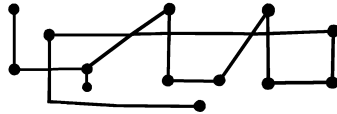


CONSTRUCTING AND FORMALIZING TILING RHYTHMIC CANONS: A HISTORICAL SURVEY OF A “MATHEMUSICAL” PROBLEM



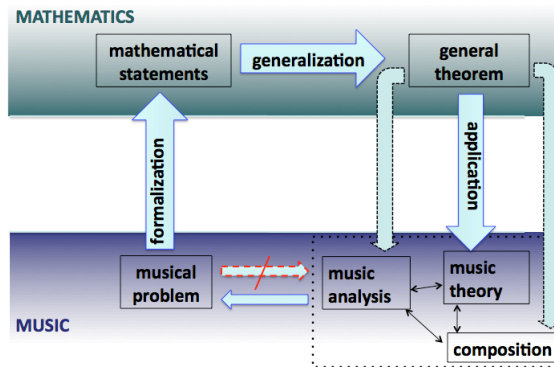
MORENO ANDREATTA

IN THIS PAPER we trace back the history of the construction and algebraic formalization of Tiling Rhythmic Canons as a “mathemusical” problem. This neologism, which is now commonly used in mathematical music theory, was introduced by the author in the early nineties with the aim of suggesting that, although the history of relations between music and mathematics offers many examples of applications of mathematical models to music, there are also several examples showing that music can be sometimes a very important source of inspirational ideas for mathematicians. One may thus balance the usual Leibnizian perspective of music as an *exercitium arithmeticae* by proposing that the reverse hypothesis also holds, according to

which mathematics can be considered, in some special cases, as an *exercitium musicae*.¹

A “mathemusal” problem is characterized by a dynamic movement between a given musical problem, the mathematical statements and general theorems obtained respectively by the formalization and generalization processes, and the musical application of these results in music theory, analysis, and/or composition. In other words, setting the originally musical problem in an appropriate mathematical framework not only gives rise, eventually, to new mathematical results, but also paves the way to new music-theoretical, analytical, or compositional constructions that would have been impossible to conceive without the process of mathematization. Although generalized mathematical results can be applied to the music domain with a focus on music theory, analysis, or composition, in the case of the algebraic approach, it is often very difficult to distinguish between these three domains, as we discussed in a previous study (see Andreatta 2003). It is this double movement, from music to mathematics and back, which makes a “mathemusal” problem so intriguing to both mathematicians and musicians. Example 1 shows the dynamical movement underlying a “mathemusal” activity.

In this paper I will discuss some historical aspects of the construction of Tiling Rhythmic Canons from the previously sketched “mathemusal” perspective. I will present two independent developments of



EXAMPLE 1: THE DOUBLE MOVEMENT OF A “MATHEMUSAL” ACTIVITY, FROM A MUSICAL PROBLEM TO ITS FORMALIZATION, GENERALIZATION, AND FINAL APPLICATION TO THE MUSIC DOMAIN

music-theoretical constructions connected with tiling rhythmic canon structures: the first arising from Messiaen's practice as "rythmicien," and the second exploring the rhythmic analogies of Anatol Vieru's modal theory.

Olivier Messiaen (1908–1992) made undoubtedly one of the most significant efforts to study canons by abstracting the pitch content and focusing on the underlying rhythmic structure. A brief description of Messiaen's compositional practice shows that the starting point is a genuine compositional problem related to some apparently very different theoretical constructions, such as Anatol Vieru's composition of modal classes or Pierre Boulez's chord multiplication.

The history of tiling rhythmic canons is particularly interesting because it intersects a parallel history of mathematics, from number theory and the geometry of tiling to operator theory in functional analysis. We will discuss some aspects of these developments by sketching the Minkowski-Hajós problem, from the original formulation by Hermann Minkowski (1896), concerning the approximation of real numbers by means of rational numbers, through its successive interpretation from a geometric perspective again by Minkowski (1907), and up to the definitive solution provided by Hajós in the forties (Hajós 1941). Although the Minkowski-Hajós problem is totally solved, there are some weak versions of the Minkowski conjecture that are still open, and whose musical application seems potentially interesting. We will briefly present some of these conjectures before focusing on a second parallel development of the theory of tiling that originated in a problem raised by Bent Fuglede in functional analysis (Fuglede 1974), and whose solution is deeply related to a special class of tiling rhythmic canons: Vuza canons.

1. THE MUSICAL PROBLEM OF THE CONSTRUCTION OF TILING RHYTHMIC CANONS AS SEEN FROM DIFFERENT COMPOSITIONAL PERSPECTIVES

There is probably no need to explain in depth the relevance of canons to music. There are few musical concepts that have been used as extensively, well beyond the boundaries of the Western classical music tradition, from the complex polyphonic structures of the *Ars Nova* (Schiltz & Blackburn 2007) to the orally transmitted musical practices of the Nzakara pygmies of the Central African Republic (Chemillier 2002). As we already mentioned, Messiaen is probably one of the first composers to abstract the canonic process from all pitch considerations by focusing only on the rhythmic domain. The second of the seven volumes of his *Traité de rythme, de couleur et d'ornithologie* (1949–

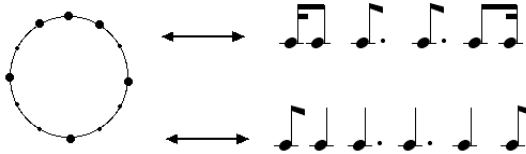
1992) is entirely devoted to the study of rhythmic structures (e.g., non-invertible rhythms, augmentation and diminution, irrational values, . . .). The problem of construction of rhythmic canons is developed in the second chapter of the volume entitled “Pedals and rhythmic canons.” As far as we know, this is the very first attempt toward a definition of the form of musical canons by considering exclusively the rhythmic organization and without starting from the melodic structure or the harmonic organization. Messiaen is particularly interested in establishing a connection between rhythmic canons and his favorite technique of non-invertible rhythms. According to the definition provided by Messiaen, non-invertible rhythms (or, equivalently, non-retrogradable rhythms) are “two groups of durations, one the retrograde of the other, surrounding [*encadrant*] a central free value which is common to the two groups. By reading the rhythm from left to right or from right to left, the order of its duration remains the same. It is a totally closed [*fermé*] rhythm” (Messiaen 1994, 7). As Messiaen rightly observes, the non-invertibility corresponds to what is traditionally called a palindrome. A non-invertible rhythm is thus a rhythm whose succession of duration values corresponds to a palindromic sequence of numbers, the central value being either a single number or the repetition of a given number. In his *Traité*, Messiaen postulates many times the analogy between non-invertible rhythms and modes with limited transpositions (*modes à transpositions limitées*). The following quotation, also reprinted in the *Traité*, originally appears in his *Technique de mon langage musical*:

These modes [of Limited Transpositions] realize in the vertical direction (transposition) what non-retrogradable rhythms realize in the horizontal direction (retrogradation). In fact, these modes cannot be transposed beyond a certain number of transpositions without falling again into the same notes, enharmonically speaking; likewise, these rhythms cannot be read in a retrograde sense without one’s finding again exactly the same order of values as in the right sense. These modes cannot be transposed because they are—without polytonality—in the modal atmosphere of several keys at once and contain in themselves small transpositions; these rhythms cannot be retrograded because they contain in themselves small retrogradations. These modes are divisible into symmetrical groups; these rhythms, also, with this difference: the symmetry of the rhythmic groups is a retrograde symmetry. Finally, the last note of each group of these modes is always *common* with the first of the following group; and the groups of these rhythms frame a

central value *common* to each group. The analogy is now complete (Messiaen 1956, 21).

The analogy is, unfortunately, far from being complete, as many music theorists have already pointed out.² If one wants to rigorously establish an analogy between non-invertible rhythms and some pitch configurations, a different property from the transpositional invariance of pitch used by Messiaen has to be taken into account. As discussed by Anatol Vieru in his *Book of Modes* (Vieru 1993), given a collection of pitch-classes, an inversion induces a retrogradation of the corresponding intervallic structure. This fact enables us to obtain a mathematical analogy, which is in fact an isomorphism, between non-invertible rhythms and, in Vieru's terminology, "self-inverse modes." One can associate to a given (periodic) non-invertible rhythm a self-inverse mode within an equal-tempered system $\mathbb{Z}/n\mathbb{Z}$, with n equal to the period of the rhythmic pattern. This correspondence is established at the level of the intervallic structure itself, which means that it is totally independent from the choice of the minimal rhythmic unit. In other words, by changing the rhythmic unit, one can attach to the same pitch collection an infinity of non-invertible rhythms, as shown in Example 2.

One of the main contributions by mathematician Dan Vuza, long before introducing the algebraic model of Complementary Unending Canons of Maximal Category in *Perspectives of New Music* (Vuza 1991–1993), was the precise algebraic formalization of this correspondence between pitch domain and rhythmic domain (Vuza 1985). We will not discuss here his model of periodic rhythm as a locally-finite, periodic subset of the set \mathbb{Q} of rational numbers. The important point is that the model enables the transfer of the algebraic structure of the equal tempered system into the rhythmic domain by canonically associating an intervallic (temporal) structure to a given



EXAMPLE 2: CHANGE OF RHYTHMIC UNIT IN THE MAPPING FROM THE CIRCULAR REPRESENTATION TO THE RHYTHMIC DOMAIN

subset of a cyclic group.³ But before providing some examples of this transfer principle with respect to the construction of tiling canons constructions, let us first analyze how Messiaen uses the property of non-invertibility of rhythms and the change of minimal unit to construct a special family of rhythmic canons.

By definition a rhythmic canon is the repetition, with a temporal translation, of a rhythmic structure or its transformations. The base rhythmic pattern, or “inner rhythm” (*pédale rythmique* in Messiaen’s terminology), is repeated, which gives the cyclic character of all rhythmic canons, and translated in time according to what we call the “outer rhythm,” providing the entries of the different voices of the canon. Messiaen considers many types of operations on this musical form. A first operation obtains a canon which is more and more tight, simply by reducing the value that establishes the distance between the voices. This value is normally the same for a given canon, which means that the voices enter regularly one after the other, each the same temporal distance from the previous one. The voices are not necessarily the same, since Messiaen explicitly includes in the definition of canonic operations transformations such as augmentations and diminutions. By means of augmentation processes, Messiaen is able to consider rhythmic canons in which the voices have a common starting point but progressively shift with respect to each other.

A particular type of rhythmic canon, recurrent in Messiaen’s examples, is obtained by considering as the inner rhythm a concatenation of non-invertible rhythms. Messiaen discusses, in particular, two examples of triple rhythmic canons (i.e., canons in three voices) based on concatenations of non-invertible rhythms. This means that from a structural point of view, the inner rhythm consists of the same canon, the only difference being the choice of the minimal value. More precisely, he starts from a concatenation of three palindromic intervallic structures providing the main rhythmic line of a three-voice canon that the composer initially uses in *Visions de l’Amen* (1943) for two pianos (in the part entitled “Amen des anges, des saints, du chant des oiseaux”) and later, with a simple change of the minimal rhythmic unit, in the piece *Harawi* (1945) for soprano and piano. In the first case, the minimal rhythmic unit is the 32nd-note, whereas in the second case is the 16th-note. Example 3 shows the rhythmic grid corresponding to the triple canon used in *Visions de l’Amen*. In the case of the piece *Harawi* (1945) for piano and soprano, the same rhythmic grid is used with, as we said, with the difference that the unit is now the 16th-note. Example 4 shows an extract of the score of Part 7 of *Harawi*, entitled “Adieu,” corresponding to the “new”



EXAMPLE 3: RHYTHMIC CANON IN THREE VOICES USED IN THE PIECE
VISIONS DE L'AMEN (1943) FOR TWO PIANOS

EXAMPLE 4: EXTRACT OF THE SCORE OF “ADIEU,”
PART 7 OF THE PIECE *HARAWI* (1945) FOR PIANO AND VOICE

rhythmic canon in three voices. As in the case of *Vision de l'Amen*, the inner rhythm used to produce the rhythmic canon in *Harawi* is a concatenation of three non-invertible rhythms (see Example 5).

A constant rhythmic value, which is the eighth-note in *Harawi* (respectively the sixteenth-note in *Visions de l'Amen*), separates the different entries of voices, according to the traditional technique of canons construction by Messiaen, who never considers irregular entries



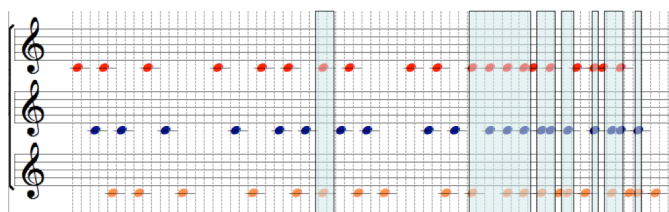
EXAMPLE 5: THE INNER RHYTHM OBTAINED BY CONTATENATION OF THREE NON-INVERTIBLE RHYTHMS AND UTILIZED FOR GENERATING THE TRIPLE CANON SHOWN IN EXAMPLE 4

of voices. In other words, in Messiaen's canonical technique, the outer voice is always of type $0, d, 2d, \dots$, for a given value d . But unlike traditional musical canons, each voice is a succession of different chords that are repeated cyclically by producing a "harmonic ostinato which is independent from the rhythmic organization" (Messiaen 1994, 46). The fundamental aspect of this compositional process is the tension between non-invertible rhythms and the regular entries of voices. This induces a global perceptive result that is a mixture of chaotic behavior and organized structure. Quoting a passage that Messiaen wrote to describe *Harawi*, but which also applies in its general conclusion to *Visions de l'Amen*:

The three non-invertible rhythms divide the durations in $5+5+7$ durations, whereas the terms of the three harmonic ostinatos always contain six sonorities for the superior voice, and three sonorities for the two other voices. Consider also that the durations are very unequal. As a result, the different sonorities mix together or contrast to each other in very different ways, never at the same moment nor at the same place. . . . It is an organized chaos" (Messiaen 1994, 46).

Although Messiaen never refers explicitly to the role of geometric and tiling processes in music, it is clear that he aims at using a special family of concatenations of non-invertible rhythms in order to canonically organize the global musical form in such a way that the onsets of the different rhythmic voices, potentially, never intersect. This is not what finally happens in the actual compositions, since mutual intersections between voices frequently occur—particularly in the second half of the canon (see Example 6).

Nevertheless, it seems reasonable enough to state that the use of the tiling concept in this special case of Messiaen's compositional technique well describes the geometric character of the canonical process



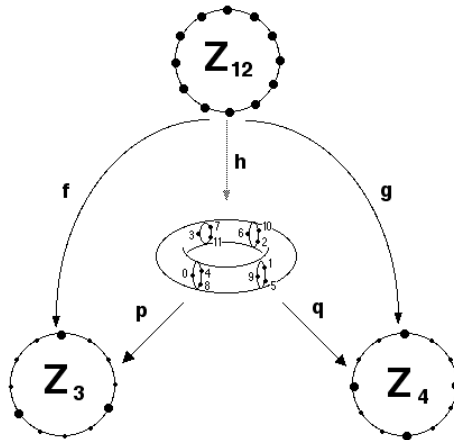
EXAMPLE 6: INTERSECTIONS BETWEEN ONSETS OF DIFFERENT VOICES IN THE TRIPLE CANON ON NON-INVERTIBLE RHYTHMS USED IN “ADIEU”

and of the final global result. It is, however, pretty clear that there is generally no connection between Messiaen’s use of palindromic structures and the tiling process. There exist non-invertible rhythms that may eventually be taken as the “inner rhythm” of a rhythmic canon rigorously realizing the tiling of the time axis, as in one of the examples shown in the next section (see Example 9), but Messiaen was, of course, unaware of the existence of concepts such as “group factorizations” or “direct sums,” which are used today to elegantly describe the construction of tiling canons. This is one of Dan Tudor Vuza’s major contributions to the subject, rooted in a long-term collaboration with composer Anatol Vieru (1926–1998) and which led to a detailed formalization of Vieru’s modal theory—in particular the composition of modal structures and the interpretation of this composition law in the time domain.

2. ANATOL VIERU’S COMPOSITION OF MODAL STRUCTURES AND DAN VUZA’S MODEL OF REGULAR COMPLEMENTARY CANONS OF MAXIMAL CATEGORY

The modal theory of composer Anatol Vieru represents, first of all, a remarkable example of an algebraically oriented perspective on the intervallic thinking in music theory, analysis, and composition. As pointed out by the composer,⁴ there are several deep connections between his theoretical model and some algebraic approaches which were developed independently by other composers/theorists during the second half of the twentieth century. For example, the concept of composition of modal structures is equivalent to the “transpositional combination” of the set-theoretical tradition (Cohn 1986), which happens to provide the general framework for serial techniques such as Pierre Boulez’s chord multiplications (Weiss 2007).

As we mentioned, starting from a formalization of Vieru's modal theory, Vuza built a mathematical model of periodic rhythms allowing what Bourbaki would call the "transfer of structures" from the modal (or pitch) universe to the time domain, as Vuza clearly explains in his critical review of David Lewin's *Generalized Musical Intervals and Transformations* (Vuza 1988). Transposing one pitch-class set (or, more generally, an ordered or unordered subset of a given cyclic group) according to the intervallic content of a second pitch-class set is the "outside-of-time" equivalent, as Iannis Xenakis would have put it, of rhythmic canons by translation.⁵ This only works once an appropriate isomorphism between the pitch and the rhythmic domain is constructed, as Vuza rigorously does in his mathematical model of periodic rhythm (Vuza 1985). The tiling property follows immediately by considering the special case of the composition of two modal structures whose cardinalities, say m and n , divide the order of the underlying cyclic group, say c , in such a way that c is equal to the product of m and n . Example 7 shows one of the simple cases one can obtain in the traditional equal-tempered system $\mathbb{Z}/12\mathbb{Z}$. It corresponds to the Sylow decomposition of the cyclic group of order twelve into the direct sum of the cyclic groups of orders three and four, respectively, a well-known music-theoretical construction that is traditionally used to represent the pitch space as a torus.

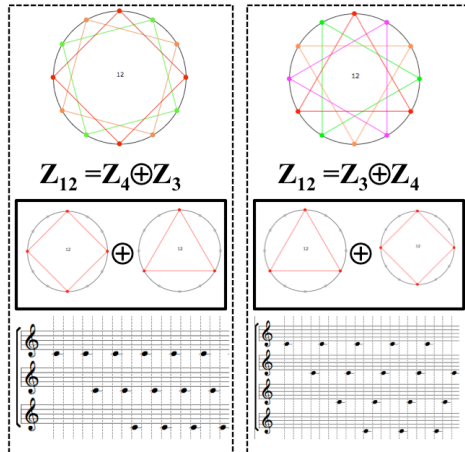


EXAMPLE 7: SYLOW DECOMPOSITION OF $\mathbb{Z}/12\mathbb{Z}$ INTO THE DIRECT SUM OF THE CYCLIC GROUPS OF ORDER 3 AND 4, RESPECTIVELY

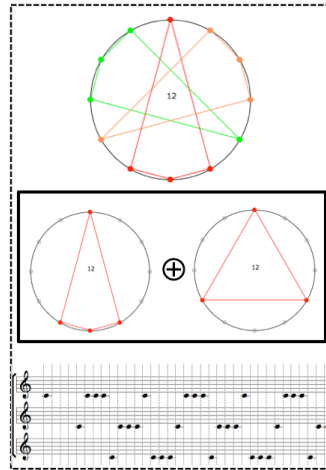
The interpretation of this Sylow decomposition in terms of a transpositional combination produces a tiling of the pitch space with four transpositions of a given pitch collection (i.e., the augmented triad) or, equivalently, three transpositions of another (i.e., the diminished seventh). From a rhythmic perspective, this decomposition leads to a tiling rhythmic canon in three or four voices, according to the choice of the factor of the group decomposition as the inner rhythm.⁶ Example 8 shows the two “dual” canons obtained through the rhythmic interpretation of the Sylow decomposition of $\mathbb{Z}/12\mathbb{Z}$.

Apart from this first family of tiling rhythmic canons directly obtained by applying Sylow decomposition theorem, other less trivial tiling rhythmic canons can be constructed, providing some examples of possible connections between Messiaen’s non-invertible rhythms and the tiling process. Example 9 shows one tiling rhythmic canon whose inner rhythm corresponds to a non-invertible rhythm and for which the voices enter in a regular way, as requested by Messiaen in his original formulation.⁷

All the previous examples, as well as all the uses of rhythmic canons in the compositional literature, as far as we know, use factorizations in which one of the two factors is a subgroup of the underlying cyclic group, which means that the voices are always entering one after the



EXAMPLE 8: TWO “DUAL” CANONS OBTAINED BY THE SYLOW DECOMPOSITION OF $\mathbb{Z}/12\mathbb{Z}$ AS A DIRECT SUM OF THE SUBGROUPS $\mathbb{Z}/3\mathbb{Z}$ AND $\mathbb{Z}/4\mathbb{Z}$

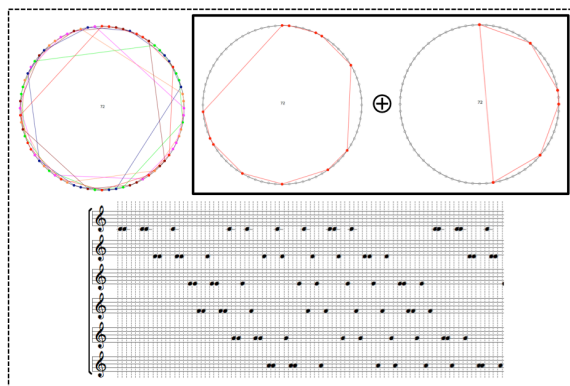


EXAMPLE 9: A TILING RHYTHMIC CANON CONSTRUCTED ON A NON-INVERTIBLE RHYTHM AND WITH THE THREE VOICES ENTERING IN A REGULAR WAY

other in a regular way. The series of papers published by Dan Vuza in *Perspectives of New Music* from 1991 to 1993 not only constitute a milestone in the development of the mathematical theory of tiling canons but also offer new possibilities for composers to free themselves from the constraint of regularity with respect to the entries of the voices in a canon. Although there are today several examples of papers containing new mathematical theorems which are published in music theory journals, Vuza's series of articles and his use of non-elementary mathematical concepts, like characters and the discrete Fourier transform applied to locally compact groups, still remains one of the most significant mathematical contributions to the musicological community. Among the rich collection of new, interesting music-theoretical models introduced by Vuza, it is undoubtedly the concept of Regular Complementary Canons of Maximal Category that has proven to be a central construction, being capable of intersecting a variety of different areas in mathematics. These canons, which we simply call nowadays "Vuza Canons," have the remarkable property of tiling the time axis without inner periodicity. From an algebraic point of view they correspond to a factorization of a cyclic group into two non-periodic subsets. These types of factorizations are fascinating

objects for mathematicians, as one may infer from the reading of the collection of remarkable numbers compiled by François Le Lionnais (1983) in collaboration with mathematician Jean Brette. The founding member of the French *Oulipo* group (*Ouvroir de littérature potentielle*) grants a special place to the number 72 by virtue of the fact that “the cyclic group with seventy-two elements can be decomposed as $S+T$, with S and T which are non-periodic” (Le Lionnais 1983). The decomposition cited by Le Lionnais and originally proposed by László Fuchs in his monograph on abelian groups (Fuchs 1960) corresponds, in fact, to one of the possible solutions for the construction of Vuza canons. Example 10 shows the Vuza canon associated to this remarkable property of $\mathbb{Z}/72\mathbb{Z}$.

Contrary to what happens for larger cyclic groups, the algorithm proposed by Vuza provides the complete catalogue of decompositions into two non-periodic factors in the case of $\mathbb{Z}/72\mathbb{Z}$. It is, nevertheless, useful to compare Vuza’s algorithm with an older and slightly more general approach which goes back to de Bruijn (1953b). As observed by Giulia Fidanza in her dissertation (Fidanza 2007), the decomposition of a given N as the product $p_1 p_2 n_1 n_2 n_3$ (with the usual conditions on the p_i as being distinct prime numbers and on $p_1 n_1$ as being relatively prime with $p_2 n_2$) can be generalized as done by de Bruijn. He



EXAMPLE 10: A VUZA CANON CONSTRUCTED BY THE TWO NON-PERIODIC FACTORS
 $S=0,1,5,6,12,25,29,36,42,48,49,53$ AND $T=0,8,16,18,26,34$
 ASSOCIATED WITH THE DECOMPOSITION OF A CYCLIC GROUP OF ORDER EQUAL TO
 THE REMARKABLE NUMBER 72

considers, for a given order N of such a cyclic group, the decomposition as nmk where:

1. $\langle n, m \rangle = 1$ (i.e., n and m are relatively prime);
2. $n = n_1 n_2$, $m = m_1 m_2$;
3. n_1, n_2, m_1, m_2, k are all greater than 1.

In order to apply the construction proposed by de Bruijn to $72 = n \cdot m \cdot k = 4 \cdot 9 \cdot 2$, one needs the following subsets:

1. $R_k = \mathbb{Z}_N / k \mathbb{Z}_N$ (i.e., $R_k = 0, 1, 2, \dots, k-1$, for every $k \in N$);
2. $C_1 = k_{n_1} R_{n_2} \oplus k_{n_1 n_2 m_1} \mathbb{Z}_N$;
3. $C_2 = k_{m_1} R_{m_2} \oplus k_{m_1 m_2 n_1} \mathbb{Z}_N$.

Since $n_1 = n_2 = 2$ and $m_1 = m_2 = 3$, one obtains the following factorization of $\mathbb{Z}/72\mathbb{Z}$ into non-periodic subsets S and T :

$$\begin{aligned} S &= C_2 \cup \{1, 2, \dots, k-1\} \oplus C_1 \\ &= k_{m_1} R_{m_2} \oplus k_{m_1 m_2 n_1} \mathbb{Z}_N \cup \{1, 2, \dots, k-1\} \oplus k_{n_1} R_{n_2} \oplus k_{n_1 n_2 m_1} \mathbb{Z}_N \\ &= 6\{0, 1, 2\} \oplus 36\mathbb{Z}/72\mathbb{Z} \cup \{1\} \oplus 4\{0, 1\} \oplus 24\mathbb{Z}/72\mathbb{Z} \\ &= \{0, 6, 12, 36, 42, 48\} \cup \{1\} \oplus \{0, 4, 24, 28, 48, 52\} \\ &= \{0, 1, 5, 6, 12, 25, 29, 36, 42, 48, 49, 53\} \end{aligned}$$

$$\begin{aligned} T &= k_{m_1 m_2} R_{n_1} \oplus k_{n_1 n_2} R_{m_1} \\ &= 18\{0, 1\} \oplus 8\{0, 1, 2\} \\ &= \{0, 8, 16, 18, 26, 34\} \end{aligned}$$

These are exactly the factors corresponding to the “remarkable” Vuza canon shown in Example 10. A critical comparison between the solutions of the factorization problem for a general non-Hajós group obtained by using the two algorithms by Vuza and de Bruijn has still to be undertaken. For a first attempt at classifying Vuza canons by using this more general approach, see Fidanza (2007, 99–102).

3. FROM MINKOWSKI’S CONJECTURE TO VUZA CANONS:

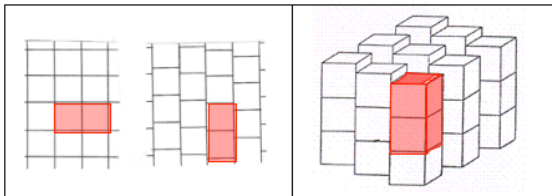
THE INTERPLAY BETWEEN NUMBER THEORY, GEOMETRY,
ALGEBRA, AND MUSIC COMPOSITION⁸

As originally shown by the author (Andreatta 1996), Vuza canons that are obtained through the factorization of a cyclic group into two non-periodic subsets are the musical metamorphosis of a tessellation

problem initially raised by Minkowski in a number-theoretical form in his *Geometrie der Zahlen* (Minkowski 1896). His original problem is a generalization of a result concerning the approximation of a real number by rational numbers which states that for every real number a and for every integer $t > 1$, there exist two integers x and y such that $|ay - x| < 1/t$ where $0 < y \leq t$. A first way of generalizing the problem is to consider an n -tuple of real numbers instead of a single real number. The new problem states that for every n -tuple of real numbers a_1, \dots, a_n , and for every integer $t > 1$, there exist $n+1$ integers x_1, \dots, x_n, y such that $|a_i y - x_i| < 1/t$ for every $i=1, \dots, n$ where $0 < y \leq t^n$. The following step is to consider the approximation problem with the parameter t being a real number instead of an integer number. This generalization led Minkowski to the following theorem, whose solution made use of geometric tools, such as the Lattice Point Theorem (Stein and Szabó 1994):

THEOREM (Minkowski). *For every tuple of $n+1$ real numbers a_1, \dots, a_n and t , with $t > 1$, there exist $n+1$ integers x_1, \dots, x_n, y such that $|a_i - x_i/y| \leq 1/(ty)$ for every $i=1, \dots, n$ where $0 < y \leq t^n$.*

This geometric technique applied to a number-theoretical problem was pushed even further by Minkowski some years later in his *Diophantische Approximationen* (Minkowski 1907), where this same problem lead the mathematician to investigate the structural properties of the tiling of Euclidean space by congruent unit cubes. In the attempt to generalize a property that is evident in two- and three-dimensional Euclidean space (see Example 11), Minkowski made the



EXAMPLE 11: MINKOWSKI'S CONJECTURE IN THE CASES OF THE LATTICE TILING OF THE TWO- AND THREE-DIMENSIONAL SPACES BY UNIT CUBES

following conjecture, which was solved almost half a century after its first formulation:

CONJECTURE (*Minkowski*). *Any lattice tiling of n -dimensional Euclidean space by unit cubes has the property that some pair of cubes must share a complete $(n-1)$ -dimensional face.*

Before discussing Hajós's reformulation of this geometric conjecture *more algebrico*, it can be useful to do a short exegesis of the content of the conjecture, which will enable us to mention two historical attempts at relaxing some conditions contained in the original formulation. As observed by Szabó (1993), there are, in fact, four conditions which are somehow hidden in Minkowski's problem:

1. the cubes are all obtained by translation;
2. the translation vectors describe a lattice;
3. the interiors of the cubes are disjoint;
4. every point of the space not belonging to a cube's boundary is contained in exactly one cube (simple-like condition).

By removing the lattice condition, as suggested by O. Keller (1930), one obtains the following weak version of Minkowski's conjecture:

CONJECTURE (*Keller*). *Any simple tiling of the n -dimensional Euclidean space by unit cubes has the property that some pairs of cubes must share a complete $(n-1)$ -dimensional face.*

Unlike Minkowski's conjecture, which is true in all dimensions, the validity of the Keller conjecture deeply depends on the dimension n of the underlying Euclidean space. It is true when $n \leq 6$ (Perron 1940) and false for $n \geq 8$ (Mackey 2002). The only case which still remains open is the dimension $n=7$. Around the same period as the Keller conjecture, there was an interesting attempt at weakening the simple-like lattice condition of Minkowski's conjecture, asking whether the conclusions still held when one allowed every point of the space not belonging to a cube's boundary to be contained in exactly a given number of cubes. Such a lattice tiling is also called "multiple," and the number k of cubes that contain every point of the space not belonging to a boundary of a cube is called the *multiplicity* of the tiling. The new version of Minkowski's conjecture in the case of multiple tiling is, therefore, the following:

CONJECTURE (*Furtwängler 1936*). *Any multiple lattice tiling of n -dimensional Euclidean space by unit cubes, with multiplicity equal to k ,*

has the property that some pair of cubes must share a complete $(n-1)$ -dimensional face.

The cases when the conjecture holds (i.e., $n \leq 3$) were established and studied by Furtwängler and, independently, by Hajós who proved that whenever $n > 3$, there exist some values of k for which the Furtwängler conjecture is not true. Hajós obtained these counterexamples while working on the original version of Minkowski's conjecture, whose reformulation in algebraic terms directly induced an algebraic version of Furtwängler's conjecture.⁹ Recalling the history of Fermat's last theorem, which was solved more than two-and-a-half centuries after its first formulation, we suggested calling this problem Minkowski's last theorem (Andreatta 2004). In fact, like his French colleague, Minkowski largely underestimated the difficulties of the n -dimensional version of a problem that he had no difficulty in proving in lower dimensions. And indeed, the problem turned out to be so difficult that its solution, provided by Hajós some forty years later (Hajós 1941), has been described as "the most dramatic work in factoring" (Stein 1974). As Szabó points out by referring to Stein's enthusiastic perspective, the reformulation by Hajós of Minkowski's conjecture as an algebraic problem is no less remarkable than the metamorphosis of a caterpillar into a butterfly (Szabó 2004, 4). Hajós's result is sometimes referred to as the second main theorem for finite abelian groups (Rédei 1967), and it can be easily formulated by using the notion of a "cyclic subset" A of a group G (i.e., a subset A of G with elements $e, a, a^2, \dots, a^{r-1}$ for a given integer $r \geq 2$). In other words, a cyclic subset consists of the first r elements of the cyclic group generated by the element a .

THEOREM (Minkowski/Hajós). *If a finite abelian group G is a direct sum of its cyclic subsets A_1, A_2, \dots, A_k then there exists at least one i such that A_i is a subgroup.*

The reason the Minkowski/Hajós theorem has a central place in the theory of finite abelian groups also relates to the logical duality between this result and the Frobenius/Stickenberger theorem, expressing the fact that every finite abelian group can be factored into the direct sum of some of its cyclic subgroups. This logical duality was first observed by Laszlo Rédei (1965a) who, in the same year, generalized Minkowski/Hajós by means of the concept of "normalized factorization" of a finite abelian group (i.e., decomposition into the direct sum of subsets all containing the identity element).

THEOREM (Rédei 1965b) *If a finite abelian group G admits a normalized factorization into subsets A_1, A_2, \dots, A_k where each A_i has a prime number of elements, then at least one of the subsets is a subgroup of G .*

For a detailed description of the proof of Rédei’s theorem, as well as Hajós’s original algebraic version of Minkowski’s geometric conjecture, we refer the reader to the monograph by Stein and Szabó (1994), currently the most comprehensive study of the Minkowski/Hajós problem. As amply documented by the authors, Hajós’s solution to Minkowski’s conjecture not only clearly showed the interplay between the geometry of tiling and the algebra of group factorizations, but also paved the way for the classification of groups featuring the property that they cannot be factored into a direct sum of subsets without at least one of the factors being periodic. This factorization property, compelling one of the factors to be periodic, is also called the “Hajós property.” It distinguishes the family of the so-called Hajós (or *good*) groups, whose exhaustive classification engaged many mathematicians, including Hajós himself, for more than twenty years. Example 12 contains the complete list of good groups in chronological order of discovery. The notation (a, b, \dots, z) is an abbreviation for the product $\mathbb{Z}/a\mathbb{Z} \times \mathbb{Z}/b\mathbb{Z} \times \dots \times \mathbb{Z}/z\mathbb{Z}$ where p, q, r, s are distinct primes and α is an integer greater than 1.

Surprisingly, the proof that the group $G = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ is a good group historically precedes the proof of Minkowski’s conjecture by Hajós. As we can see from Example 12, it took several years to fully

Rédei 1947	(p, p)	Sands 1962	$(p, 3, 3)$
Hajós 1950	\mathbb{Z}		$(p, 2^2, 2)$
	$\mathbb{Z}/n\mathbb{Z}$ with $n = p^\alpha$		$(p, 2, 2, 2, 2)$
de Bruijn 1953	(p^α, q)		$(p^2, 2, 2, 2)$
	(p, q, r)		$(p^3, 2, 2)$
Sands 1957	(p^2, q^2)		$(p, q, 2, 2)$
	(p^2, q, r)	Sands 1964	\mathbb{Q}
	(p, q, r, s)		$\mathbb{Z} + \mathbb{Z}/p\mathbb{Z}$
Sands 1959	$(2^2, 2^2)$		$\mathbb{Q} + \mathbb{Z}/p\mathbb{Z}$
	$(3^2, 3)$		
	$(2^n, 2)$		

EXAMPLE 12: THE COMPLETE LIST OF GOOD GROUPS IN CHRONOLOGICAL ORDER. THE OUTLINED REGION CORRESPONDS TO THE SPECIAL CASE OF CYCLIC GROUPS

classify the good groups in the special case of the cyclic groups. This classification result immediately offers the list of non-Hajós (or *bad*) cyclic groups, which correspond exactly to the algebraic structures enabling us to obtain Vuza's rhythmic tiling canons.

Starting from a purely musical problem, Vuza not only unexpectedly came across a domain of group-theoretical research with profound intersections with the geometry of tessellations, but he also provided an original contribution in the field of non-Hajós groups. To mention only one of these new results, he proved that the tiling property in rhythmic canons is invariant up to musical augmentations, which can be mathematically expressed by saying that if a non-Hajós cyclic group $\mathbb{Z}/n\mathbb{Z}$ admits a factorization into two subsets A and B , it also admits the factorizations into the subsets kA and B (or equivalently A and kB) for any k coprime with the period n of the group. As we will see in the next section, this property suggests that the paradigmatic approach, which is commonly used in mathematical music theory for classifying chords as orbits under the action of different groups on the powerset $\wp(\mathbb{Z}/n\mathbb{Z})$, can also be applied to the classification of Vuza canons and, more generally, any type of tiling rhythmic canon.¹⁰

Published in *Perspectives of New Music*, this result had almost no chance to be noticed by professional mathematicians, which explains that it was rediscovered several years later as a purely mathematical problem and proved in different ways, first by Robert Tijdeman (1995) who calls this result the "Fundamental Lemma" and later by Ethan M. Coven and Aaron Meyerowitz (1999). The first employed a combinatorial argument, while the latter provided a proof of this property in a polynomial ring. Note that Vuza's original proof makes use of convolutions and discrete Fourier transforms, following the path which David Lewin opened in the late fifties (Lewin 1959). These Fourier-based algebraic tools are now commonly applied in mathematical music theory and they provide the most elegant framework in which to study the constructions of tiling rhythmic canons.¹¹

4. THE RELEVANCE OF THE COMPUTATIONAL PERSPECTIVE, AND THE CONNECTIONS BETWEEN VUZA CANONS AND ONE OPEN MATHEMATICAL CONJECTURE.

The implementation of Vuza's algorithm in the OpenMusic visual programming language, which was done at IRCAM together with computer-scientist Carlos Agon,¹² raised several questions that were not explicitly addressed in the original theory: is the algorithm exhaustive for any non-Hajós cyclic group $\mathbb{Z}/n\mathbb{Z}$? How big is the

solution space? Can the same computational model be taken as a starting point for stylistically different compositional processes?

After the initial belief that Vuza's algorithm was exhaustive, we soon realized that the family of factorizations of non-Hajós cyclic groups into two non-periodic subsets is generally larger than the family of Regular Complementary Canons of Maximal Category obtained by the algorithm published in *Perspectives*. The catalogue for the smallest non-Hajós group (i.e., $\mathbb{Z}/72\mathbb{Z}$) using Vuza's algorithm, happens to be complete (Andreatta 1999). This has been verified by Harald Friperntinger by exhaustive testing of all possible factorizations. But already for the next order (i.e., $n=108$), Emmanuel Amiot and Harald Friperntinger found, independently, 252 new canons that were not constructible using Vuza's original algorithm. Again, by exhaustive testing Friperntinger showed that these are all the tiling canons that have to be added to the catalogue of Vuza canons in order to complete the list of Vuza canons for $n=108$. At this point we had the complete catalogue of Vuza canons corresponding to the first two non-Hajós cyclic groups. By using the invariance of the tiling process up to affine transformations, as we mentioned before, we computed the 30 non-isomorphic solutions for the subsets with eighteen elements (usually representing the voices or "inner rhythms" of the canon) and three non-isomorphic solutions for the subsets with six elements (called "outer rhythms" and providing the entries of the voices of the canon). There are, thus, 90 non-isomorphic Vuza canons for the non-Hajós group $\mathbb{Z}/108\mathbb{Z}$. The number of solutions dramatically decreases for $\mathbb{Z}/72\mathbb{Z}$, with only one non-isomorphic, twelve-element inner rhythm and two non-isomorphic solutions for outer rhythms with six elements. This leads to the astonishing result that there are only two different Vuza canons with period 72 (always up to affine transformations).

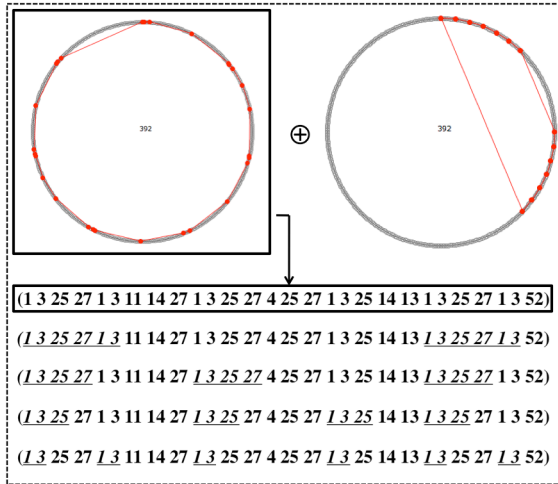
By giving such a detailed description of computational results concerning the thoroughness of Vuza's original model of tiling canons, we would like to stress one of the major ingredients of contemporary "mathemusal" research. Building computational models of formal constructions may radically change the perspective on a given music-theoretical problem by emphasizing its experimental component. In the case of the construction of tiling canons, having a computer-aided model made evident a series of properties that would have been difficult to perceive by relying purely on the original theoretical model. For example, one can show computationally that in non-Hajós groups almost all "outer rhythms" obtained by Vuza's algorithm have the property of being palindromes, which establishes an unexpected connection with Olivier Messiaen's original attempt at constructing

canons based on non-invertible rhythms. This is the case, for example, for all the solutions that we obtained for $n=72$ and $n=108$. Although transpositional symmetry is forbidden in both factors, making these canonic forms very difficult to grasp for the listener, the palindromic character of one of the factors, as well as some local periodicities, can eventually become a structural element in a compositional application of the model. This is what actually happened once the catalogue of solutions for Vuza canons was made accessible to composers.

Surprisingly, in spite of the rigid form of tiling rhythmic canons, the catalogue suggests to composers a large variety of possible applications of the model in compositional practice. The first compositional application of Vuza canons was made by composer Fabien Lévy in his orchestral piece *Coincidence* (1999). Complex musical objects filled the underlying rhythmic grid provided by a Vuza canon in such a way that the global perceptual result is not heard contrapuntally but rather as a continuous information flow where timber melodies spontaneously emerge via the combinatorial play of the different voices of the canon. Some years later, composer Georges Bloch proposed a very different use of the catalogues of Vuza canons in several compositional projects. These range from the piece *Empreinte sonore pour la Fondation Beyeler* (2001), a guided musical tour for an exhibition of the Beyeler Foundation in Basel, Switzerland (Bloch 2006), to the recent experiments in computer-aided improvisations using the OMax program developed at IRCAM and combining OpenMusic formal models and Max/MSP real-time functions.

A third example of a composer having benefitted from a computational model of Vuza canons is Mauro Lanza, who was inspired by the local periodicity of some of the factors in the case of cyclic groups with large cardinality. In his piece entitled *La descrizione del diluvio* (Ricordi 2007), for choir and electronics, he uses a particular Vuza canon of period 392 built on an inner rhythm of cardinality 28 and in which the fourteen voices enter according to a non-invertible rhythm. Although, according to the theory of Vuza canons, the inner rhythm has no inner periodicity, its structure is locally highly redundant (i.e., it is possible to find local repetitions of smaller rhythmic patterns of different length, which suggested to the composer that he select the notes and the durations by emphasizing these quasi-periodicities of the Vuza canon, which provides some redundancy within each voice) (Example 13).

This short description of compositional applications of Vuza's model illustrates the importance of closely connecting theoretical research and computational modeling. Moreover, as we suggested earlier in our



EXAMPLE 13: LOCAL PERIODICITIES IN ONE FACTOR OF A VUZA CANON WHICH WAS USED BY MAURO LANZA IN HIS PIECE *LA DESCRIZIONE DEL DILUVIO* (2007)

description of “mathematical problems,” a single music-theoretical construction may intersect with a number of different mathematical problems. This is precisely what is happening in the case of Vuza’s model, which, beyond the Minkowski/Hajós problem,¹³ calls upon a still open conjecture in functional analysis: the Fuglede, or spectral, conjecture (Fuglede 1974). This conjecture deals with the relation between the spectral property of a domain in n -dimensional Euclidean space and its tiling character. It states that such a domain admits a *spectrum* if it tiles \mathbb{R}^n by translation. But conversely to the case of the Minkowski/Hajós problems, the cases which are the most difficult to tackle and still remain unsolved correspond to the two lowest dimensions: $n=1$ and $n=2$.

Surprisingly, despite the tiling character of the Fuglede conjecture, its connection with the Minkowski/Hajós problem was not at all evident at the beginning. One may say that there are two parallel developments of tiling theory, the first one in the extension of the Minkowski/Hajós conjecture and the second one which focuses on spectral theory, without directly referring—at least initially—to the theory of Hajós and non-Hajós groups. As an example, simply consider that there is no mention of the Minkowski/Hajós problem in

Fuglede's paper on the spectral conjecture and, conversely, Coven and Meyerowitz (1999) were not aware of Fuglede's conjecture when they tried to establish necessary and sufficient conditions for a subset of the integers to tile by translation. One may argue that the two domains are different in nature, since Fuglede's conjecture concerns continuous spaces, whereas the theory of Hajós groups is essentially discrete. In fact, although the original Minkowski conjecture deals with the tiling of \mathbb{R}^n , Hajós (1938) has shown that there is no loss of generality in considering rational lattices, which leads to the algebraic reformulation of the problem in terms of group factorization. With the same argument, one may show that one of the two cases of Fuglede's conjecture which still remain open—the equivalence between spectrality and the tiling property in the case of subsets of the real axis \mathbb{R} —can be reduced to the tiling of the integers problem, which itself turns out to be equivalent to the factorization of a cyclic group into subsets, under the hypothesis that the factors have finite length.¹⁴ The connection between the Minkowski/Hajós problem and Fuglede conjecture became clear once Izabella Łaba showed the relationships between the two conditions T1 and T2 established by Coven and Meyerowitz¹⁵ and the property of a set being spectral (Łaba 2002). Using the fact that the two conditions are sufficient and that the condition T1 is also necessary for the 0-1 polynomial $A(x)$ associated with a subset A of \mathbb{Z} to tile the integers by translation, Izabella Łaba showed that T1 and T2 are sufficient for A to be spectral and that the condition T1 is necessary in some special cases. In both cases, the tiling property and spectrality, the open problem is the necessity of the condition T2. Once again, Vuza canons are precisely the musical constructions that could help mathematicians give an answer to this open problem and, as a consequence, to the Fuglede conjecture in the one-dimensional case. More precisely, Emmanuel Amiot has shown that if there exists a subset R of a cyclic group $\mathbb{Z}/n\mathbb{Z}$ that tiles the space by translation without being spectral, then the tiling rhythmic canon generated by R either is a Vuza canon or can be reduced to one whose inner voice is not spectral either. In other words, as we suggested elsewhere (Andreatta and Agon 2009), a possible counter-example of the spectral conjecture may already exist within the yet unwritten pages of the catalogues of all possible (and still unheard) Vuza canons.

NOTES

1. For a more detailed presentation of this research perspective in contemporary mathematical music theory, see Andreatta (2010).
2. See, for example, Guerino Mazzola's criticisms raised in *Geometrie des Töne* (Mazzola 1990, 98–99), further developed in *The Topos of Music* (Mazzola 2002, 150–152).
3. For a discussion of this model with respect to similar music-theoretical construction from the American set-theoretical and transformational tradition, see Vuza (1988). The model has also been described by the present author within the module-theoretic framework of Guerino Mazzola's mathematical music theory in Mazzola (2002, 380–382).
4. See Vieru (1998).
5. The reference to Xenakis is more than simply anecdotic. His Sieve Theory (Xenakis 1990) is a further example of theoretical construction based on the assumption that intervallic (or relational) thinking has a major role in twentieth-century composition. For an epistemological discussion of this aspect within Gilles-Gaston Granger's "operational/objectal" duality, see Andreatta (2011).
6. In Andreatta (2004) we suggested using the term "duality" to describe the change of role of the factors in the process of the construction of tiling rhythmic canons. Note that the compatibility of the tiling property with the duality operation, which comes from the commutativity of the direct sum, raises many interesting music-theoretical questions. Duality is also one of the two main operations which can be performed in order to structurally reduce a given tiling rhythmic canon either to a Vuza canon or to the trivial canon $0 \oplus 0 = 0$ (Fidanza 2007).
7. This corresponds to the case of k -asymmetric rhythms (Hall and Klinsberg 2006), a concept which generalizes the property of "imparité rythmique" [rhythmic odditivity] introduced by ethnomusicologist Simha Arom and mathematically formalized by Marc Chemillier (see Chemillier 2007 for a larger description of the odditivity property from the perspective of "natural" mathematics).
8. In this section we will follow the presentation contained in Giulia Fidanza's dissertation (Fidanza 2007), which is one of the most

recent and most detailed accounts of the Minkowski/Hajós problem in connection with the theory of tiling rhythmic canons.

9. See Zong (2006) for one of the most recent accounts of the Furtwängler conjecture and its interplay with the Keller conjecture and the Minkowski/Hajós problem. In Andreatta (2004) we suggested that the Furtwängler conjecture could be taken as a starting point for a more general theory of tiling rhythmic canons allowing precise control over the number of intersections between voices. A recent conversation with mathematicians Máté Matolcsi and Sinai Robins seems to suggest that this approach would also open new mathematical questions, since, crucially, it generalizes the traditional theory of tiling rhythmic canons (in which the multiplicity of the tiling is equal to 1). Unfortunately we are not able at present to provide more elements confirming this hypothesis.
10. See Fripertinger (2001) for the first attempt at applying combinatorial algebraic techniques adapted from Polya enumeration theory in order to provide an explicit formulation for counting the number of non-isomorphic tiling rhythmic canons. The combinatorial explosion, however, prevented the author from considering the case of Vuza canons.
11. See, for example, the special issue of the *Journal of Mathematics and Music* devoted to Tiling Problems in Music (Andreatta and Agon 2009). Also, see Emmanuel Amiot's contribution in the present issue of *Perspectives of New Music* for a Fourier-based approach to the Fundamental Lemma, using the fact that the augmentation is a Galoisian automorphism in the cyclotomic field \mathbb{Q}_n .
12. For a more detailed presentation of the implementational aspects of Vuza canons, see Agon and Andreatta's article "Modeling and Implementing Tiling Rhythmic Canons in the OpenMusic Visual Programming Language" in the present issue of *Perspectives*.
13. For a recent discussion on the connections between Vuza's model and the Minkowski/Hajós problem, see Fidanza (2007).
14. The fact that a tiling of the integers with translates of itself on a finite set is periodic was established simultaneously and independently by Hajós (1950a) and de Bruijn (1950).
15. For a description of these two properties, see the contribution by Emmanuel Amiot in the present issue of *Perspectives*. Since conditions T1 and T2 are sufficient in order for a subset to tile the integers by translation, we implemented an algorithm to obtain all

tiling rhythmic canons starting from a subset of a cyclic group satisfying the two conditions by Coven and Meyerowitz. This leads to a special class of tiling rhythmic canons that we call “Cyclotomic Canons,” and whose construction is described in our paper “Modeling and Implementing Tiling Rhythmic Canons in OpenMusic Visual Programming Language,” also in the present issue.

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