A Categorical Generalization of Klumpenhouwer Networks

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Abstract. This article proposes a functorial framework for generalizing some constructions of transformational theory. We focus on Klumpenhouwer Networks for which we propose a categorical generalization via the concept of set-valued poly-K-nets (henceforth PK-nets). After explaining why K-nets are special cases of these category-based transformational networks, we provide several examples of the musical relevance of PK-nets as well as morphisms between them. We also show how to construct new PK-nets by using some topos-theoretical constructions.

Keywords: Transformational theory · K-nets · PK-nets · Category theory

1 Introduction

Since the publication of pioneering work by David Lewin [1,2] and Guerino Mazzola [3,4] respectively in the American and European formalized music-theoretical tradition, transformational approaches have established themselves as an autonomous field of study in music analysis. Surprisingly, although group action-based theoretical constructions, such as Lewin's "Generalized Interval Systems" (GIS), are naturally described in terms of categories and functors, the categorical approach to transformational theory remains relatively marginal with respect to the major trend in math-music community [5–7].

Within the transformational framework, which progressively shifted the music-theoretical and analytical focus from the "object-oriented" musical content to the operational musical process, Klumpenhouwer Networks (henceforth K-nets), as observed by many scholars, have stressed the deep synergy between set-theoretical and transformational approaches thanks to their anchoring in both group and graph theory [10]. Following David Lewin's [11] and Henry Klumpenhouwer's [12] original group-theoretical description, theoretical studies have mostly focused until now on the underlying algebra dealing with the automorphisms of the T/I group or of the more general T/M affine group [11,13].

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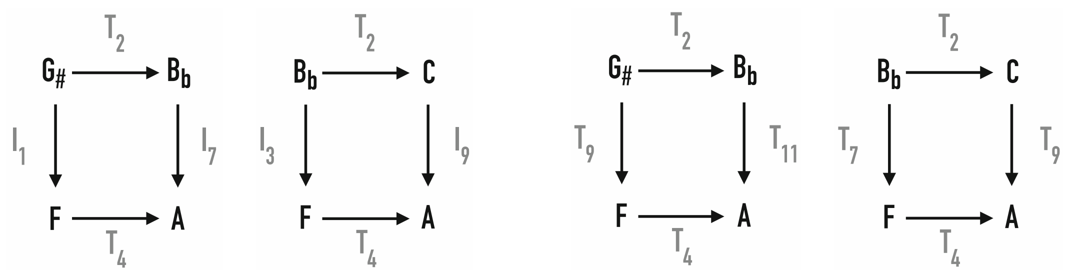
This enables one to define the main notions of positive and negative isographies, a notion which can easily be extended to more isographic relations by taking into account the group generated by affine transformations, together with high-order isographies via the recursion principle. This surely provides a computational framework for building networks of networks (and so on, in a recursive way) but it somehow misses the interplay between algebra and graph theory, which is well captured by category theory and the functorial approach. Since a prominent feature of K-nets is their ability of instantiating an in-depth multi-level model of musical structure, category theory seems nowadays the most suitable mathematical framework to capture this recursive potentiality of the graph-theoretical construction.

Following the very first attempt at formalizing K-nets as limits of diagrams within the framework of denotators [14], we propose in this article a categorical construction, called poly-K-nets and taking values in **Sets** (henceforth PK-nets). This construction generalizes the notion of K-nets in various ways. K-nets theory usually distinguishes two main types of isography: positive and negative isography. Positive isography corresponds to two directed graphs having the same transpositions, but having inversion operators that differ by a constant value. Negative isography corresponds to two directed graphs in which the subscripts of the transpositional and inversional operators sum to constant values, respectively equal to 0 and different from 0. Figure 1 shows four K-nets, the first two of which (K-nets (a) and (b)) are positively isographic, whereas the other two (K-nets (c) and (d)) have the same node-content as the K-nets (a) and (b), but their arrows are labeled in such a way that they are not isographic.



(a) (b) (c) (d)

Fig. 1. Four K-nets, the first two of which ((a) and (b)) are (positively) isographic since the transpositions in K-net (b) are the same as those in (a), while the respective inversions in (b) have subscripts 2 more than those of (a). The second two of the four K-nets ((c) and (d)) do not have any isographic relation, although their node content is the same as K-nets (a) and (b)

This clearly suggests that the concept of isography is highly dependent on the selection of specific transformations and asks for more general settings in which isographic networks remain isographic when the nodes are preserved and the family of transformations between the nodes is changed. This has been one of the main motivations for introducing PK-nets as natural extensions of K-nets (Sect. 2). Moreover, PK-nets enable one to compare in a categorical framework digraphs with different cardinalities, a fact that occurs very often in many musically interesting analytical situations. They also realize Lewin's intuition that transformational networks do not necessarily have groups as support spaces, since one can define PK-nets in any category. In this article we focus on the category $\mathbf{PKN_R}$ of PK-nets of constant form R whose (co)limit structure is described in Theorem 1. Morphisms of $\mathbf{PKN_R}$ correspond to a change of a musical context, as in the case of transformations between elements of a cyclic group. Morphisms of PK-nets clearly show the structural role of natural transformations by which one can generalize the case of isographic K-nets (Sect. 3). In particular they enable us to define K-nets which remain isographic for any choice of transformations between the original pitch-classes (or pitch-class sets). The problem of determining the morphisms between PK-nets naturally leads to the topos-theoretic formalization of the main construction we have introduced in this paper, as it will be detailed in the final section.

2 From K-Nets to PK-nets

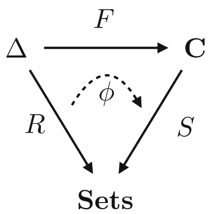
We begin this section by giving the definition of a PK-net. In the rest of this paper, all functors are covariant.

2.1 Definition of PK-nets

Let $\mathbf{C}$ be a category, and S a functor from $\mathbf{C}$ to the category **Sets** with non empty values. Such a functor corresponds to an action of the category $\mathbf{C}$ on the disjoint union of the sets $S(c)$ for the objects $c \in \mathbf{C}$ [15].

Definition 1. *Let Δ be a small category and R a functor from Δ to* **Sets** *with non empty values. A PK-net of form R and with support S is a 4-tuple (R, S, F, ϕ), in which F is a functor from Δ to $\mathbf{C}$, together with a natural transformation $\phi\colon R \to SF$.*

The definition of a PK-net is summed up by the following diagram:



The usual K-nets are a particular case of PK-nets in which

1. $\mathbf{C}$ is the group T/I of transpositions and inversions, considered as a single-object category and the functor $S\colon T/I \to$ **Sets** defines the usual action of T/I on the set $\mathbb{Z}_{12}$ of the twelve pitch classes,
2. Δ is the graph describing the K-nets, the functor R associates the singleton $\{X\}$ to each object X of Δ, and the natural transformation ϕ reduces to a map from the objects of Δ to the image of SF.

Within the framework of denotators, Guerino Mazzola and Moreno Andreatta have proposed in [14] a generalized definition of a K-net as an element of the limit of a diagram R of sets (or modules). We can compare the notion of PK-net with this notion of K-net as follows: if (R, S, F, ϕ) is a PK-net, the functor $\lim\colon \mathbf{Sets}^{\Delta} \to \mathbf{Sets}$ maps the natural transformation $\phi : R \to SF$ to the map $\lim_{\phi}\colon \lim\, R \to \lim\, SF$ from the set of K-nets of form R to the set of K-nets of form SF. Thus a PK-net does not represent a unique K-net, but the set of K-nets associated to SF and a way to 'name' them (via $\lim_{\phi}$) by the K-sets of R. The PK-net reduces to a K-net if $R(X)$ is a singleton for each object X of Δ.

Remark 1. In Definition 1 and in the sequel, the category **Sets** can be replaced by any category H to obtain the notion of a P(oly-)K-net in H (developed in a paper in preparation). Let us note two interesting cases:

i. H is a category of presheaves: the networks considered in [14] correspond to PK-nets in $\mathbf{Mod}^{@}_{\mathbf{Z}}$ of the form (R, S, F, ϕ) where $\mathbf{C} = T/I$ (p. 104), and to PK-nets in a category of presheaves, with F an identity, which they call "network of networks" (pp. 106–107) and they show how to define iterated networks using powerset constructions.
ii. H is the category $Diag(\mathbf{C})$ of diagrams in a category $\mathbf{C}$; in particular, if $\mathbf{C}$ is the category $\mathbf{PKN_R}$ of morphisms of PK-nets, a PK-net in $Diag(\mathbf{PKN_R})$ gives a notion of "PK-net of PK-nets", and by iteration of the $Diag$ construction we can define a hierarchy of PK-nets of increasing orders (without recourse to powerset constructions as in [14]).

In the more general case, the category $\mathbf{C}$ provides the musically relevant transformations, whose action on some musical objects (pitch classes, chords, etc.) is given by the functor S. The form of a PK-net, given by the functor R, provides a diagram of elements which are identified to the musical objects by the functor F and the natural transformation ϕ. The definition of PK-nets provides advantages over K-nets, some of which are detailed in the examples below.

Example 1. The functor R allows one to consider sets $R(X)$, $X \in \Delta$, whose cardinality $|R(X)|$ is greater than 1.

For example, let $\mathbf{C}$ be the group T/I, considered as a single-object category, and consider its natural action on the set $\mathbb{Z}_{12}$ of the twelve pitch classes, which defines a functor $S\colon T/I \to \mathbf{Sets}$. Let Δ be the interval category, i.e. the category with two objects X and Y and precisely one non-trivial morphism $f\colon X \to Y$, and consider the functor $F\colon \Delta \to T/I$ which sends f to T_4.

Consider now a functor $R\colon \Delta \to \mathbf{Sets}$ such that $R(X) = \{x_1, x_2, x_3\}$ and $R(Y) = \{y_1, y_2, y_3, y_4\}$, and such that $R(f)(x_i) = y_i$, for $1 \leq i \leq 3$. Consider the natural transformation ϕ such that $\phi(x_1) = 0$, $\phi(x_2) = 4$, $\phi(x_3) = 7$, and $\phi(y_1) = 4$, $\phi(y_2) = 8$, $\phi(y_3) = 11$, and $\phi(y_4) = 2$. Then (R, S, F, ϕ) is a PK-net of form R and support S which describes the transposition of the C major triad to the E major triad subset of the dominant seventh E^7 chord. This functorial construction is shown in Fig. 2.

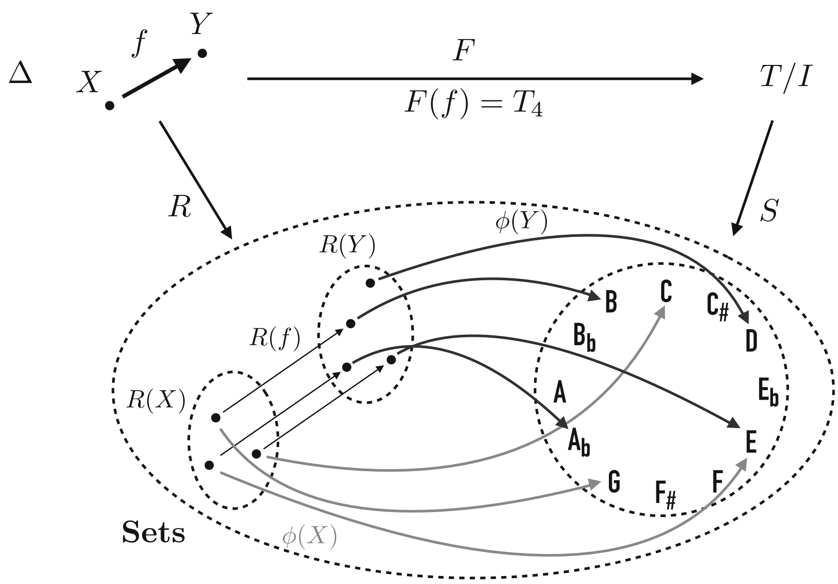


Fig. 2. Diagram showing the functorial construction underlying the definition of PK-nets as applied to the Example 1

Example 2. The definition of PK-nets allows one to consider networks of greater generality than the usual K-nets.

Consider the category $\mathbf{C} = T/I$ and the functor $S \colon T/I \to \mathbf{Sets}$ as in the previous example, and consider the category Δ with one single-object X and one non-trivial morphism $f \colon X \to X$ such that $f^2 = id_X$. Consider now the functor $F \colon \Delta \to T/I$ which sends f to $I_1 \in T/I$.

If we restrict ourselves to functors $R \colon \Delta \to \mathbf{Sets}$ such that $R(X)$ is a singleton, then there exists no natural transformation $\phi \colon R \to SF$, since the equation $\phi(x) = 1 - \phi(x)$ has no solution in $\mathbb{Z}_{12}$. However, it is possible to consider a functor R such that $R(X) = \{x_1, x_2\}$, with $R(f)(x_1) = x_2$ and *vice-versa*, and a natural transformation ϕ which sends x_1 to 0 and x_2 to 1. Then (R, S, F, ϕ) is a valid PK-net of form R and support S.

Example 3. In addition to groups, the definition of PK-nets allows one the use of any category $\mathbf{C}$. Thus, PK-nets can describe networks of musical objects being transformed by the image morphisms of $\mathbf{C}$ through S.

Consider for example the monoid $M = \{(u, 2^v) \mid u \in \mathbb{Z}[\frac{1}{2}], u \geq 0, v \in \mathbb{Z}\}$ whose multiplication law is given by the following equation:

$$(u_1, 2^{v_1}) * (u_2, 2^{v_2}) = (u_2 + u_1 \cdot 2^{v_2}, 2^{v_1+v_2}). \tag{1}$$

The monoid M is generated by the elements $a = (1, 1)$ and $b = (0, 1/2)$ and has presentation $M = \langle a, b \mid a^2 b = ba \rangle$. It can be considered as a discrete monoid version of Lewin's continuous group of time-span transformations.

Recall that a time-span, in the sense of Lewin [2], is a pair (t, d), where $t \in \mathbb{R}$ is called the *onset* of the time-span, and $d \in \mathbb{R}, d > 0$, is called its *duration*. Consider the set $T = \{(t, 2^\delta) \mid t \in \mathbb{Z}[\frac{1}{2}], \delta \in \mathbb{Z}\}$ of *dyadic time-spans*, equipped with the action $M \times T \to T$ given by the following equation:

$$(u, 2^v) \cdot (t, 2^\delta) = (t + 2^\delta \cdot u, 2^{\delta+v}) \quad (2)$$

This action defines a functor $S\colon M \to \mathbf{Sets}$. Let then the category $\mathbf{C}$ be the monoid M, and S be the functor as defined above. Let Δ be the interval category, and consider the functor $F\colon \Delta \to M$ which sends the non-trivial morphism $f\colon X \to Y$ to $(2, 1/2) \in M$. Consider the functor $R\colon \Delta \to \mathbf{Sets}$ such that $R(X)$ and $R(Y)$ are singletons, and the natural transformation $\phi\colon R \to SF$ which sends $R(X)$ to $\{(1, 1)\} \subset T$ and $R(Y)$ to $\{(3, 1/2)\} \subset T$. Then the PK-net (R, S, F, ϕ) describes the transformation of the dyadic time-span $(1, 1)$ into $(3, 1/2)$ by the element $(2, 1/2)$ of the monoid M. Observe that, contrary to the group of Lewin, no element of M can describe the transformation of the time-span $(3, 1/2)$ to the time-span $(1, 1)$, since the action of the elements of M only translates time-spans by a positive amount of time.

2.2 The Category of PK-nets with Constant Form

Let Δ be a small category and R a functor from Δ to $\mathbf{Sets}$ with non empty values. One can form a category $\mathbf{PKN_R}$ of PK-nets of constant form R, according to the following definition.

Definition 2. *The category* $\mathbf{PKN_R}$ *has*

- *objects which are PK-nets* (R, S, F, ϕ) *of form* $R\colon \Delta \to \mathbf{Sets}$*, and*
- *morphisms between PK-nets* (R, S, F, ϕ) *and* (R, S', F', ϕ') *which are pairs* (L, λ)*, where* L *is a functor from* $\mathbf{C}$ *to* $\mathbf{C}'$*, and* λ *is a natural transformation from* S *to* $S'L$ *such that* $\phi' = (\lambda F) \circ \phi$*.*

The following theorem describes part of the structure of $\mathbf{PKN_R}$. We omit here the proof, which is rather technical.

Theorem 1. *The category* $\mathbf{PKN_R}$ *is complete, and has all connected colimits.*

Musically speaking, a morphism (L, λ) of PK-nets of constant form R can be interpreted as a change of musical context, through the change of functor from S to S'. We give some examples of such morphisms below.

Example 4. Let the category $\mathbf{C}$ be the cyclic group $G = \mathbb{Z}_{12}$, generated by an element t of order 12. Consider the action of t on the set $\mathbb{Z}_{12}$ of the twelve pitch classes given by $t \cdot x = x + 1, \forall x \in \mathbb{Z}_{12}$. This defines a functor $S : G \to \mathbf{Sets}$, which corresponds to the traditional action of $\mathbb{Z}_{12}$ by transpositions by semitones. Consider now the action of t on the set $\mathbb{Z}_{12}$ of the twelve pitch classes given by $t \cdot x = x + 5$, $\forall x \in \mathbb{Z}_{12}$. This defines another functor $S' : G \to \mathbf{Sets}$, which corresponds to the action of $\mathbb{Z}_{12}$ by transpositions by fourths. Let L be the automorphism of G which sends $t^p \in G$ to t^{5p} in G, $\forall p \in 1, \ldots, 12$, and let λ be the identity function on the set $\mathbb{Z}_{12}$. It is easily checked that λ is a natural transformation from S to $S'L$.

Let (R, S, F, ϕ) be the PK-net wherein Δ is the interval category, F is the functor from Δ to G which sends the non-trivial morphism $f\colon X \to Y$ of Δ

to t^{10} in G, R is the functor from Δ to **Sets** which sends the objects of Δ to singletons, and ϕ is the natural transformation which sends $R(X)$ to $\{0\} \subset \mathbb{Z}_{12}$ and $R(Y)$ to $\{10\} \subset \mathbb{Z}_{12}$. This PK-net describes the transformation of C to $B\flat$ by a transposition of ten semitones.

By the morphism of PK-nets (L, λ) introduced above, one obtains a new PK-net (R, S', F', ϕ'), wherein the functor $F' = LF$ sends $f \in \Delta$ to $t^2 \in G$, and the natural transformation $\phi' = (\lambda F) \circ \phi$ sends $R(X)$ to $\{0\} \subset \mathbb{Z}_{12}$ and $R(Y)$ to $\{10\} \subset \mathbb{Z}_{12}$. This new PK-net describes the transformation C to $B\flat$ by a transposition of two fourths.

Example 5. We give here an example of a morphism between a PK-net of beats and a PK-net of pitches. Figure 3 shows a passage from the final movement of Chopin's Piano Sonata Nr. 3, op. 58 in B minor, wherein the initial six-notes motive is raised by two semitones every half-bar.

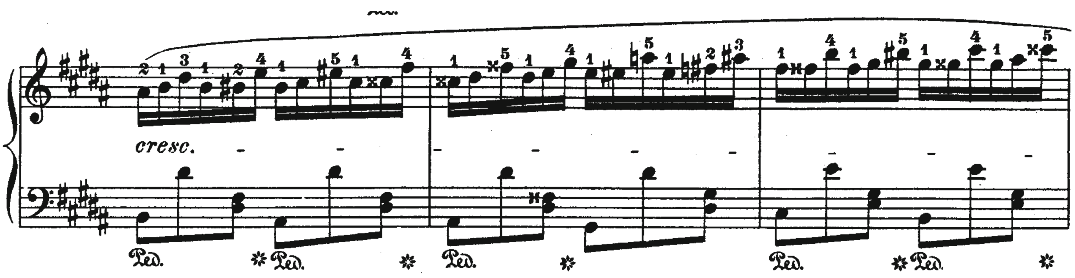


Fig. 3. A passage from the final movement of Chopin's Piano Sonata Nr. 3 op. 58

Let the category **C** be the infinite cyclic group $G = \mathbb{Z}$, generated by an element t. Let $\mathbb{Z}$ be the set of equidistant beats of a given duration and consider the action of t on this set given by $t \cdot x = x + 1$, $\forall x \in \mathbb{Z}$. This action defines a functor $S : G \to$ **Sets**.

Let the category $\mathbf{C}'$ be the cyclic group $G' = \mathbb{Z}_{12}$, generated by an element t' of order 12. Consider the set $U = \{u_i \mid i \in \mathbb{Z}_{12}\}$ of the twelve successive transpositions of the pitch class set $u_0 = \{10, 11, 0, 3, 4\}$, and consider the action of t' on U given by $t' \cdot u_i = u_{i+1 \text{ (mod 12)}}$, $\forall i \in \mathbb{Z}_{12}$. This defines a functor $S' : G' \to$ **Sets**.

Let (R, S, F, ϕ) be the PK-net wherein

- Δ defines the order of the ordinal number 4 (whose objects are labelled X_i),
- F is the functor from Δ to G which sends the non-trivial morphisms $f_{i,i+1} : X_i \to X_{i+1}$ of Δ to t in G,
- R is the functor from Δ to **Sets** which sends the objects X_i of Δ to singletons $\{x_i\}$, and
- ϕ is the natural transformation which sends $R(X_i)$ to $\{i\} \subset \mathbb{Z}$.

This PK-net describes the successive transformations of the initial set by translation of one half-bar in time. Let (R, S', F', ϕ') be the PK-net wherein

- F' is the functor from Δ to G' sending the non-trivial morphisms $f_{i,i+1} \colon X_i \to X_{i+1}$ of Δ to t'^2 in G',
- ϕ' is the natural transformation which sends $R(X_i)$ to $\{u_{2i}\} \subset U$.

This PK-net describes the successive transformations of the initial set $\{10, 11, 0, 3, 4\}$ by transpositions of two semitones.

Consider the functor $L \colon G \to G'$ which sends t to t'^2, together with the natural transformation $\lambda \colon \mathbb{Z} \to U$ given by $\lambda(x) = u_{2x \pmod{12}}$. As compared to a traditional K-net approach which would typically focus on the pitch-class set transformation, the morphism of PK-nets (L, λ) allows us to describe the relation between the translation in time and the transposition in pitch.

Given the knowledge of two functors $S \colon \mathbf{C} \to \mathbf{Sets}$ and $S' \colon \mathbf{C}' \to \mathbf{Sets}$, and a functor $L \colon \mathbf{C} \to \mathbf{C}'$, there may not always exist a natural transformation $\lambda \colon S \to S'L$. The following theorem gives a sufficient condition on S for the existence of the natural transformation $\lambda \colon S \to S'L$.

Theorem 2. *Let $S \colon \mathbf{C} \to \mathbf{Sets}$, $S' \colon \mathbf{C}' \to \mathbf{Sets}$, and $L \colon \mathbf{C} \to \mathbf{C}'$ be three functors, where S' has non empty values. If S is a representable functor, then there exists at least one natural transformation $\lambda \colon S \to S'L$.*

Proof. If S is a representable functor, then there exists a natural isomorphism $\mu \colon S \to \mathrm{Hom}(c, -)$ for some object c of $\mathbf{C}$. We therefore need to prove that there exists at least one natural transformation $\lambda' \colon \mathrm{Hom}(c, -) \to S'L$, as the composition $\lambda' \circ \mu$ will give the desired natural transformation λ. By Yoneda Lemma, the natural transformations from $\mathrm{Hom}(c, -)$ to $S'L$ are in bijection with the elements of $S'L(c)$, thus there exists at least one λ' since $S'L(c)$ is supposed to be non empty. □

An immediate corollary of this result is that, given a PK-net (R, S, F, ϕ) where S is a Generalized Interval System (GIS), a functor $S' \colon \mathbf{C}' \to \mathbf{Sets}$, and a functor $L \colon \mathbf{C} \to \mathbf{C}'$, one can always form a new PK-net $(R, S', F' = LF, \phi')$. Indeed, from a result of Vuza [16] and Kolman [8], a GIS is known to be equivalent to a simply transitive group action on a set, which is in turn equivalent to a representable functor from the group (as a single-object category) to **Sets**. The previous theorem can then be used to form the new PK-net.

Note that given a functor $S \colon \mathbf{C} \to \mathbf{Sets}$ and a functor $L \colon \mathbf{C} \to \mathbf{C}'$, it is known that there always exists a functor $S_K \colon \mathbf{C}' \to \mathbf{Sets}$ and a natural transformation $\kappa \colon S \to S_K L$ obtained by the Kan extension [17] of S along L. Any other natural transformation $\lambda \colon S \to S'L$, where S' is a functor from $\mathbf{C}'$ to **Sets**, factors through it.

3 Application of PK-net Morphisms to Isographic Networks

We have seen previously that, given two functors $S \colon \mathbf{C} \to \mathbf{Sets}$ and $S' \colon \mathbf{C}' \to \mathbf{Sets}$, a morphism of PK-nets is a pair (L, λ) where L is a functor from $\mathbf{C}$ to $\mathbf{C}'$, and λ is a natural transformation from S to $S'L$ such that

$\phi' = (\lambda F) \circ \phi$. One notable feature which is directly derived from the definition of the natural transformation λ in a PK-Net morphism is that, given two objects X and Y in $\mathbf{C}$, and two elements $x \in S(X)$, $y \in S(Y)$ such that $y = S(f)(x)$ for some morphism $f \in \mathbf{C}$, we have

$$\lambda(y) = \lambda(S(f)(x)) = S'L(f)(\lambda(x)), \tag{3}$$

In other words, whatever the transformation f in $\mathbf{C}$ which relates the elements x and y, their images by λ are related by the image transformation $L(f)$.

This property is all the more interesting in the case $S = S'$, which covers the case of isomorphic networks (see below). However, in the general case, the problem of determining the existence of a natural transformation $\lambda\colon S \to SL$ given the functors L and S has no obvious solution. It can nevertheless be solved for some particular cases: we consider here the case when $\mathbf{C}$ is a "generalized" group of transpositions and inversions, with an application to isographic networks.

Let $\mathbf{C}$ be the dihedral group D_{2n} of order $2n$ whose presentation is given by $\langle T, I \mid T^n = I^2 = ITIT^{-1} = 1\rangle$. By analogy with the T/I group, the elements of D_{2n} are designated by $T_n = T^n$, and by $I_n = T^n I$. Consider the set $\mathbb{Z}_n$ of pitch classes in n-equal temperament (n-TET), equipped with the action of D_{2n} given by $T \cdot x = x + 1$, and $I \cdot x = -x$, for all $x \in \mathbb{Z}_n$. This defines a functor $S\colon \mathbf{C} \to \mathbf{Sets}$. The following theorem establishes the existence of natural transformations $\lambda\colon S \to SL$, for a given automorphism L of D_{2n}.

Theorem 3. *Let L be an automorphism of $\mathbf{C} = D_{2n}$. Then:*

- *if n is even, there exists either 0 or 2 natural transformations $\lambda\colon S \to SL$.*
- *if n is odd, there exists exactly one natural transformation $\lambda\colon S \to SL$.*

Proof. From known results about dihedral groups, an automorphism L of D_{2n} sends, for all $p \in \mathbb{Z}_n$, the elements $T_p \in D_{2n}$ to T_{kp}, and the elements $I_p \in D_{2n}$ to I_{kp+l}, where $k, l \in \mathbb{Z}_n$, with $\gcd(k, n) = 1$. Assume that there exists a natural transformation $\lambda\colon S \to SL$, which defines (by an abuse of notation) a function $\lambda\colon \mathbb{Z}_n \to \mathbb{Z}_n$. Given any element x of the set $\mathbb{Z}_n$, the definition of a natural transformation imposes $T_{kp} \cdot \lambda(x) = \lambda(T_p \cdot x)$ for all $p \in \mathbb{Z}_n$, which leads to the equation $kp + \lambda(x) = \lambda(p + x)$. By setting $x = 0$, we have that $\lambda(p) = kp + \lambda(0),$ for all $p \in \mathbb{Z}_n$. Similarly, given any element x of the set $\mathbb{Z}_n$, the definition of λ imposes the equation $I_{kp+l} \cdot \lambda(x) = \lambda(I_p \cdot x)$, which leads to $kp + l - \lambda(x) = \lambda(p - x)$, for all $p \in \mathbb{Z}_n$. We therefore obtain a condition on $\lambda(0)$ given by the equation $kp + l - \lambda(0) = kp + \lambda(0)$, which reduces to $l - \lambda(0) = \lambda(0)$. If n is odd, this equation has exactly one solution, given by $\lambda(0) = l/2$ if l is even, and by $\lambda(0) = (n + l)/2$ if l is odd. If n is even and l is odd, then the equation has no solution. Finally, if n and l are even, this equation has two solutions $\lambda(0) = l/2$ and $\lambda(0) = (n + l)/2$. □

We now give an application to isographic networks. We have previously introduced two isographic K-nets (see the Fig. 1, networks (a) and (b)). These can be considered as PK-nets (R, S, F, ϕ) and $(R, S, F' = LF, \phi')$, wherein the category $\mathbf{C}$ corresponds to the usual T/I group, the functor $S\colon \mathbf{C} \to \mathbf{Sets}$ corresponds to the usual action of T/I on the set $\mathbb{Z}_n$ of pitch classes, and the functor

$L\colon T/I \to T/I$ is the automorphism which sends $T_p \in T/I$ to T_p, and $I_p \in T/I$ to I_{p+2}. Figure 1 presents one PK-net (R, S, F'', ϕ'') with the same pitch classes, wherein the functor F'' labels the arrows with transpositions (Fig. 1, networks (c)). Figure 1 also shows the PK-net (b) where arrows are labelled with transpositions: the two PK-nets (c) and (d) of Fig. 1 are not isographic, which could be deduced from the fact that, in this particular case, we have $F'' = LF''$.

By the previous theorem, there exists two natural transformations from S to SL, given by the functions $\lambda_1(x) = x+1$ and $\lambda_2(x) = x+7$, for $x \in \mathbb{Z}_{12}$. By the PK-net morphisms (L, λ_1) and (L, λ_2) applied to (R, S, F, ϕ), two new PK-nets are obtained, which are represented in Fig. 4. The reader can verify that, for any other choice of transformations between the original pitch-classes, these PK-nets remain isographic to the initial one.

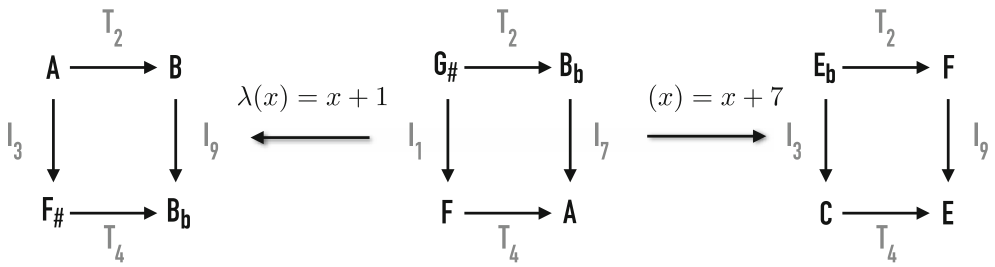


Fig. 4. Isographic PK-nets

4 PK-nets and Topoi

It is a well-known result that for any small category $\mathbf{C}$, the category of functors $\mathbf{Sets}^{\mathbf{C}}$ is a topos. The category $\mathbf{Sets}^{\mathbf{C}}$ therefore has a subobject classifier Ω, and for any subobject $A \in \mathbf{Sets}^{\mathbf{C}}$ of an object $B \in \mathbf{Sets}^{\mathbf{C}}$, there exists a *characteristic map* $\chi_A\colon B \to \Omega$. Topoi have found applications in music theory, for example in the work of Mazzola [5] and more recently in the work of Noll, Fiore and Satyendra [6,9].

In the context of PK-nets, the characteristic map can be considered as a morphism of PK-nets. Let (R, S, F, ϕ) be a PK-net of form R and of support $S \in \mathbf{Sets}^{\mathbf{C}}$. Let A be a subobject of S: this defines a characteristic map $\chi_A : S \to \Omega$ which is equivalent to a morphism of PK-nets $(\mathrm{id}_{\mathbf{C}}, \chi_A)$. This morphism thus defines a new PK-net (R, Ω, F, ϕ'). We detail below a concrete example based on the monoid introduced in Example 3.

Example 6. Consider the monoid $M = \{(u, 2^v) \mid u \in \mathbb{Z}[\frac{1}{2}], u \geq 0, v \in \mathbb{Z}\}$ introduced above, acting on the set $T = \{(t, 2^\delta) \mid t \in \mathbb{Z}[\frac{1}{2}], \delta \in \mathbb{Z}\}$ of *dyadic time-spans*, which defines a functor $S \in \mathbf{Sets}^M$. For a given $k \in \mathbb{Z}[\frac{1}{2}]$, the set $T_k = \{(t, 2^\delta) \mid t \in \mathbb{Z}[\frac{1}{2}], t \geq k, \delta \in \mathbb{Z}\}$ equipped with the same action of M is a subobject A of S. Let $\mathbb{Z}[\frac{1}{2}]_{\geq 0}$ be the set $\{p \in \mathbb{Z}[\frac{1}{2}] \mid p \geq 0\}$. The reader can verify that the subobject classifier of $\mathbf{Sets}^M$ is the union of $\mathbb{Z}[\frac{1}{2}]_{\geq 0}$ and a singleton $\{x\}$, equipped with the following action of the generators of M

$$(1,1)\cdot p = \begin{cases} p-1 \text{ if } p \geq 1 \\ 0 \qquad \text{otherwise} \end{cases} \tag{4}$$

and

$$(0,1/2)\cdot p = 2p \tag{5}$$

for all $p \in \mathbb{Z}[\frac{1}{2}]_{\geq 0}$, and where the singleton $\{x\}$ is a fixed point by the action of M. The characteristic map χ_A then sends any element $(t, 2^\delta) \in T$ to $\dfrac{k-t}{2^\delta}$ if $k \geq t$, or 0 otherwise. In other words, the characteristic map measures the time period $t-k$ from a time-span $(t, 2^\delta)$ in units of 2^δ. Consider for example the PK-net (R, S, F, ϕ) defined in Example 3, and the subobject A defined from the set $T_{9/2}$. The morphism of PK-nets (id_M, χ_A) sends the PK-net (R, S, F, ϕ) to the new PK-net (R, Ω, F, ϕ'), where the natural transformation $\phi' = \chi_A \circ \phi$ sends $R(X)$ to $\{7/2\}$ and $R(Y)$ to $\{3\}$.

5 Conclusions and Perspectives

We have presented a generalized framework of Klumpenhouwer Networks based on category theory. In order to show the richness of this new framework we have chosen some pedagogical examples by focusing on the concept of set-valued PK-nets of constant form R and the category $\mathbf{PKN_R}$ they form. This construction stresses the categorical description of musical objects based on the synergy between algebra and graph-theory, as it is the case for Klumpenhouwer Networks and other constructions within transformational theory. The category $\mathbf{PKN_R}$ is the category of objects under R of the category *Diag*(**Sets**) of diagrams in Sets which we denote by **PKN**; the morphisms of **PKN** are all the PK-nets (where the diagram Δ and/or the functor R may vary). Our current research addresses the question of the musical relevance of **PKN**, of *Span*(**PKN**), and of different constructions based on **PKN**, such as the construction of PK-nets of higher order (see Remark 1), or the characterization and musical applications of a notion of PK-homographies generalizing the problem of isographies.

We are also planning to better study some computational aspects underlying PK-nets, once they are integrated into some programming languages for computer-aided music theory and analysis, such the MathTools environment in OpenMusic [18]. This will probably enable one to better understand the cognitive and perceptual relevance of transformational theory and contribute to the programmatic research area of a categorical approach to creativity [19].

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