On two open mathematical problems in music theory: Fuglede spectral conjecture and discrete phase retrieval

Algebra Seminar, TU Dresden, 29 November 2012

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The mixed research lab UMR9912 brings together the CNRS, the UPMC, the French Ministry of Culture, and IRCAM around the theme of multidisciplinary research on sciences and technologies for music and sound.

The lab is associated with the CNRS Institutes for Information Sciences and Technologies (INS2I), for Engineering and Systems Sciences (INSIS), for Humanities and Social Sciences (INSHS) and of Biological Sciences (INSB). It is also a part of the UPMC’s faculty of engineering (UFR 919) in the Research Pole for Modeling and Engineering.

Director: Gérard Assayag (IRCAM)
Deputy Director: Hugues Vinet (IRCAM)

As of January 1, 2012, the laboratory consisted of the following teams:

- Instrumental Acoustics
- Acoustic and Cognitive Spaces
- Perception and Sound Design
- Analysis/Synthesis
- Music Representation
- Analysis of Musical Practices
- Real-Time Musical Interactions
- IRCAM Resource Center
# Mathematics/Music…a recent history!


- **2000-2003**: International Seminar on *MaMuTh (Perspectives in Mathematical and Computational Music Theory)* (Mazzola, Noll, Luis-Puebla eds, epOs, 2004)

- **2003**: *The Topos of Music* (G. Mazzola et al.)

- **2004-…**: *MaMuX Seminar at Ircam*

- **2004-…**: *mamuphi Seminar (Ens/Ircam)*

- **2006**: *Mathematical Theory of Music* (F. Jedrzejewski), Coll. ‘Musique/Sciences’

- **2007**: *La vérité du beau dans la musique* (G Mazzola), Coll. ‘Musique/Sciences’

- **2007**: *Journal of Mathematics and Music* (Taylor & Francis) and SMCM

- **2007**: First MCM 2007 (Berlin) and Proceedings by Springer

- **2007-…**: AMS Special Session on Mathematical Techniques in Musical Analysis


- **2009**: MCM 2009 (Yale University) and Proceedings by Springer

- **2010**: Mathematics Subject Classification : 00A65 Mathematics and music

- **2011**: MCM 2011 (Ircam, 15-17 June 2011) and Proceedings LNCS Springer

- **2013**: MCM 2013 (McGill University, Canada, 12-14 June 2013)
“…the role of mathematics, which at the beginning was considered as a part of physics, has become – thanks to modern mathematics – a kind of substitution of philosophy with respect to the creation of concepts” (Alain Connes).

http://agora.ircam.fr/971.html?event=1002
The double movement of a ‘mathemusical’ activity
Some examples of ‘mathemusical’ problems

⇒ http://repmus.ircam.fr/moreno/hdr

- The construction of Tiling Rhythmic Canons
- The Z relation and the theory of homometric sets
- Set Theory and Transformational Theory
- Diatonic Theory and Maximally-Even Sets
- Periodic sequences and finite difference calculus
- Block-designs and algorithmic composition
Periodic sequences and finite difference calculus

Every periodic sequence $f$ can be decomposed in a unique way as a sum $f_1 + f_2$ of a reducible sequence $f_1$ and a reproducible sequence $f_2$.

Reducible sequences: $\exists \ k \geq 1 \text{ such that } D^k f = 0$

Reproducible sequences: $\exists \ k \geq 1 \text{ such that } D^k f = f$

- **Decomposition theorem**
  (Vuza & Andreatta, *Tatra M.*, 2001)

Block-designs and algorithmic composition

Fano plane

Tom Johnson’s *Twelve for piano*, part III, based on (12,4,3)

Block-design (11,6,3)

Algebraic structures and geometrical transformations

\[ Z_{12} = \langle T_k \mid (T_k)^{12} = T_0 \rangle \]

\[ T_k : x \mapsto x + k \text{ mod } 12 \]

\[ D_{12} = \langle T_k, I \mid (T_k)^{12} = I^2 = T_0, ITI = I(IT)^{-1} \rangle \]

\[ I : x \mapsto -x \text{ mod } 12 \]
A group action based classification of musical structures

Paradigmatic architecture

A second algebraic invariant: the interval content

\[ IC_A(k) = \text{Card}\{(x, y) \in A \times A | x + k = y\} \]

\[ IC_A = [4, 3, 2, 3, 2, 1] = [4, 3, 2, 3, 2, 1] = IC_{A'} \]

Babbitt’s Hexachord Theorem:
A hexacord and its complement have the same interval content

(Proofs by Wilcox, Ralph Fox (?), Chemillier, Lewin, Mazzola, Schaub, …, Amiot, …)
From Euler’s *Speculum musicum* to the *Tonnetz*

\[
\rho = \langle L, R \mid L^2 = (LR)^{12} = 1, LRL = L(LR)^{-1} \rangle
\]

\[
\rho \subseteq C_{\text{Sym}}(D_{12})
\]

\[
D_{12} \subseteq C_{\text{Sym}}(\rho)
\]

Hamiltonian cycles and spatial computing

L. Bigo, PhD (ongoing), Ircam / University of Paris 12
⇒ http://www.lacl.fr/~lbigo/louis_bigo
Extract of the 2\textsuperscript{nd} movement of the Symphony No. 9
(L. van Beethoven)
Extract of the 2\textsuperscript{nd} movement of the Symphony No. 9
(L. van Beethoven)

\begin{hexgrid}
\cell{C\#}{D}{Eb}{E}{F}{D}
\cell{G}{B}{Bb}{E}{C}{C\#}
\cell{E}{C\#}{B}{F}{A}{D}
\cell{F}{G\#}{F\#}{D}{G}{G\#}
\cell{Bb}{A}{C\#}{F\#}{D}{Bb}
\cell{A}{Bb}{E}{G}{C}{F\#}
\cell{D}{G\#}{B}{C}{E}{F}
\cell{B}{G}{E}{E}{A}{D}
\cell{G}{A}{D}{F\#}{C}{Bb}
\cell{F\#}{C\#}{E}{Bb}{F}{G\#}
\cell{F\#}{D}{B}{G}{C}{Bb}
\cell{F\#}{C\#}{E}{Bb}{F}{G\#}
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Tiling the line (or the circle) with translates of one tile
Some examples of traditional rhythms

El cinquillo

El trecillo
A very particular rhythmic pattern

Bembé

Rhythmic Tiling Canons as group factorisations

$A < Z_n$

$B < Z_n$

$ACZ_n$

$B < Z_n$

$ACZ_n$

$B < Z_n$

$B$ periodic subset

$Z_n = A \oplus B$
Factorisations without inner periodicity (Vuza Canons)
A Tiling rhythmic canon with unpredictable melody
In any simple lattice tiling of the $n$-dimensional Euclidean space by unit cubes, some pair of cubes must share a complete $(n-1)$-dimensional face.

A Vuza Canon is a factorization of a cyclic group in a direct sum of two non-periodic subsets.

$$\mathbb{Z}/n\mathbb{Z} = \mathbb{R} \oplus S$$

Hajós groups (good groups)

$$\mathbb{Z}/n\mathbb{Z} \text{ with } n \in \{p^\alpha, pq, pq, p^2q, p^2qr, p^2qrs\}$$

where $p, q, r, s$, are distinct prime numbers

Non-Hajós group (bad groups)

72
108 120 144 168 180
200 216 240 252 264 270 280 288
300 312 324 336 360 378 392 396
400 408 432 440 450 456 468 480
500 504 520 528 540 552 560 576 588 594
600 612 616 624 648 672 675 680 684 696
700 702 720 728 744 750 756 760 784 792
800 810 816 828 864 880 882 888...

(Sloane’s sequence A102562)

From Minkowski Conjecture to Hajós groups

**Minkowski Conjecture (1907)**
In any simple lattice tiling of the $n$-dimensional Euclidean space by unit cubes, some pair of cubes must share a complete $(n-1)$-dimensional face.

**Hajós Theorem (1941)**
Given a finite abelian group $G$ and given $n$ elements $a_1, a_2, ..., a_n$ of $G$. If $G$ is a direct product of cyclic subsets $A_1, ..., A_n$ where

\[ A_i = \{ e, a_i, a_i^2, \ldots, a_i^{q_i-1} \} \]

with $q_i > 0$ for all $i=1, 2, ..., n$, then $A_k$ is a group for a given $k$.

**Hajós groups (good groups)**

<table>
<thead>
<tr>
<th>Year</th>
<th>Group Description</th>
</tr>
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<tbody>
<tr>
<td>Rédei 1947</td>
<td>$(p, p)$</td>
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<tr>
<td>Hajós 1950</td>
<td>$\mathbb{Z}$ with $n=p^a$</td>
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<tr>
<td>De Bruijn 1953</td>
<td>$(p^a, q)$</td>
</tr>
<tr>
<td>De Bruijn 1953</td>
<td>$(p, q, r)$</td>
</tr>
<tr>
<td>Sands 1957</td>
<td>$(p^2, q^2)$</td>
</tr>
<tr>
<td>Sands 1957</td>
<td>$(p^2, q, r)$</td>
</tr>
<tr>
<td>Sands 1959</td>
<td>$(p, q, r, s)$</td>
</tr>
<tr>
<td>Sands 1962</td>
<td>$(p, 3, 3)$</td>
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<tr>
<td>Sands 1962</td>
<td>$(p, 2, 2, 2)$</td>
</tr>
<tr>
<td>Sands 1962</td>
<td>$(p^2, 2, 2, 2)$</td>
</tr>
<tr>
<td>Sands 1962</td>
<td>$(p^3, 2, 2)$</td>
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<tr>
<td>Sands 1964</td>
<td>$(p, q, 2, 2)$</td>
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<td>Sands 1964</td>
<td>$\mathbb{Z}/p\mathbb{Z}$</td>
</tr>
<tr>
<td>Sands 1964</td>
<td>$\mathbb{Q}/p\mathbb{Z}$</td>
</tr>
</tbody>
</table>

Weak versions of Minkowski Conjecture

Minkowski Conjecture (1907)
In any simple lattice tiling of the \(n\)-dimensional Euclidean space by unit cubes, some pair of cubes must share a complete \((n-1)\)-dimensional face.

The four conditions of Minkowski Conjecture

1. The cubes are translates of each other
2. The translation vectors form a lattice
3. The interiors of the cubes are disjoint
4. Each point of the space which is not on the boundary of any cube is contained in exactly one cube


- True for \(n \leq 6\) (Perron, 1940)
- False for \(n \geq 8\) (Lagarias et Shor, 1992; Mackey, 2000)
- Open for \(n = 7\)

Furtwängler Conjecture (1936) = Minkowski – [3, 4] + new \(k\)-fold tiling condition:

- Each point not lying on the boundary of any cube is contained in exactly \(k\) cubes
- True for \(n \leq 2\) (Hajos 1941)
- False in general, since for every \(k > 1\) and every \(n > 2\) there is a \(k\)-fold tiling of \(n\)-dimensional space by cubes with no shared faces (Szabó 1982).
Minkowski Conjecture (1907)
In any simple lattice tiling of the $n$-dimensional Euclidean space by unit cubes, some pair of cubes must share a complete $(n-1)$-dimensional face.

Hajós quasi-periodic Conjecture (1950)
Every factorisation of a finite abelian group $G = A+B$ is quasi-periodic i.e. $B$ is decomposable in subsets $B_1, \ldots, B_k$ and there exists a periodic subset $\{g_1, \ldots, g_k\}$ of $G$ such that $A+B_i=g_i+A+B_1$.

- False for $\mathbb{Z}_5 \times \mathbb{Z}_{25}$ (Sands, 1974)
- True for $\mathbb{Z}_p \times \mathbb{Z}_p$ (Steinberger, 2008)
- Open in general

S. Stein, S. Szabó: Algebra and Tiling, Carus Math. Mon. 1994
Fuglede Spectral Conjecture

A subset of the \( n \)-dimensional Euclidean space tiles by translation iff it is spectral.

(J. Func. Anal. 16, 1974)

\[ n=1 \]

**Definition 6** A subset \( A \) of some vector space (say \( \mathbb{R}^n \)) is spectral iff it admits a Hilbert base of exponentials, i.e. if any map \( f \in L^2(A) \) can be written

\[ f(x) = \sum_{k} f_k \exp(2i\pi \lambda_k x) \]

for some fixed family of vectors \( (\lambda_k)_{k \in \mathbb{Z}} \) where the maps \( e_k : x \mapsto \exp(2i\pi \lambda_k x) \) are mutually orthogonal (i.e. \( \int_A \overline{e_k} e_j = 0 \) whenever \( k \neq j \)).

**Definition 8.** A subset \( A \in \mathbb{Z} \) is spectral if there exists a spectrum \( \Lambda \subset [0,1] \) (i.e., a subset with the same cardinality as \( A \)) such that \( e^{2i\pi \lambda_i \lambda_j} \) is a root of \( A(X) \) for all distinct \( \lambda_i, \lambda_j \in \Lambda \).

**Example:**

\( A = \{0,1,6,7\} \Rightarrow \Lambda = \{0,1/12,1/2,7/12\} \)

since \( \exp(\pi i), \exp(\pi i/6), \exp(-\pi i/6), \exp(5\pi i/6), \exp(-5\pi i/6) \) are the roots of the associated polynomial \( A(X) = 1 + X + X^6 + X^7 \)

\[ \mathbb{Z}_n = B \oplus A \]
Minkowski/Hajos Problem and Fuglede Conjecture

**Minkowski/Hajos Problem**

In any simple lattice tiling of the $n$-dimensional Euclidean space by unit cubes, some pair of cubes must share a complete $(n-1)$-dimensional face.

  A tiles $\mathbb{Z}_n \Rightarrow pA$ tiles $\mathbb{Z}_n$ when $<p,n>=1$

  T1 + T2 => tile
  Tile => T1

**Fuglede Spectral Conjecture**

A subset of the $n$-dimensional Euclidean space tiles by translation iff it is *spectral*.

  - T1 + T2 => spectral
  - T2 => spectral
  - spectral => T1

Minkowski/Hajos Problem and Fuglede Conjecture?
Polynomial Representations of Tiling Canons

\[ Z_n \rightarrow 1 + X + X^2 + \ldots + X^{n-1} \]

\[ A = \{0, 5, 6, 7\} \rightarrow A(X) = 1 + X^5 + X^6 + X^7 \]

\[ B = \{0, 4, 8\} \rightarrow B(X) = 1 + X^4 + X^8 \]

\[ \Delta_{12} = 1 + X + \ldots + X^{11} = A(X) \times B(X) \mod X^{12} - 1 \]
Using cyclotomic polynomials to build tiling canons

\[ \Phi_n(X) = \frac{\varphi(n)}{\prod_{k=1}^{n} (X - z_k)} \]

\[ X^n - 1 = \prod_{d|n} \Phi_d(X). \]

\[ \Phi_1(X) = X - 1 \quad \Phi_2(X) = 1 + X \quad \Phi_3(X) = 1 + X + X^2 \]
\[ \Phi_4(X) = 1 + X^2 \quad \Phi_5(X) = 1 + X + X^2 + X^3 + X^4 \]
\[ \Phi_6(X) = 1 - X + X^2 \]

\[ \Delta_n = 1 + X + X^2 + ... + X^{n-1} = \prod_{d|n} \Phi_d(X) \]
\[ \Delta_4 = 1 + X + X^2 + X^3 = \Phi_2(X) \times \Phi_4(X) \]

\[ A(x) \times B(x) = (A \oplus B)(x) \equiv 1 + x + \ldots x^{n-1} \pmod{X^n - 1}. \]
Good and bad decompositions

\[ \Delta_n = 1 + X + X^2 + \ldots + X^{n-1} = \prod_{d \mid n, \ d \neq 1} \Phi_d(X) \]

\[ \Phi_2(X) = 1 + X \]
\[ \Phi_3(X) = 1 + X + X^2 \]
\[ \Phi_4(X) = 1 + X^2 \]
\[ \Phi_6(X) = 1 - X + X^2 \]

\[ \Delta_{12} = 1 + X + \ldots + X^{11} = \Phi_2 \times \Phi_3 \times \Phi_4 \times \Phi_6 \times \Phi_{12} \]

\[ A(X) = \Phi_2 \times \Phi_3 \times \Phi_6 \times \Phi_{12} = 1 + X + X^4 + X^5 + X^8 + X^9 \]

\[ B(X) = \Phi_4 = 1 + X^2 \]

\[ S = \{0, 2\} \]
\[ R = \{0, 1, 4, 5, 8, 9\} \]

\[ \Phi_2 \ (X) = 1 + X \]
\[ \Phi_3 \ (X) = 1 + X + X^2 \]

(T1) \( A(1) = 6 = \Phi_2(1) \times \Phi_3(1) = 2 \times 3 \)

(T2) \( \Phi_2 \mid A(X) \) et \( \Phi_3 \mid A(X) \) \( \Rightarrow \) \( \Phi_{2 \times 3} \mid A(X) \)
Coven and Meyerowitz Conditions


There is no loss of generality in restricting attention to translates of a finite set $A$ of nonnegative integers. Then $A(x) = \sum_{a \in A} x^a$ is a polynomial such that $\# A = A(1)$. Let $S_A$ be the set of prime powers $s$ such that the $s$-th cyclotomic polynomial $\Phi_s(x)$ divides $A(x)$. Consider the following conditions on $A(x)$.

(T1) $A(1) = \prod_{s \in S_A} \Phi_s(1)$.
(T2) If $s_1, \ldots, s_m \in S_A$ are powers of distinct primes, then $\Phi_{s_1, \ldots, s_m}(x)$ divides $A(x)$.

**Theorem A.** If $A(x)$ satisfies (T1) and (T2), then $A$ tiles the integers.

**Theorem B1.** If $A$ tiles the integers, then $A(x)$ satisfies (T1).

**Theorem B2.** If $A$ tiles the integers and $\# A$ has at most two prime factors, then $A(x)$ satisfies (T2).

**Corollary.** If $\# A$ has at most two prime factors, then $A$ tiles the integers if and only if $A(x)$ satisfies (T1) and (T2).
A bad decomposition

\[ \Delta_n = 1 + X + X^2 + \ldots + X^{n-1} = \prod \Phi_d(X) \quad d \mid n \quad d \neq 1 \]

\[ \Phi_2(X) = 1 + X \]
\[ \Phi_3(X) = 1 + X + X^2 \]
\[ \Phi_4(X) = 1 + X^2 \]
\[ \Phi_6(X) = 1 - X + X^2 \]

\[ \Delta_{12} = 1 + X + \ldots + X^{11} = \Phi_2 \times \Phi_3 \times \Phi_4 \times \Phi_6 \times \Phi_{12} \]

\[ A(X) = \Phi_2 \times \Phi_3 \times \Phi_{12} = 1 + 2X + X^2 - X^3 - X^4 + X^5 + 2X^6 + X^7 \]

\[ B(X) = \Phi_4 \times \Phi_6 = 1 - X + 2X^2 - X^3 + X^4 \]

\[ S = ? \]
\[ R = ? \]

Coven & Meyerowitz conditions

\[ (T1) \quad A(1) = 7 \neq \Phi_2 \left(1\right) \times \Phi_3 \left(1\right) = 2 \times 3 \]
\[ (T2) \quad \Phi_2 \mid A(X) \text{ et } \Phi_3 \mid A(X) \Rightarrow \Phi_{2 \times 3} \mid A(X) \]
C&M Conditions, tiling and spectrality

(T1) \( A(1) = \prod_{g \in S_A} \Phi_g(1) \).
(T2) If \( s_1, \ldots, s_m \in S_A \) are powers of distinct primes, then \( \Phi_{s_1 \cdots s_m}(x) \) divides \( A(x) \).

C&M (1999)
- T1 + T2 => tiling
- tiling => T1
- tiling \( \mathbb{Z}_n \) where \( n \) has at most two prime factors => T1 + T2

Laba (2002)
- T1 + T2 => spectral
- T2 => spectral
- spectral => T1

Amiot (2009)
- Tiling of a Hajós group => T2.
  This means that if \( A \) tiles \( \mathbb{Z}_n \) without being spectral => \( A \) is the rhythm of a Vuza Canon


Is condition T2 mandatory for tiling?
Paradigmatic classification of Vuza Canons

Vuza Canons of period $n$

- $n=p_1 p_2 n_1 n_2 n_3$
- $<p_1 n_1, p_2 n_2 >=1$
- $n_3 > 1$

$n=72$

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Tijdeman’s ‘Fundamental Lemma’:
$R$ tiles $\mathbb{Z}_n$ $\Rightarrow$ $aR$ tiles $\mathbb{Z}_n$
$<a,n>=1$

Conclusion:
There are only two « types » of Vuza Canons of period 72
(up to an affine transformation)
Tiling and Discrete Fourier Transform

\[ IC_A(k) = \text{Card}\{(x, y) \in A \times A \mid x + k = y\} \]

\[ IC_A(k) = (1_A * 1_{1-A})(k) \]

\[ F_A : t \mapsto \sum_{k \in A} e^{-2\pi i kt/c} \]

TILING

Let \( Z_A = \{ t \in \mathbb{Z}_c, F_A(t) = 0 \} \)

A tiles \( \mathbb{Z}_c \) when equivalently:

- There exists \( B, A \oplus B = \mathbb{Z}_c \)
- \( 1_A * 1_B = 1 \)
- \( F_A \times F_B(t) = 1 + e^{-2\pi i t/c} + \ldots + e^{-2\pi i (c-1)/c} \) (0 unless \( t = 0 \))
- \( Z_A \cup Z_B = \{1, 2, \ldots, c-1\} \) AND \( \text{Card } A \times \text{Card } B = c \)
- \( IC_A * IC_B = IC(Z_c) = c \) and \( \text{Card } A \times \text{Card } B = c \)

(D. Lewin, JMT, 1958)
A short (and elegant) proof of Babbitt’s Theorem

$IC_A = [4, 3, 2, 3, 2, 1] = [4, 3, 2, 3, 2, 1] = IC_A$.

$IC_A(k) = (1_A \ast 1_{-A})(k)$

$\mathcal{F}_A : t \mapsto \sum_{k \in A} e^{-2i\pi kt/c}$

$\forall k \mathcal{F}(IC_{Z_c \setminus A})(k) = \mathcal{F}(IC_A)(k)$

Z-relation, homometry and phase retrieval problem

Two sets are Z-related if they have the same module of the DFT

\[
IC_A(k) = (1_A \ast 1_{-A})(k)
\]

\[
\mathcal{F}_A: t \mapsto \sum_{k \in A} e^{-2i\pi kt/c}
\]

\[
\mathcal{F}(IC_A) = \mathcal{F}_A \times \mathcal{F}_{-A} = |\mathcal{F}_A|^2
\]

Paradigmatic classification and homometry

• Is there a (non-trivial) group action whose orbits are the equivalence classes of homometric sets?
• Is there an enumeration formula for homometric sets?

• John Mandereau, Étude des ensembles homométriques et leur application en théorie mathématique de la musique et en composition assistée par ordinateur, Master Thesis, ATIAM, Ircam/Université Paris 6, juin 2009
High-order ‘interval’ content: Lewin’s $mv^k$ vector

3-set (prime forms):
- $\{0, 1, 2\} \rightarrow 0$ copies
- $\{0, 1, 3\} \rightarrow 2$ copies
- $\{0, 1, 4\} \rightarrow 3$ copies
- $\{0, 1, 5\} \rightarrow 0$ copies
- $\{0, 1, 6\} \rightarrow 1$ copy
- $\{0, 2, 4\} \rightarrow 0$ copies
- $\{0, 2, 5\} \rightarrow 0$ copies
- $\{0, 2, 6\} \rightarrow 1$ copy
- $\{0, 2, 7\} \rightarrow 0$ copies
- $\{0, 3, 6\} \rightarrow 1$ copy
- $\{0, 3, 7\} \rightarrow 2$ copies
- $\{0, 4, 8\} \rightarrow 0$ copies

$mv^3(\{0,1,3,4,7\}) = [0, 2, 3, 0, 1, 0, 0, 1, 0, 1, 2, 0]$

- Daniele Ghisi, "From Z-relation to homometry: an introduction and some musical examples", Séminaire MaMuX, 10 décembre 2010 [http://repmus.ircam.fr/mamux/saisons/saison10-2010-2011/2010-12-10]
High-order $Z$-relation and $k$-homometric nesting

$A = \{0, 1, 2, 3, 5, 6, 7, 9, 13\}$

$B = \{0, 1, 4, 5, 6, 7, 8, 10, 12\}$

$mv^3(A) = mv^3(B) \Rightarrow 3$-homometry ($\Rightarrow 2$-homometry)

Where does this nesting stop?
The extended phase retrieval and related open problems

The reconstruction index $R(n)$ is the minimum integer $k$ for which there exist no $k$-Homometric sets in $\mathbb{Z}_n$ (i.e. $mv^3$ provides enough information to reconstruct the set).

**Proposition 4.8**

$$
R(n) = \begin{cases} 
1 & \text{if } n = 1, 2, 3 \\
2 & \text{if } n = 4, 5, 6, 7, 9, 11 \\
3 & \text{if } n = 8, 10, 12, 13, 14, 15, 16, 17, 19, 22, 23, 25, 29, 31, 37 \\
4 & \text{if } n = 18, 20, 21, 24, 26, 27, 28, 30, 32, 33, 34, 35 \\
5 & \text{if } n = 36 
\end{cases}
$$

$A = \{0, 1, 2, 3, 4, 5, 7, 10, 12, 15, 19, 20, 22, 23, 24, 25, 27, 28\}$

$B = \{0, 1, 2, 3, 4, 5, 6, 9, 14, 17, 18, 19, 21, 22, 24, 26, 27, 29\}$

$mv^4(A) = mv^4(B) \Rightarrow 4$-homometry
Homometry and Tiling Rhythmic Canons

**Tiling**

Let $Z_A = \{ t \in \mathbb{Z}_c, F_A(t) = 0 \}$

$A$ tiles $\mathbb{Z}_c$ when equivalently:

- There exists $B$, $A \oplus B = \mathbb{Z}_c$
- $1_A \star 1_B = 1$
- $F_A \times F_B(t) = 1 + e^{2i\pi t/c} + \cdots + e^{2i\pi (c-1)/c}$ (0 unless $t=0$)
- $Z_A \cup Z_B = \{1, 2, \ldots, c-1\}$ AND Card $A \times$ Card $B = c$
- $IC_A \star IC_B = IC(\mathbb{Z}_c) = c$ and Card $A \times$ Card $B = c$

**A musical offering:**

**Theorem:**
If $A$ tiles with $B$ and $A'$ has the same IC, then $A'$ tiles with $B$, too.

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**$\mathcal{F}_A : t \mapsto \sum_{k \in A} e^{-2i\pi kt/c}$**

**$IC_A(k) = (1_A \star 1_{\neg A})(k)$**

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Thank you for your attention!