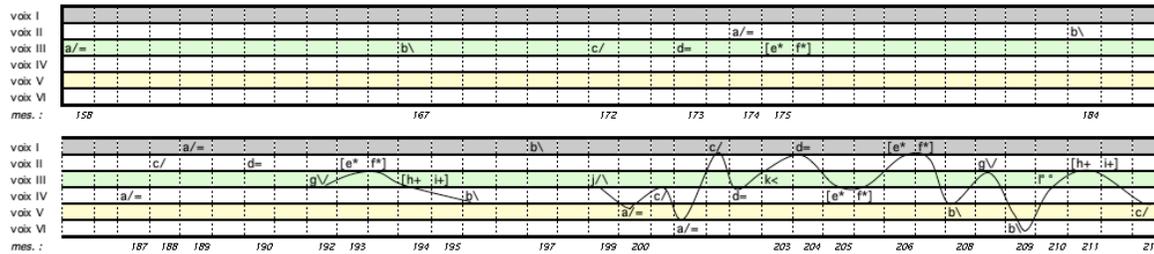
- All non-Vuza canons are spectral.
- Fuglede Conjecture is true (or false) iff it is true (or false) for Vuza Canons



# Vuza Canons in OpenMusic 'MathTool' environment



# Some compositional applications of the Vuza Canons model



a/= : montée vers accord puis "mise en pulsation"  
 b\ : "mise en pulsation" superposé à un gliss. descendant  
 c/ : montée vers accord (tête de a/-)  
 d= "mise en pulsation" en diminuendo (fin de a/-)  
 [e\* f\*] : accord mis en "cross rhythm" (durée double)  
 gV : gliss. descendant puis ascendant  
 [Hj+] : accord mis en "cross rhythm" (durée double)  
 JV : gliss. ascendant puis descendant avec accent  
 k< : "son à l'envers"  
 P\* : deux impacts brefs et piano



F. Lévy

## Coïncidences (1999)

Coincidence - Fabien Levy : déroulement du canon (mes. 158 à 226)  
 (chaque impact fait 3 temps)



M. Lanza

## La bataille de caesme et de charnage

(pour violoncelle et accompagnement, 2012)



G. Bloch

## A piece based on Monk (2007) (« Well You Need'nt »)

## La notte poco prima della foresta

(opéra de chambre pour acteur, mezzo-soprano, baryton, ensemble et électronique, 2009)



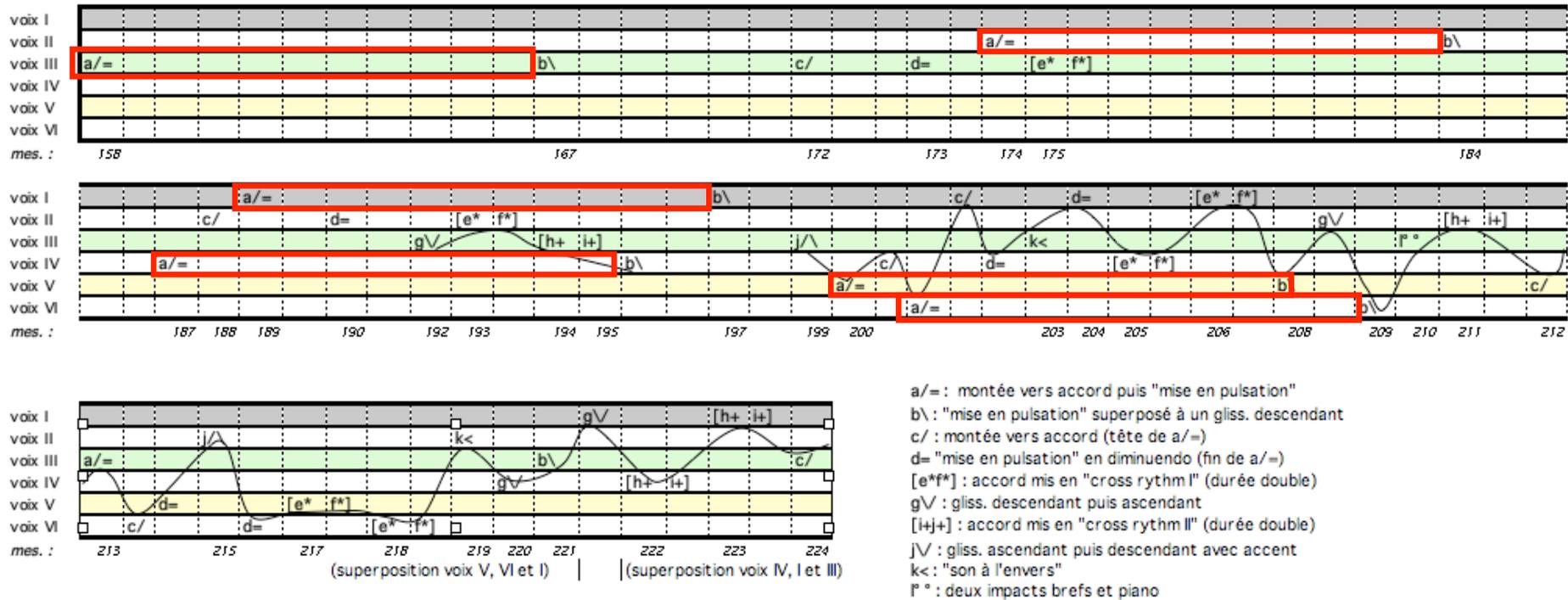
D. Ghisi

# Fabien Lévy

## Morphological Tiling Canons



- *Coïncidences* (pour 33 musiciens, 1999-2007)



Coïncidences - Fabien Lévy : déroulement du canon (mes. 158 à 226)  
(chaque impact fait 3 temps)



Tokyo Symphony Orchestra, Dir.: Kazuyoshi Akiyama, 05/09/2007, Suntory Hall, Tokyo, Japan

F. Lévy, « Three uses of Vuzza Canons », *Perspectives of New Music*, 49(2), 2011, p. 23-31.

# Georges Bloch

## Several compositional strategies

- **Metrical organization of a tiling canon**
- **Self-similarity processes**
- **Metrical modulations between canons**

- *Projet Beyeler* (2001) 
- *Projet Hitchcock*
- *Visite des tours de la cathédrale de Reims*
- *Noël des Chasseurs*
- *Canons à marcher*
- *Canon à eau*
- *Harawun* (2004)
- *L'Homme du champ* (2005)
- *A piece based on Monk* (2007)
- *Peking Duck Soup* (2008)



V1

V2

mp

pp

mf

V5

V6

pp

f

- *A piece based on Monk*, 2007 (« Well You Need'nt ») 

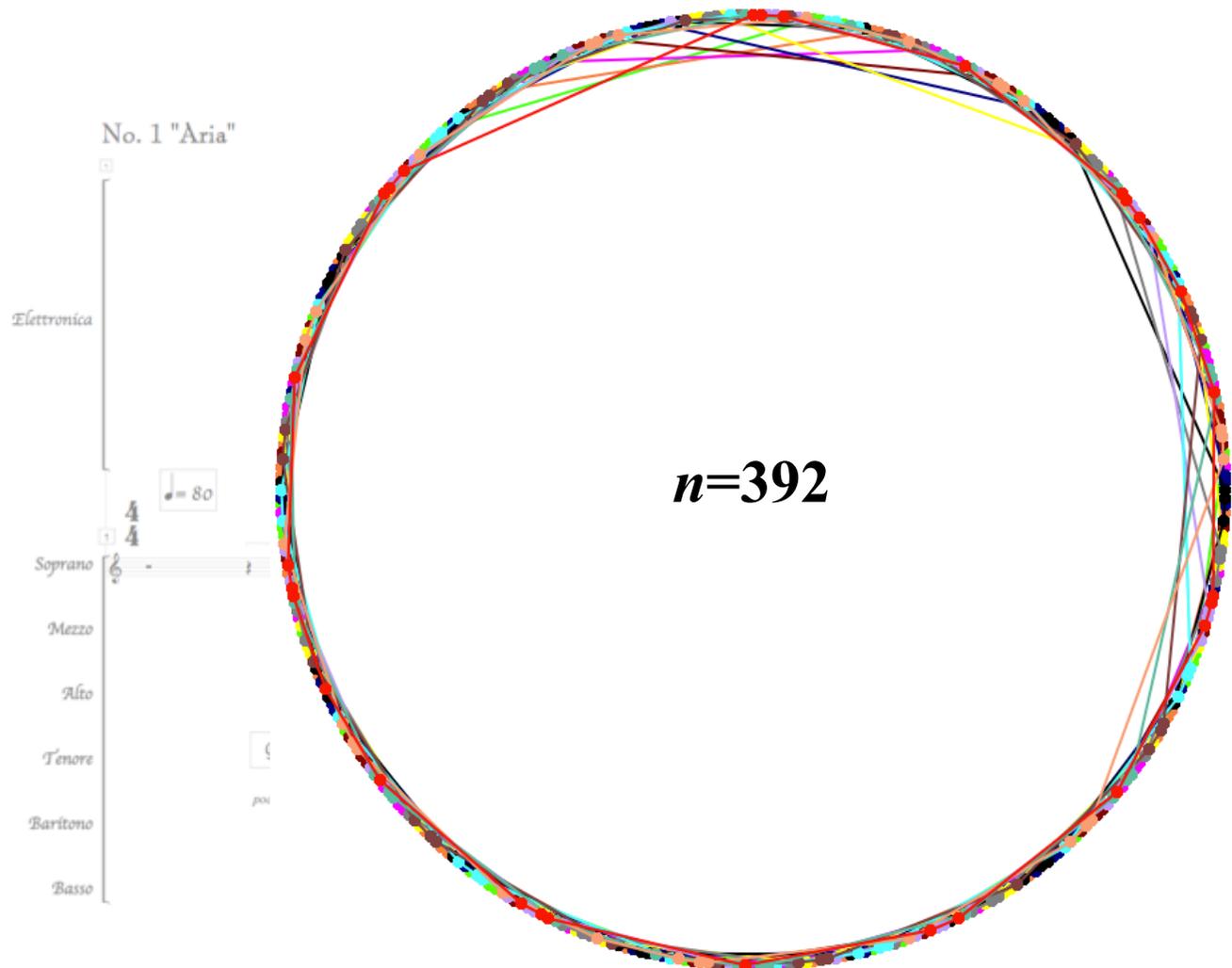
G. Bloch, « Vuza Canons into the museum », *The OM Composers' Book*, 2006

# Mauro Lanza

## Vuza Canons and local periodicities



- *La descrizione del diluvio* (Ricordi, 2007-2008)



*“6 voices are live and 8 are in the electronic part. The choice of the notes and the durations is made in order to stress some quasi-periodicities of the underlying Vuza canon and this gives to each voice a much more “redundant” character”.*

# Mauro Lanza

## Vuza Canons and local periodicities

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- *La descrizione del diluvio* (Ricordi, 2007-2008)

*“The choice of the notes and the durations is made in order to stress some **quasi-periodicities** of the underlying Vuza canon [...]*”

(1 3 25 27 1 3 11 14 27 1 3 25 27 4 25 27 1 3 25 14 13 1 3 25 27 1 3 52)

(1 3 25 27 1 3 11 14 27 1 3 25 27 4 25 27 1 3 25 14 13 1 3 25 27 1 3 52)

(1 3 25 27 1 3 11 14 27 1 3 25 27 4 25 27 1 3 25 14 13 1 3 25 27 1 3 52)

(1 3 25 27 1 3 11 14 27 1 3 25 27 4 25 27 1 3 25 14 13 1 3 25 27 1 3 52)

(1 3 25 27 1 3 11 14 27 1 3 25 27 4 25 27 1 3 25 14 13 1 3 25 27 1 3 52)



Mauro Lanza

# LA BATAILLE DE CARESME ET DE CHARNAGE

per violoncello e accompagnamento (2012)

(min. 10'15"")



190  $\frac{3}{4}$

*Canone sulla 11 corda*  $\frac{4}{4}$

1

pp

pizz

This musical score block covers measures 190 to 193. It features a grand staff with a treble clef on the left and a bass clef on the right. The left hand part is marked with a piano-piano (*pp*) dynamic and includes a downward bowing stroke above the first measure. The right hand part includes a pizzicato (*pizz*) instruction above the final measure. The music is in 3/4 time.

194

1

pizz

This musical score block covers measures 194 to 197. It features a grand staff with a treble clef on the left and a bass clef on the right. The left hand part includes several pizzicato (*pizz*) instructions above the notes. The right hand part includes downward bowing strokes above the first and third measures. The music is in 3/4 time.

# *La notte poco prima della foresta* (2009)

(opéra de chambre pour acteur, mezzo-soprano, baryton, ensemble et électronique)



D. Ghisi

457 AC

FL.

CL.

Fis.

Pf.

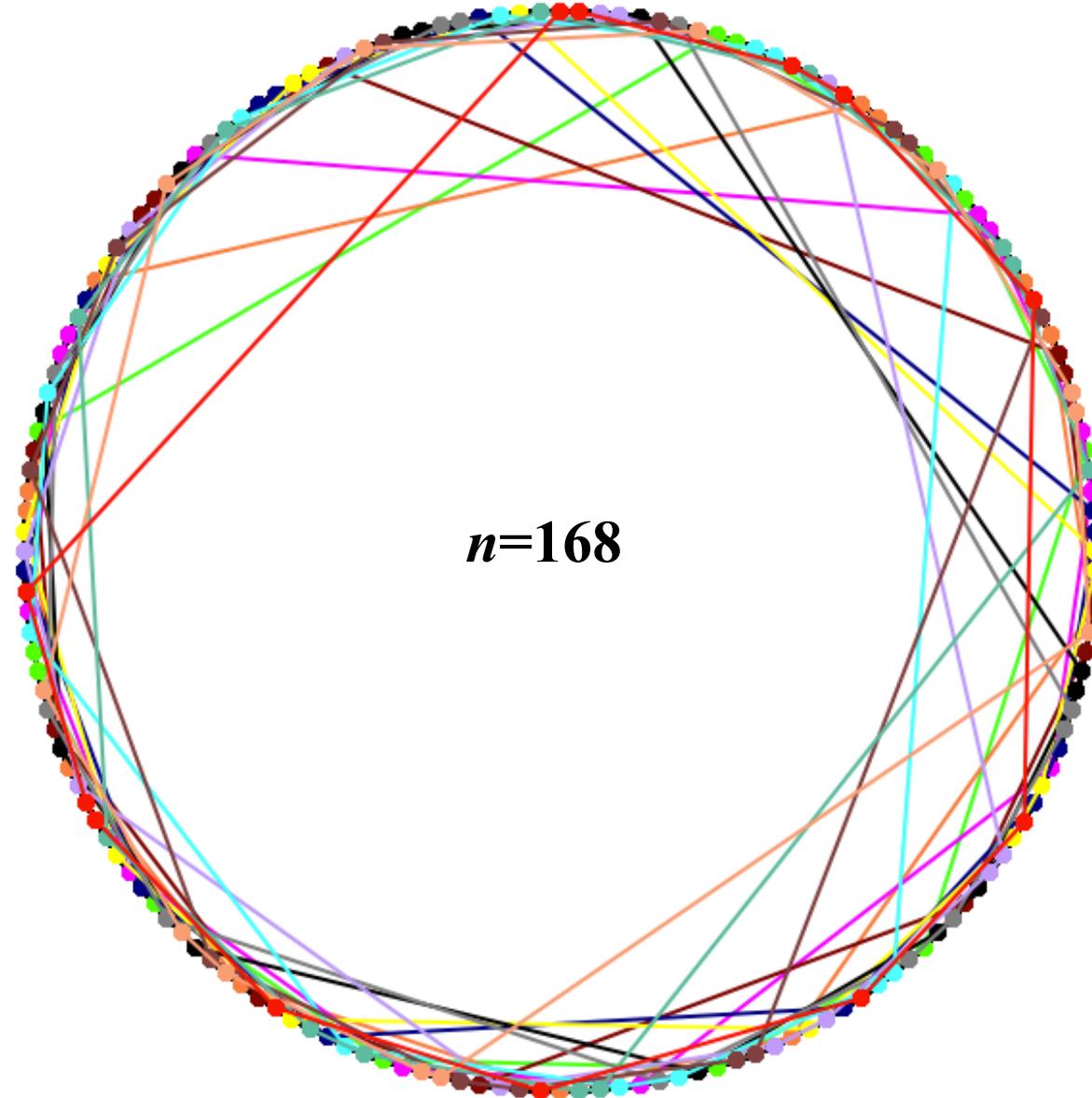
Att.

Mz.

Bar.

mama ti amo

ma - ma



/ma/

*p*

*pp*

*pp*

mama ti amo,

*pp*

ma - ma

# La notte poco prima della foresta (2009)



(opéra de chambre pour acteur, mezzo-soprano, baryton, ensemble et électronique)



D. Ghisi

457

AC

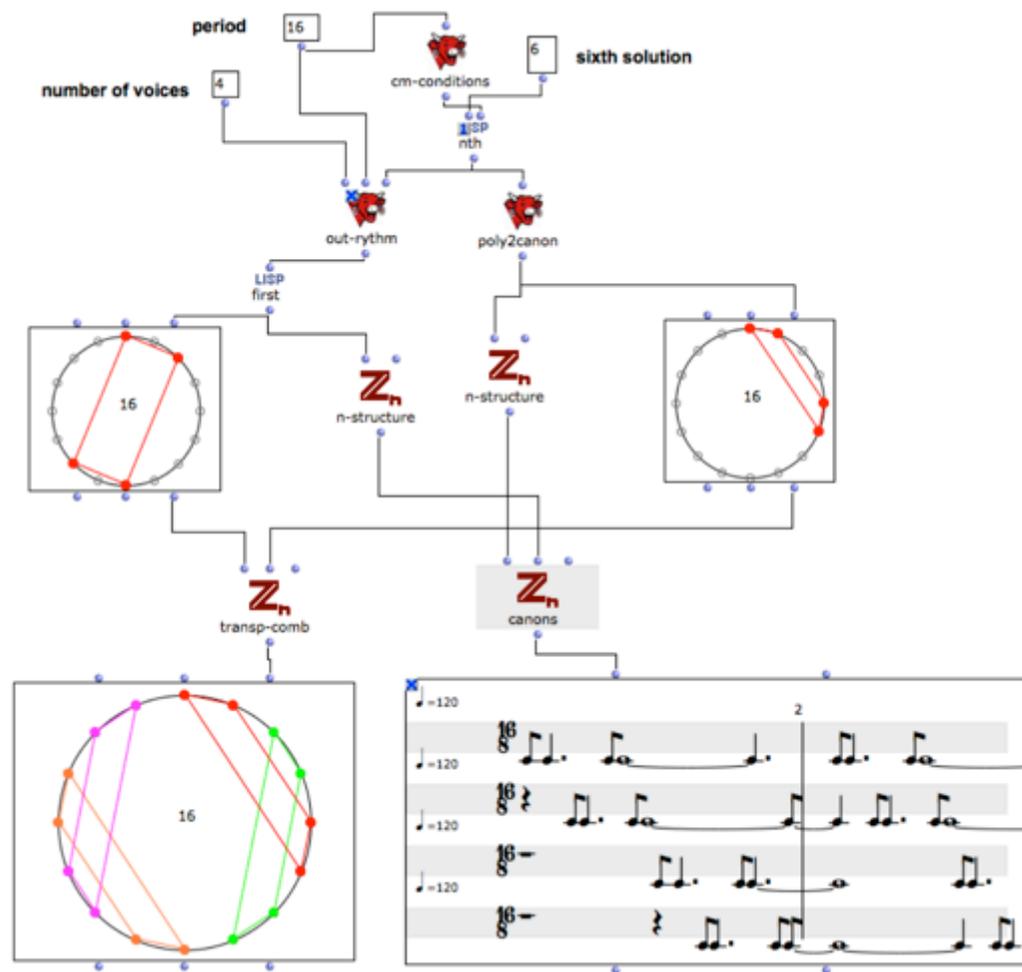
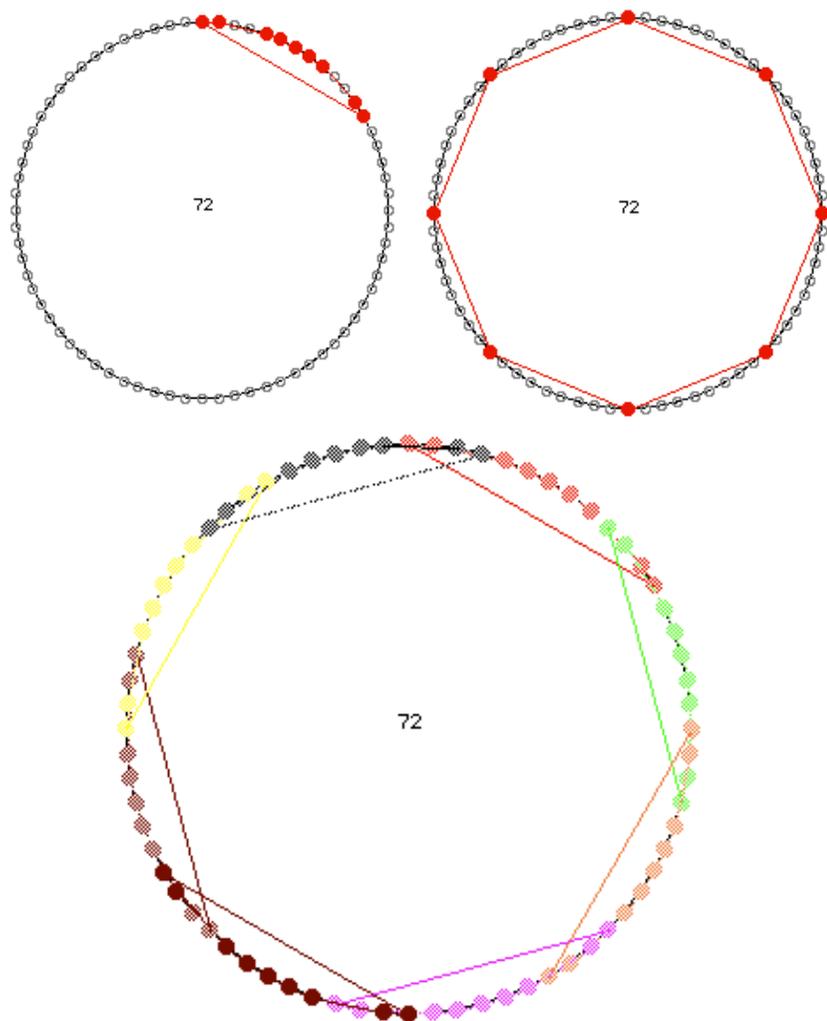
Sprechgesang  
sommesso, dolce (senza suonare)

The musical score is arranged in a system with the following parts from top to bottom:

- Fl.** (Flute): Sprechgesang with lyrics "ma - ma" and "/mal/". Dynamics include *p* and *s*.
- Cl.** (Clarinet): Sprechgesang with lyrics "ma - ma" and "/mal/". Dynamics include *p*, *pp*, *mp*, and *p*.
- Fis.** (Fagotto): Sprechgesang with lyrics "ma - ma" and "/mal/". Dynamics include *pp* and *nV*.
- Pf.** (Piano): Instrumental accompaniment with various dynamics like *pp*, *p*, *ppp*, and *pp*.
- Att.** (Attore): Actor's vocal line with lyrics "mama ti amo," "mama ti amo," "su tutti i muri, di modo che non potesse non vederlo," and "mama ti amo,".
- Mz.** (Mezzo-soprano): Sprechgesang with lyrics "ma - ma" and "/mal/". Dynamics include *p* and *s*.
- Bar.** (Baryton): Sprechgesang with lyrics "ma - ma". Dynamics include *p* and *pp*.

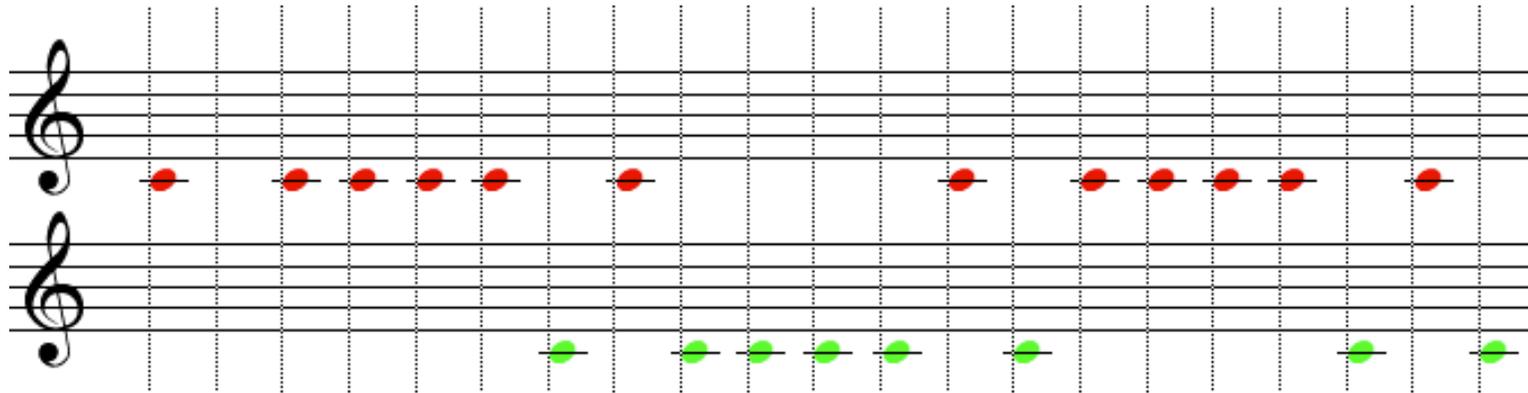
The score includes various musical notations such as slurs, accents, and dynamic markings. The lyrics are written below the vocal staves.

# Using (cyclotomic) polynomials to tile the line

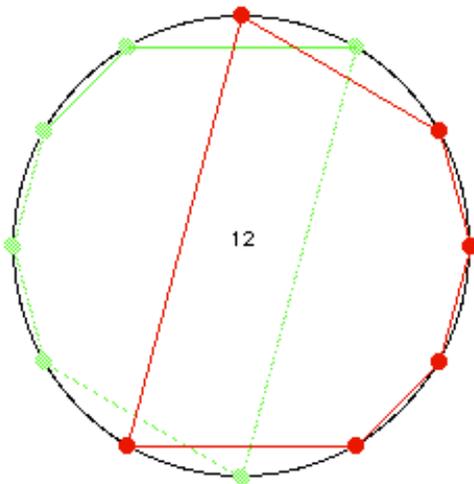


- E. Amiot, M. Andreatta, C. Agon, « Tiling the (musical) line with polynomial: some theoretical and implementational aspects », *ICMC*, Barcelona, 2005, 227-230
- C. Agon, M. Andreatta, « Modelling and Implementing Tiling Rhythmic Canons in OpenMusic Visual Programming Language », *Perspectives of New Music*, Special Issue, vol. 1-2, n° 49.

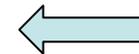
# Tiling canons by translation and augmentation



Thomas Noll

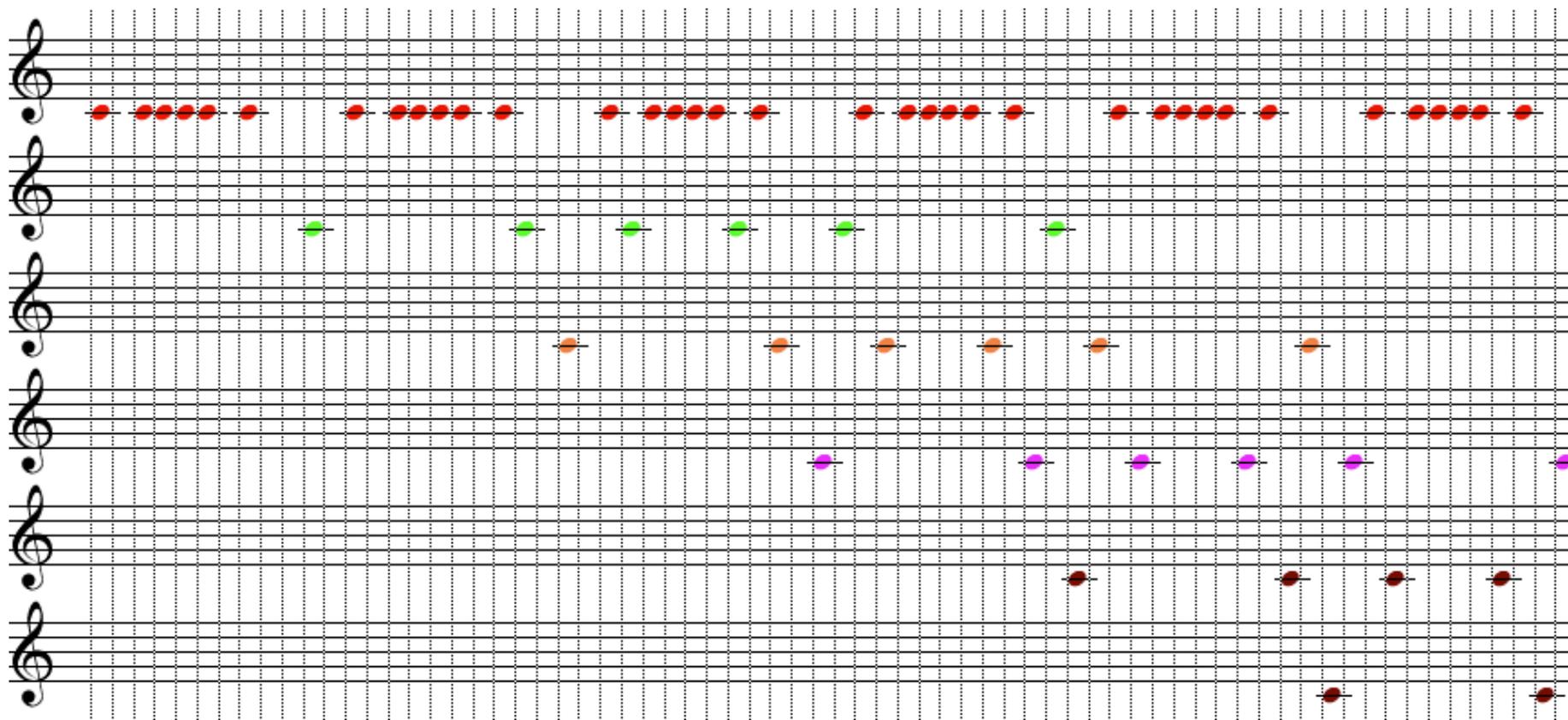
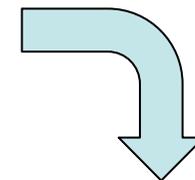
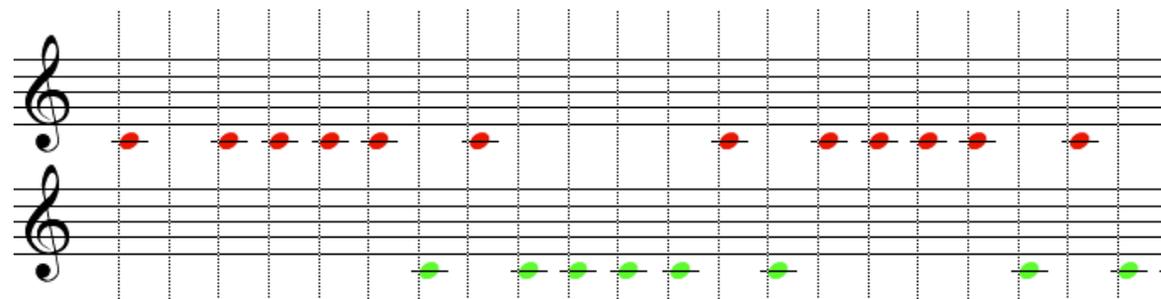


- $(((\emptyset 1 2 3 4 6) ((1 11))))$
- $(((\emptyset 1 2 3 4 5) ((1 11) (1 1))))$
- $(((\emptyset 1 2 3 5 7) ((1 11) (1 7))))$
- $(((\emptyset 1 3 4 7 8) ((1 5))))$
- $(((\emptyset 1 2 3 6 7) ((1 11))))$
- $(((\emptyset 1 3 4 6 9) ((1 11) (1 5))))$
- $(((\emptyset 1 3 6 7 9) ((1 11) (1 5))))$
- $(((\emptyset 1 2 6 7 8) ((1 11) (1 7) (1 5) (1 1))))$
- $(((\emptyset 1 4 5 8 9) ((1 11) (1 7) (1 5) (1 1))))$
- $(((\emptyset 1 2 5 6 7) ((1 7) (1 5))))$
- $(((\emptyset 2 3 4 5 7) ((1 11) (1 7) (1 5) (1 1))))$
- $(((\emptyset 1 4 5 6 8) ((1 11) (1 7))))$
- $(((\emptyset 1 2 4 5 7) ((1 5))))$
- $(((\emptyset 1 3 4 5 8) ((1 5) (1 1))))$
- $(((\emptyset 1 2 4 5 8) ((1 11))))$
- $(((\emptyset 1 2 4 6 8) ((1 11) (1 7))))$
- $(((\emptyset 2 3 4 6 8) ((1 11))))$
- $(((\emptyset 2 4 6 8 10) ((1 11) (1 7) (1 5) (1 1))))$



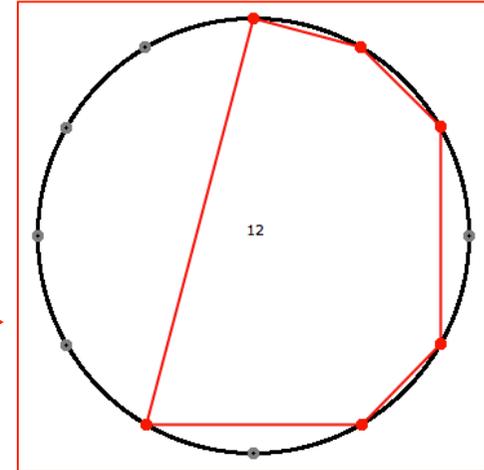
# Augmented Tiling Canons (Noll Canons)

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# Augmented Tiling Canons (Noll Canons)

(((0 1 2 3 4 6) ((1 11)))  
 ((0 1 2 3 4 5) ((1 11) (1 1)))  
 ((0 1 2 3 5 7) ((1 11) (1 7)))  
 ((0 1 3 4 7 8) ((1 5)))  
 ((0 1 2 3 6 7) ((1 11)))  
 ((0 1 3 4 6 9) ((1 11) (1 5)))  
 ((0 1 3 6 7 9) ((1 11) (1 5)))  
 ((0 1 2 6 7 8) ((1 11) (1 7) (1 5) (1 1)))  
 ((0 1 4 5 8 9) ((1 11) (1 7) (1 5) (1 1))) ((0 1 2 5 6 7) ((1 7) (1 5)))  
 ((0 2 3 4 5 7) ((1 11) (1 7) (1 5) (1 1))) ((0 1 4 5 6 8) ((1 11) (1 7)))  
**((0 1 2 4 5 7) ((1 5)))**  
 ((0 1 3 4 5 8) ((1 5) (1 1)))  
 ((0 1 2 4 5 8) ((1 11)))  
 ((0 1 2 4 6 8) ((1 11) (1 7)))  
 ((0 2 3 4 6 8) ((1 11)))  
 ((0 2 4 6 8 10) ((1 11) (1 7) (1 5) (1 1)))

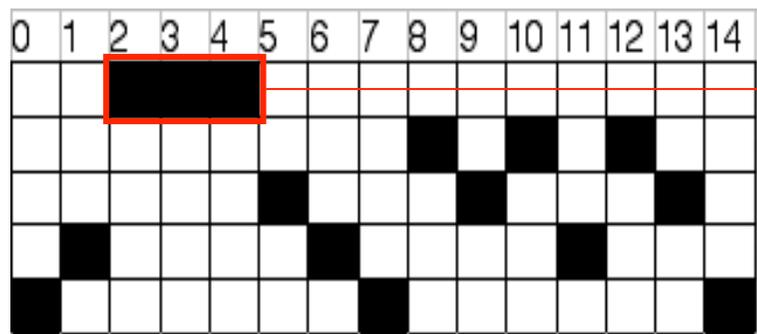



# Tom Johnson's Perfect Tilings

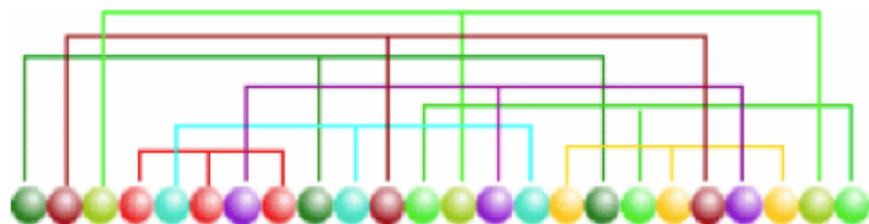
## Tilework for Piano

perfect triplet tilings, 5th order

with thanks to Jon Wild and Erich Neuwirth

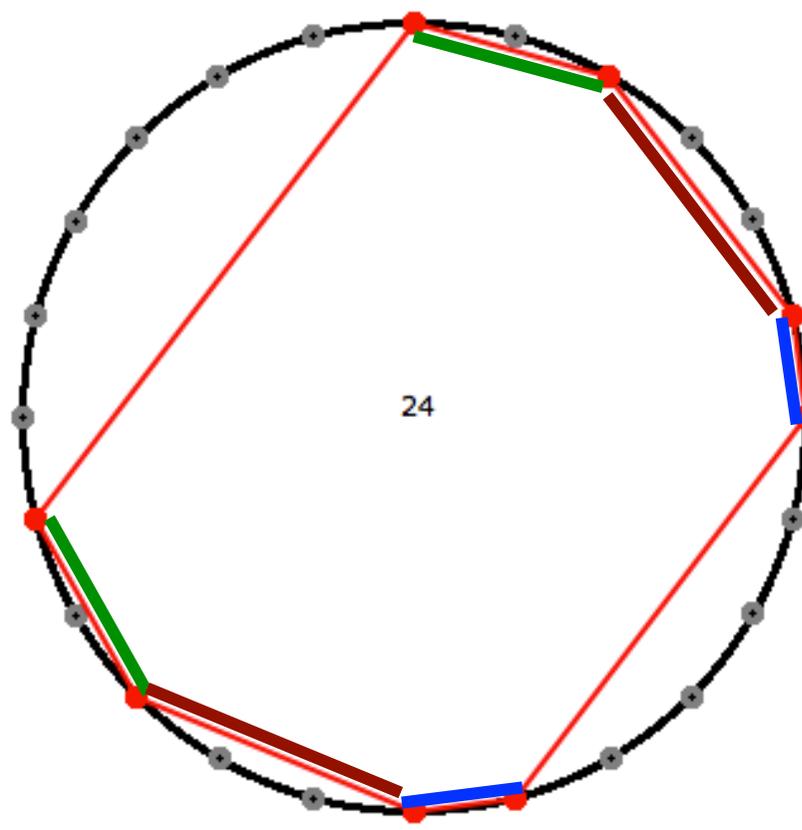
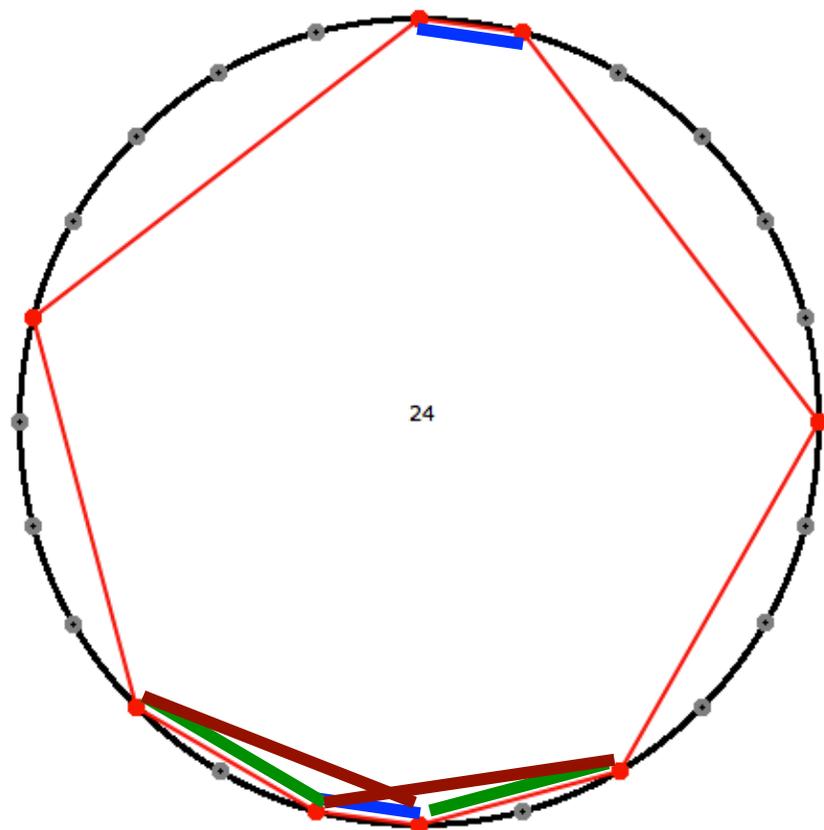


(min. 08'15'')



# Tiling Rhythmic Canons and Homometry

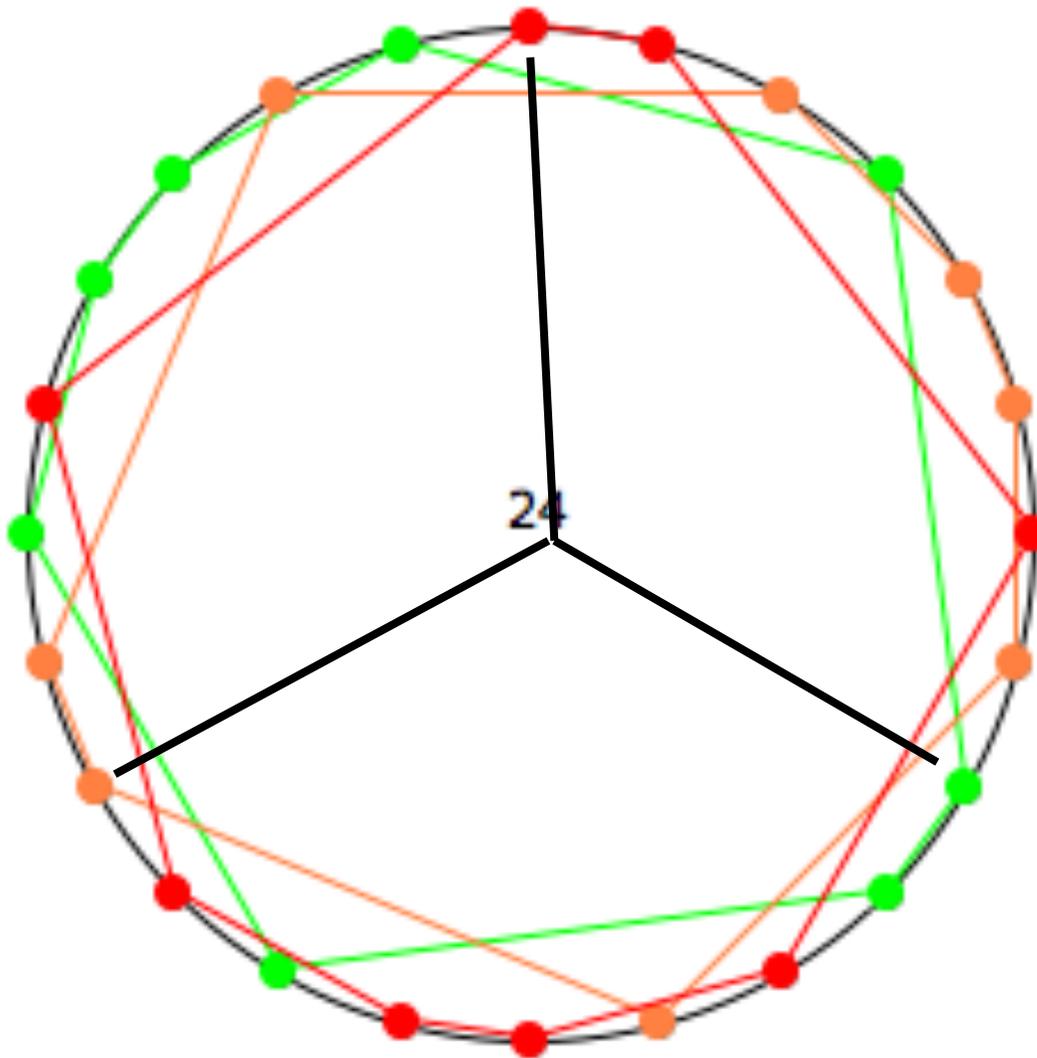
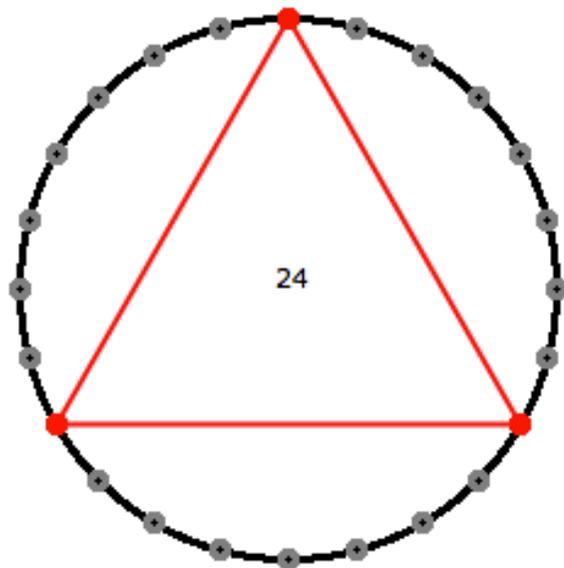
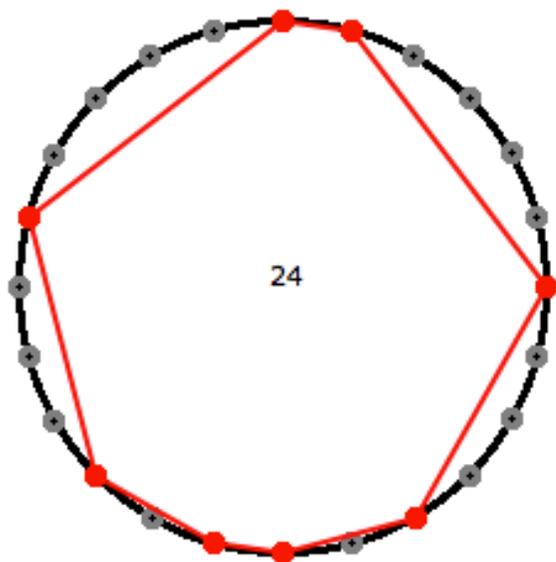
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**Can you hear the shape of a canon?**

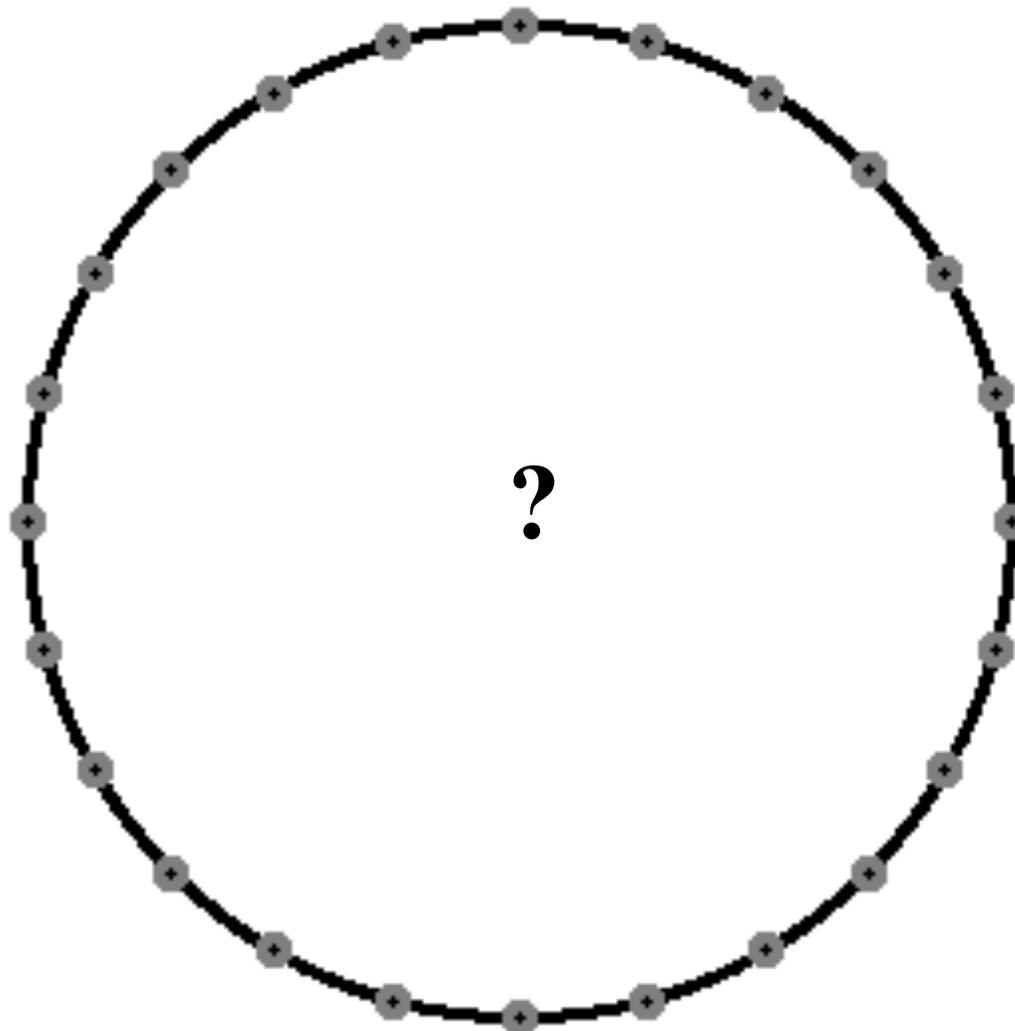
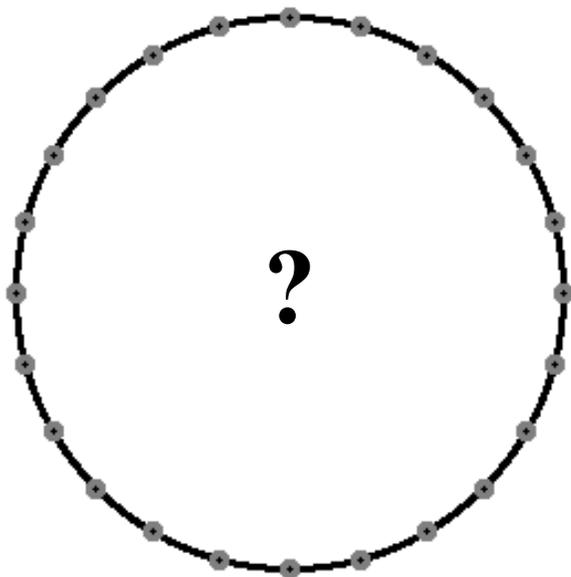
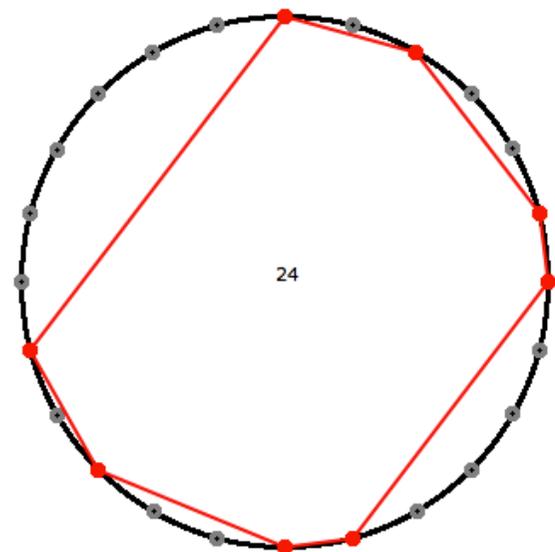
# Tiling Rhythmic Canons and Homometry

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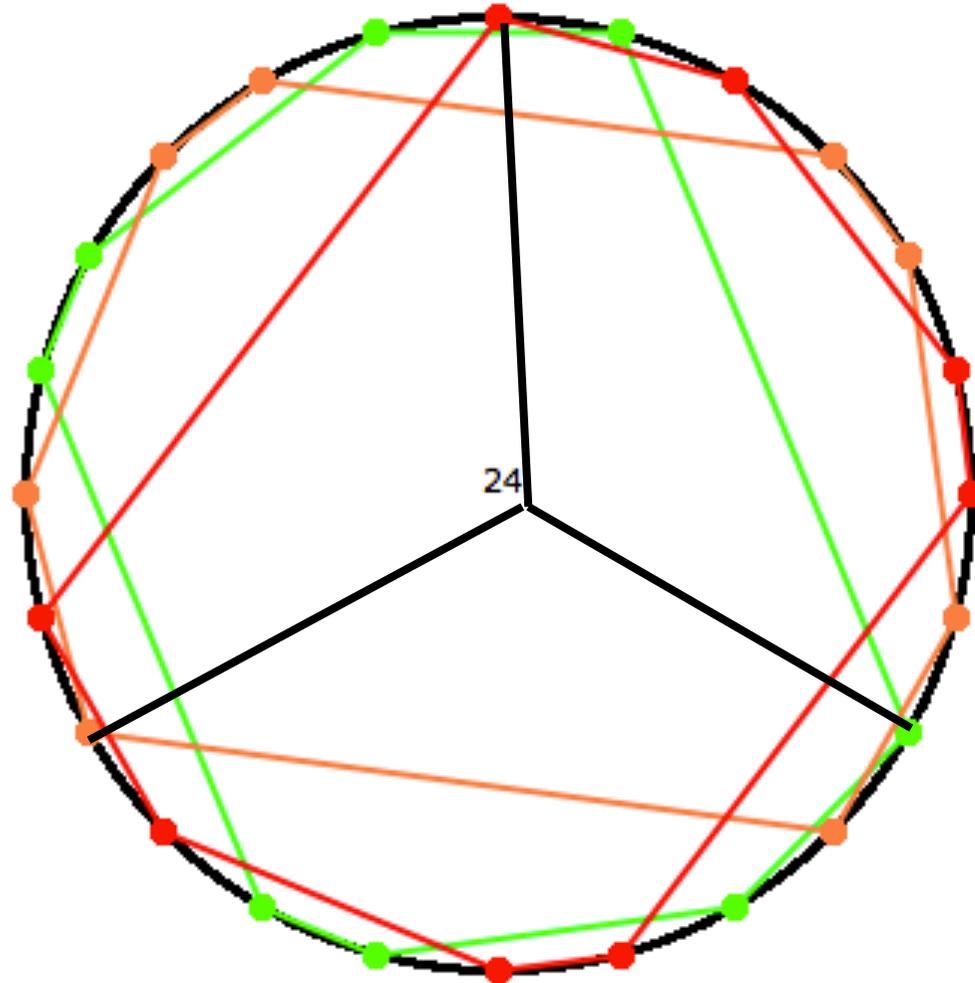
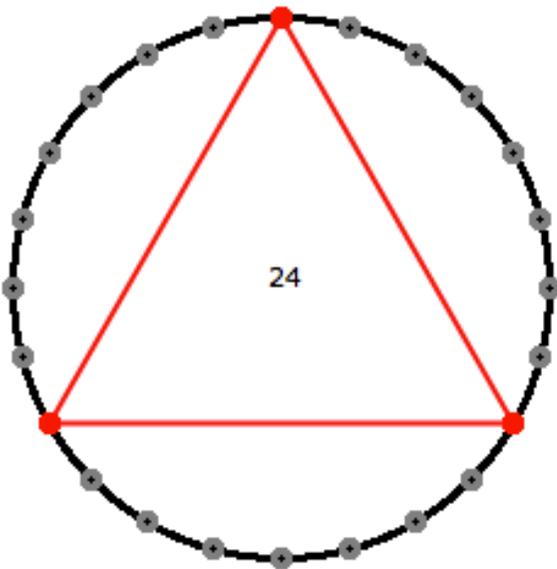
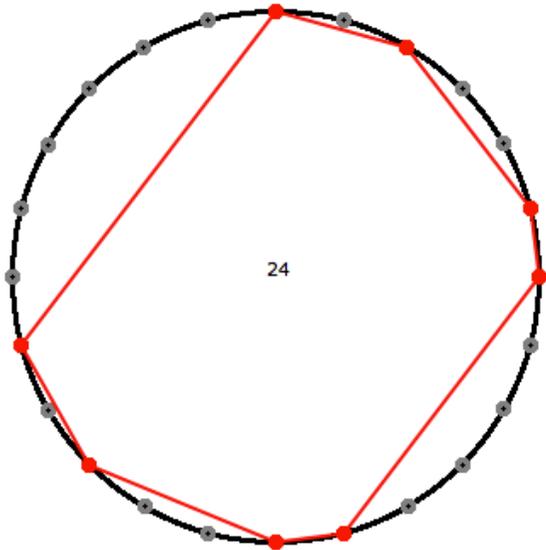
# Tiling Rhythmic Canons and Homometry

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# Tiling Rhythmic Canons and Homometry

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# Homometry and Tiling Rhythmic Canons

## TILING

Let  $Z_A = \{ t \in \mathbb{Z}_c, F_A(t)=0 \}$

$A$  tiles  $\mathbb{Z}_c$  when equivalently:

- There exists  $B$ ,  $A \oplus B = \mathbb{Z}_c$
- $1_A \star 1_B = 1$
- $F_A \times F_B(t) = 1 + e^{-2i\pi t/c} + \dots + e^{-2i\pi t(c-1)/c}$  (0 unless  $t=0$ )
- $Z_A \cup Z_B = \{1, 2 \dots c-1\}$  AND  $\text{Card } A \times \text{Card } B = c$
- $IC_A \star IC_B = IC(\mathbb{Z}_c) = c$  and  $\text{Card } A \times \text{Card } B = c$

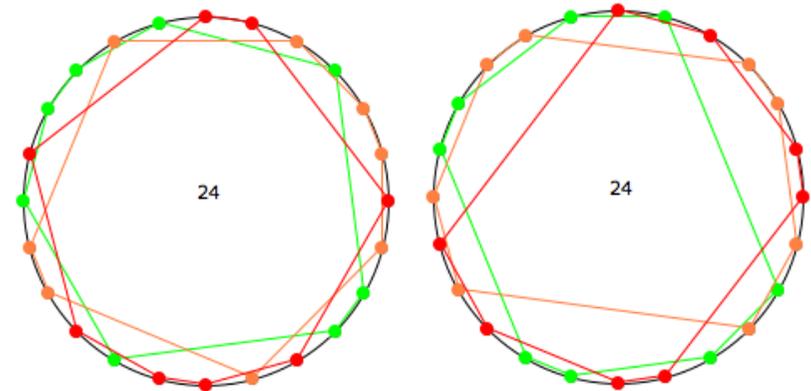
$$\mathcal{F}_A : t \mapsto \sum_{k \in A} e^{-2i\pi kt/c}$$

$$IC_A(k) = (1_A \star 1_{-A})(k)$$

## *A musical offering:*

### • *Theorem:*

If  $A$  tiles with  $B$  and  $A'$  has the same IC, then  $A'$  tiles with  $B$ , too.



Thank you for your attention!

