On two open mathematical problems in music theory: Fuglede spectral conjecture and discrete phase retrieval

Algebra Seminar, TU Dresden, 29 November 2012

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Music Representation Team
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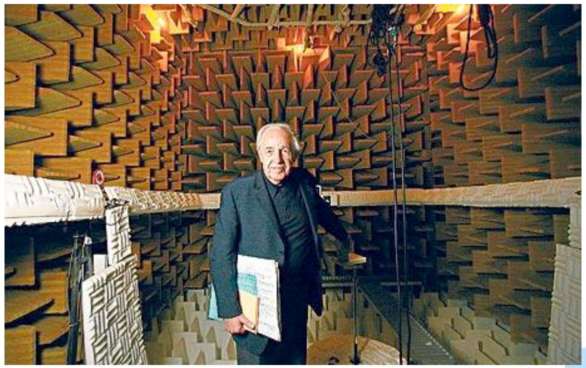
IRCAM = Institut de Recherche et Coordination Acoustique/Musique



UMR STMS

www.ircam.fr

The mixed research lab UMR9912 brings together the CNRS, the UPMC, the French Ministry of Culture, and IRCAM around the theme of multidisciplinary research on **sciences and technologies for music and sound.**

The lab is associated with the CNRS Institutes for Information Sciences and Technologies (INS2I), for Engineering and Systems Sciences (INSIS), for Humanities and Social Sciences (INSHS) and of Biological Sciences (INSB). It is also a part of the UPMC's faculty of engineering (UFR 919) in the Research Pole for Modeling and Engineering.

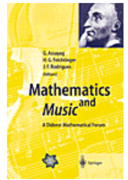
Director: Gérard Assayag (IRCAM)
Deputy Director: Hugues Vinet (IRCAM)

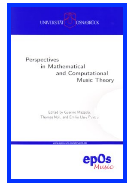
As of January 1, 2012, the laboratory consisted of the following teams:

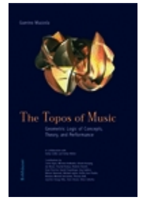
- Instrumental Acoustics
- Acoustic and Cognitive Spaces
- Perception and Sound Design
- Analysis/Synthesis
- Music Representation
- Analysis of Musical Practices
- Real-Time Musical Interactions
- IRCAM Resource Center

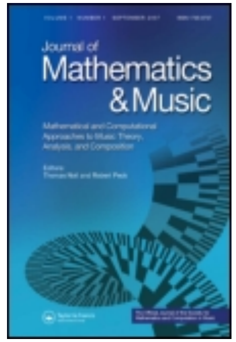
Mathematics/Music…a recent history!

- 1999: 4[e] Forum Diderot (Paris, Vienne, Lisbonne), *Mathematics and Music* (G. Assayag, H.G. Feichtinger, J.F. Rodrigues, Springer, 2001)
- 2000-2001: *MaMuPhi Seminar, Penser la musique avec les mathématiques ?* (Assayag, Mazzola, Nicolas éds., Coll. 'Musique/Sciences', Ircam/Delatour, 2006)
- 2000-2003: International Seminar on *MaMuTh* (*Perspectives in Mathematical and Computational Music Theory* (Mazzola, Noll, Luis-Puebla eds, epOs, 2004)
- 2003: *The Topos of Music* (G. Mazzola et al.)
- 2001-…: *MaMuX* Seminar at Ircam
- 2004-…: *mamuphi* Seminar (Ens/Ircam)
- 2006: *Mathematical Theory of Music* (F. Jedrzejewski), Coll. 'Musique/Sciences'
- 2007: *La vérité du beau dans la musique* (G Mazzola), Coll. 'Musique/Sciences'
- 2007: Journal of Mathematics and Music (Taylor & Francis) and SMCM
- 2007: First MCM 2007 (Berlin) and Proceedings by Springer
- 2007-…: AMS Special Session on Mathematical Techniques in Musical Analysis
- 2009: *Computational Music Science* (eds: G. Mazzola, M. Andreatta, Springer)
- 2009: MCM 2009 (Yale University) and Proceedings by Springer
- 2010: Mathematics Subject Classification : 00A65 Mathematics and music
- 2011: MCM 2011 (Ircam, 15-17 June 2011) and Proceedings LNCS Springer
- 2013: MCM 2013 (McGill University, Canada, 12-14 June 2013)









Société Mathématique de France
SMF

Modern Mathematics and the process of concepts creation

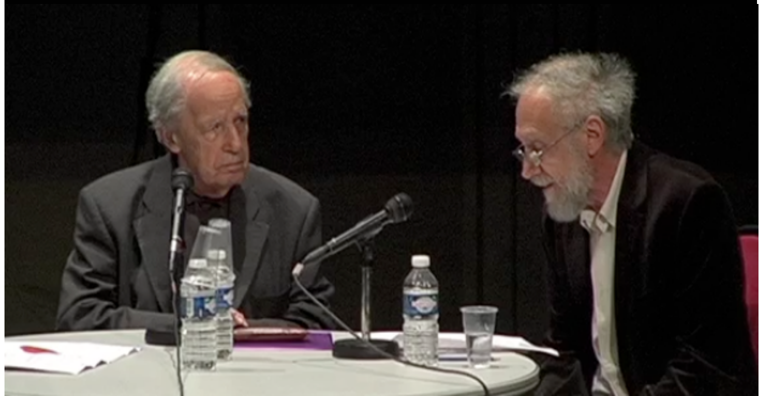
MATH / MUSiC MEETiNGS

Creativity in Music and Mathematics

Pierre Boulez & Alain Connes

Encounter with two major figures of musical creation and contemporary mathematical research: Pierre Boulez and Alain Connes.

What is the role of intuition in mathematical reasoning and in artistic activities? Is there an aesthetic dimension to mathematical activity? Does the notion of elegance of a mathematical demonstration or of a theoretical construction in music play a role in creativity?



Gérard Assayag, director of the CNRS/IRCAM Laboratory for The Science and Technology of Music and Sound, will lead this dialogue on invention in the two disciplines.

Photo: Pierre Boulez © Jean Radel

Wednesday, June 15, 2011, 6:30pm / IRCAM, Espace de projection

Creativity in Mathematics and the Arts

Do mathematics have a unique place within scientific disciplines, as music does within artistic practices?

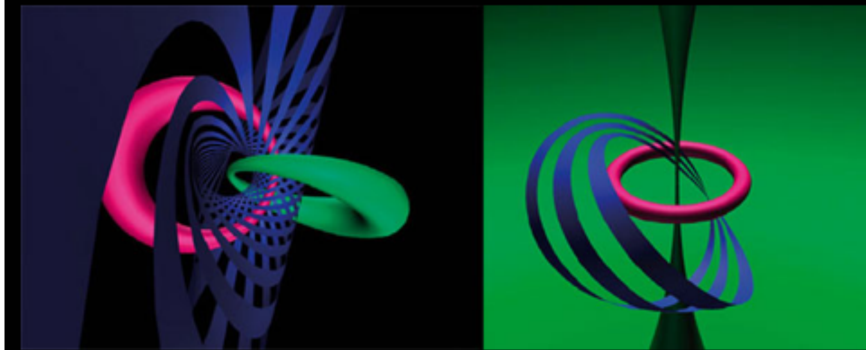
Starting from the mathematics/music relationship, this final round-table will raise the issue of the connections between art and science.

11am-1pm Round-Table Discussion

Palais de la découverte, Salle de conférences

With **Jean-Paul Allouche**, mathematician I **Claude Bruter**, mathematician and president of the ESMA I **Yves Hellegouarch**, mathematician I **Tom Johnson**, composer I **Jean-Marc Lévy-Leblond**, physicist and author I **Jacques Mandelbrojt**, painter and theoretical physicist I **Jean-Claude Risset**, physicist and composer

Session led by **Moreno Andreatta** (researcher IRCAM/CNRS and vice president of the Society for Mathematics and Computation in Music).



"…the role of mathematics, which at the beginning was considered as a part of physics, has become – thanks to modern mathematics – a kind of substitution of philosophy with respect to the creation of concepts" (Alain Connes).

➔ `http://agora.ircam.fr/971.html?event=1002`

ircam Centre Pompidou AGORA FESTiVAL 8-18 JUiN 2011

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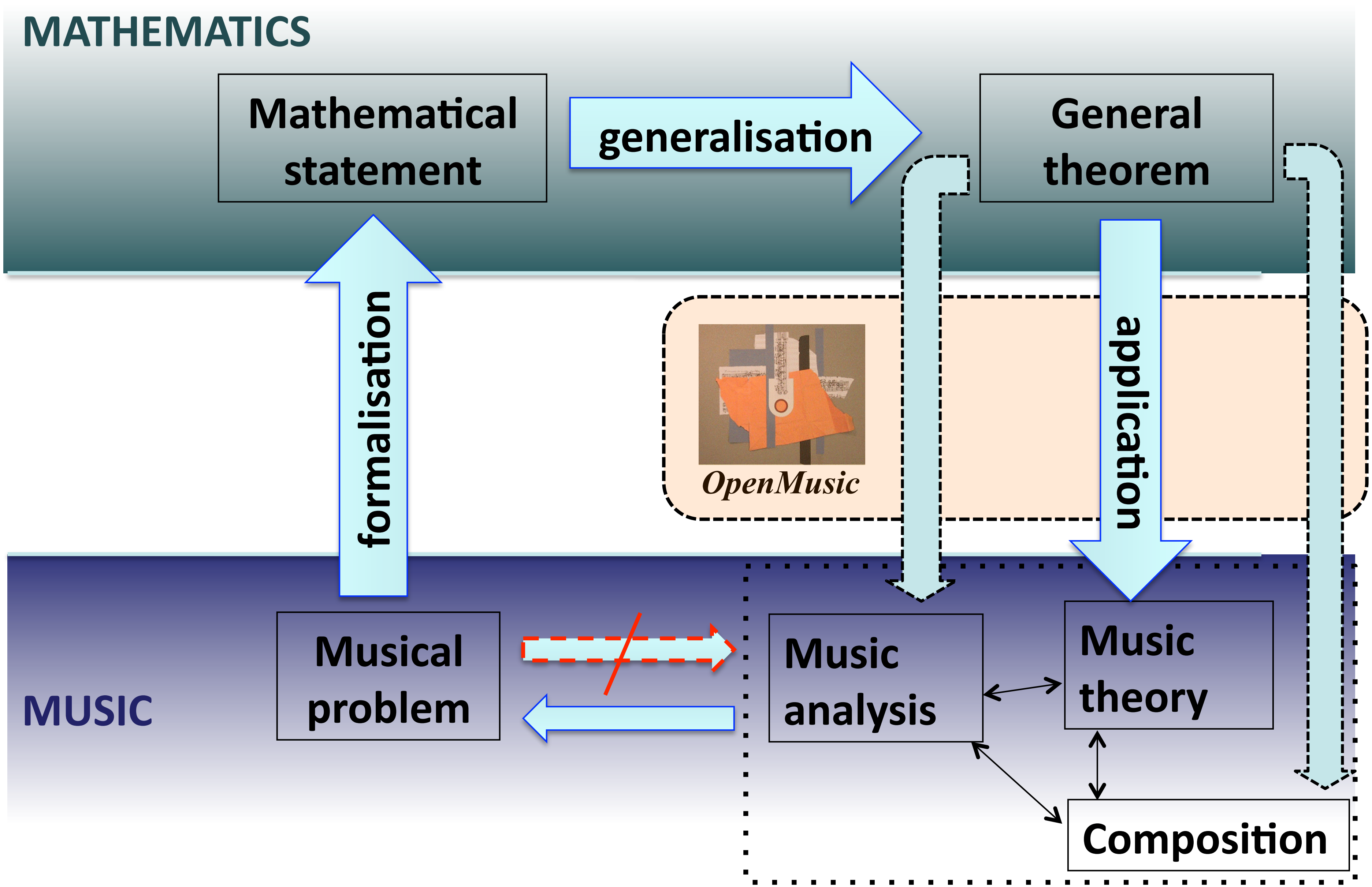
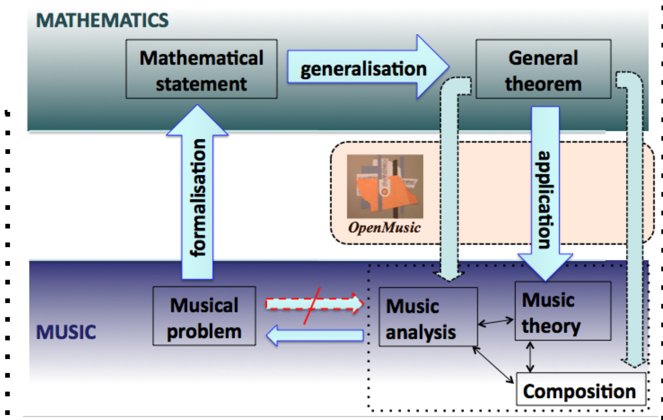
The double movement of a ‘mathemusical’ activity

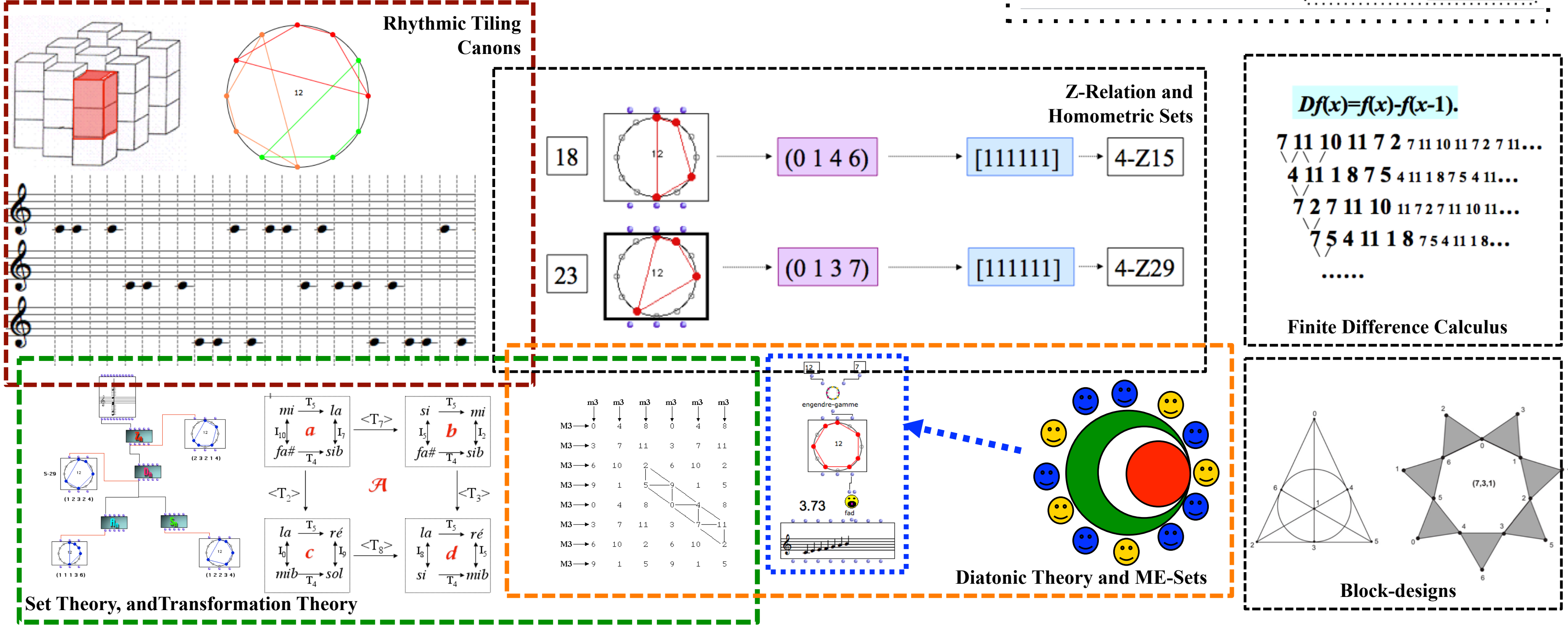


Some examples of ‘mathemusical’ problems

M. Andreatta : *Mathematica est exercitium musicae*, Habilitation Thesis, IRMA University of Strasbourg, 2010
➔ `http://repmus.ircam.fr/moreno/hdr`

- The construction of Tiling Rhythmic Canons
- The *Z* relation and the theory of homometric sets
- *Set Theory* and Transformational Theory
- Diatonic Theory and Maximally-Even Sets
- Periodic sequences and finite difference calculus
- Block-designs and algorithmic composition

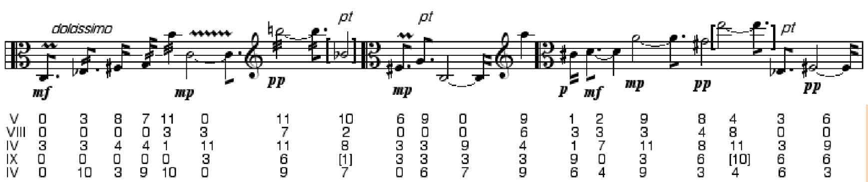




Periodic sequences and finite difference calculus

$Df(x)=f(x)-f(x-1)$.

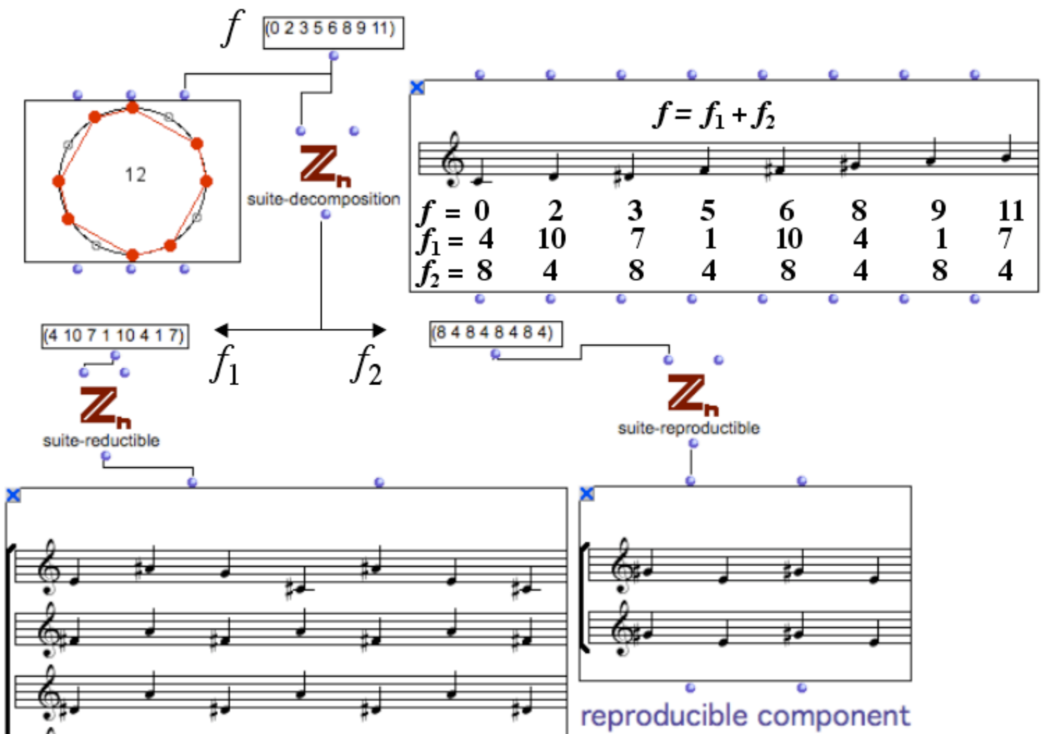
7 11 10 11 7 2 7 11 10 11 7 2 7 11…
4 11 1 8 7 5 4 11 1 8 7 5 4 11…
7 2 7 11 10 11 7 2 7 11 10 11…
7 5 4 11 1 8 7 5 4 11 1 8…
……



Anatol Vieru: *Zone d'oubli* for viola (1973)

Reducible sequences:
$\exists\, k \geq 1$ such that $D^k f = 0$

Reproducible sequences:
$\exists\, k \geq 1$ such that $D^k f = f$

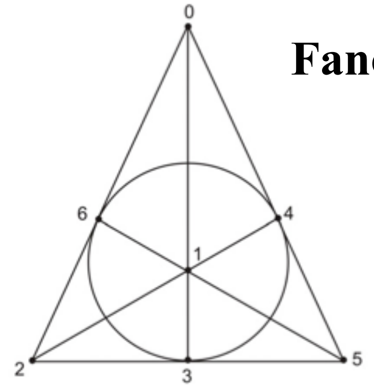


- **Decomposition theorem**

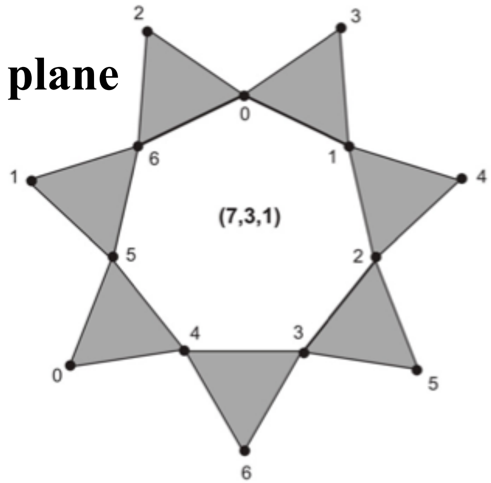
(Vuza & Andreatta, *Tatra M.*, 2001)
Every periodic sequence f can be decomposed in a unique way as a sum $f_1 + f_2$ of a reducible sequence f_1 and a reproducible sequence f_2

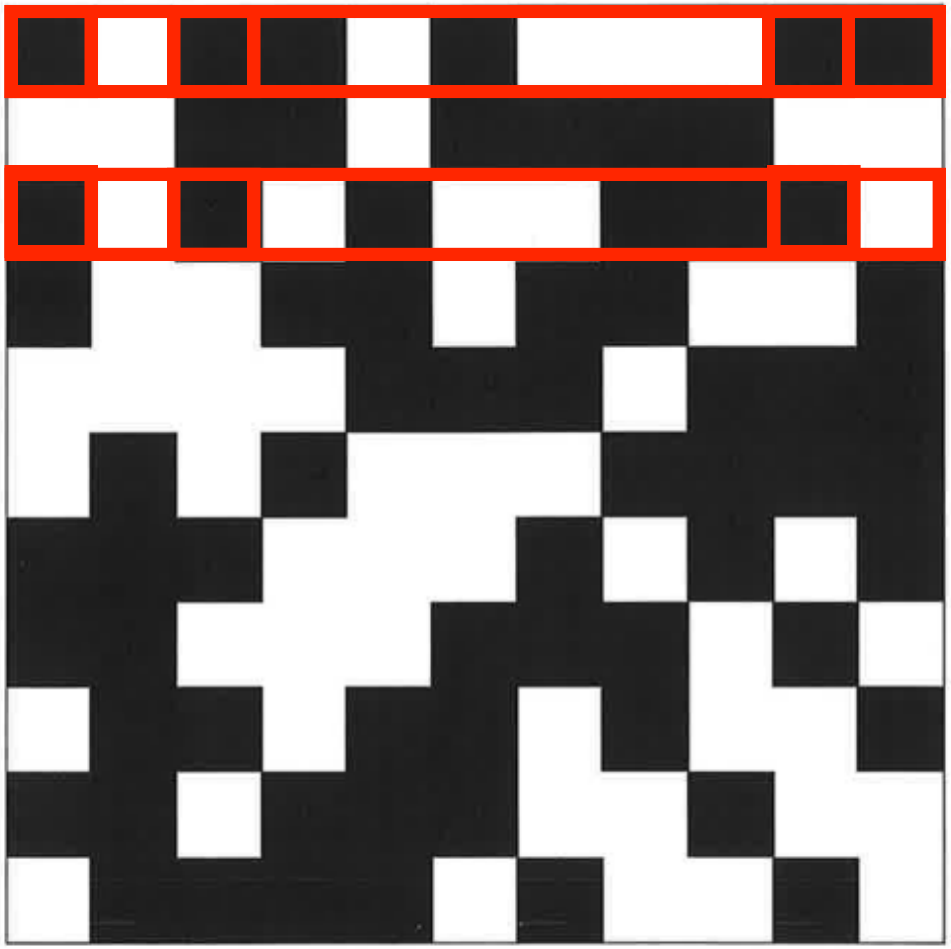
D. Vuza, M. Andreatta (2001), « On some properties of periodic sequences in Anatol Vieru's modal theory », Tatra Mountains Mathematical Publications, Vol. 23, p. 1-15

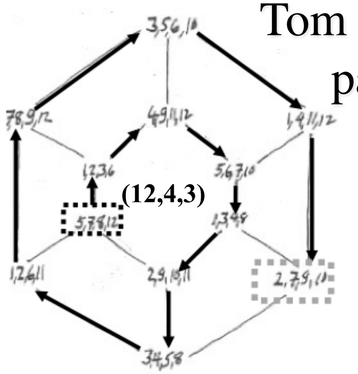
Block-designs and algorithmic composition



Fano plane

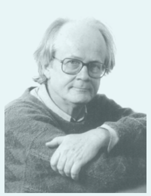






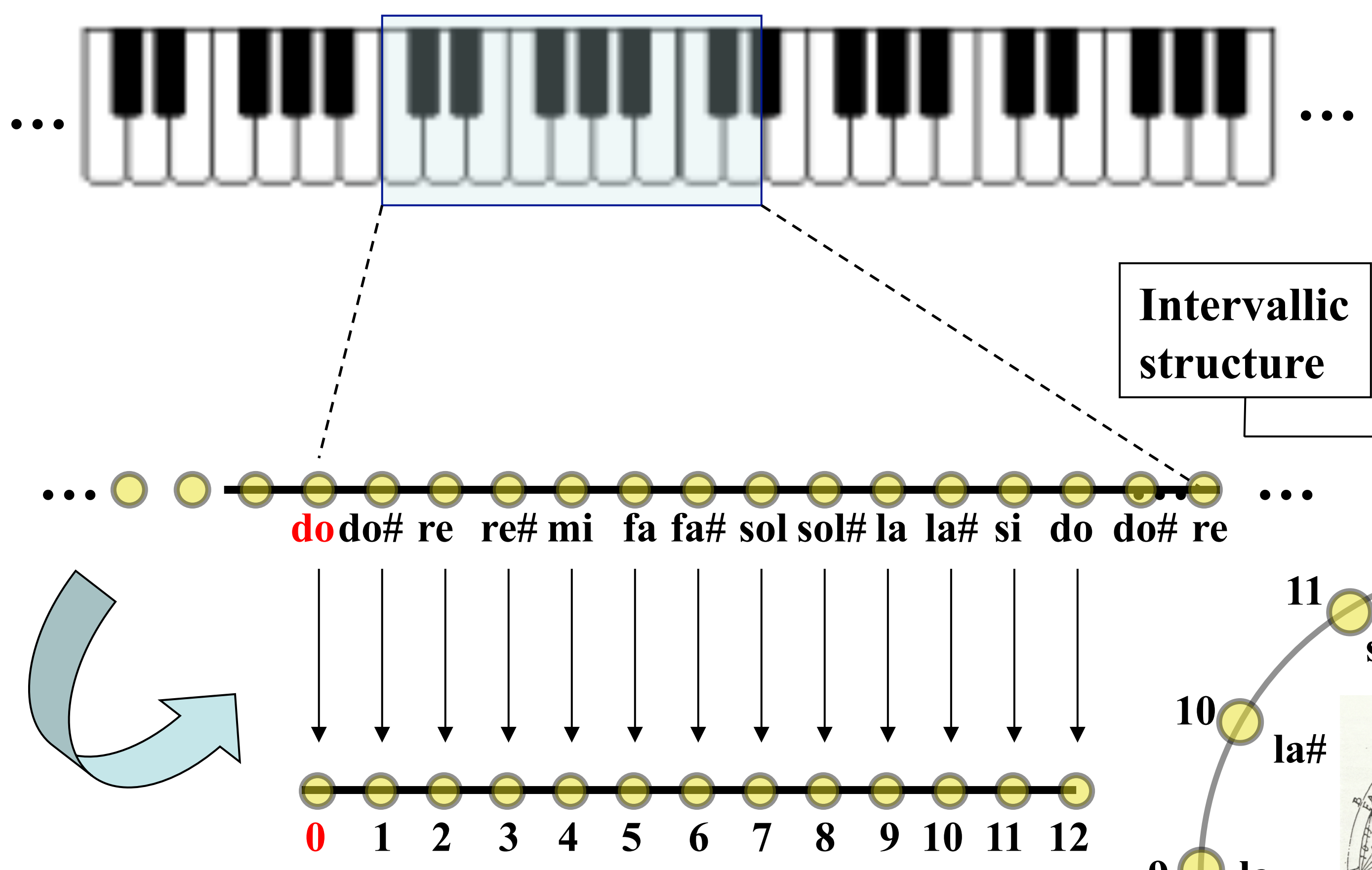
Tom Johnson's *Twelve for piano*, part III, based on (12,4,3)

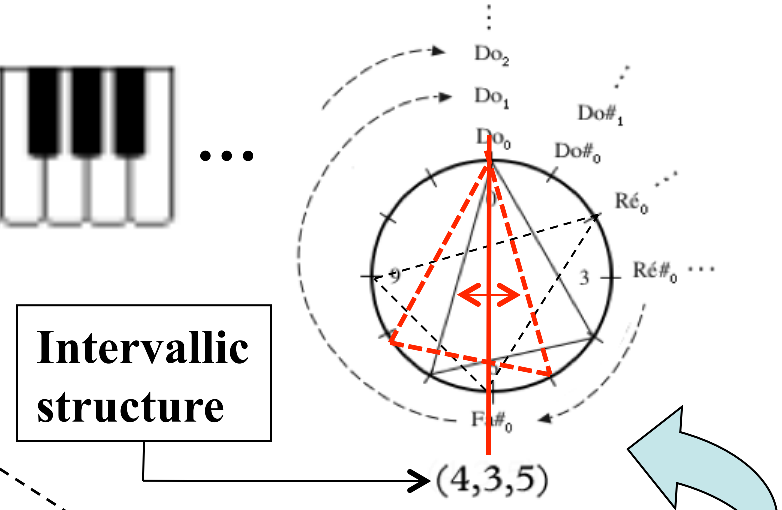
Block-design (11,6,3)

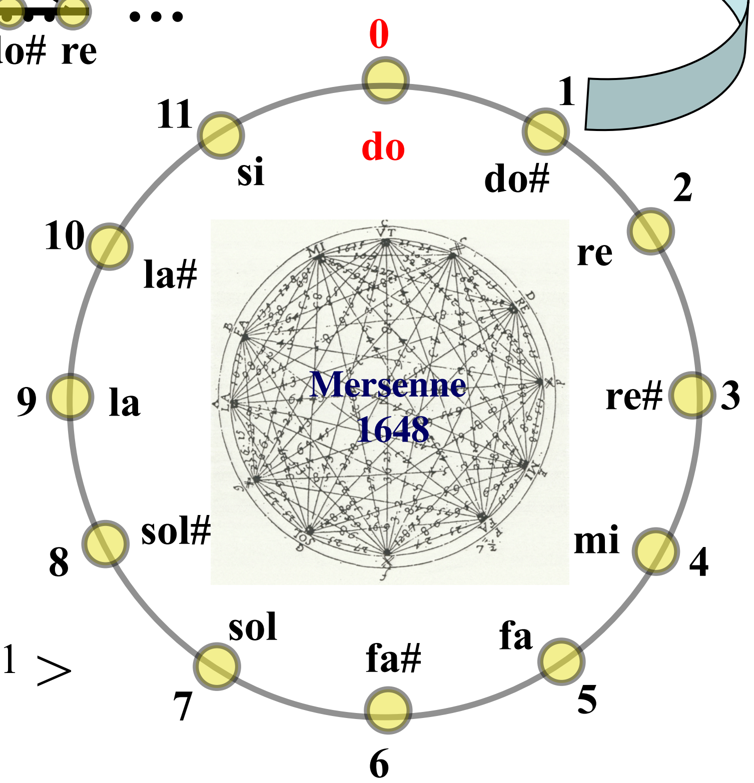


Jedrzejewski, F., M. Andreatta, T. Johnson (2009), « Musical experiences with Block Designs », Second International Conference MCM 2009, vol. 38, New Haven, p. 154-165

Algebraic structures and geometrical transformations







$$\mathbf{Z_{12}} = < T_k \mid (T_k)^{12} = T_0 >$$

$$T_k : x \rightarrow x + k \bmod 12$$

$$\mathbf{D_{12}} = < T_k , I \mid (T_k)^{12} = I^2 = T_0 , ITI = I(IT)^{-1} >$$

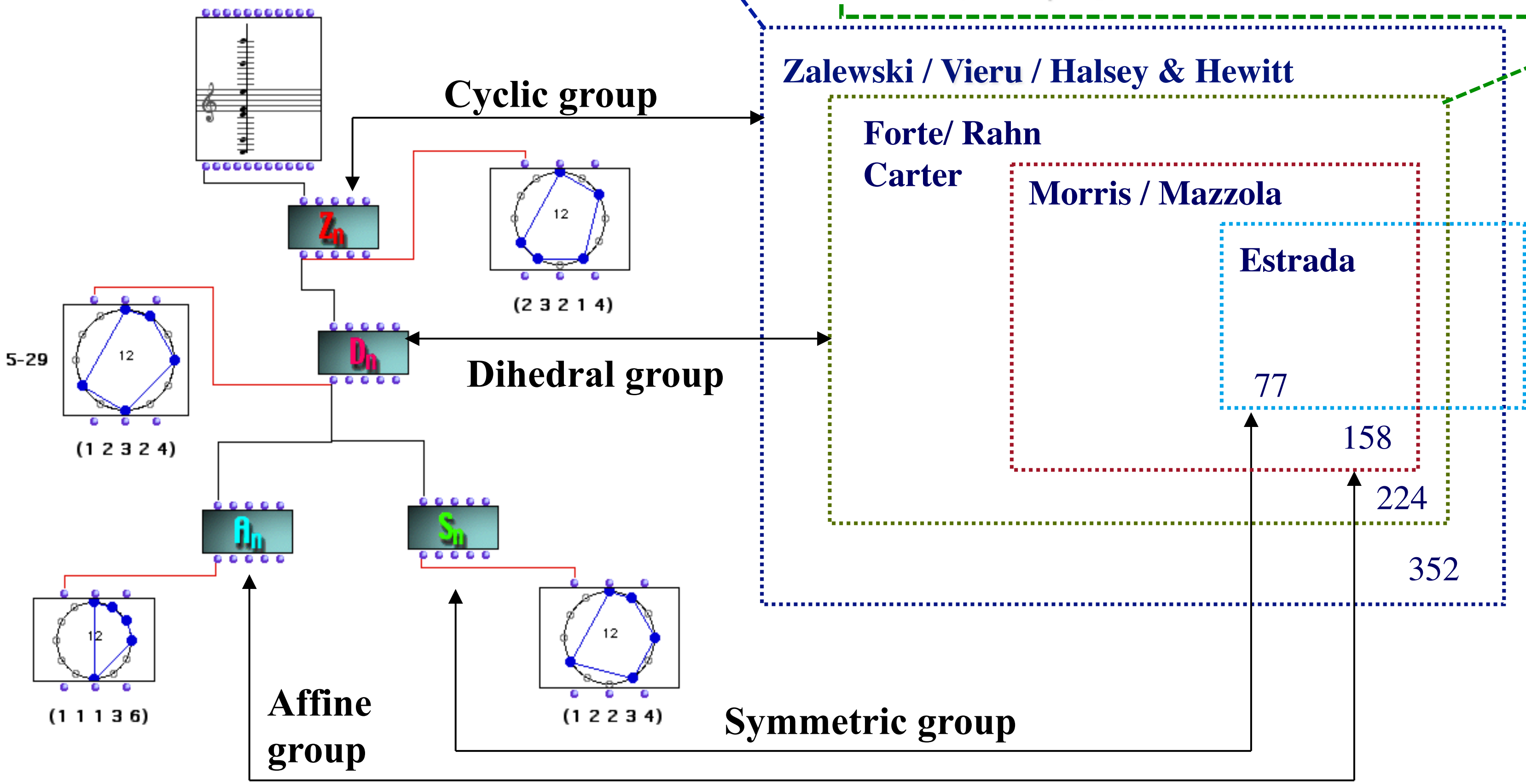
$$I : x \rightarrow -x \bmod 12$$

A group action based classification of musical structures

$$\#\text{ of } k\text{-chords} = \frac{1}{n}\sum_{j|(n,k)} \phi(j)\binom{n/j}{k/j} = \frac{1}{n}\Phi_n(k)$$

$$\#\text{ of } k\text{-chords} = \begin{cases} \frac{1}{2n}\left[\Phi_n(k) + n\binom{(n-1)/2}{[k/2]}\right], & \text{if } n \text{ is odd,} \\ \frac{1}{2n}\left[\Phi_n(k) + n\binom{n/2}{k/2}\right], & \text{if } n \text{ is even and } k \text{ is even,} \\ \frac{1}{2n}\left[\Phi_n(k) + n\binom{(n/2)-1}{[k/2]}\right], & \text{if } n \text{ is even and } k \text{ is odd.} \end{cases}$$

Paradigmatic architecture



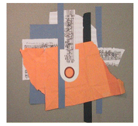

F. Klein

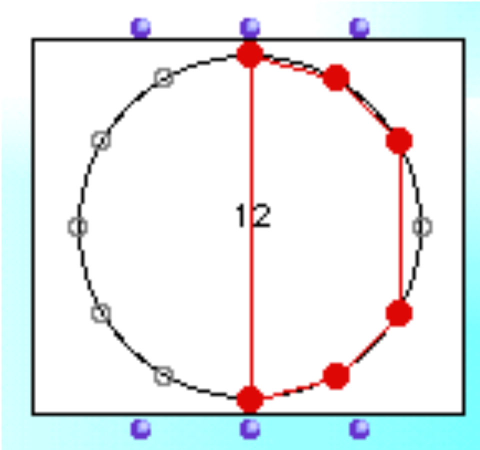

W. Burnside


G. Polya

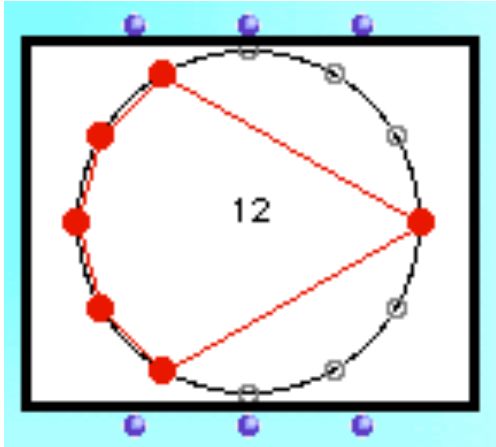
- D. Halsey & E. Hewitt, "Eine gruppentheoretische Methode in der Musik-theorie", *Jahr. der Dt. Math.-Vereinigung*, 80, 1978
- D. Reiner, "Enumeration in Music Theory", *Amer. Math. Month*. 92:51-54, 1985
- H. Fripertinger, "Enumeration in Musical Theory", *Beiträge zur Elektr. Musik*,1,1992
- R.C. Read, "Combinatorial problems in the theory of music", *Discrete Mathematics* 1997
- H. Fripertinger, "Enumeration of mosaics", *Discrete Mathematics*, 1999
- H. Fripertinger, "Enumeration of non-isomorphic canons", *Tatra Mt. Math. Publ.*, 2001

A second algebraic invariant: the interval content

$$IC_A(k) = \mathrm{Card}\{(x, y) \in A \times A \mid x + k = y\}$$ ➔ 

A 

Z-relation
⟷
homometry

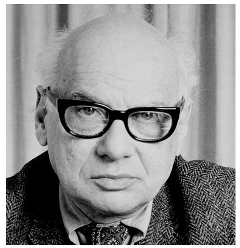


A'

$$IC_A = [4, 3, 2, 3, 2, 1\ ] \;\; = \;\; [4, 3, 2, 3, 2, 1\ ] = IC_{A'}$$

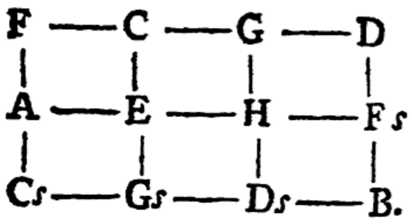
Babbitt's Hexachord Theorem:

A hexacord and its complement have the same interval content

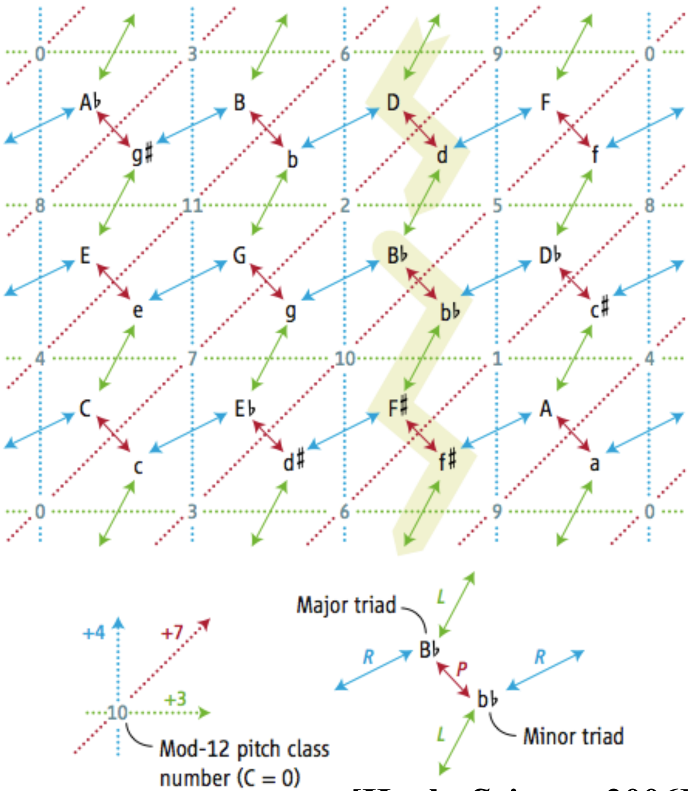


(Proofs by Wilcox, Ralph Fox (?), Chemillier, Lewin, Mazzola, Schaub, …, Amiot, …)

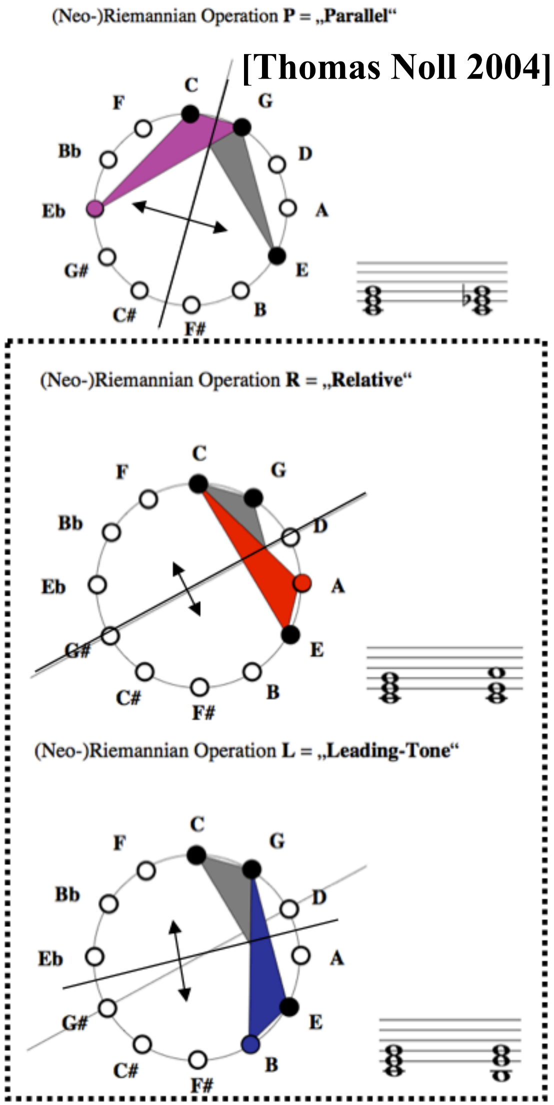
From Euler's *Speculum musicum* to the *Tonnetz*



Euler : *Speculum musicum*, 1773



[Hook, *Science*, 2006]



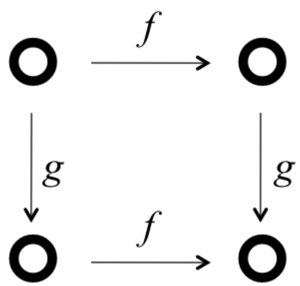
[Thomas Noll 2004]

$\rho = \langle L, R \mid L^2 = (LR)^{12} = 1$
$LRL = L(LR)^{-1} \rangle$

$\rho \subseteq C_{Sym}(D_{12})$

$D_{12} \subseteq C_{Sym}(\rho)$

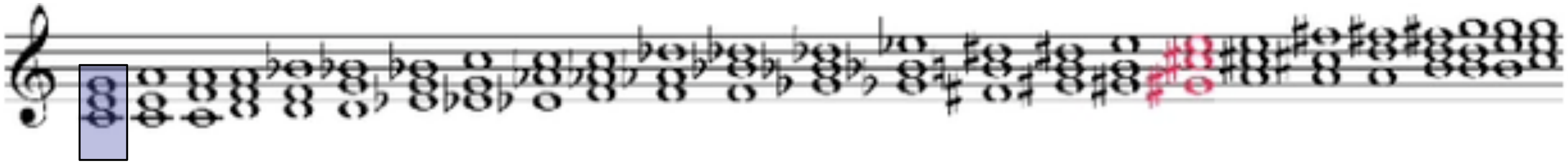
$D_{12} = \langle I, T \mid I^2 = T^{12} = 1$
$ITI = I(IT)^{-1} \rangle$

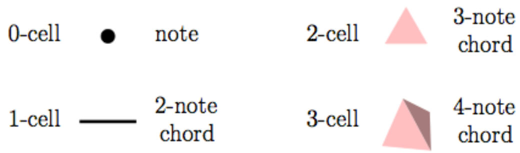


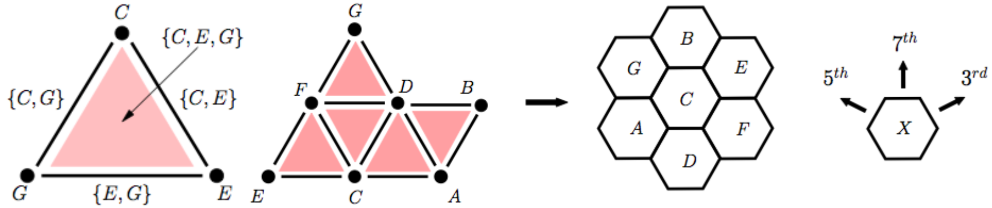
$\forall f \in D_{12}$
$\forall g \in \rho$

Crans A., Fiore T., and Satyendra R. "Musical Actions of Dihedral Groups." *The American Mathematical Monthly*, Vol. 116 (2009), No. 6: 479 – 495

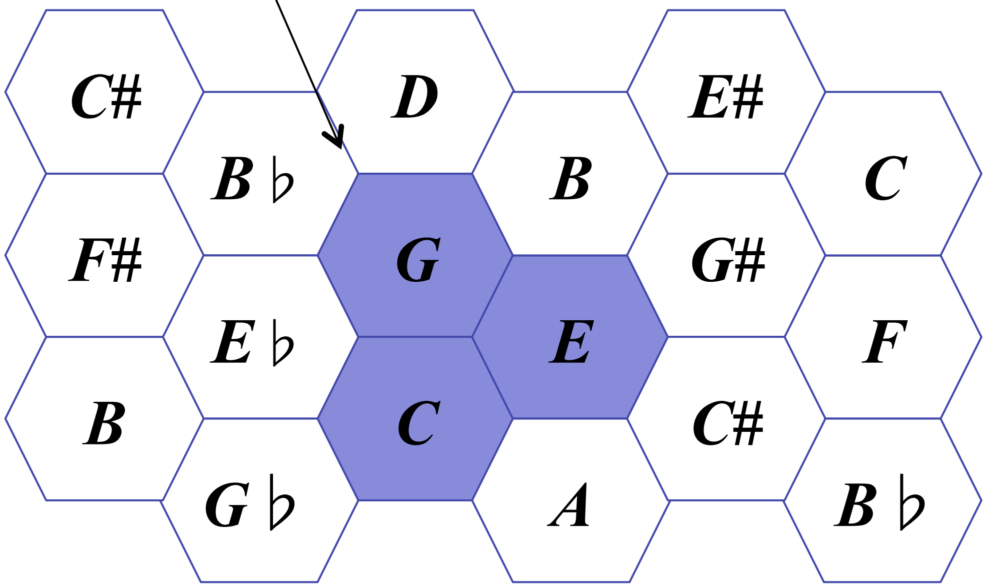
Hamiltonian cycles and spatial computing

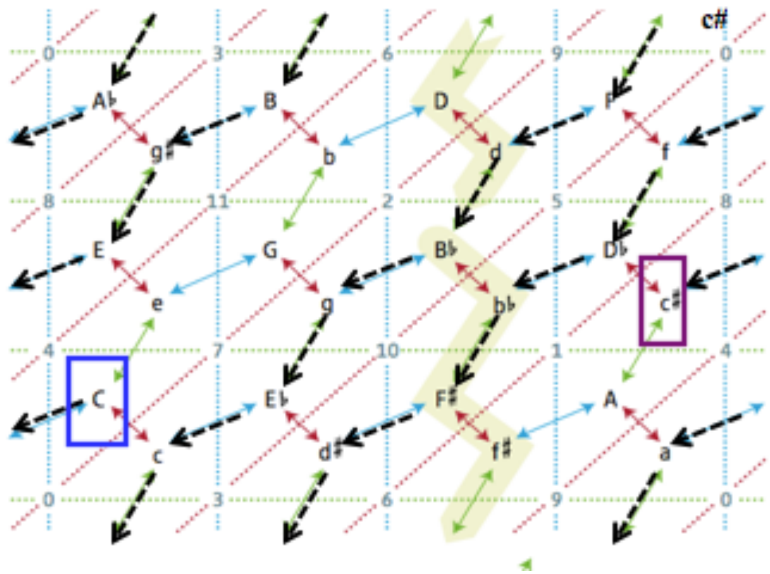




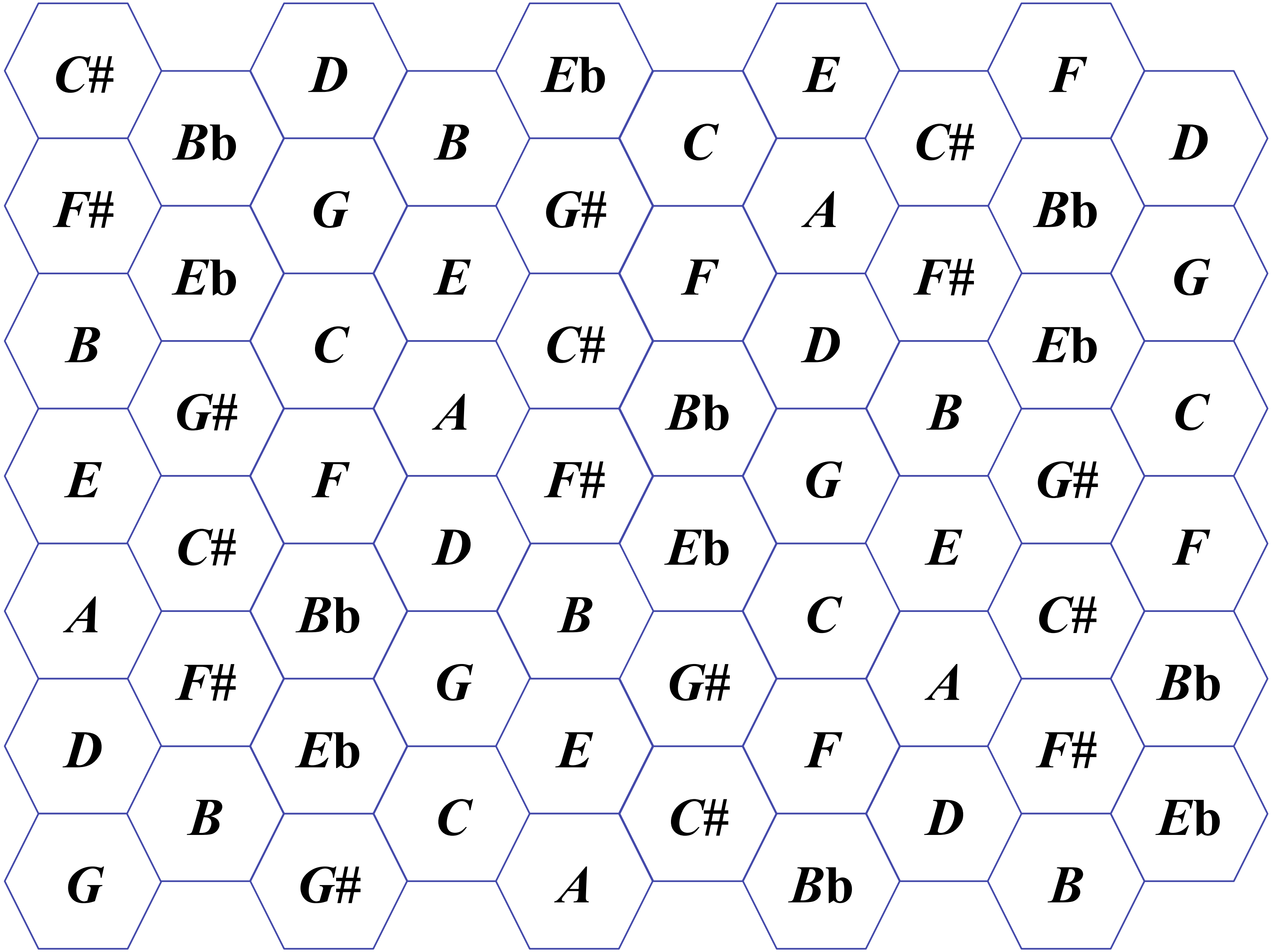


L. Bigo, J.-L. Giavitto, A. Spicher, "Building Topological Spaces for Musical Objects", MCM 2011
L. Bigo, PhD (ongoing), Ircam / University of Paris 12
➔ `http://www.lacl.fr/~lbigo/louis_bigo`





Extract of the 2nd movement of the Symphony No. 9
(L. van Beethoven)



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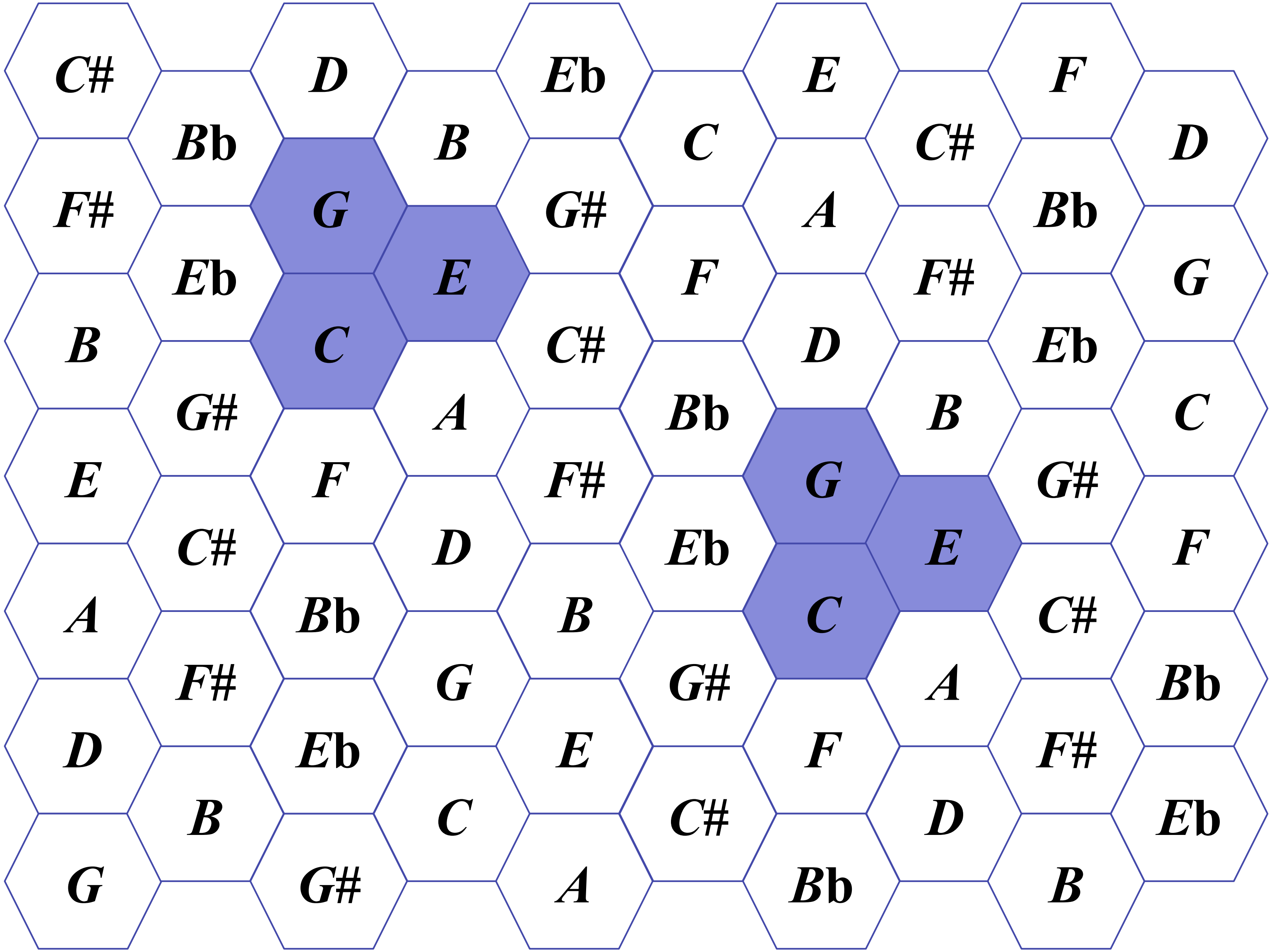

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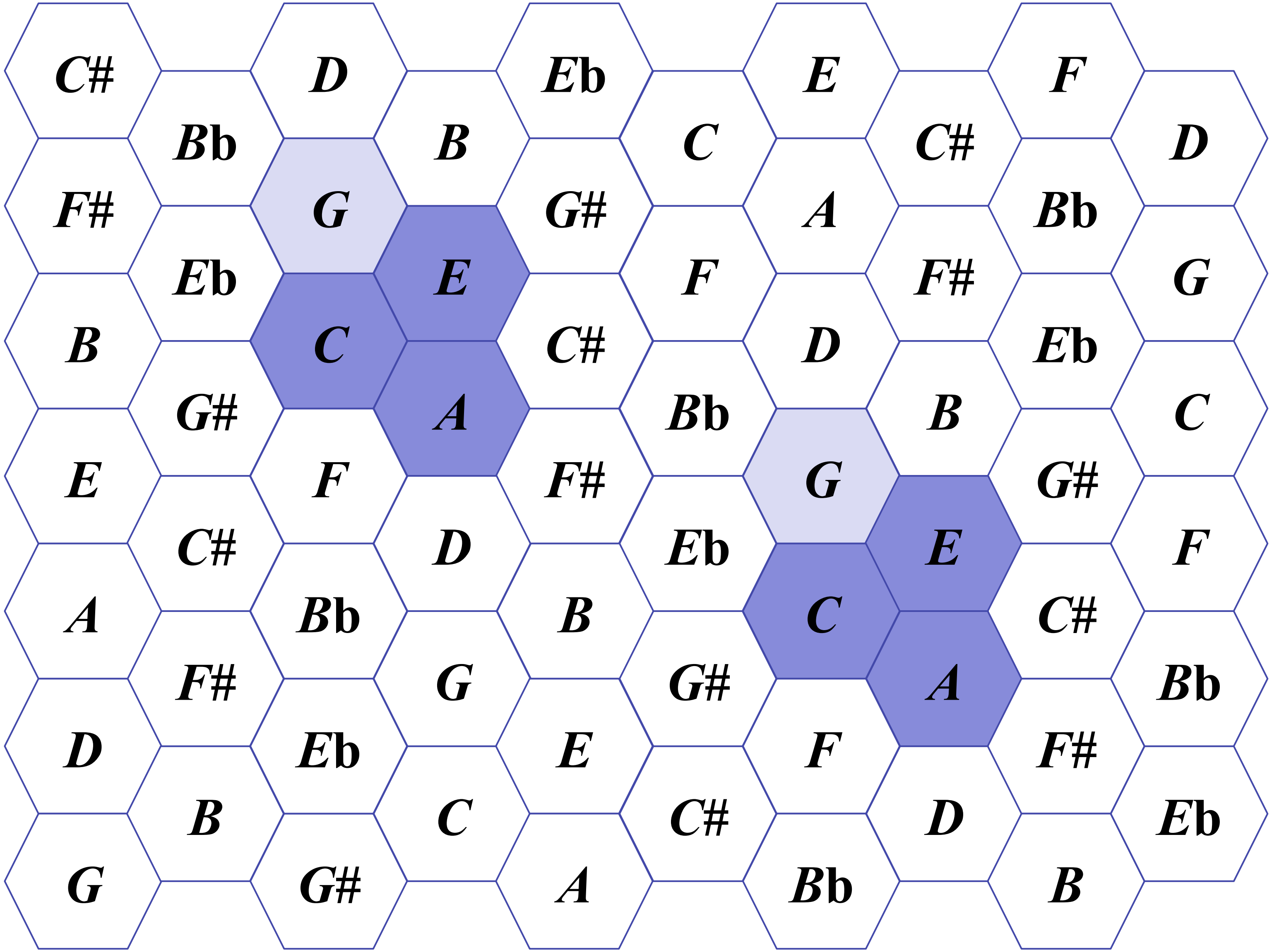

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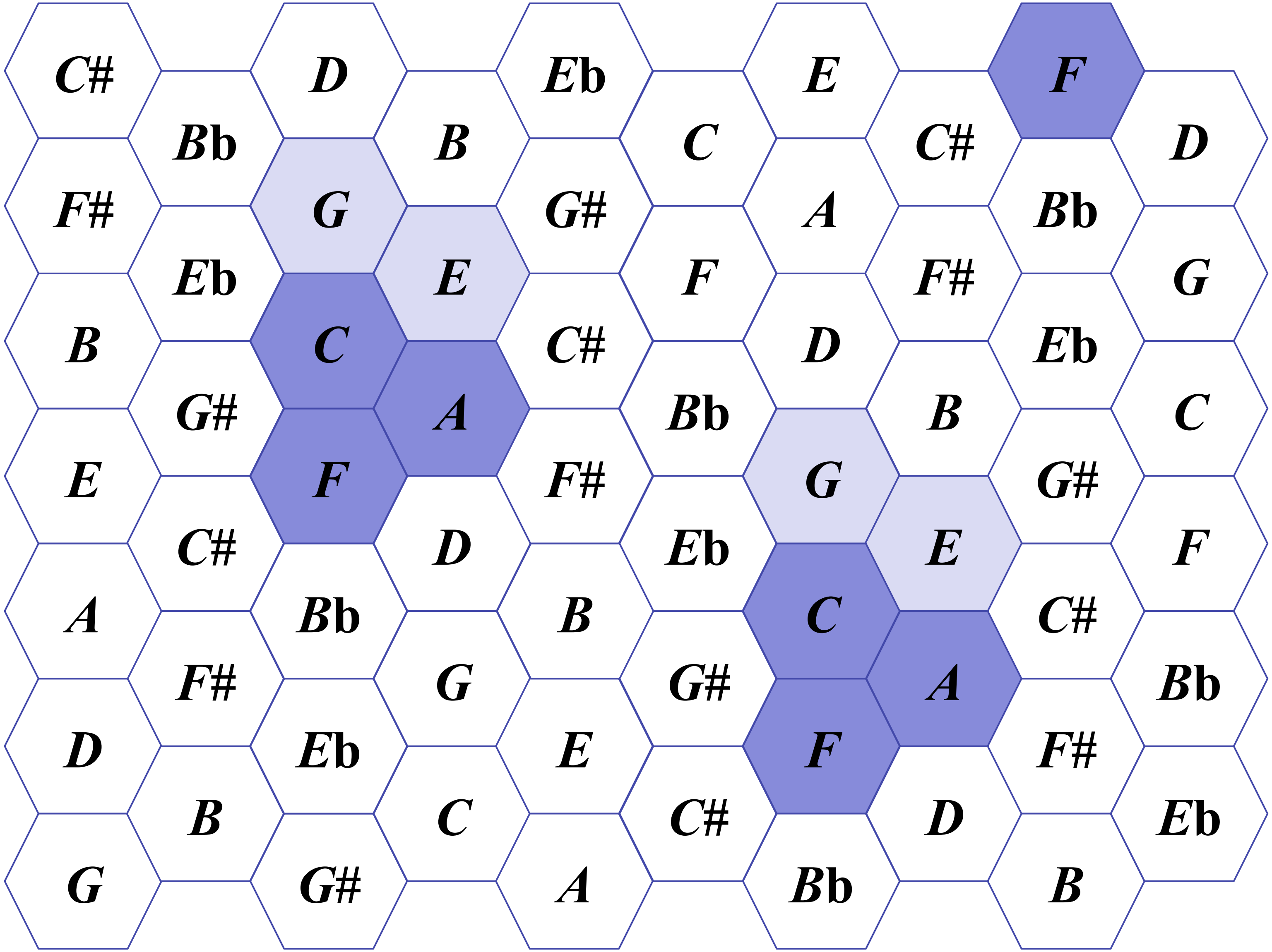

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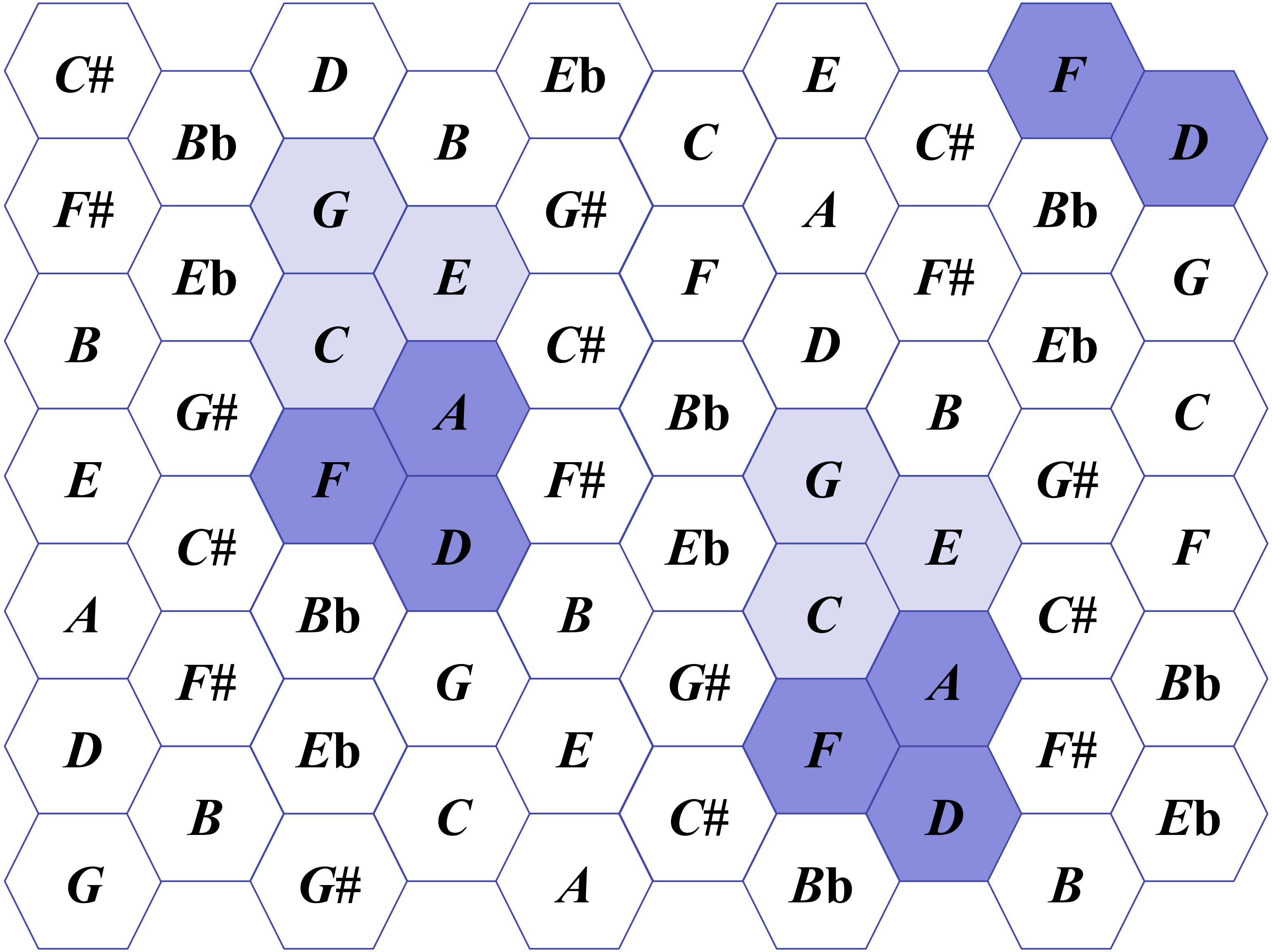

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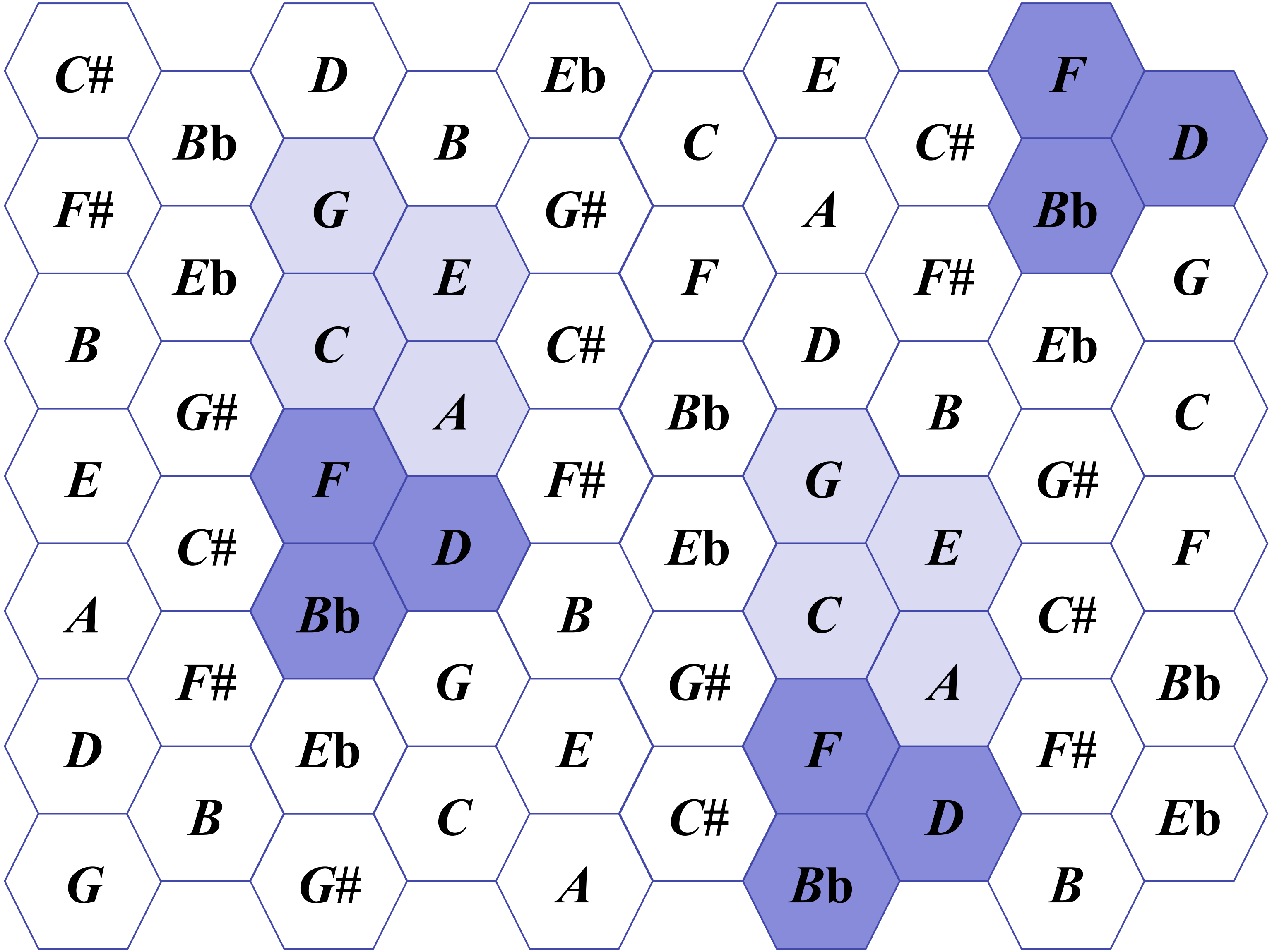

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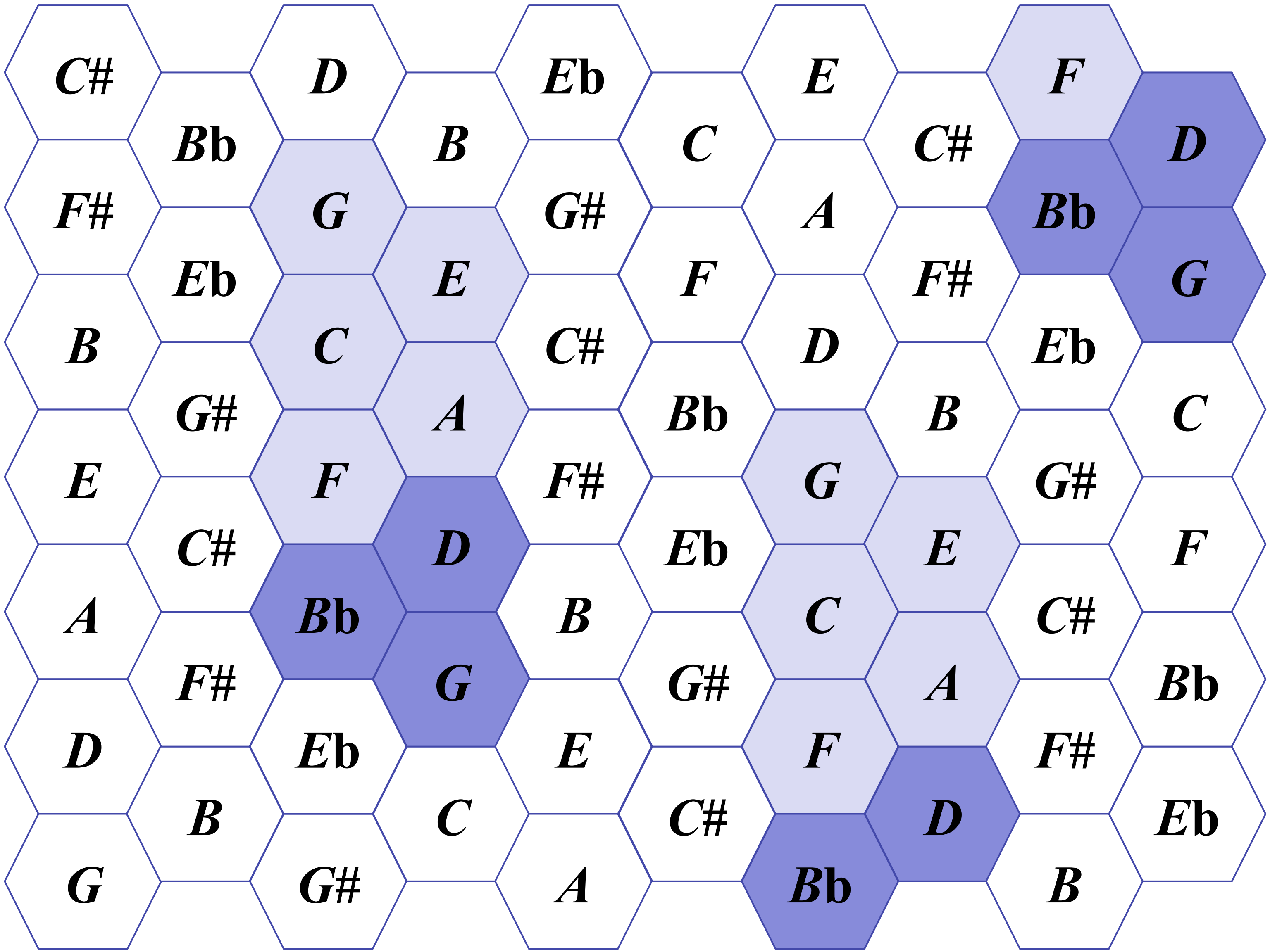

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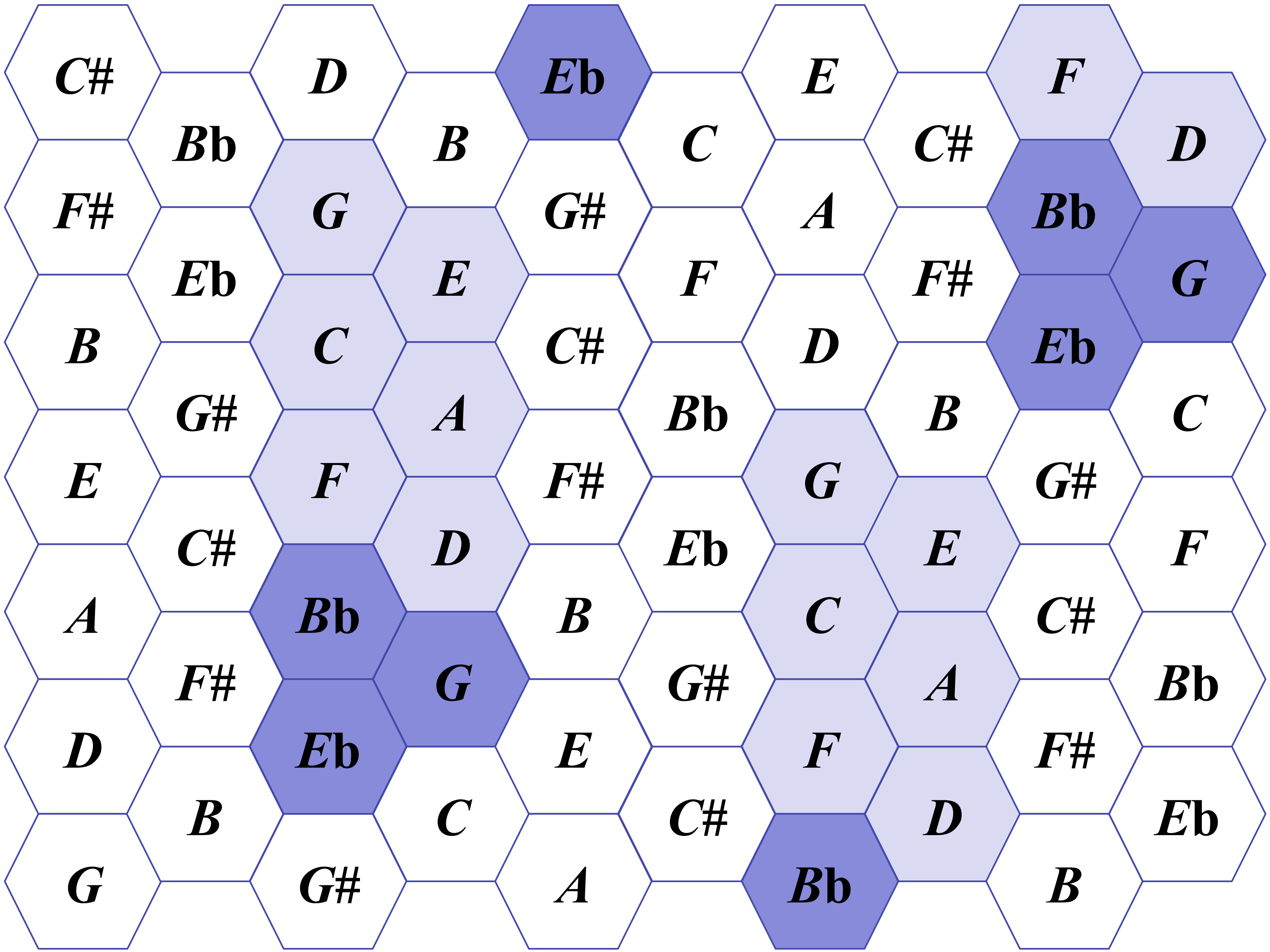
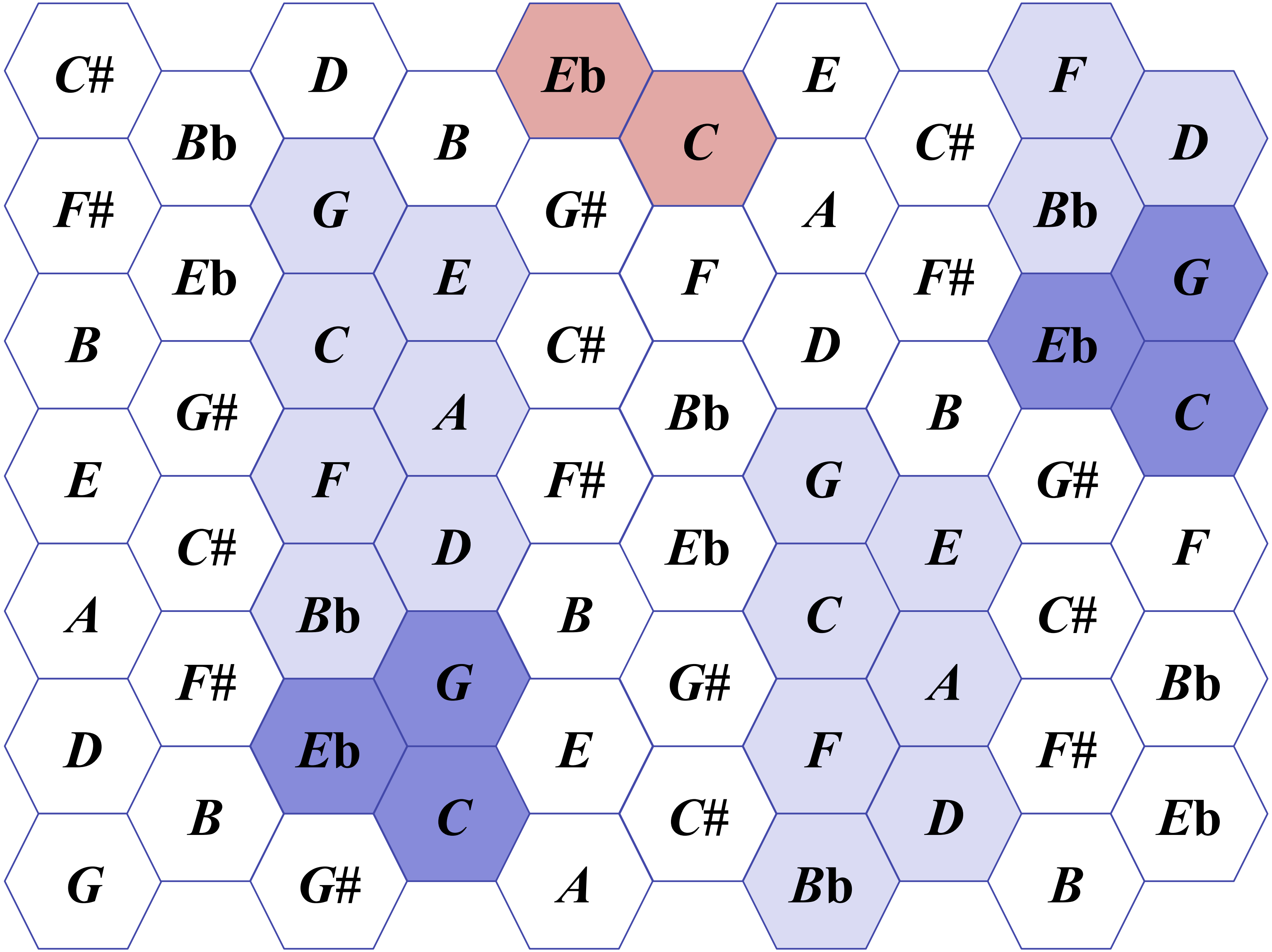


Extract of the 2nd movement of the Symphony No. 9 (L. van Beethoven)



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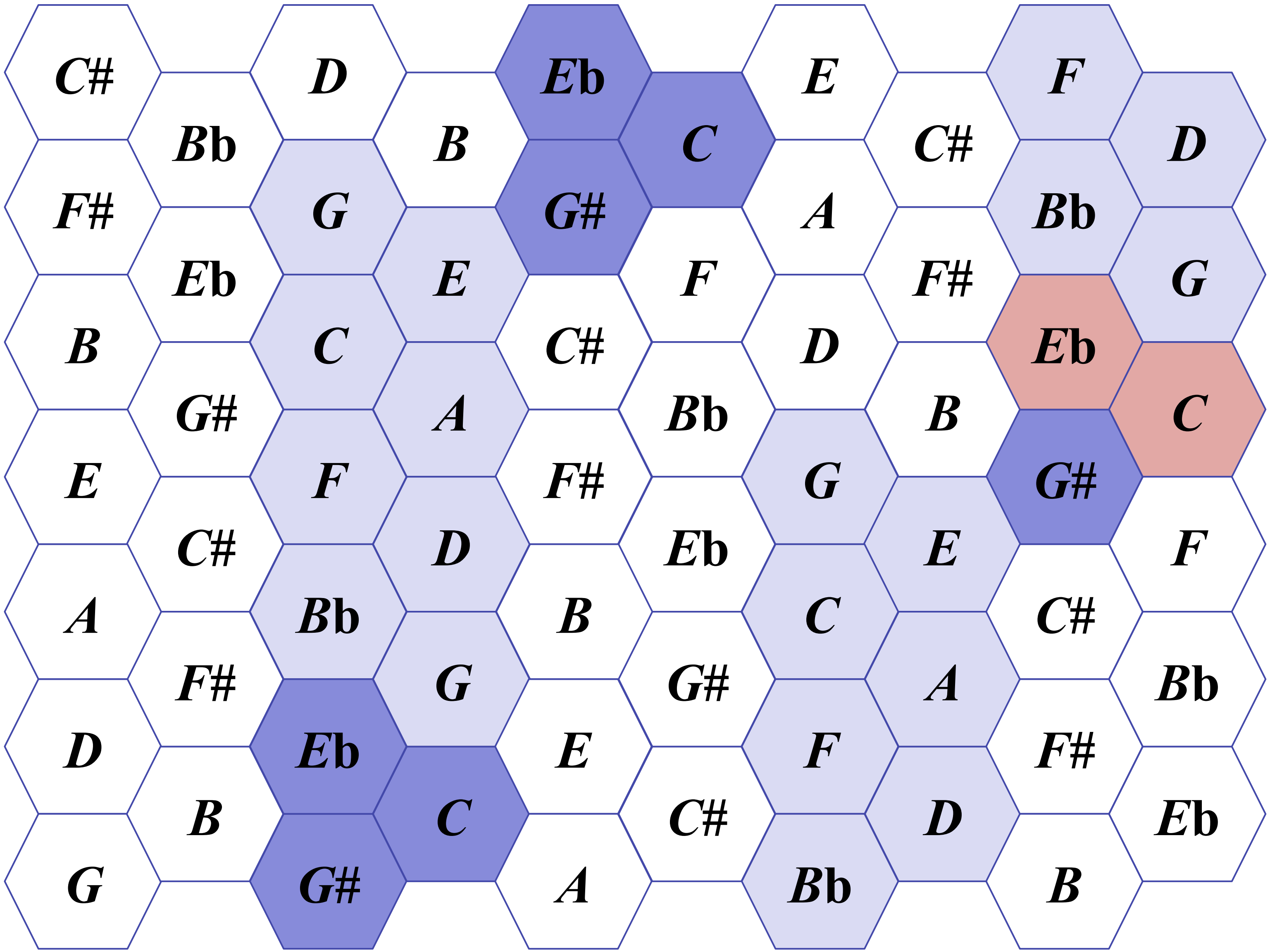

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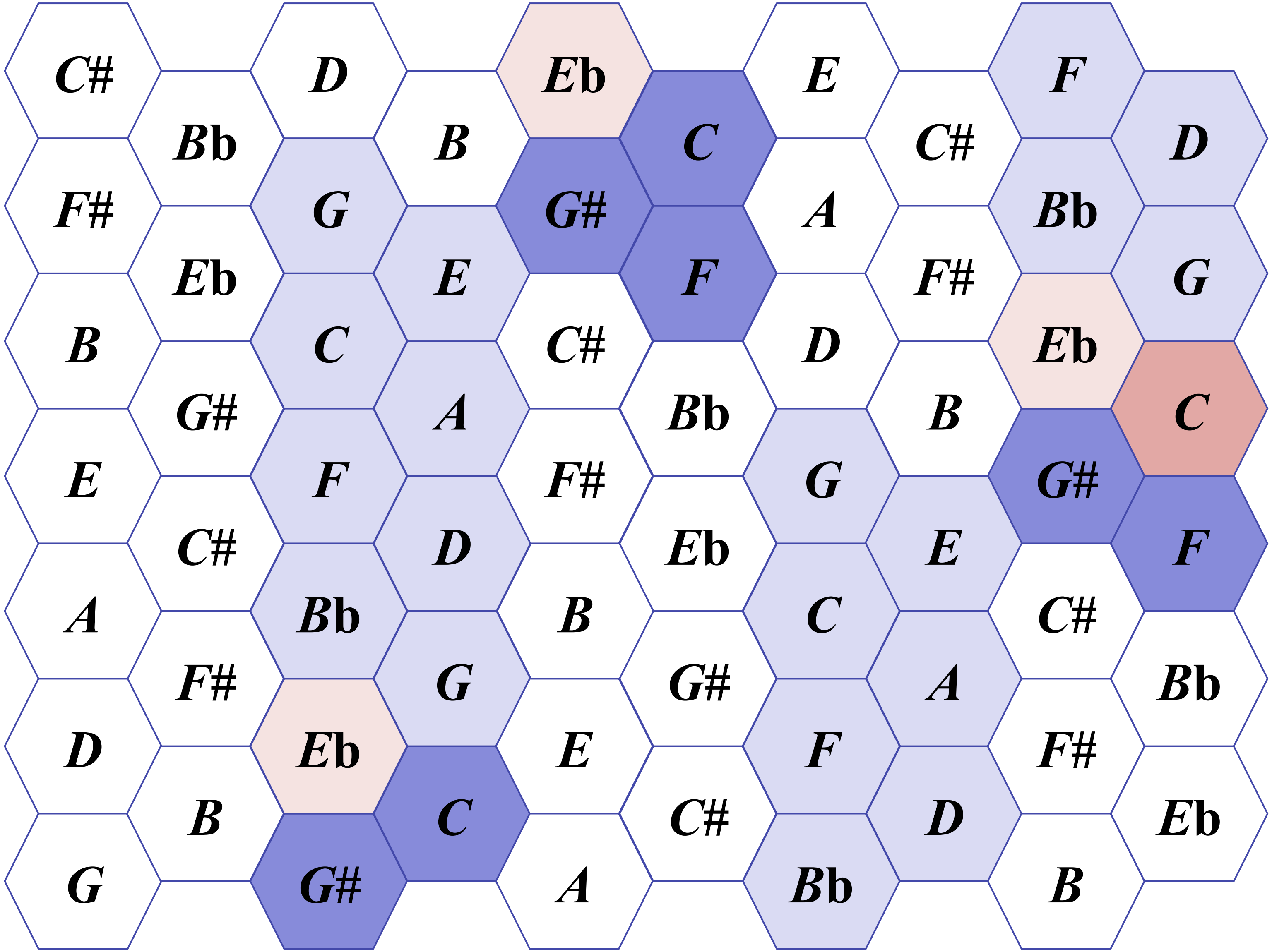

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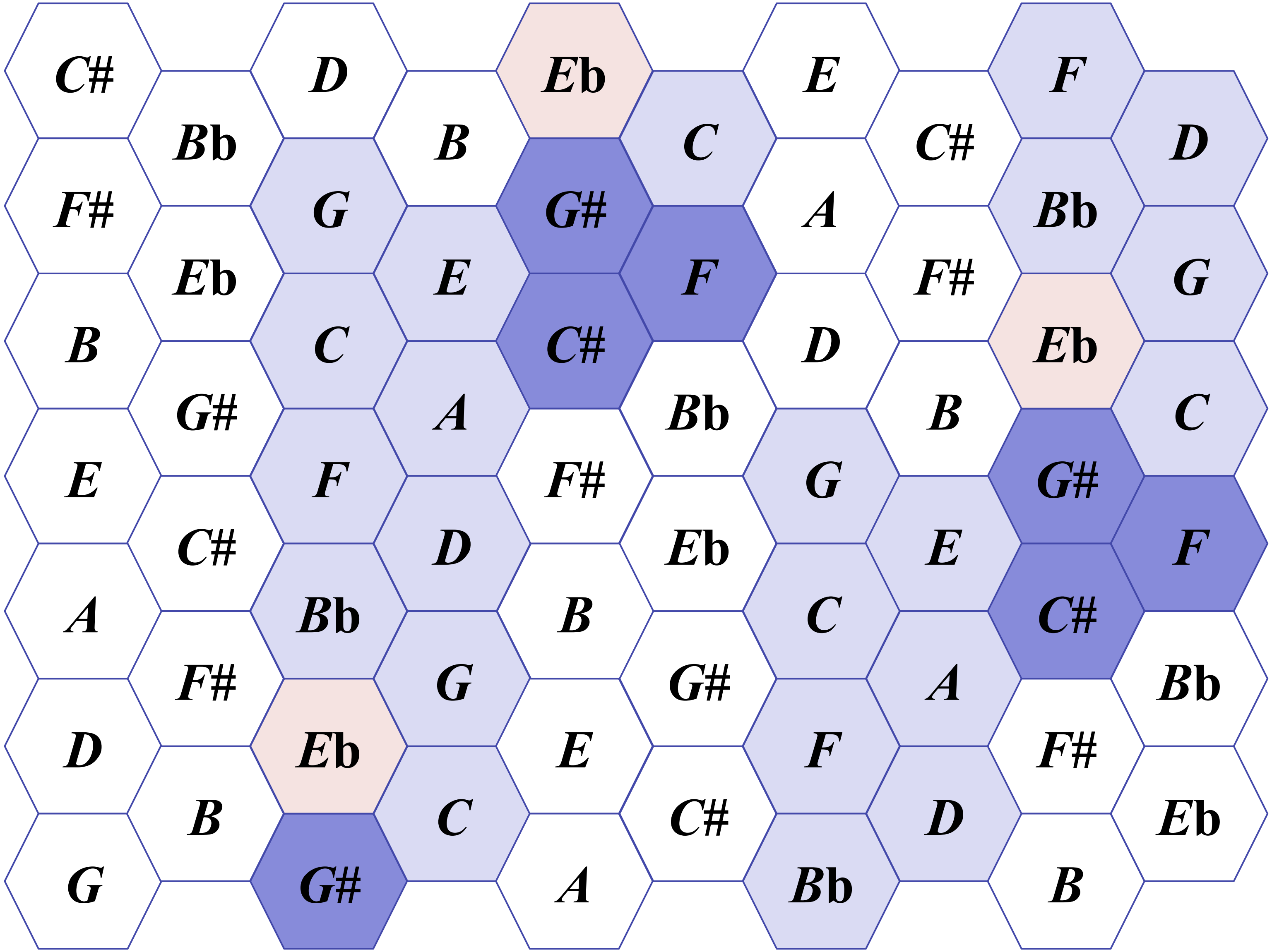

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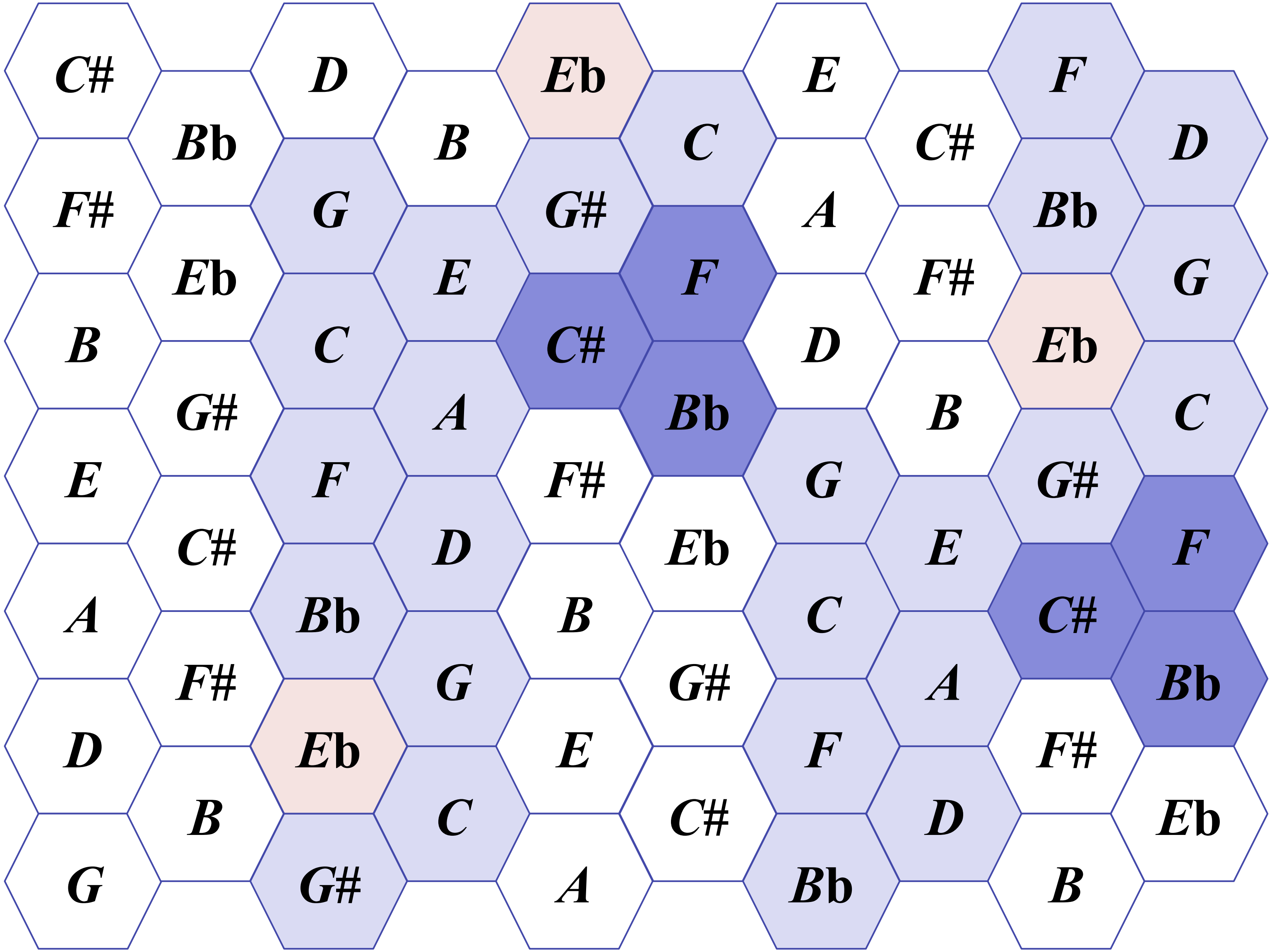
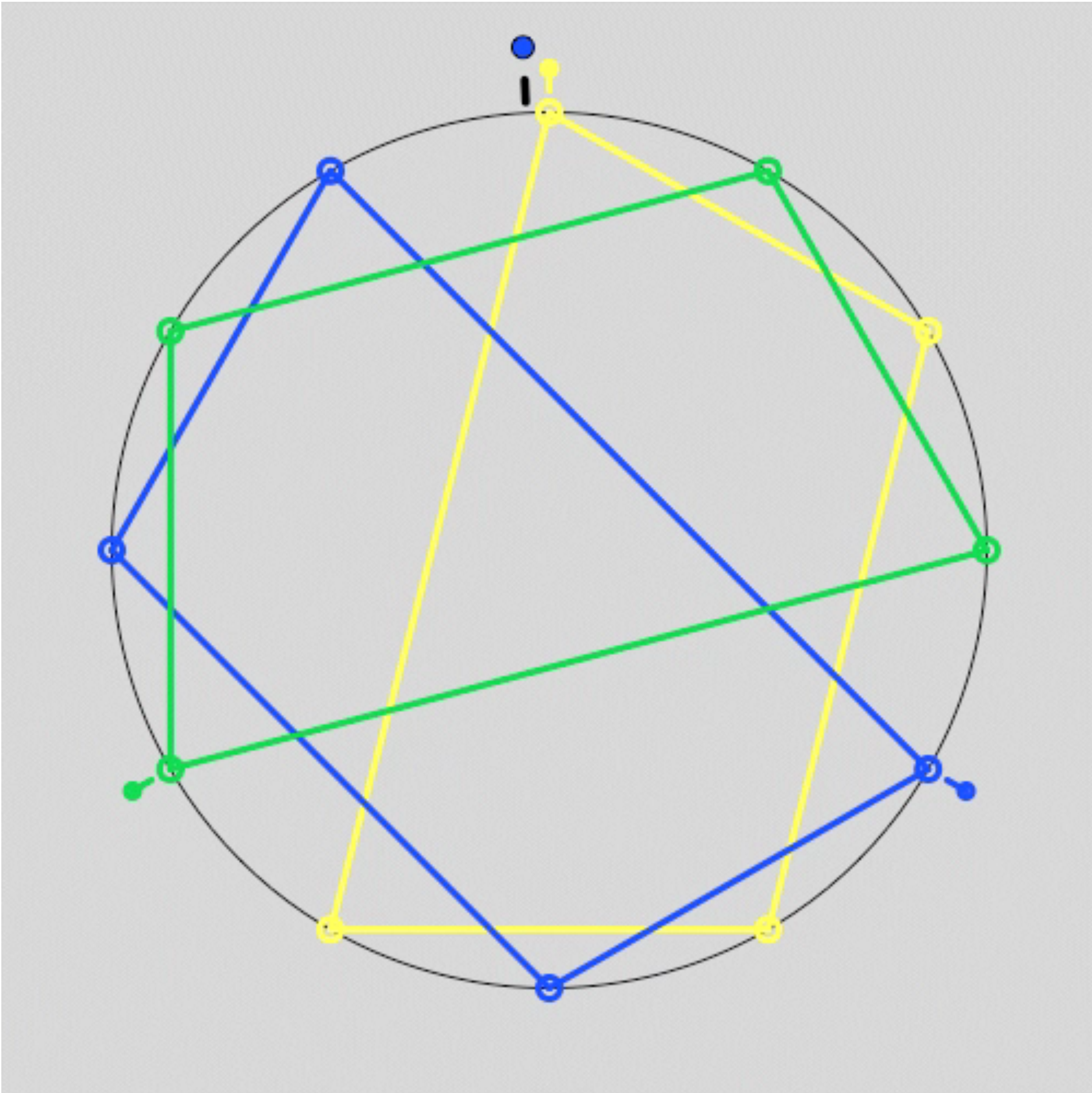
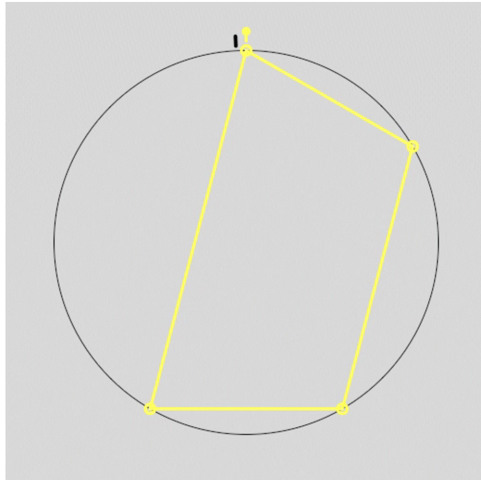


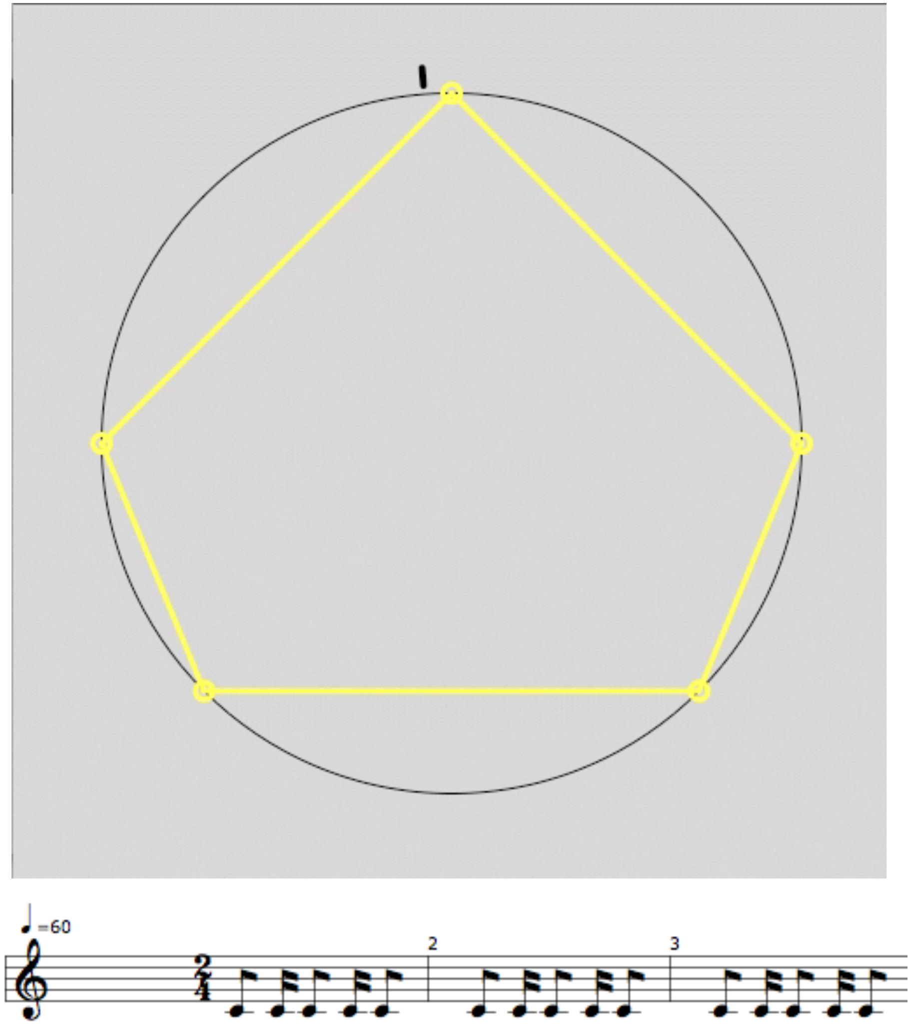
Tiling the line (or the circle) with translates of one tile



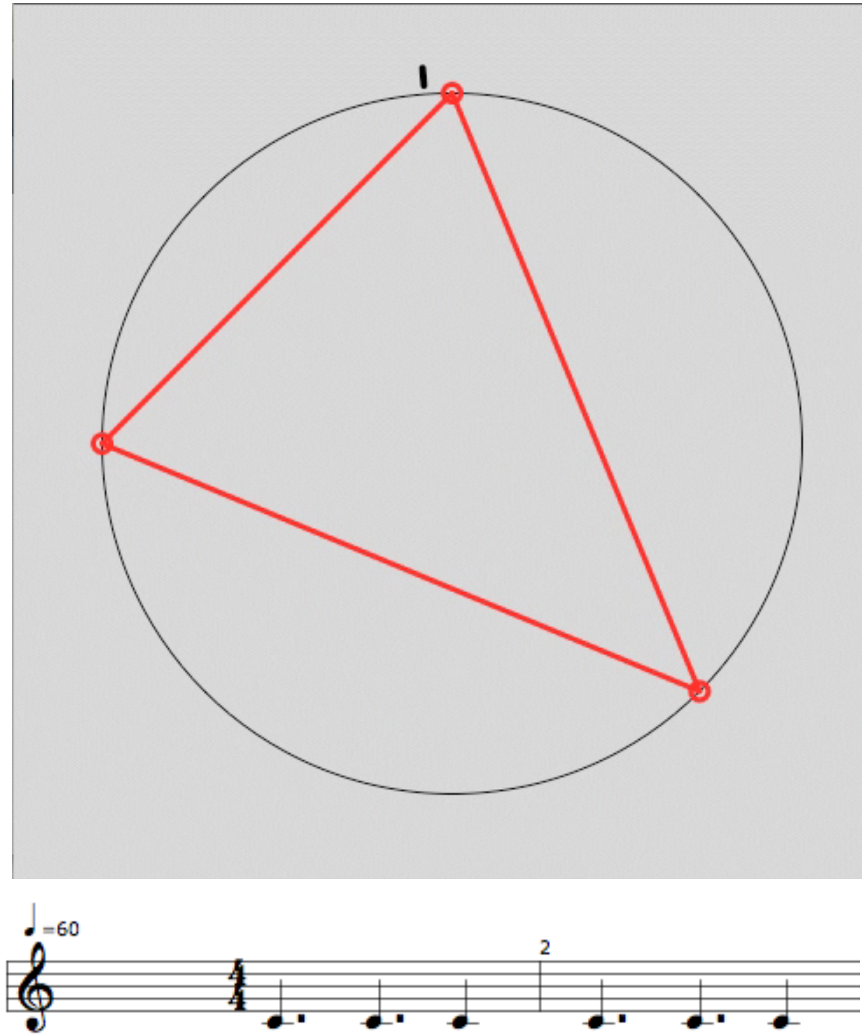


Some examples of traditional rhythms

El cinquillo

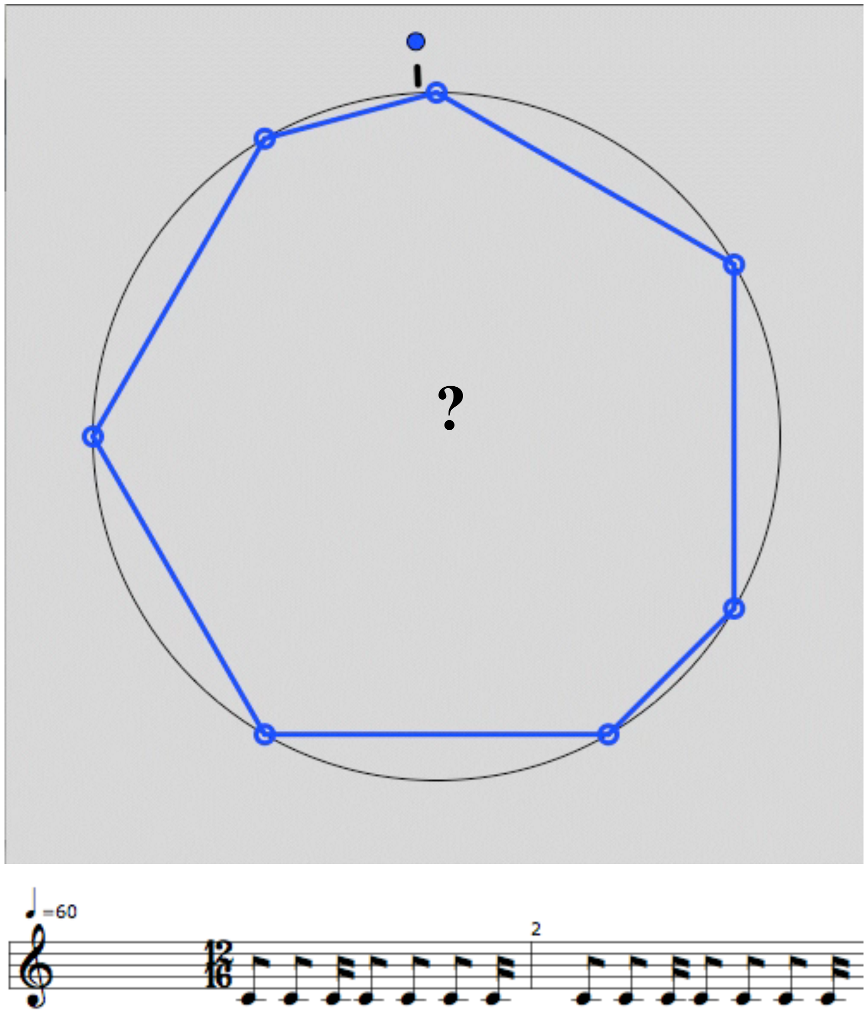


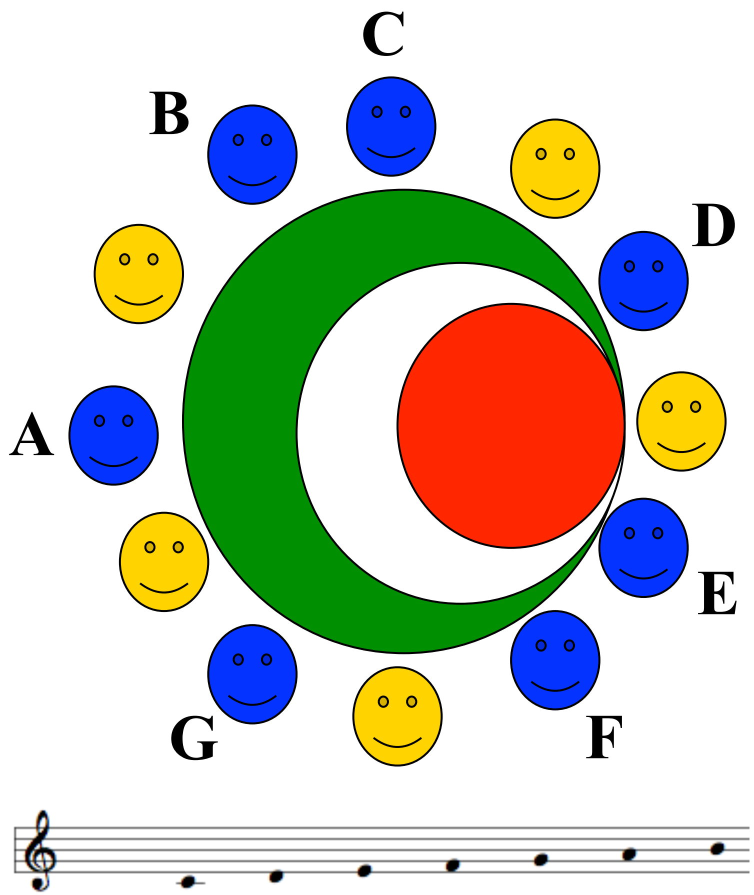
El trecillo



A very particular rhythmic pattern

Bembé



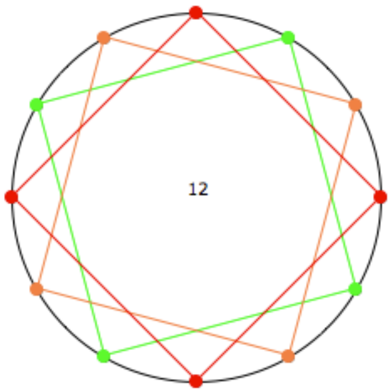


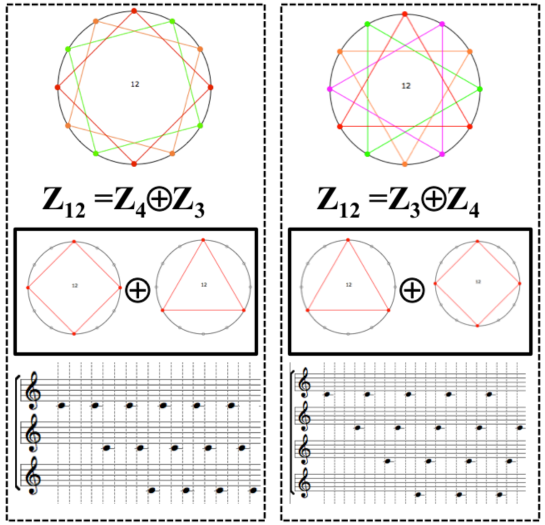
Jack Douthett & Richard Krantz, “Energy extremes and spin configurations for the one-dimensional antiferromagnetic Ising model with arbitrary-range interaction”, *J. Math. Phys.* 37 (7), July 1996

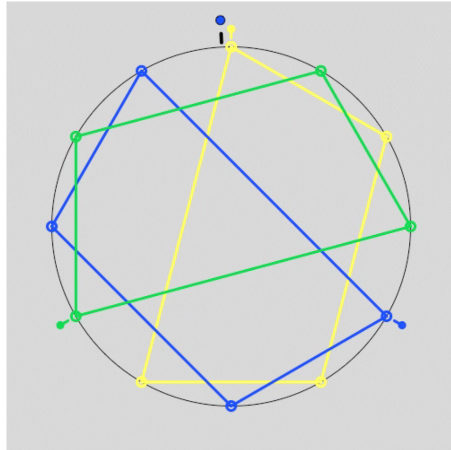
Rhythmic Tiling Canons as group factorisations

A<Z_n

B<Z_n

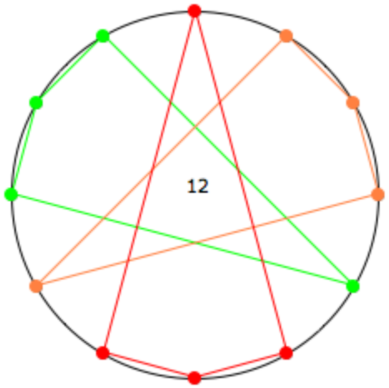


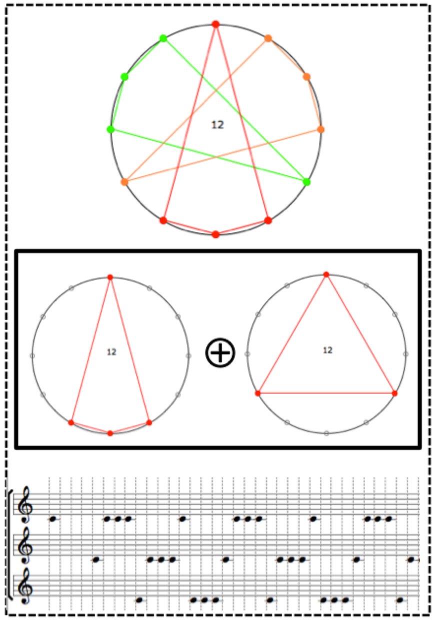




A⊂Z_n

B<Z_n

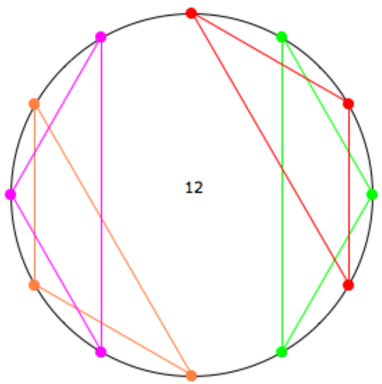


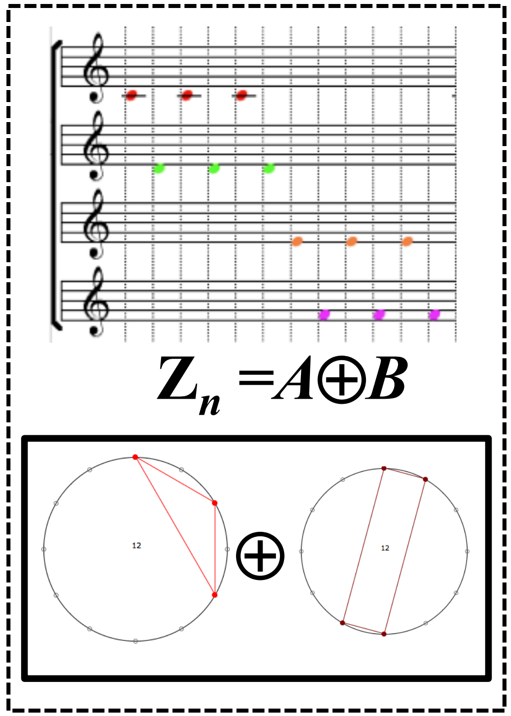


A⊂Z_n

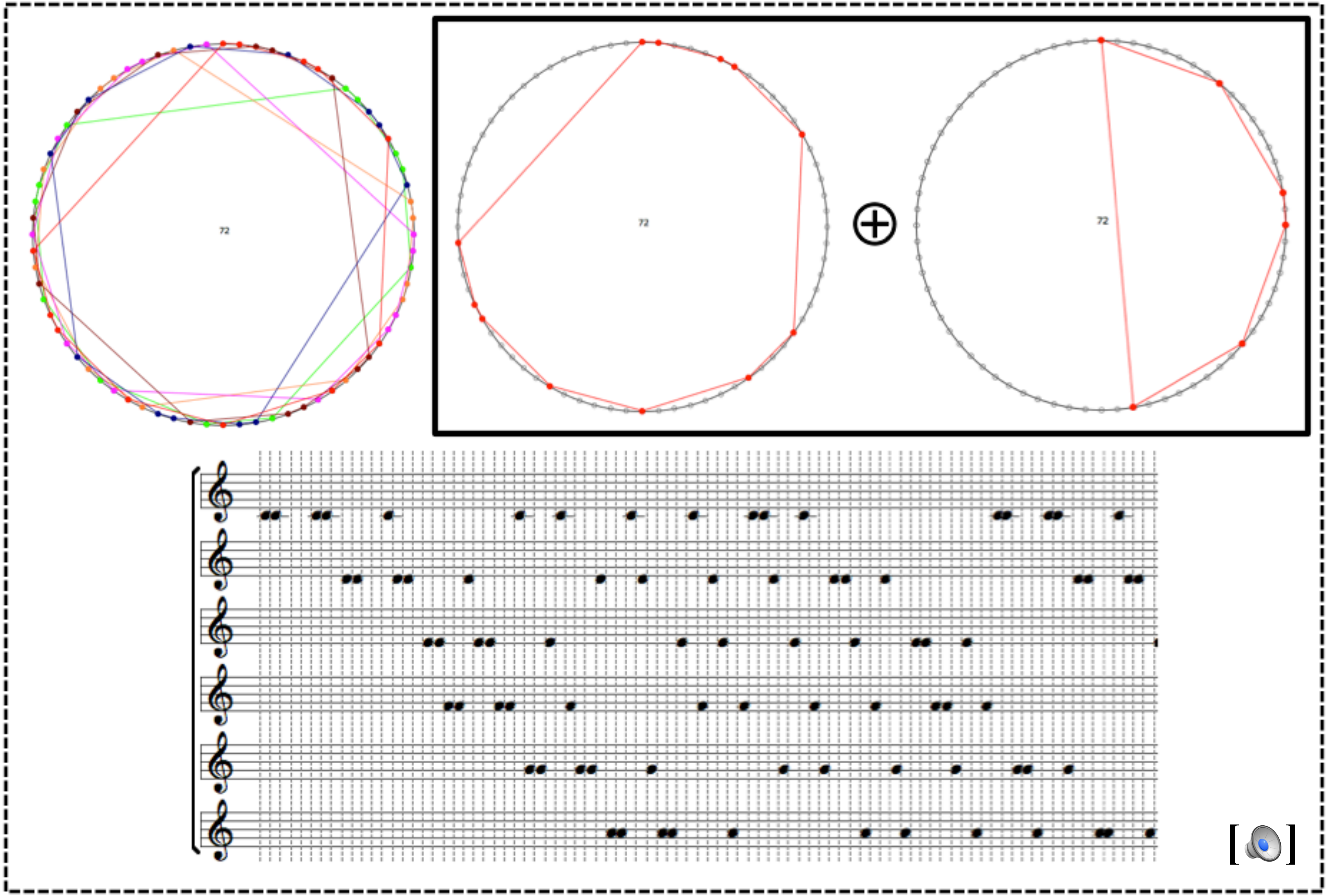
B⊂Z_n

B **periodic subset**

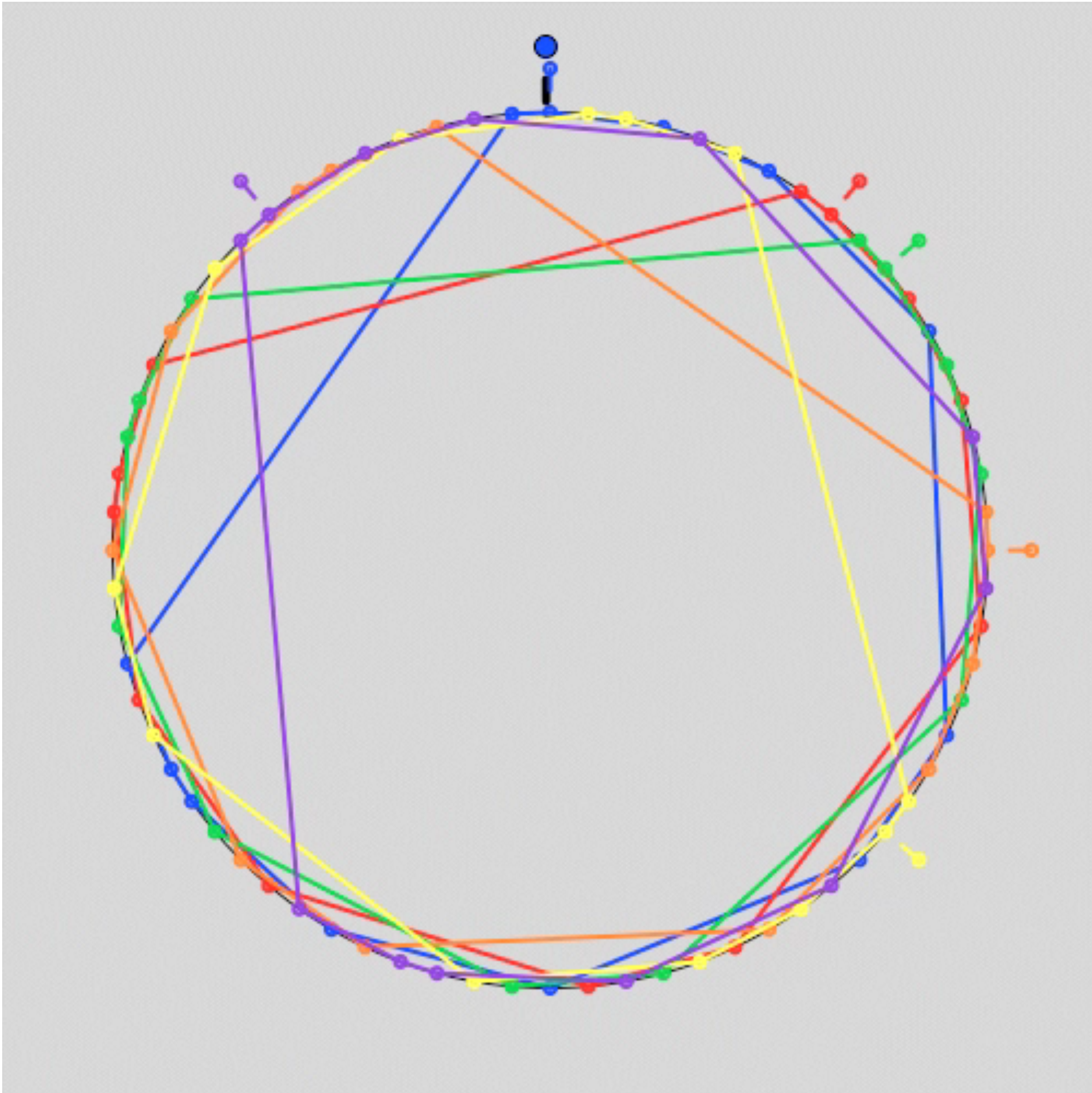


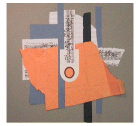


Factorisations without inner periodicity (Vuza Canons)



A Tiling rhythmic canon with unpredictable melody

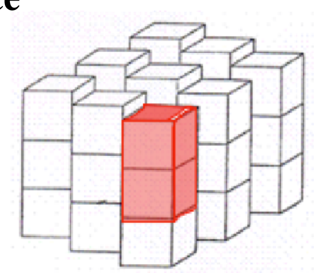
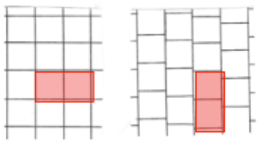




Tiling Rhythmic Canons as a ‘mathémusical’ problem

Minkowski/Hajós Problem (1907-1941)

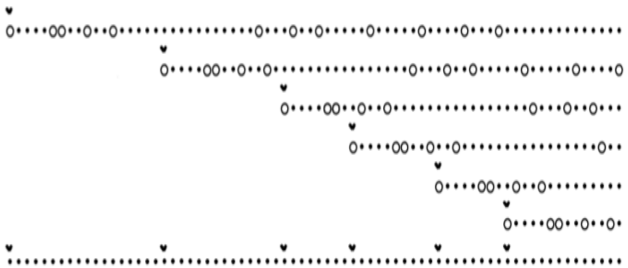

In any simple lattice tiling of the *n*-dimensional Euclidean space by unit cubes, some pair of cubes must share a complete (*n*-1)-dimensional face

Vieru’s problem and Vuza’s formalization (PNM, 1991)


A Vuza Canon is a factorization of a cyclic group in a direct sum of two non-periodic subsets

$$Z/nZ = R \oplus S$$



Link between Minkowski problem and Vuza Canons (Andreatta, Master diss. 1996)

Hajós groups (*good groups*)

Z/nZ with $n \in \{p^\alpha, p^\alpha q, pqr, p^2q^2, p^2qr, pqrs\}$ where p, q, r, s, are distinct prime numbers

Non-Hajós group (*bad groups*)

72
108 120 144 168 180
200 216 240 252 264 270 280 288
300 312 324 336 360 378 392 396
400 408 432 440 450 456 468 480
500 504 520 528 540 552 560 576 588 594
600 612 616 624 648 672 675 680 684 696
700 702 720 728 744 750 756 760 784 792
800 810 816 828 864 880 882 888…

(Sloane’s sequence A102562)

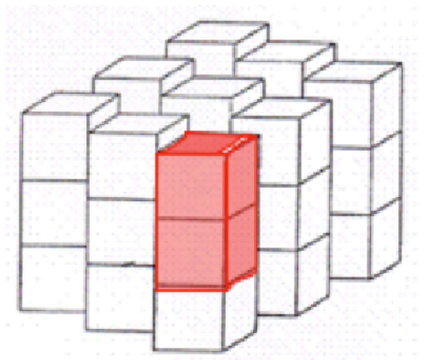
M. Andreatta, “Constructing and Formalizing Tiling Rhythmic Canons : A Historical Survey of a ‘Mathemusical’ Problem,” Perspectives of New Music, Special Issue, 49(1-2), 2011

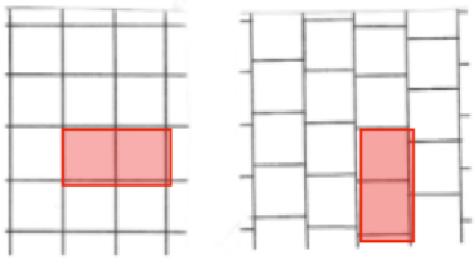
Perspectives of New Music

From Minkowski Conjecture to Hajós groups

Minkowski Conjecture (1907)

In any simple lattice tiling of the n-dimensional Euclidean space by unit cubes, some pair of cubes must share a complete $(n$-$1)$-dimensional face



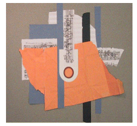


S. Stein, S. Szabó: *Algebra and Tiling,* Carus Math. Mon. 1994

Hajós Theorem (1941)

Given a finite abelian group G and given n elements $a_1, a_2, \ldots,$ a_n of G. If G is a direct product of *cyclic subsets* $A_1,\ldots, A_n$ where

$$A_i = \{e, a_i, a_i^2, \ldots, a_i^{q_i - 1}\}$$



with $q_i > 0$ for all i=1, 2, …, n, then A_k is a group for a given k

Hajós groups (good groups)

Rédei 1947 (p, p)

Hajós 1950 Z

Z/nZ with $n = p^\alpha$

De Brujin 1953 (p^α, q)

(p, q, r)

Sands 1957 (p^2, q^2)

(p^2, q, r)

(p, q, r, s)

Sands 1959 $(2^2, 2^2)$

$(3^2, 3)$

$(2^n, 2)$

Sands 1962 $(p, 3, 3)$

$(p, 2^2, 2)$

$(p, 2, 2, 2, 2)$

$(p^2, 2, 2, 2)$

$(p^3, 2, 2)$

$(p, q, 2, 2)$

Sands 1964 Q

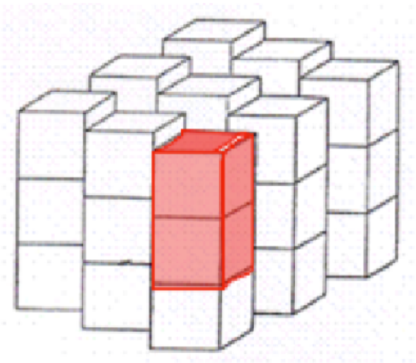
Z+Z/pZ

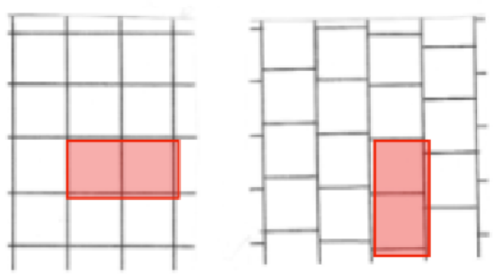
Q+Z/pZ

Weak versions of Minkowski Conjecture

Minkowski Conjecture (1907)

In any simple lattice tiling of the n-dimensional Euclidean space by unit cubes, some pair of cubes must share a complete (n-1)-dimensional face





S. Stein, S. Szabó: *Algebra and Tiling*, Carus Math. Mon. 1994

- ***The four conditions of Minkowski Conjecture***

  [1] The cubes are translates of each other

  [2] The translation vectors form a lattice

  [3] The interiors of the cubes are disjoint

  [4] Each point of the space which is not on the boundary of any cube is contained in exactly one cube

S. Szabó: "Cube Tilings as contribution of Algebra to Geomery", *Beiträge zur Algebra und Geometrie*, 34(1), 1993

- ***Keller Conjecture* (1930) = *Minkowski* – [2]**
  - ➔ **True** for $n \leq 6$ (Perron, 1940)
  - ➔ **False** for $n \geq 8$ (Lagarias et Shor, 1992; Mackey, 2000)
  - ➔ **Open for n=7**

- ***Furtwängler Conjecture* (1936) = *Minkowski* – [3, 4] + new k-fold tiling condition :**

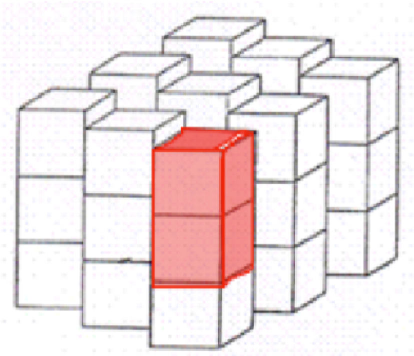
  [4'] Each point not lying on the boudary of any cube is contained in exactly k cubes

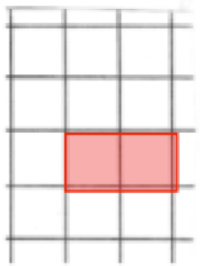
  - ➔ **True** for $n \leq 2$ (Hajos 1941)
  - ➔ **False** in general, since for every $k > 1$ and every $n > 2$ there is a k-fold tiling of n-dimensional space by cubes with no shared faces (Szabó 1982).

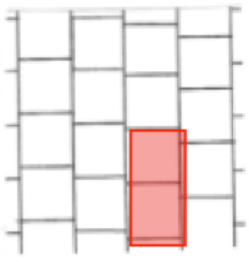
Hajós quasi-periodic Conjecture

Minkowski Conjecture (1907)

In any simple lattice tiling of the n-dimensional Euclidean space by unit cubes, some pair of cubes must share a complete (n-1)-dimensional face







S. Stein, S. Szabó: *Algebra and Tiling,* Carus Math. Mon. 1994

SUR LE PROBLÈME DE FACTORISATION DES GROUPES CYCLIQUES

Par

G. HAJÓS (Budapest), correspondant de l'Académie

6. Toutes les factorisations des groupes abéliens finis connues jusqu'à présent sont *quasipériodiques* dans le sens suivant du mot : Nous disons que la factorisation $C = AB$ est quasipériodique si l'un des facteurs, p. ex. B, est décomposable en ensembles $B_1, B_2, \ldots, B_k$ de la manière que les ensembles $AB_1, AB_2, \ldots, AB_k$ résultent en multipliant l'un d'eux par des éléments d'un sous-ensemble périodique. C'est à dire, il existe un ensemble périodique $(b_1, b_2, \ldots, b_k)$, pour lequel les sous-ensembles AB_i et $AB_1 b_i$ ne diffèrent que dans leur ordre. Il est une question ouverte si toutes les factorisations des groupes abéliens finis sont quasipériodiques. La réponse n'est pas connu même pour le cas des groupes cycliques finis.

• ***Hajos quasi-periodic Conjecture*** **(1950)**

Every factorisation of a finite abelian group $G = A+B$ is *quasi-periodic* i.e. B is decomposable in subsets $B_1, \ldots, B_k$ and there exists a periodic subset $\{g_1, \ldots, g_k\}$ of G such that $A+B_i=g_i+A+B_1$.

➔ **False** for $\mathbf{Z_5} \times \mathbf{Z_{25}}$ (Sands, 1974)

➔ **True** for $\mathbf{Z}_p \times \mathbf{Z}_p$ (Steinberger, 2008)

➔ **Open** in general

Fuglede Spectral Conjecture

A subset of the *n*-dimensional Euclidean space tiles by translation *iff* it is *spectral*.
(*J. Func. Anal.* 16, 1974)

➔**False in dim. *n*≥3**
(Tao, Kolountzakis, Matolcsi, Farkas and Mora)

➔ **Open in dim. 1 et 2**

DEFINITION 6 *A subset A of some vector space (say $\mathbb{R}^n$) is spectral iff it admits a Hilbert base of exponentials, i.e. if any map $f \in L^2(A)$ can be written*

$$f(x) = \sum f_k \exp(2i\pi\lambda_k.x)$$

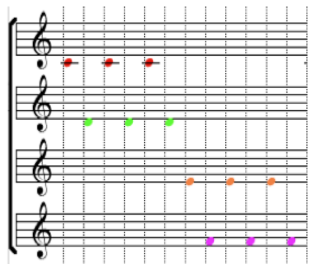
for some fixed family of vectors $(\lambda_k)_{k\in\mathbb{Z}}$ where the maps $e_k : x \mapsto \exp(2i\pi\lambda_k.x)$ are mutually orthogonal (i.e. $\int_A \overline{e_k} e_j = 0$ whenever $k \neq j$).

↓ ***n*=1**

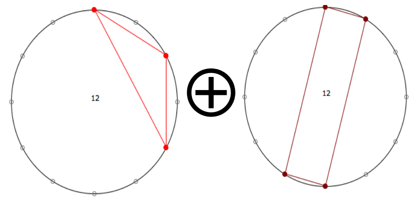
DEFINITION 8. *A subset $A \in \mathbb{Z}$ is* spectral *if there exists a spectrum $\Lambda \subset [0,1]$ (i.e., a subset with the same cardinality as A) such that $e^{2i\pi(\lambda_i - \lambda_j)}$ is a root of $A(X)$ for all distinct $\lambda_i, \lambda_j \in \Lambda$.*

Example:
$A=\{0,1,6,7\}$ ➔ $\Lambda=\{0,1/12,1/2,7/12\}$
since exp(πi), exp(πi/6), exp(-πi/6), exp(5πi/6), exp(-5πi/6) are the roots of the associated polynomial
$A(X)=1+X+X^6+X^7$

→

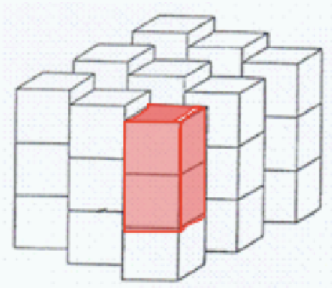


$\mathbf{Z}_n = B \oplus A$



Minkowski/Hajos Problem and Fuglede Conjecture

Minkowski/Hajos Problem



In any simple lattice tiling of the *n*-dimensional Euclidean space by unit cubes, some pair of cubes must share a complete (*n*-1)-dimensional face

?

Fuglede Spectral Conjecture

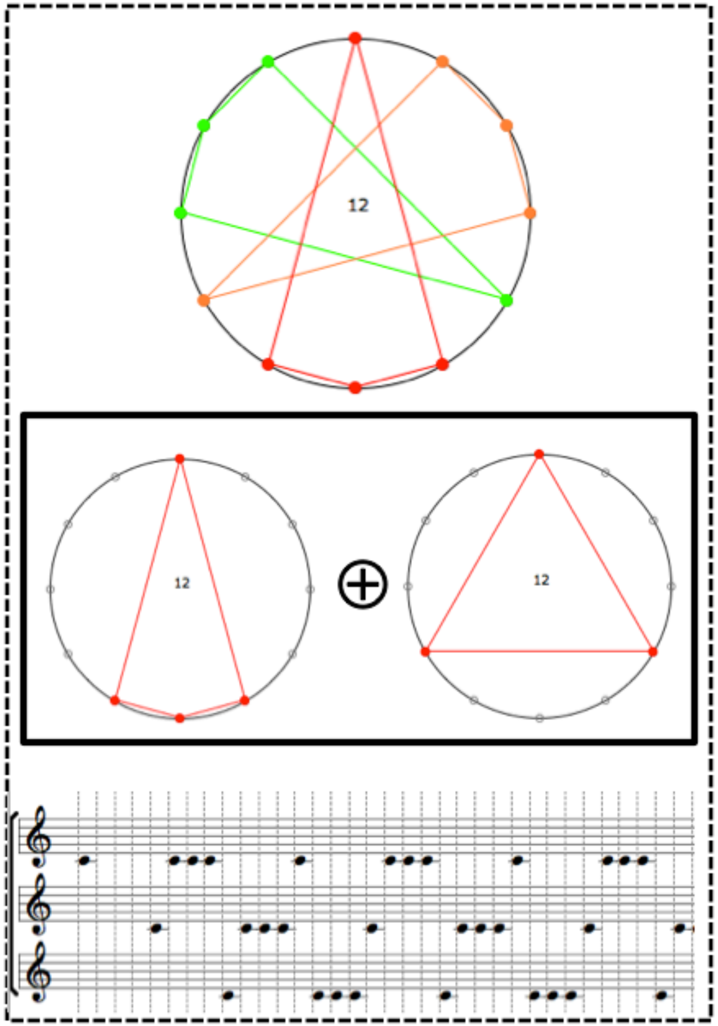
A subset of the *n*-dimensional Euclidean space tiles by translation iff it is *spectral*

- R. Tijdeman: “Decomposition of the Integers as a direct sum of two subsets”, *Number Theory*, Cambridge University Press, 1995. The fundamental Lemma:
  A tiles $\mathbf{Z}_n$ => *p*A tiles $\mathbf{Z}_n$ when $<p,n>=1$

- E. Coven & A. Meyerowitz: “Tiling the integers with translates of one finite set”, *J. Algebra*, 212, pp.161-174, 1999
  T1 + T2 => tile
  Tile => T1

- I. Laba : “The spectral set conjecture and multiplicative properties of roots of polynomials”, *J. Lond Math Soc*, 2002
  - T1 + T2 => spectral
  - T2 => spectral
  - spectral => T1

Polynomial Representations of Tiling Canons



$Z_n \longleftrightarrow 1 + X + X^2 + \ldots + X^{n-1}$

$A=\{0,5,6,7\} \longleftrightarrow A(X)=1 + X^5 + X^6 + X^7$

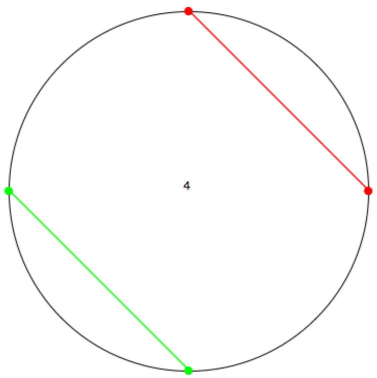
$B=\{0,4,8\} \longleftrightarrow B(X)=1 + X^4 + X^8$

$$\Delta_{12} = 1 + X + \ldots + X^{11} = A(X)\times B(X) \bmod X^{12}-1$$

Using cyclotomic polynomials to build tiling canons

$$\Phi_n(X) = \prod_{k=1}^{\varphi(n)} (X - z_k) \longleftrightarrow X^n - 1 = \prod_{d|n} \Phi_d(X).$$

$\Phi_1(X) = X - 1$	$\longleftrightarrow$	?
$\Phi_2(X) = 1 + X$	$\longleftrightarrow$	{0,1}
$\Phi_3(X) = 1 + X + X^2$	$\longleftrightarrow$	{0,1,2}
$\Phi_4(X) = 1 + X^2$	$\longleftrightarrow$	{0,2}
$\Phi_5(X) = 1 + X + X^2 + X^3 + X^4$	$\longleftrightarrow$	{0,1,2,3,4}
$\Phi_6(X) = 1 - X + X^2$	$\longleftrightarrow$	?



$$\Delta_n = 1 + X + X^2 + \ldots + X^{n-1} = \prod_{\substack{d|n \\ d \neq 1}} \Phi_d(X)$$

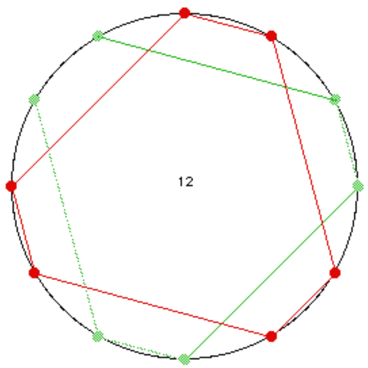
$$\Delta_4 = 1 + X + X^2 + X^3 = \Phi_2(X) \times \Phi_4(X)$$

$$A(x) \times B(x) = (A \oplus B)(x) \equiv 1 + x + \ldots x^{n-1} \pmod{X^n - 1}$$

Good and bad decompositions

$$\Delta_n = 1 + X + X^2 + \ldots + X^{n-1} = \prod \Phi_d(X) \qquad \begin{matrix} d \mid n \\ d \neq 1 \end{matrix}$$

$$\Phi_2(X) = 1 + X$$
$$\Phi_3(X) = 1 + X + X^2$$
$$\Phi_4(X) = 1 + X^2$$
$$\Phi_6(X) = 1 - X + X^2$$



$$\Delta_{12} = 1 + X + \ldots + X^{11} = \Phi_2 \times \Phi_3 \times \Phi_4 \times \Phi_6 \times \Phi_{12}$$

$$A(X) = \Phi_2 \times \Phi_3 \times \Phi_6 \times \Phi_{12} = 1 + X + X^4 + X^5 + X^8 + X^9$$

$$B(X) = \Phi_4 = 1 + X^2$$

$$S = \{0, 2\}$$

$$R = \{0, 1, 4, 5, 8, 9\}$$

Coven & Meyerowitz conditions

$$\Phi_2(X) = 1 + X$$
$$\Phi_3(X) = 1 + X + X^2$$

(T1) $A(1) = 6 = \Phi_2(1) \times \Phi_3(1) = 2 \times 3$

(T2) $\Phi_2 \mid A(X)$ et $\Phi_3 \mid A(X) \Rightarrow \Phi_{2\times 3} \mid A(X)$

Coven and Meyerowitz Conditions

- E. Coven & A. Meyerowitz : "Tiling the integers with translates of one finite set", *J. Algebra*, 212, pp.161-174, 1999

There is no loss of generality in restricting attention to translates of a finite set A of *nonnegative* integers. Then $A(x) = \sum_{a \in A} x^a$ is a polynomial such that $\#A = A(1)$. Let S_A be the set of prime powers s such that the s-th cyclotomic polynomial $\Phi_s(x)$ divides $A(x)$. Consider the following conditions on $A(x)$.

(T1) $A(1) = \prod_{s \in S_A} \Phi_s(1)$.

(T2) If $s_1, \ldots, s_m \in S_A$ are powers of distinct primes, then $\Phi_{s_1 \cdots s_m}(x)$ divides $A(x)$.

Theorem A. *If $A(x)$ satisfies (T1) and (T2), then A tiles the integers.*

Theorem B1. *If A tiles the integers, then $A(x)$ satisfies (T1).*

Theorem B2. *If A tiles the integers and $\#A$ has at most two prime factors, then $A(x)$ satisfies (T2).*

Corollary. *If $\#A$ has at most two prime factors, then A tiles the integers if and only if $A(x)$ satisfies (T1) and (T2).*

A bad decomposition

$$\Delta_n = 1 + X + X^2 + \ldots + X^{n-1} = \prod_{\substack{d \mid n \\ d \neq 1}} \Phi_d(X)$$

$$\Phi_2(X) = 1 + X$$
$$\Phi_3(X) = 1 + X + X^2$$
$$\Phi_4(X) = 1 + X^2$$
$$\Phi_6(X) = 1 - X + X^2$$

$$\Delta_{12} = 1 + X + \ldots + X^{11} = \Phi_2 \times \Phi_3 \times \Phi_4 \times \Phi_6 \times \Phi_{12}$$

$$A(X) = \Phi_2 \times \Phi_3 \times \Phi_{12} = 1 + 2X + X^2 - X^3 - X^4 + X^5 + 2X^6 + X^7$$

$$B(X) = \Phi_4 \times \Phi_6 = 1 - X + 2X^2 - X^3 + X^4$$

$S = ?$

$R = ?$

Coven & Meyerowitz conditions

$$\Phi_2(X) = 1 + X$$
$$\Phi_3(X) = 1 + X + X^2$$

(T1) $A(1) = 7 \neq \Phi_2(1) \times \Phi_3(1) = 2 \times 3$

(T2) $\Phi_2 \mid A(X)$ et $\Phi_3 \mid A(X) \Rightarrow \Phi_{2\times 3} \mid A(X)$

C&M Conditions, tiling and spectrality

(T1) $A(1) = \prod_{s \in S_A} \Phi_s(1)$.
(T2) If $s_1, \ldots, s_m \in S_A$ are powers of distinct primes, then $\Phi_{s_1 \cdots s_m}(x)$ divides $A(x)$.

C&M (1999)

- T1 + T2 => tiling
- tiling => T1
- tiling Z_n where n has at most two prime factors => T1 + T2

Laba (2002)

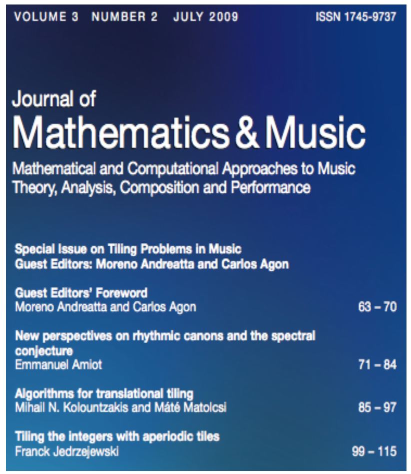
- T1 + T2 => spectral
- T2 => spectral
- spectral => T1

Amiot (2009)

- Tiling of a Hajos group => T2.

This means that if A tiles $\mathbf{Z}_n$ without being spectral => A is the rhythm of a Vuza Canon

E. Amiot, "New perspectives on rhythmic canons and the spectral conjecture", Special Issue 'Tiling Problems in Music', *Journal of Mathematics and Music*, July 2009, 3(2).



Is condition T2 mandatory for tiling?

Paradigmatic classification of Vuza Canons

Vuza Canons of period *n*

- $n=p_1p_2n_1n_2n_3$
- $<p_1n_1, p_2n_2>=1$
- $n_3 > 1$

72
108 120 144 168 180
200 216 240 252 264 270 280 288
300 312 324 336 360 378 392 396
400 408 432 440 450 456 468 480
500 504 520 528 540 552 560 576 588 594
…

***n*=72**

R
(1 3 3 6 11 4 9 6 5 1 3 20)
(20 3 1 5 6 9 4 11 6 3 3 1)
(1 4 1 19 4 1 6 6 7 4 13 6)
(6 13 4 7 6 6 1 4 19 1 4 1)
(1 5 15 4 5 6 6 3 4 17 3 3)
(3 3 17 4 3 6 6 5 4 15 5 1)

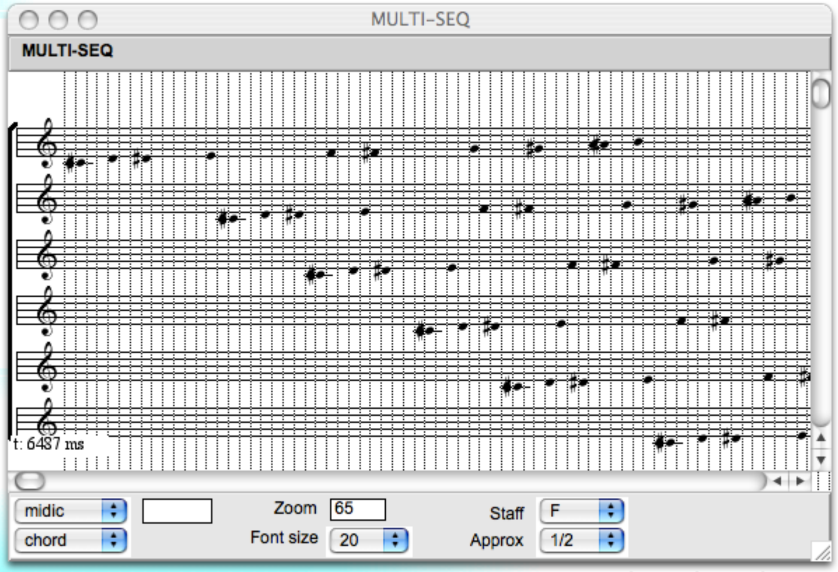
S
(8 8 2 8 8 38)
(16 2 14 2 16 22)
(14 8 10 8 14 18)

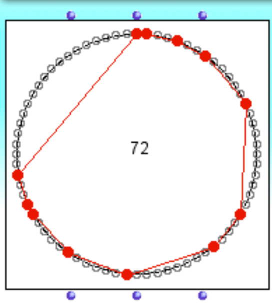
R
(1 3 3 6 11 4 9 6 5 1 3 20)
(1 4 1 19 4 1 6 6 7 4 13 6)
(1 5 15 4 5 6 6 3 4 17 3 3)

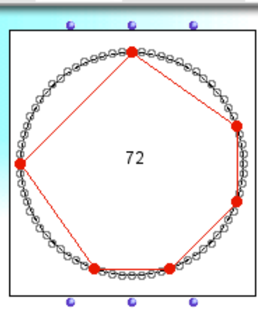
S
(8 8 2 8 8 38)
(16 2 14 2 16 22)
(14 8 10 8 14 18)

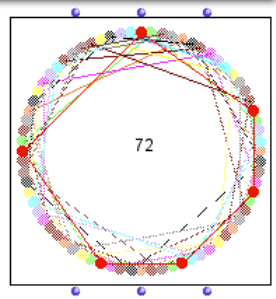
R
(1 3 3 6 11 4 9 6 5 1 3 20)
(1 4 1 19 4 1 6 6 7 4 13 6)

S
(14 8 10 8 14 18)









- **Tijdeman's 'Fundamental Lemma':**
  ***R* tiles Z_n => *aR* tiles Z_n**
  $<a,n>=1$

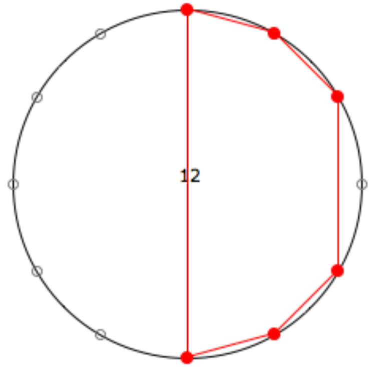
Conclusion:
There are only two « types » of Vuza Canons of period 72 (up to an affine transformation)

Tiling and Discrete Fourier Transform

$$IC_A(k) = \mathrm{Card}\{(x,y) \in A \times A \mid x + k = y\}$$

$$IC_A(k) = (1_A \star 1_{-A})(k)$$

(D. Lewin, *JMT*, 1958)



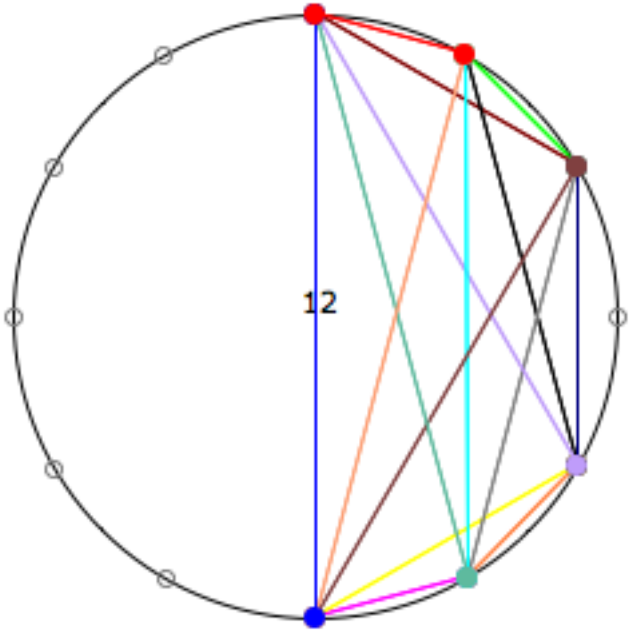
$$\mathcal{F}_A : t \mapsto \sum_{k \in A} e^{-2i\pi kt/c}$$

TILING

Let $Z_A = \{ t \in \mathbb{Z}_c, F_A(t)=0 \}$

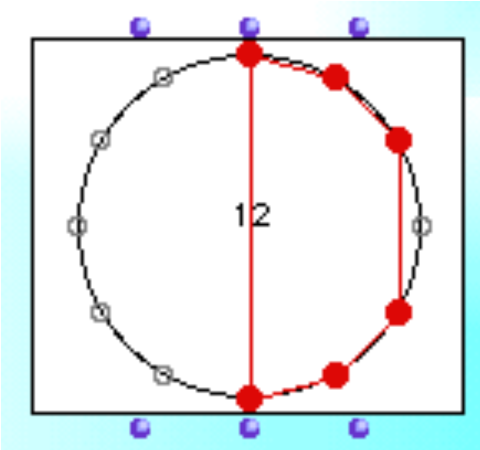
A tiles $\mathbb{Z}_c$ when equivalently:

- There exists B, $A \oplus B = \mathbb{Z}_c$
- $1_A \star 1_B = 1$
- $F_A \times F_B\ (t) = 1 + e^{-2i\pi t/c} + \ldots\ e^{-2i\pi t(c-1)/c}$ (0 unless t=0)
- $Z_A \cup Z_B = \{1,2 \ldots c-1\}$ AND Card A × Card B = c
- $IC_A \star IC_B = IC(\mathbb{Z}_c)=c$ and Card A × Card B = c

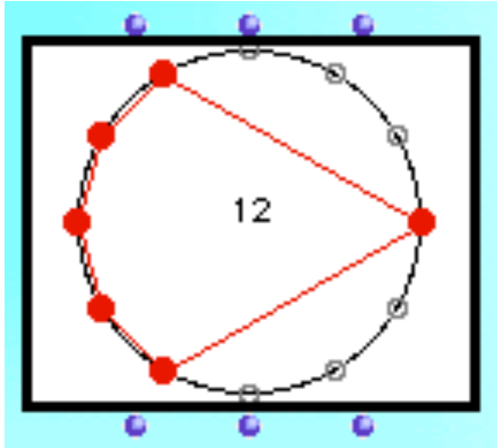


A short (and elegant) proof of Babbitt's Theorem

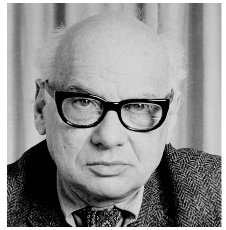
A



A hexacord and its complement have the same interval content



A'



$$IC_A = [4, 3, 2, 3, 2, 1] \quad = \quad [4, 3, 2, 3, 2, 1] = IC_{A'}$$

$$IC_A(k) = (1_A \star 1_{-A})(k)$$

$$\mathcal{F}_A : t \mapsto \sum_{k \in A} e^{-2i\pi kt/c}$$

$$\forall k\ \mathcal{F}(IC_{\mathbb{Z}_c \setminus A})(k) = \mathcal{F}(IC_A)(k)$$

E. Amiot : « Une preuve élégante du théorème de Babbitt par transformée de Fourier discrète », Quadrature, 61, 2006.

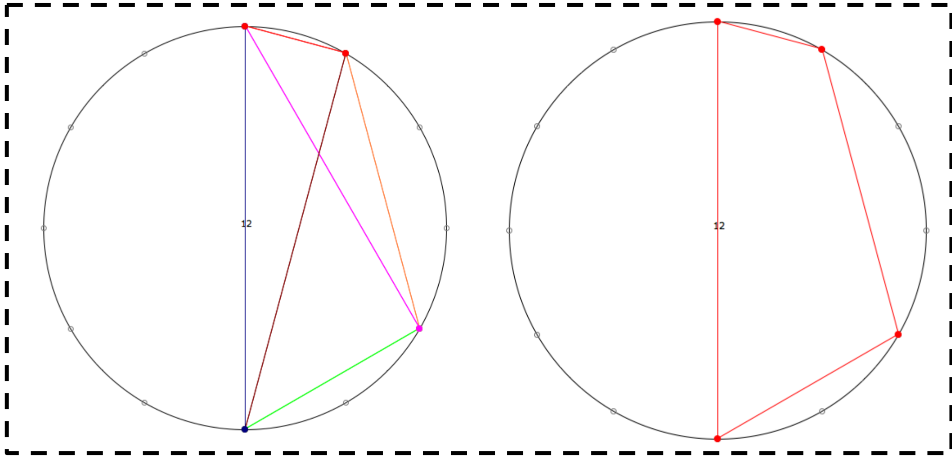
Z-relation, homometry and phase retrieval problem

- Two sets are Z-related if they have the same module of the DFT

$$IC_A(k) = (1_A \star 1_{-A})(k)$$

$$\mathcal{F}_A : t \mapsto \sum_{k \in A} e^{-2i\pi kt/c}$$

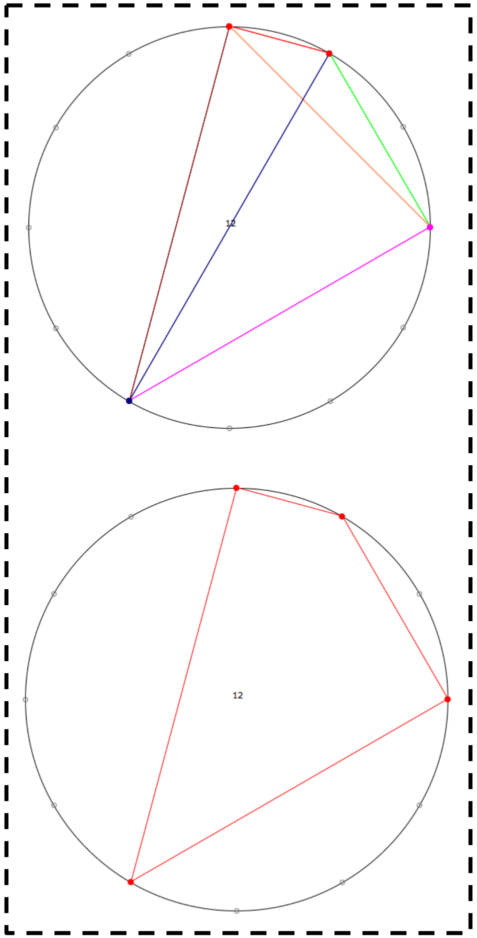
$$\mathcal{F}(IC_A) = \mathcal{F}_A \times \mathcal{F}_{-A} = |\mathcal{F}_A|^2$$



Z-relation

⟷

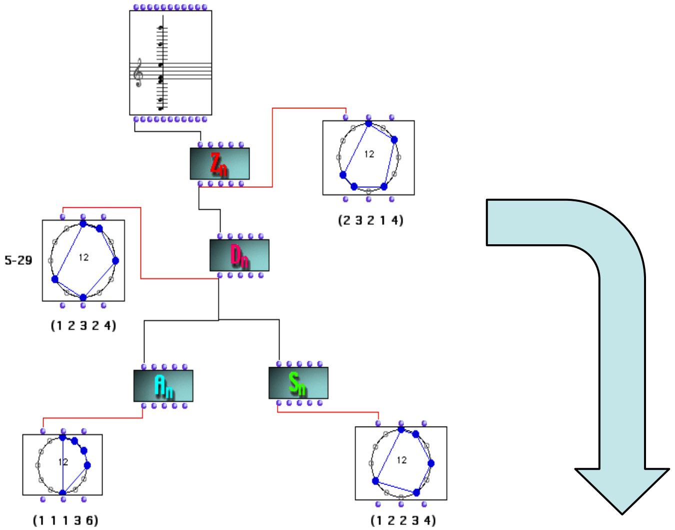
homometry



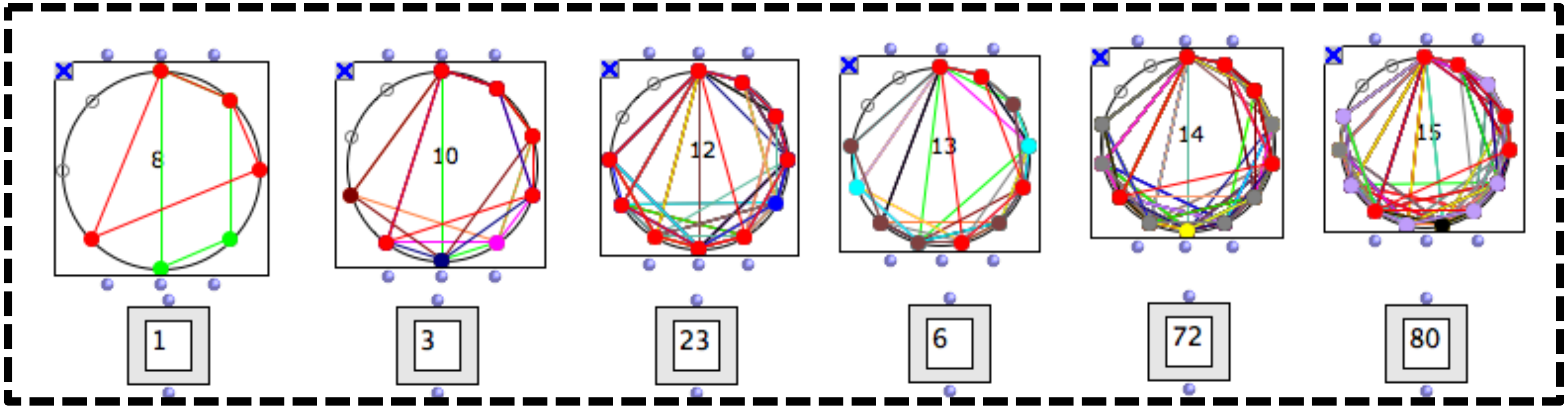
• Mandereau J., D. Ghisi, E. Amiot, M. Andreatta, C. Agon, (2011), « Z-relation and homometry in musical distributions », *Journal of Mathematics and Music*, vol. 5, n° 2, p. 83-98.

• Mandereau J, D. Ghisi, E. Amiot, M. Andreatta, C. Agon (2011), « Discrete phase retrieval in musical structures », *Journal of Mathematics and Music*, vol. 5, n° 2, p. 99-116.

Paradigmatic classification and homometry

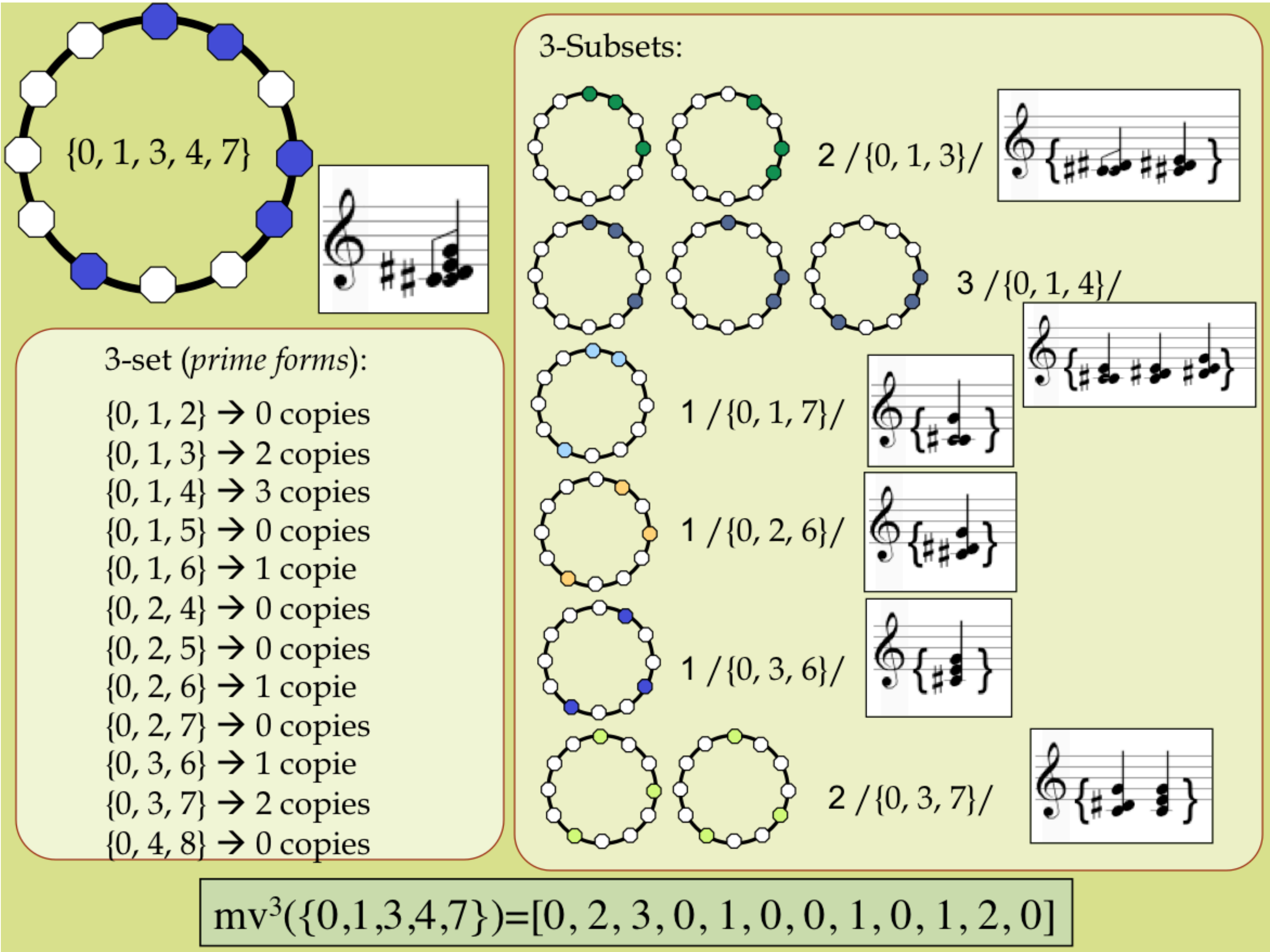


- *Is there a (non-trivial) group action whose orbits are the equivalence classes of homometric sets?*
- *Is there an enumeration formula for homometric sets?*



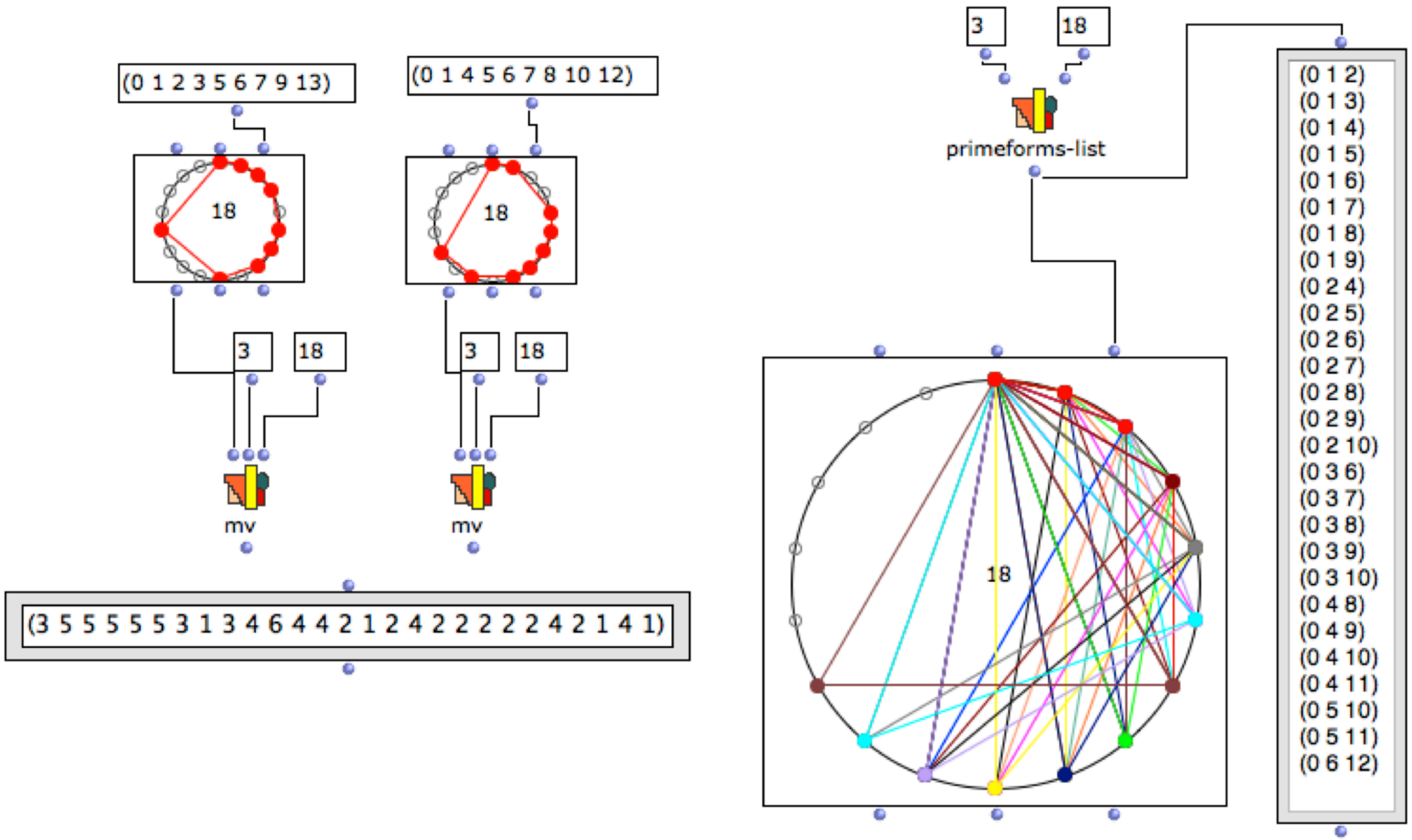
• John Mandereau, *Étude des ensembles homométriques et leur application en théorie mathématique de la musique et en composition assistée par ordinateur*, Master Thesis, ATIAM, Ircam/Université Paris 6, juin 2009

High-order ‘interval’ content: Lewin’s mv^k vector



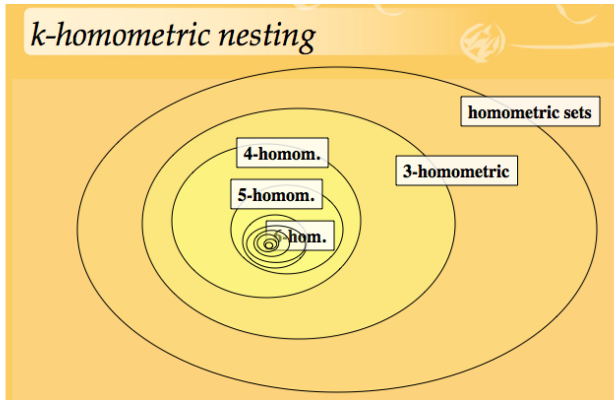
• Daniele Ghisi, "From Z-relation to homometry: an introduction and some musical examples", Séminaire MaMuX, 10 décembre 2010 [http://repmus.ircam.fr/mamux/saisons/saison10-2010-2011/2010-12-10]

High-order Z-relation and *k*-homometric nesting



$\mathbf{Z}_{18}$ A = {0, 1, 2, 3, 5, 6, 7, 9, 13}
B = {0, 1, 4, 5, 6, 7, 8, 10, 12}

$mv^3(A) = mv^3(B)$ ➔ 3-homometry (➔ 2-homometry)



Where does this nesting stop?

The extended phase retrieval and related open problems

- The reconstruction index $R(n)$ is the minimum integer k for which there exist no k-Homometric sets in $\mathbf{Z}_n$ (i.e. $\mathbf{mv}^3$ provides enough information to reconstruct the set)

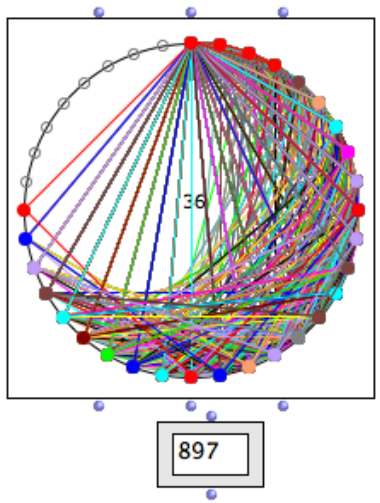
Proposition 4.8

$$R(n) = \begin{cases} 1 & \text{if } n = 1, 2, 3 \\ 2 & \text{if } n = 4, 5, 6, 7, 9, 11 \\ 3 & \text{if } n = 8, 10, 12, 13, 14, 15, 16, 17, 19, 22, 23, 25, 29, 31, 37 \\ 4 & \text{if } n = 18, 20, 21, 24, 26, 27, 28, 30, 32, 33, 34, 35 \\ 5 & \text{if } n = 36 \end{cases}$$

$\mathbf{Z}_{36}$

A = {0, 1, 2, 3, 4, 5, 7, 10, 12, 15, 19, 20, 22, 23, 24, 25, 27, 28}
B = {0, 1, 2, 3, 4, 5, 6, 9, 14, 17, 18, 19, 21, 22, 24, 26, 27, 29}

mv^4(A)= mv^4(B) ➔ 4-homometry



Homometry and Tiling Rhythmic Canons

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- $IC_A \star IC_B = IC(\mathbb{Z}_c) = c$ and Card A × Card B = c

$$\mathcal{F}_A : t \mapsto \sum_{k \in A} e^{-2i\pi kt/c}$$

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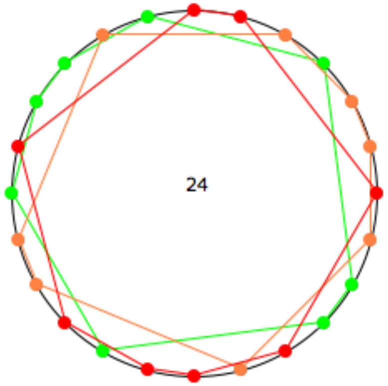
A musical offering:

- ***Theorem***:
  If A tiles with B and A' has the same IC, then A' tiles with B, too.



E. Amiot, « New Perspectives on Rhythmic Canons and the Spectral Conjecture », *JMM*, 2009.

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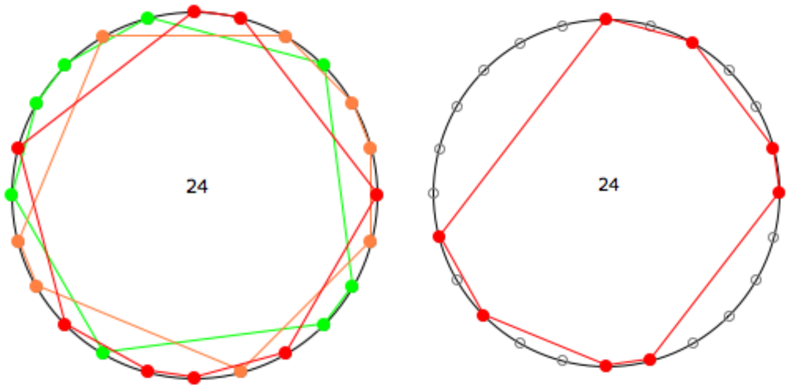

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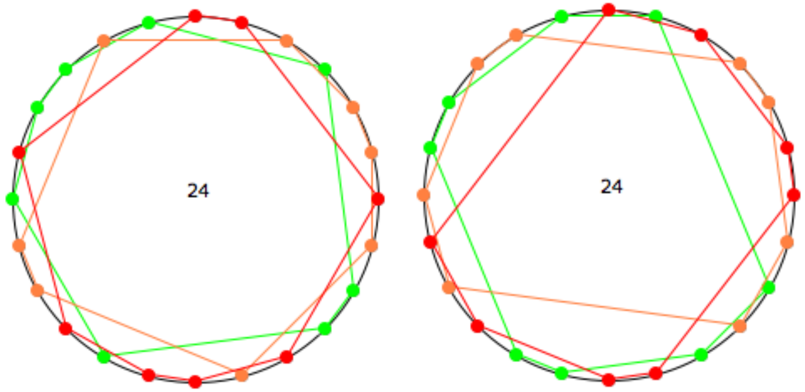

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Thank you for your attention!