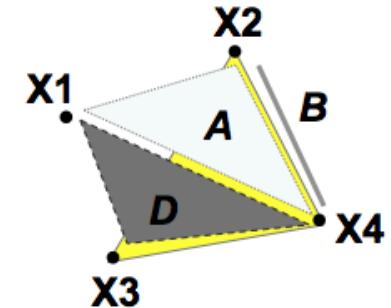
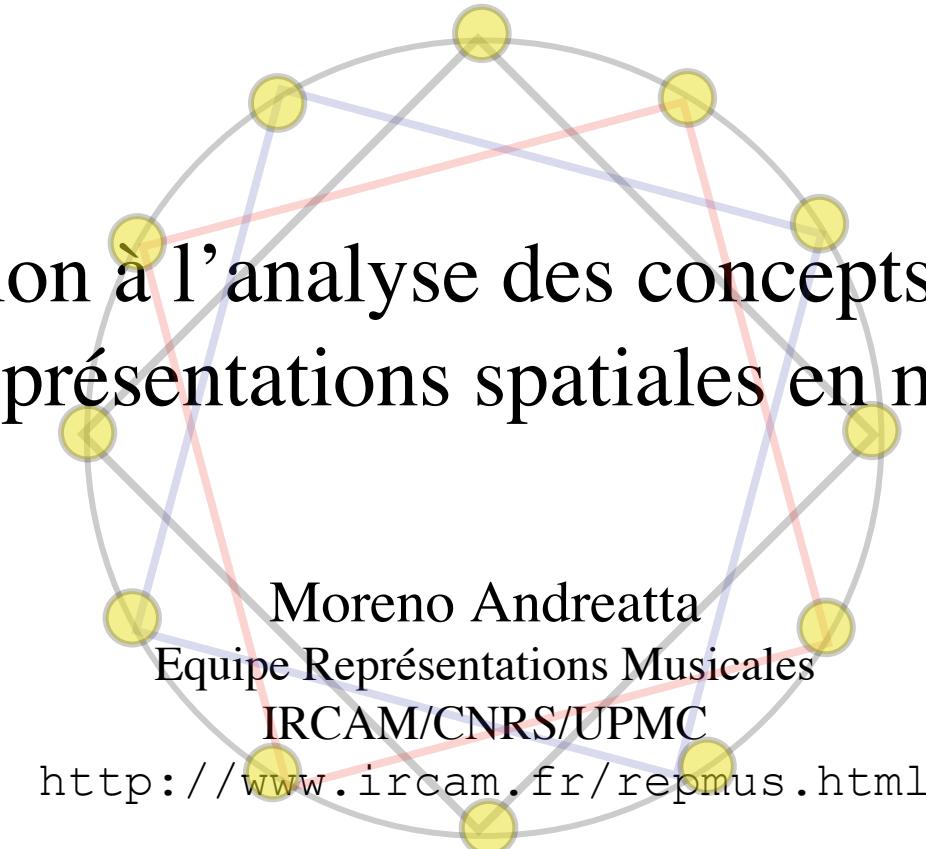


# Séminaire MaMuX

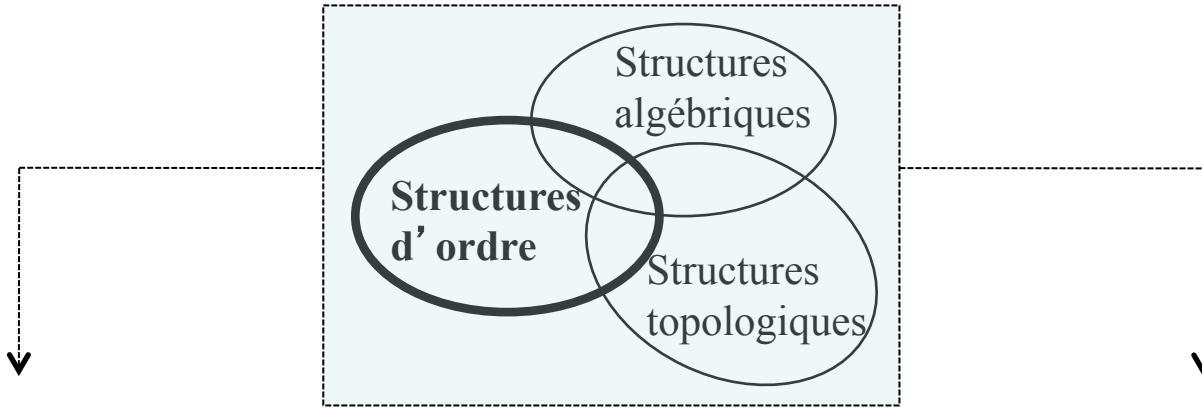
Ircam, 5 avril 2013



Introduction à l'analyse des concepts formels et aux représentations spatiales en musique



# L'Analyse des concepts formels : rappel historique

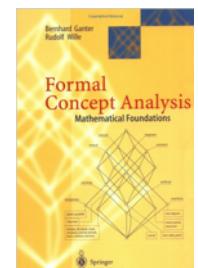


- M. Barbut, « Note sur l'algèbre des techniques d'analyse hiérarchique », in B. Matalon (éd.), *L'analyse hiérarchique*, Paris, Gauthier-Villars, 1965.
- M. Barbut, B. Monjardet, *Ordre et Classification. Algèbre et Combinatoire*, en deux tomes, 1970
- M. Barbut, L. Frey, « Techniques ordinaires en analyse des données », Tome I, *Algèbre et Combinatoire des Méthodes Mathématiques en Sciences de l'Homme*, Paris, Hachette, 1971.
- B. Leclerc, B. Monjardet, « Structures d'ordres et sciences sociales », *Mathématiques et sciences humaines*, 193, 2011, 77-97

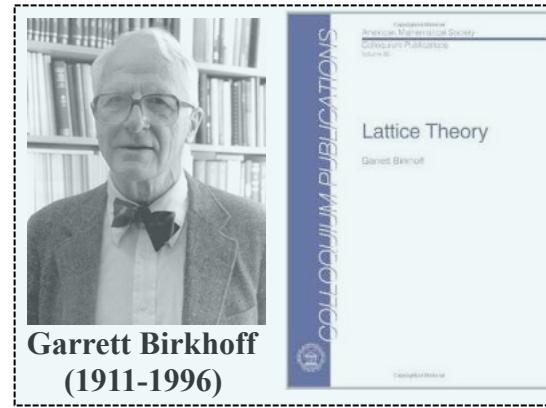
- R. Wille, « Mathematische Sprache in der Musiktheorie », in B. Fuchssteiner, U. Kulisch, D. Laugwitz, R. Liedl (Hrsg.): *Jahrbuch Überblicke Mathematik. B.I.-Wissenschaftsverlag*, Mannheim, 1980, p. 167-184.
- R. Wille, « Restructuring Lattice Theory: An approach based on Hierarchies of Concepts », I. Rival (ed.), *Ordered Sets*, 1982
- R. Wille, « Sur la fusion des contextes individuels », *Mathématiques et sciences humaines*, tome 85, 1984.
- B. Ganter & R. Wille, *Formal Concept Analysis: Mathematical Foundations*, Springer, Berlin, 1998



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# L'Analyse des concepts formels : rappel historique



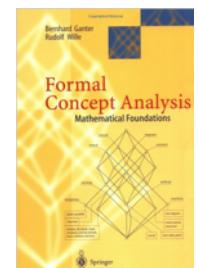
- M. Barbut, « Note sur l'algèbre des techniques d'analyse hiérarchique », in B. Matalon (éd.), L'analyse hiérarchique, Paris, Gauthier-Villars, 1965.
- M. Barbut, B. Monjardet, *Ordre et Classification. Algèbre et Combinatoire*, en deux tomes, 1970
- M. Barbut, L. Frey, « Techniques ordinaires en analyse des données », Tome I, *Algèbre et Combinatoire des Méthodes Mathématiques en Sciences de l'Homme*, Paris, Hachette, 1971.
- B. Leclerc, B. Monjardet, « Structures d'ordres et sciences sociales », *Mathématiques et sciences humaines*, 193, 2011, 77-97



- R. Wille, « Mathematische Sprache in der Musiktheorie », in B. Fuchssteiner, U. Kulisch, D. Laugwitz, R. Liedl (Hrsg.): *Jahrbuch Überblicke Mathematik. B.I.-Wissenschaftsverlag*, Mannheim, 1980, p. 167-184.
- R. Wille, « Restructuring Lattice Theory: An approach based on Hierarchies of Concepts », I. Rival (ed.), *Ordered Sets*, 1982
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- B. Ganter & R. Wille, *Formal Concept Analysis: Mathematical Foundations*, Springer, Berlin, 1998



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# FCA comme « restructuration » de la théorie des treillis

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RESTRUCTURING LATTICE THEORY:

AN APPROACH BASED ON HIERARCHIES OF CONCEPTS

Rudolf Wille

Fachbereich Mathematik

Technische Hochschule Darmstadt

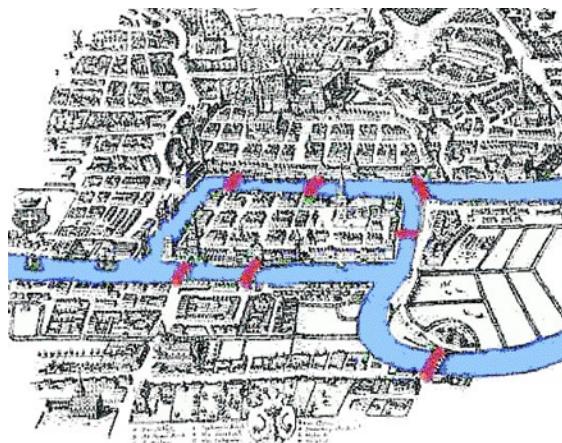
6100 Darmstadt

Federal Republic of Germany

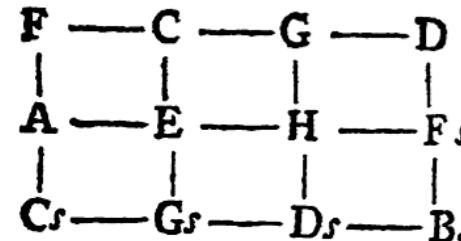
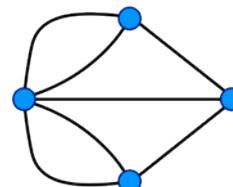
## ABSTRACT

Lattice theory today reflects the general status of current mathematics: there is a rich production of theoretical concepts, results, and developments, many of which are reached by elaborate mental gymnastics; on the other hand, the connections of the theory to its surroundings are getting weaker and weaker, with the result that the theory and even many of its parts become more isolated. Restructuring lattice theory is an attempt to reinvigorate connections with our general culture by interpreting the theory as concretely as possible, and in this way to promote better communication between lattice theorists and potential users of lattice theory.

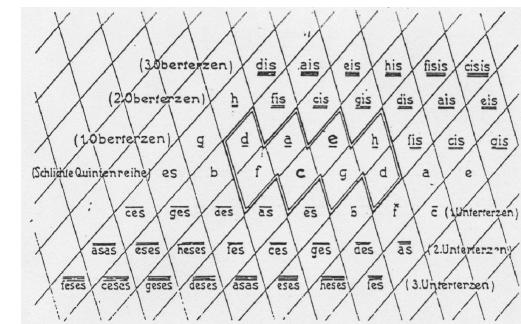
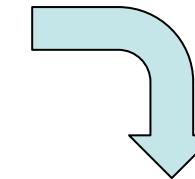
# Dynamique « mathémusicale » dans la FCA



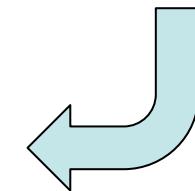
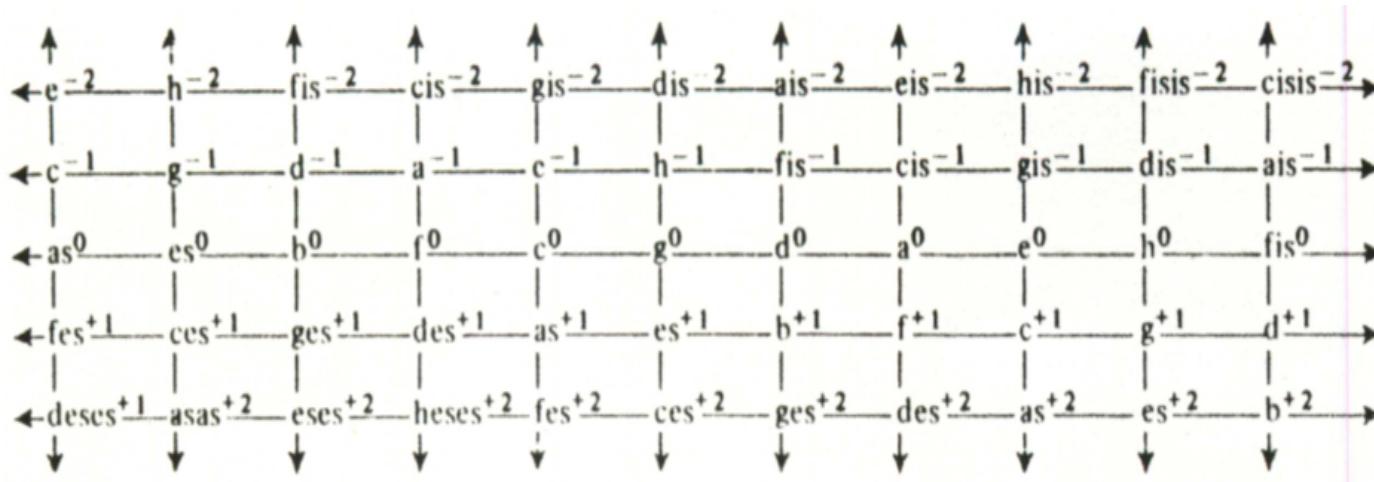
Königsberg et ses ponts



Euler, *Speculum musicum*, 1773

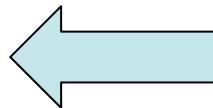
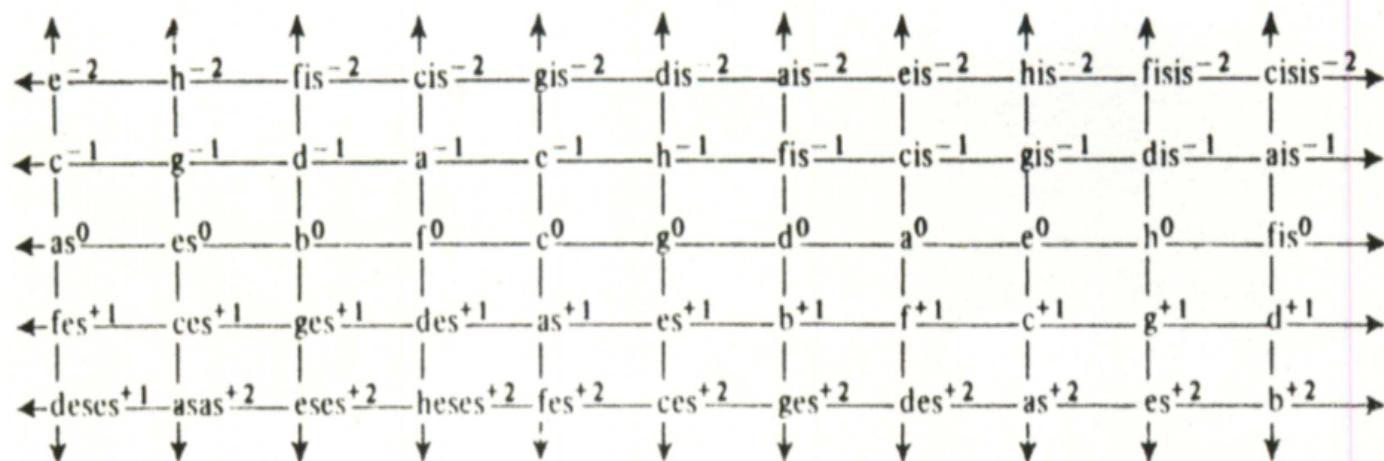


Le Tonnetz d'Hugo Riemann



R. Wille, « Mathematische Sprache in der Musiktheorie », in B. Fuchssteiner, U. Kulisch, D. Laugwitz, R. Liedl (Hrsg.): Jahrbuch Überblicke Mathematik. B.I.-Wissenschaftsverlag, Mannheim, 1980, p. 167-184.

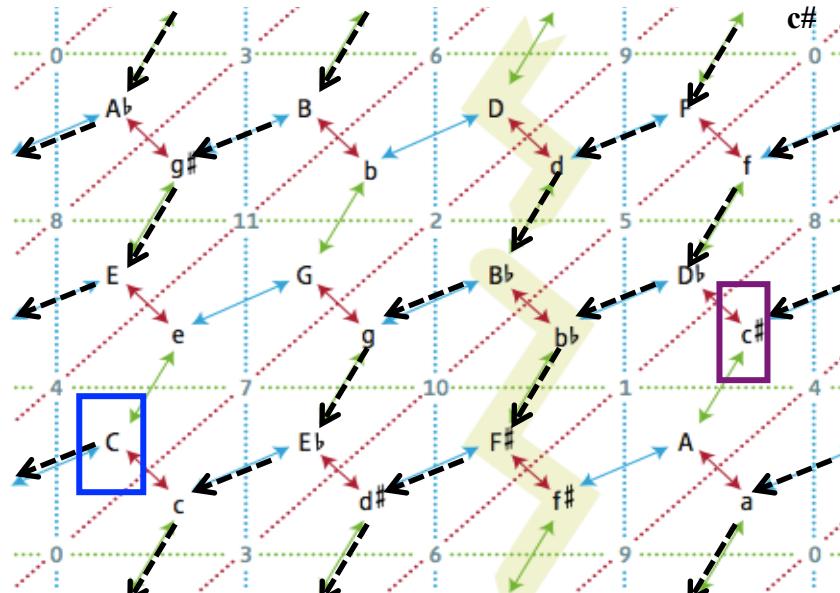
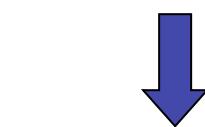
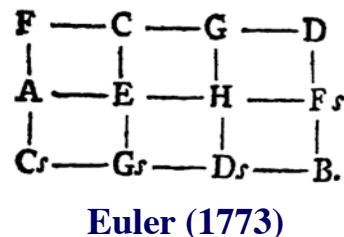
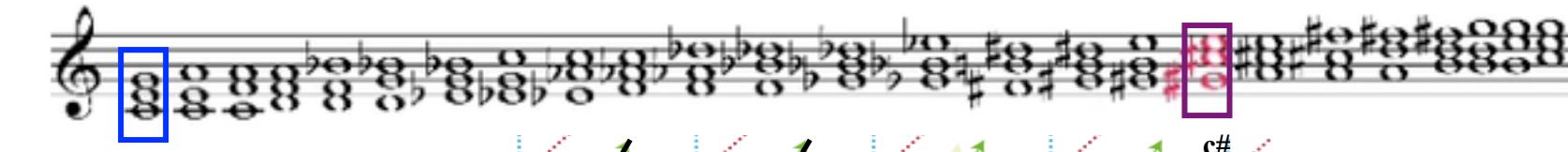
# FCA et musicologie computationnelle



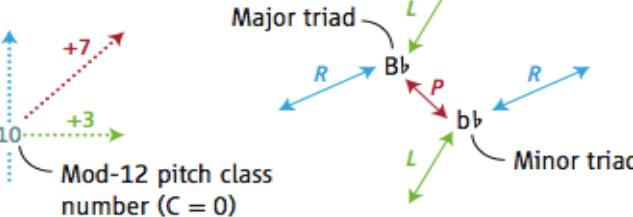
*Mutabor à Darmstadt au début des années 1980*

➔ <http://www.math.tu-dresden.de/~mutabor/>

# Le Tonnetz et l'analyse néo-riemannienne

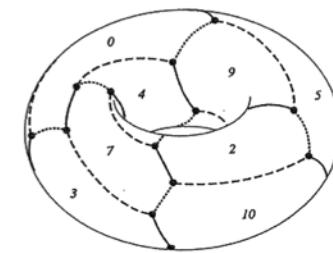


A	C#	F	A'	C#'	F'	A''	C#''	F''	A'''
D	F#	A#	D'	F#'	A#'	D''	F#''	A##'	D'''
G	B	D#	G'	B'	D#'	G"	B"	D##"	G'''
C	E	G#	C'	E'	G#'	C"	E"	G##"	C'''
F	A	C#	F'	A'	C#'	F''	A''	C##''	F'''
Bb	D	F#	Bb'	D'	F#'	Bb''	D''	F##''	Bb'''
Eb	G	B	Eb'	G'	B'	Eb''	G''	B''	Eb'''
Ab	C	E	Ab'	C'	E'	Ab''	C''	E''	Ab'''

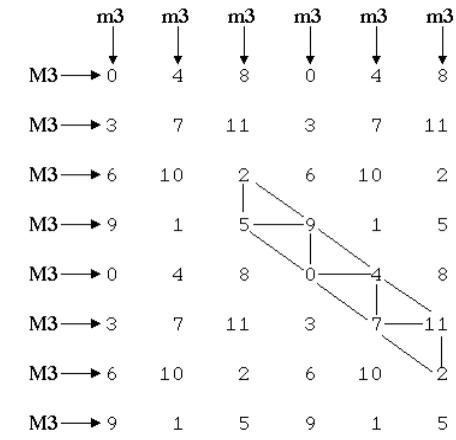
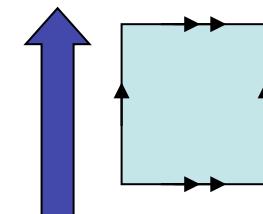


J. Hook, « Exploring Musical Space »,  
Science, 2006

Longuet-Higgins (1962)

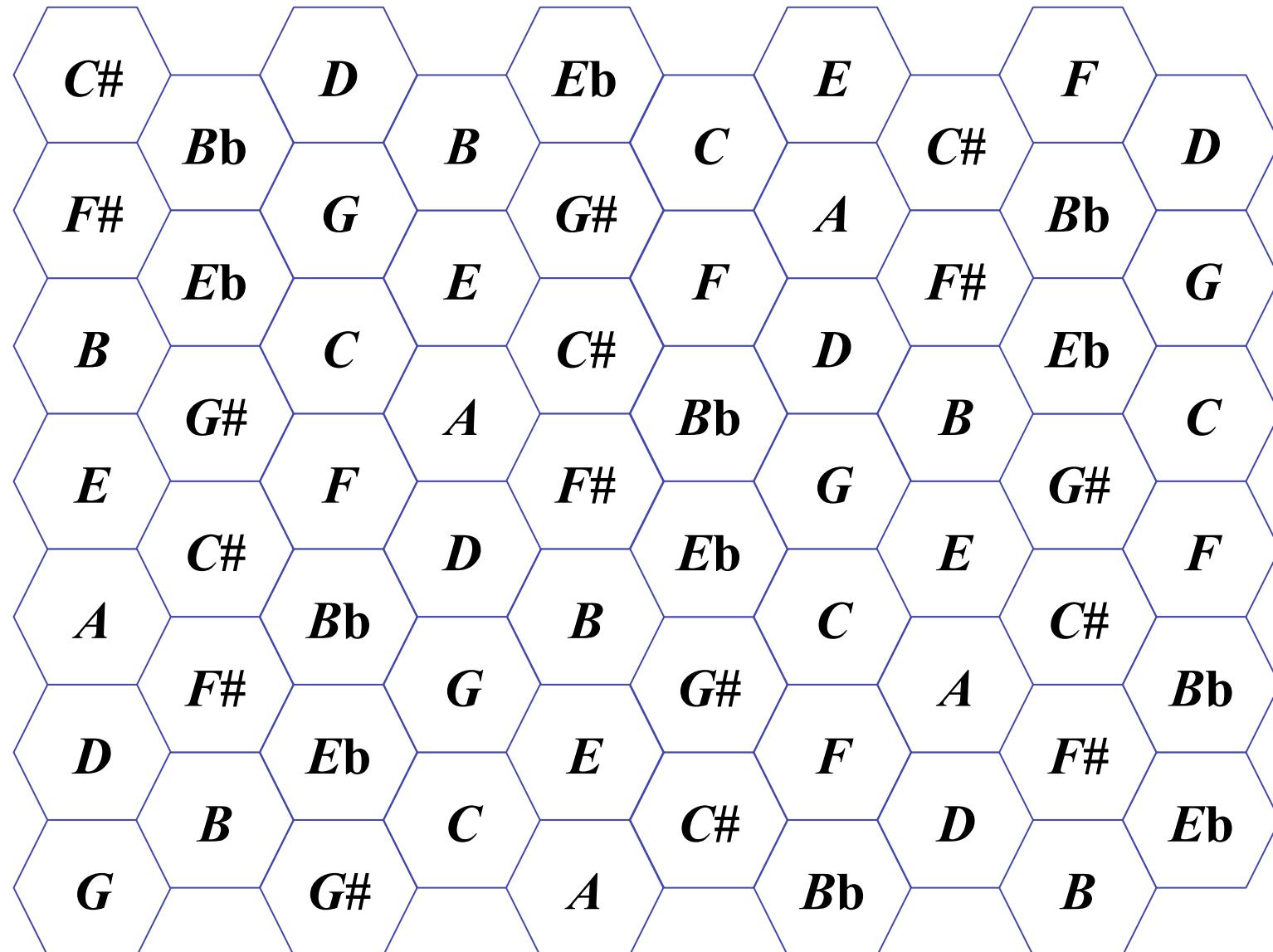


Douthett & Steinbach 1998

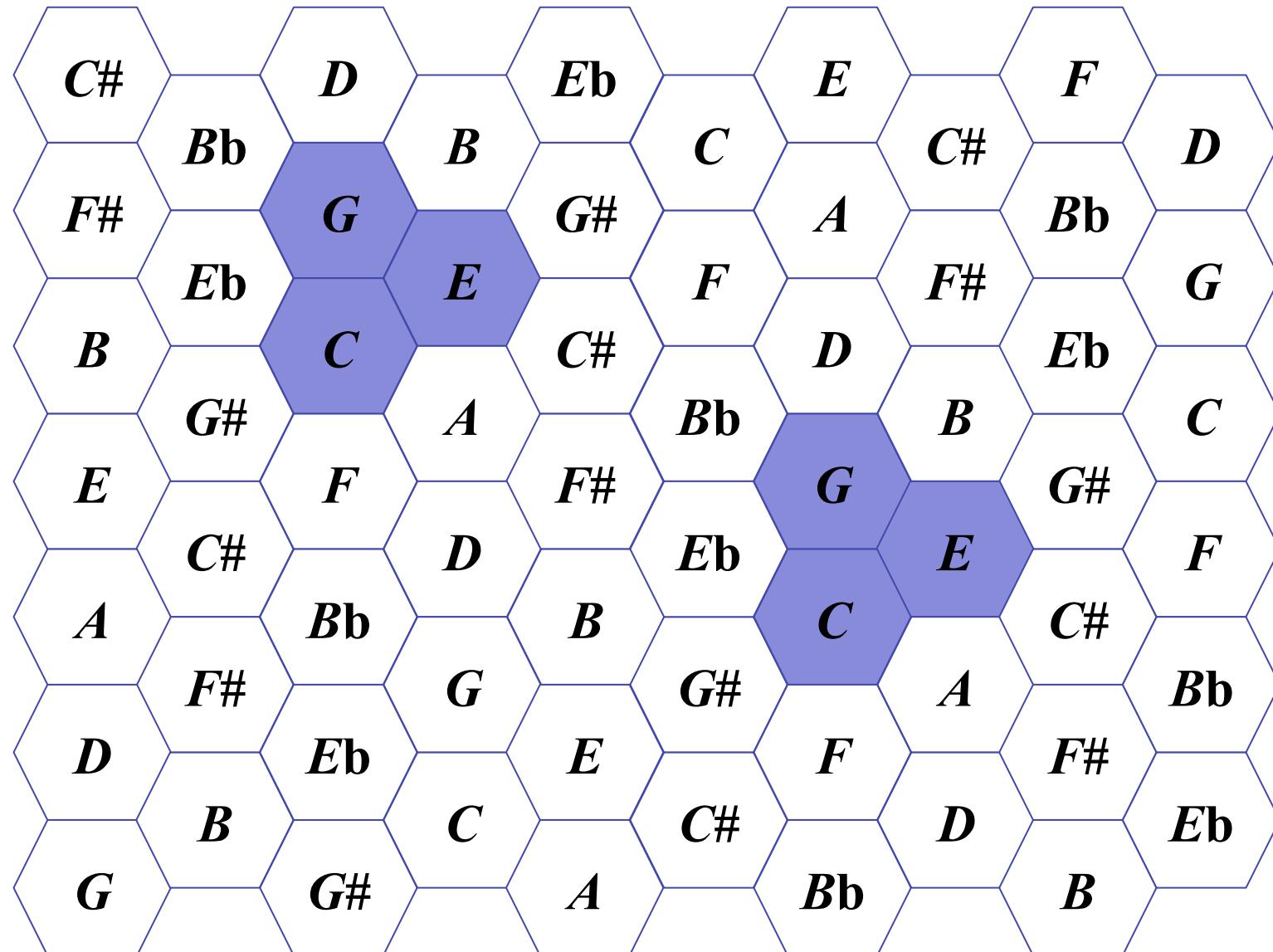


Balzano (1980)

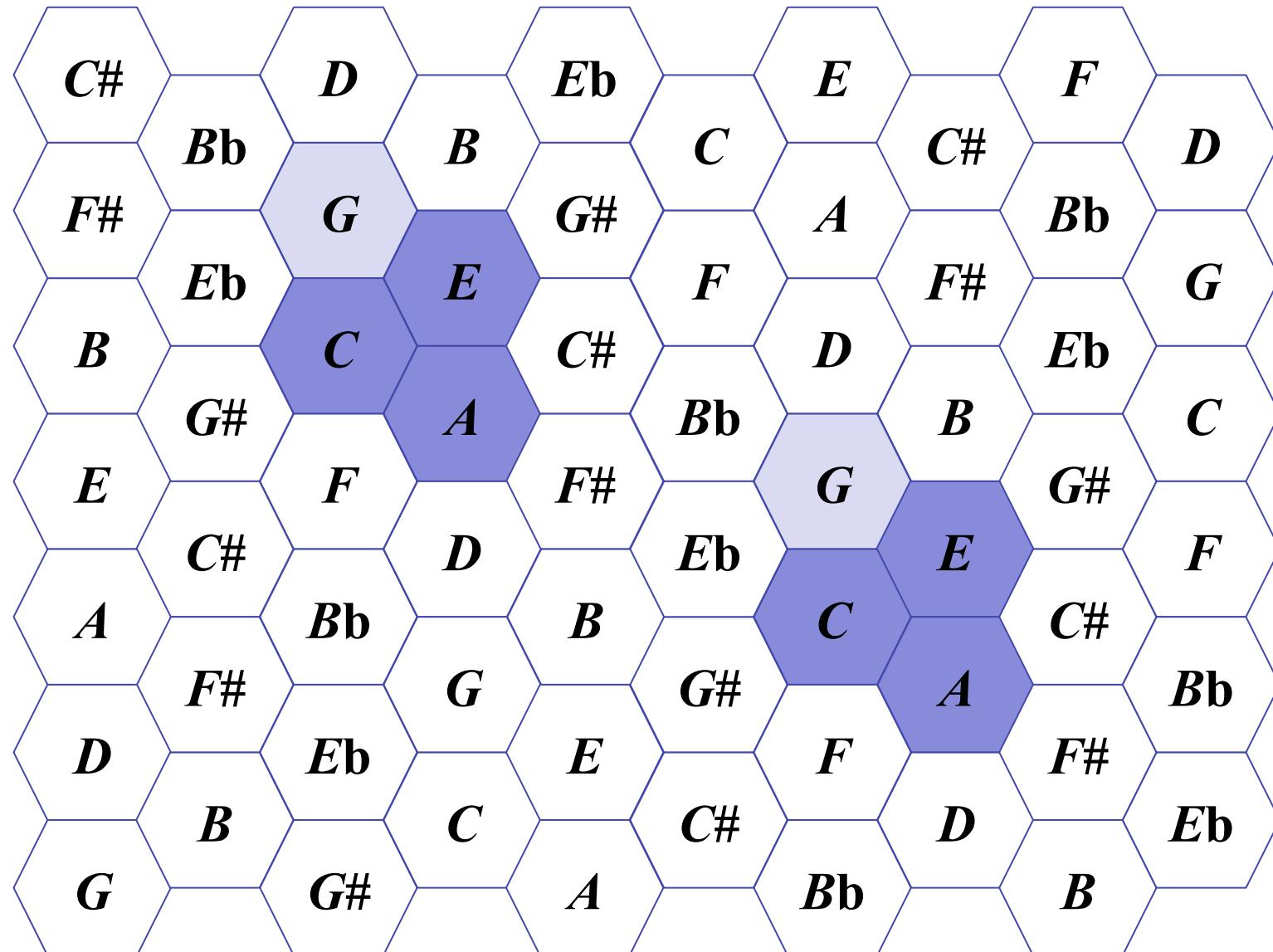
# Extract of the 2<sup>nd</sup> movement of the Symphony No. 9 (L. van Beethoven)



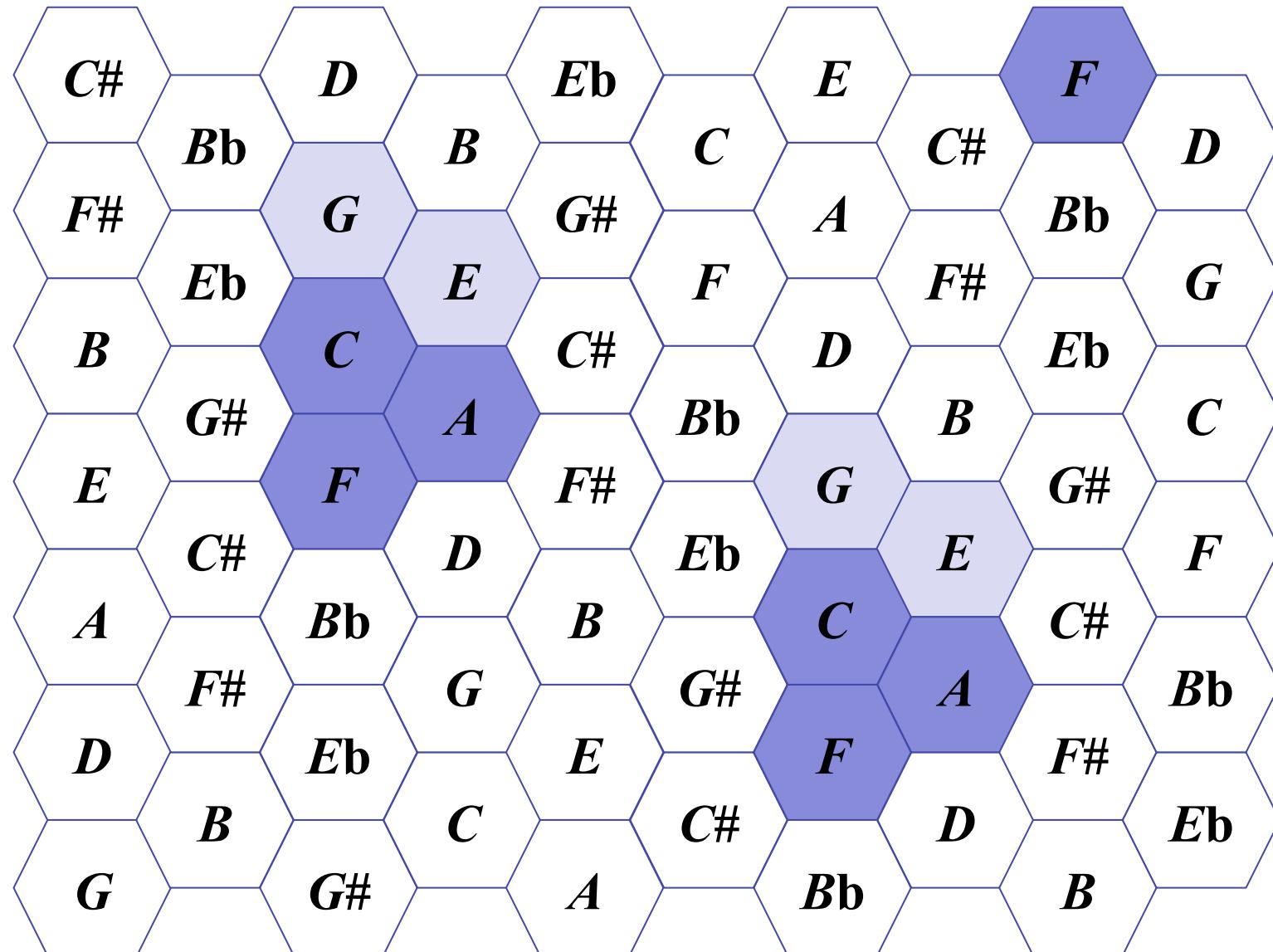
# Extract of the 2<sup>nd</sup> movement of the Symphony No. 9 (L. van Beethoven)



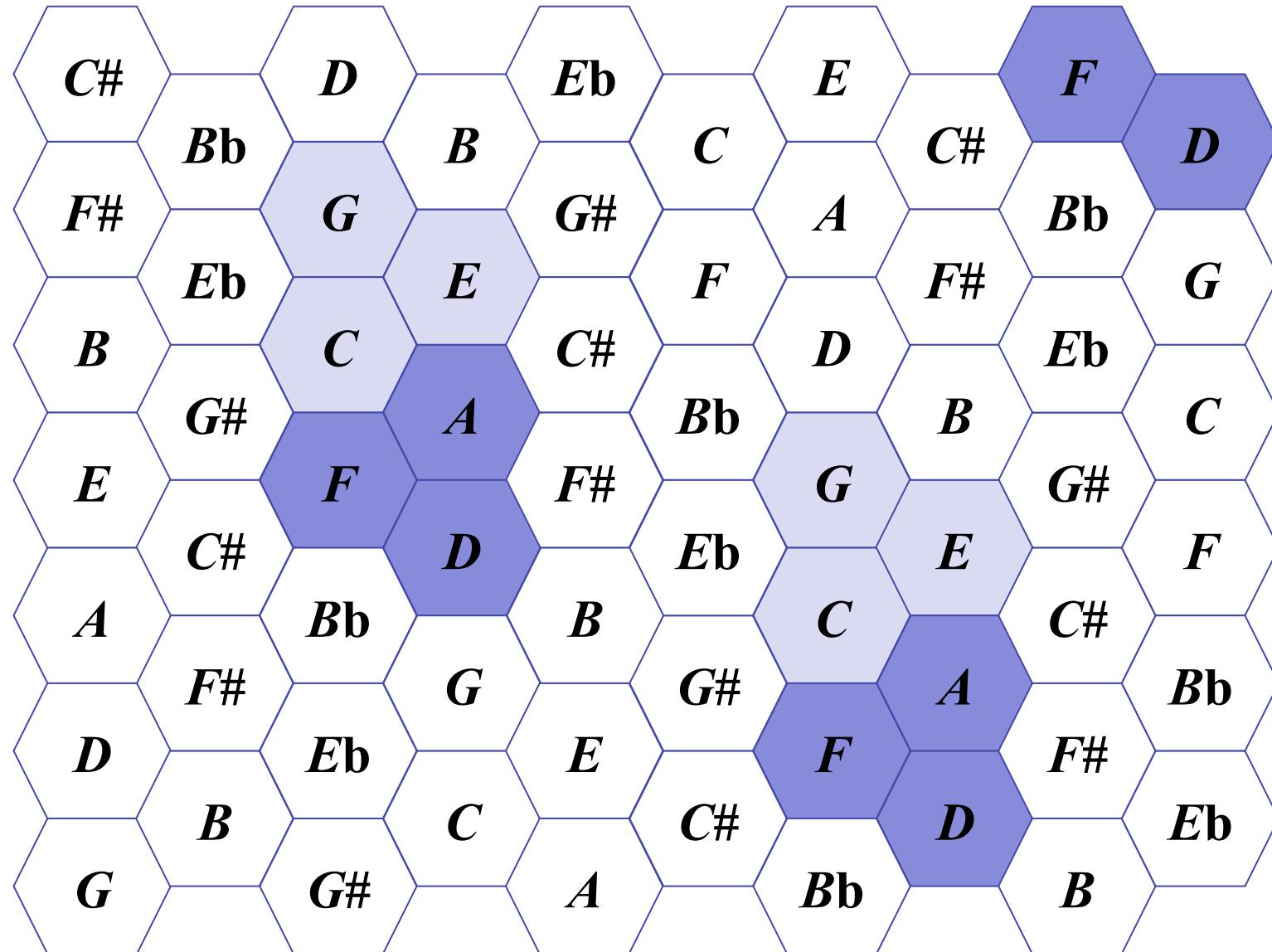
# Extract of the 2<sup>nd</sup> movement of the Symphony No. 9 (L. van Beethoven)



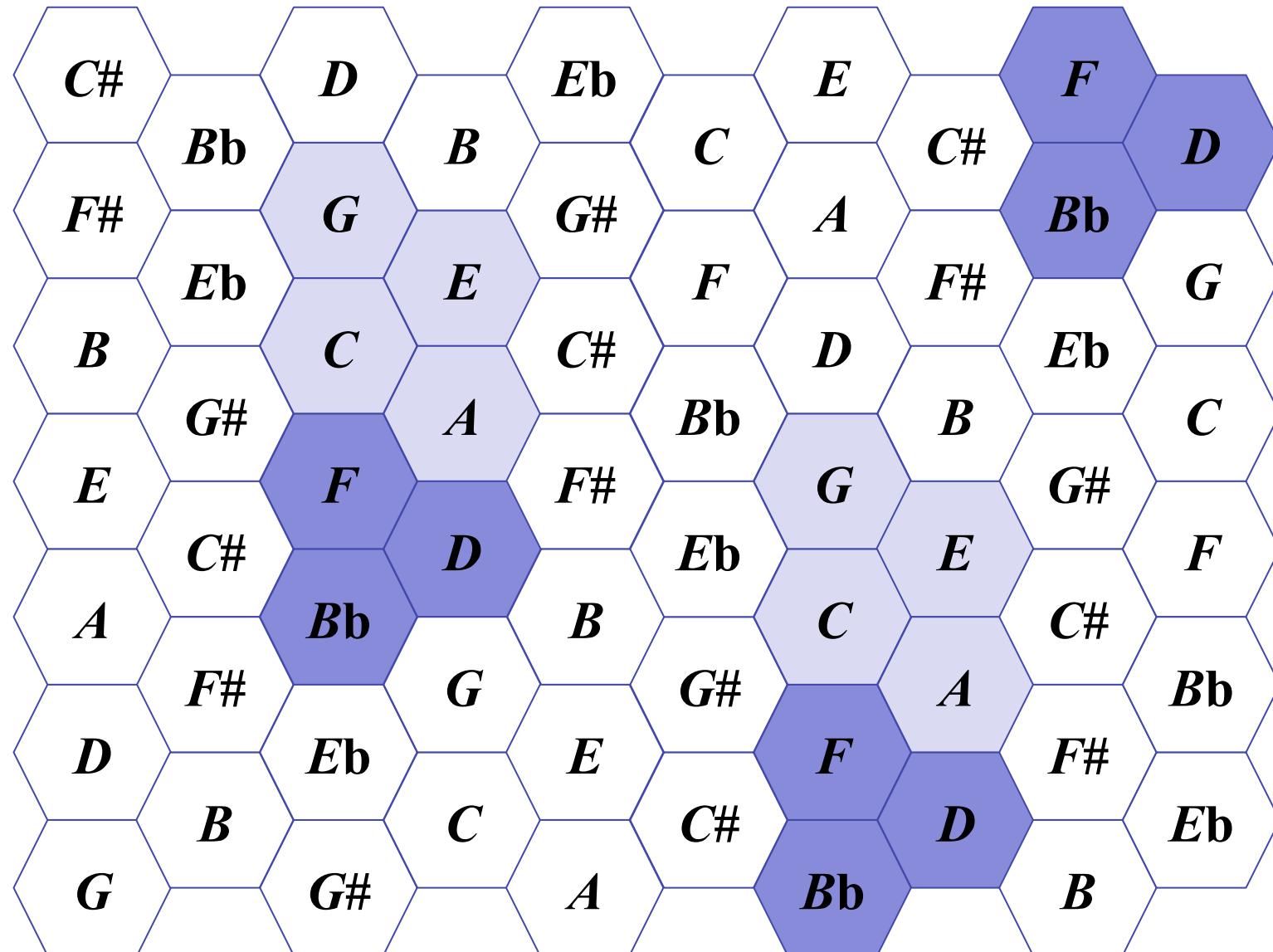
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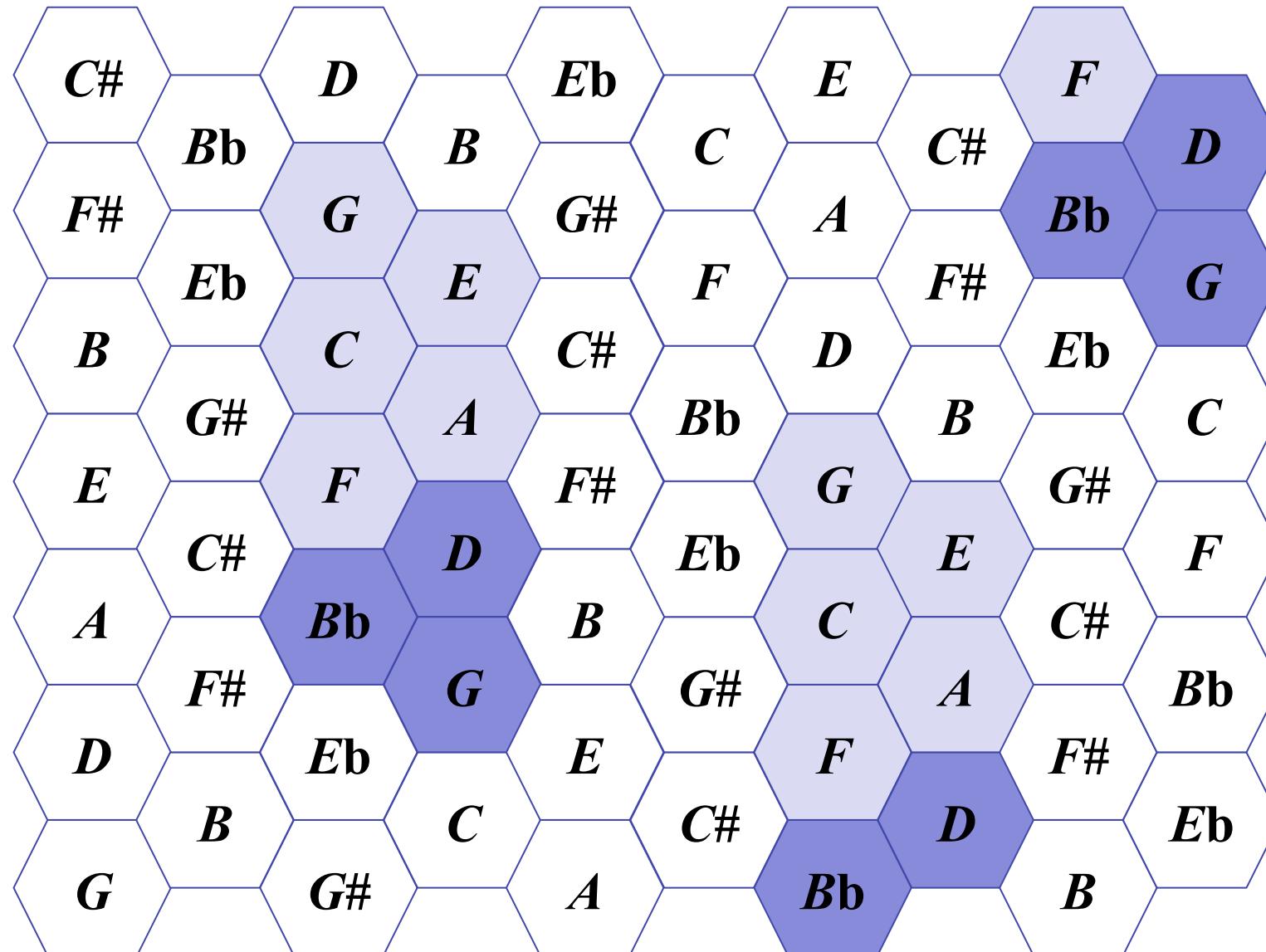
# Extract of the 2<sup>nd</sup> movement of the Symphony No. 9 (L. van Beethoven)



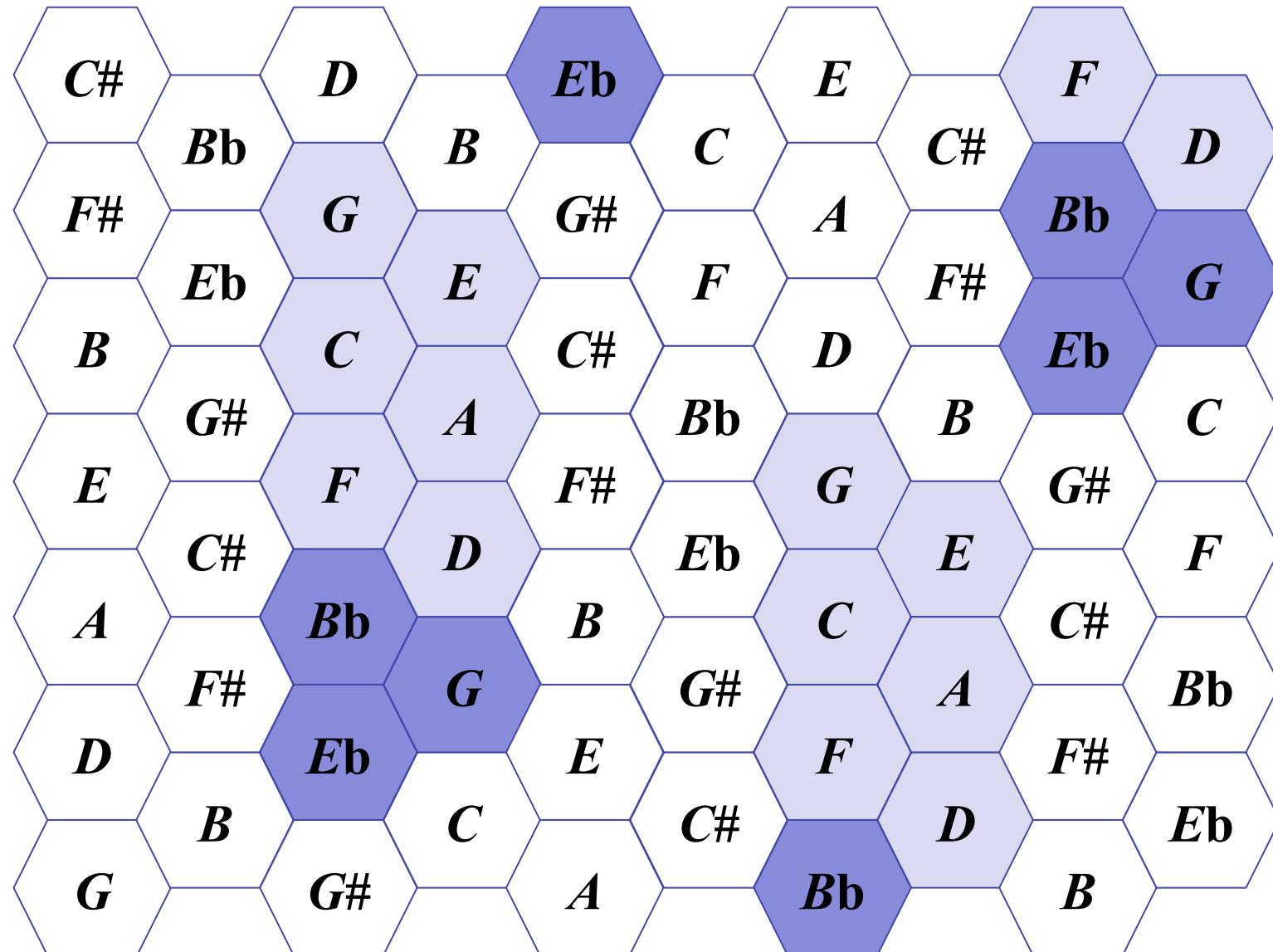
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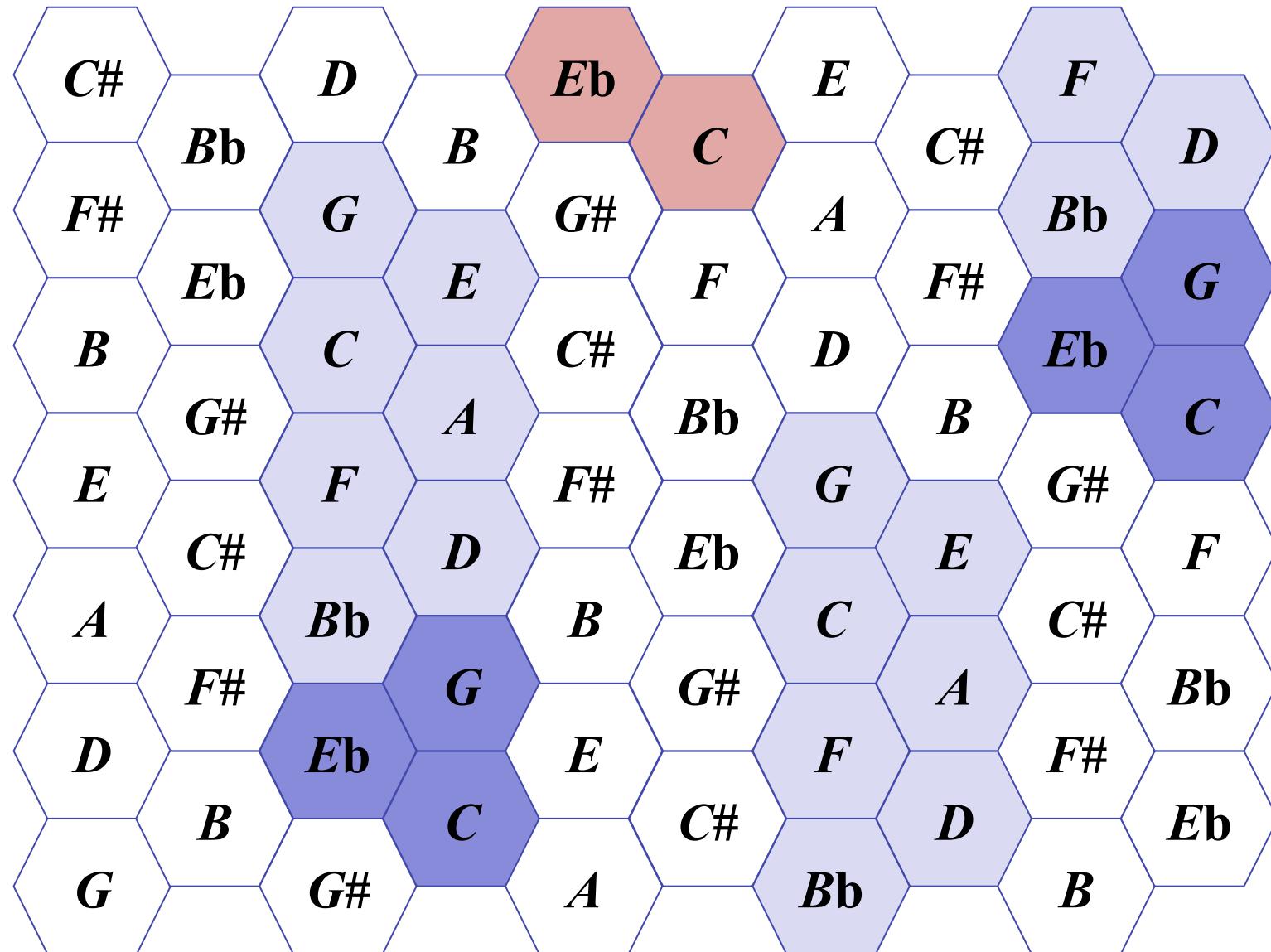
# **Extract of the 2<sup>nd</sup> movement of the Symphony No. 9 (L. van Beethoven)**



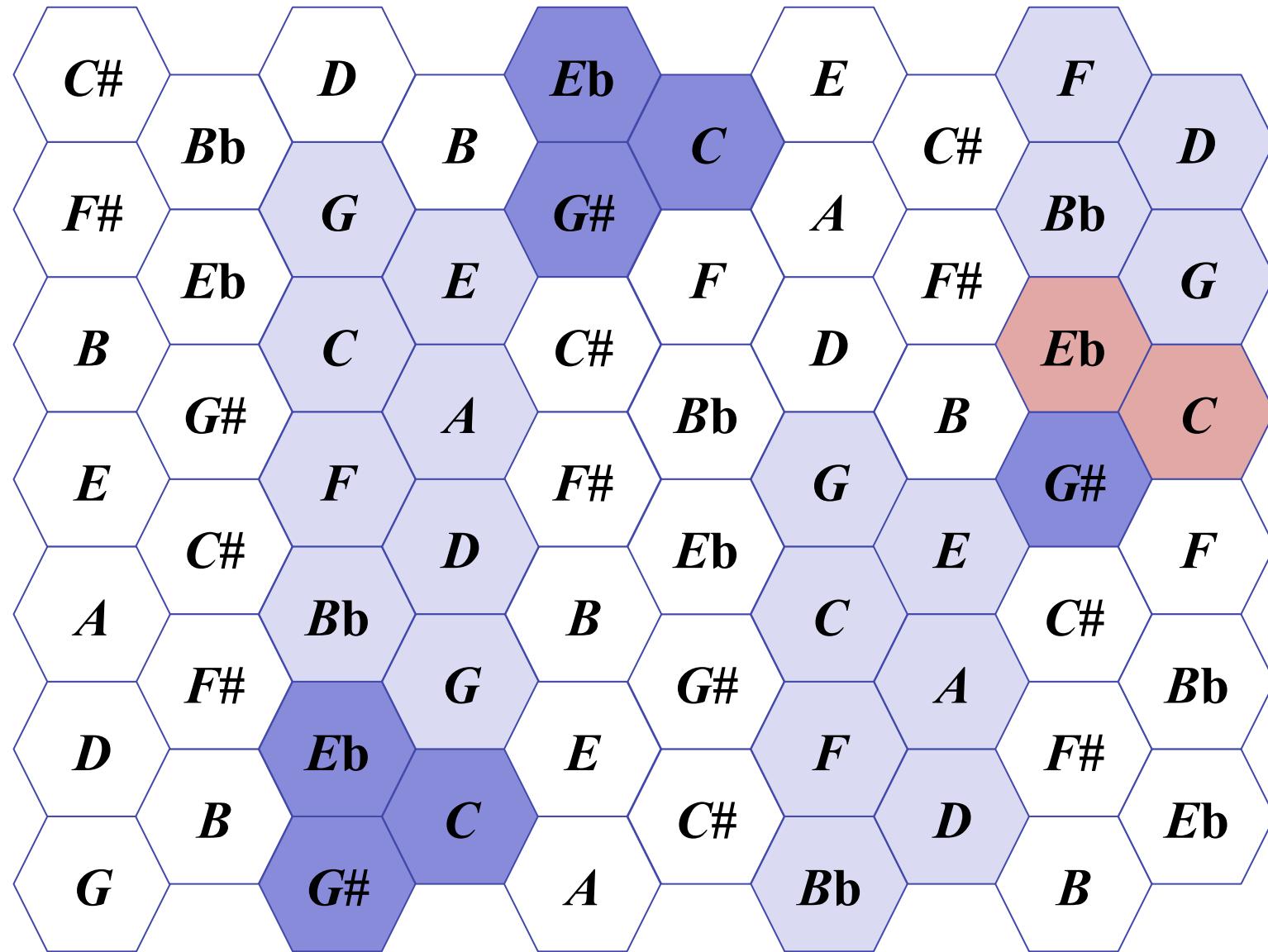
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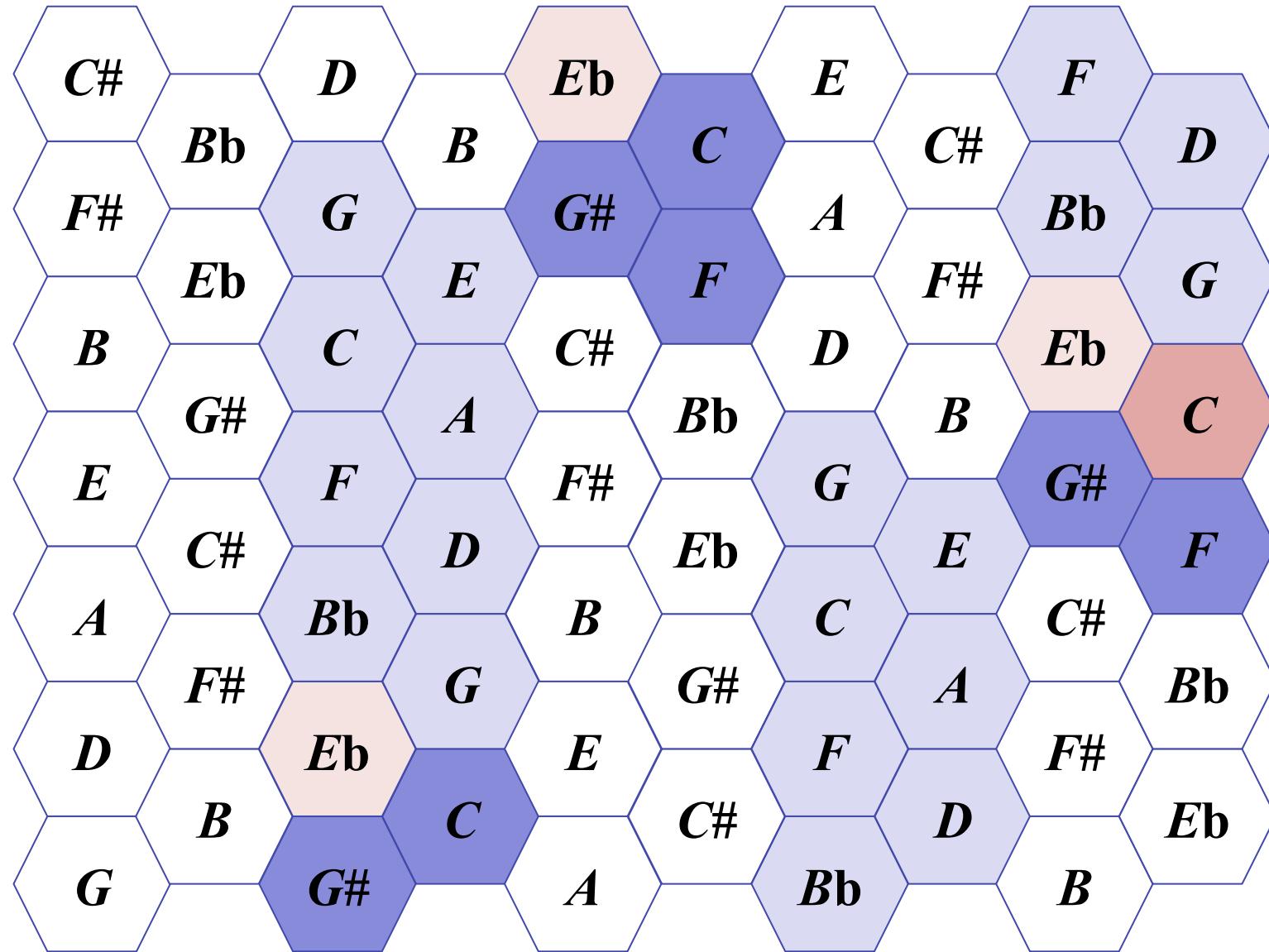
# Extract of the 2<sup>nd</sup> movement of the Symphony No. 9 (L. van Beethoven)



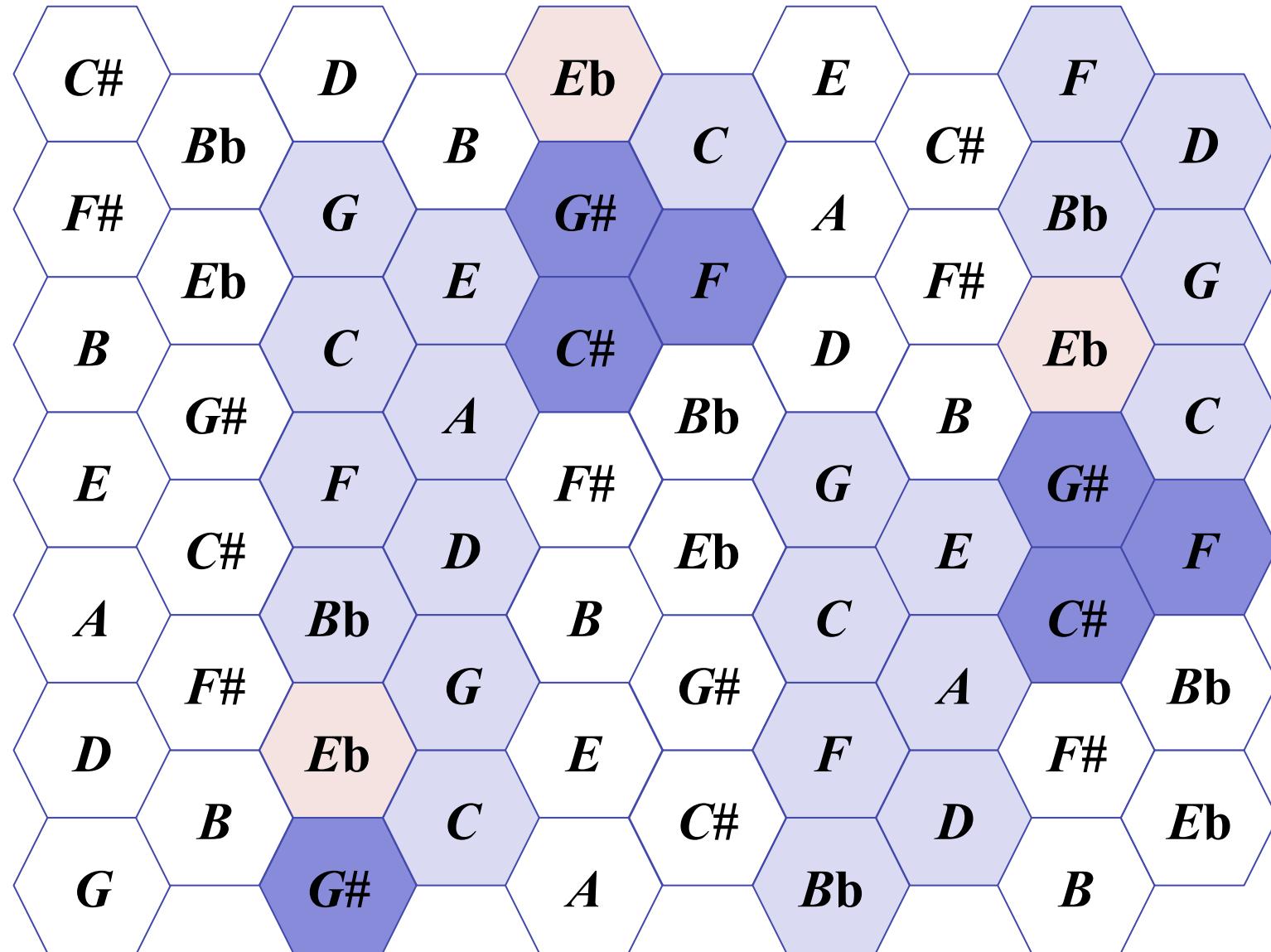
# Extract of the 2<sup>nd</sup> movement of the Symphony No. 9 (L. van Beethoven)



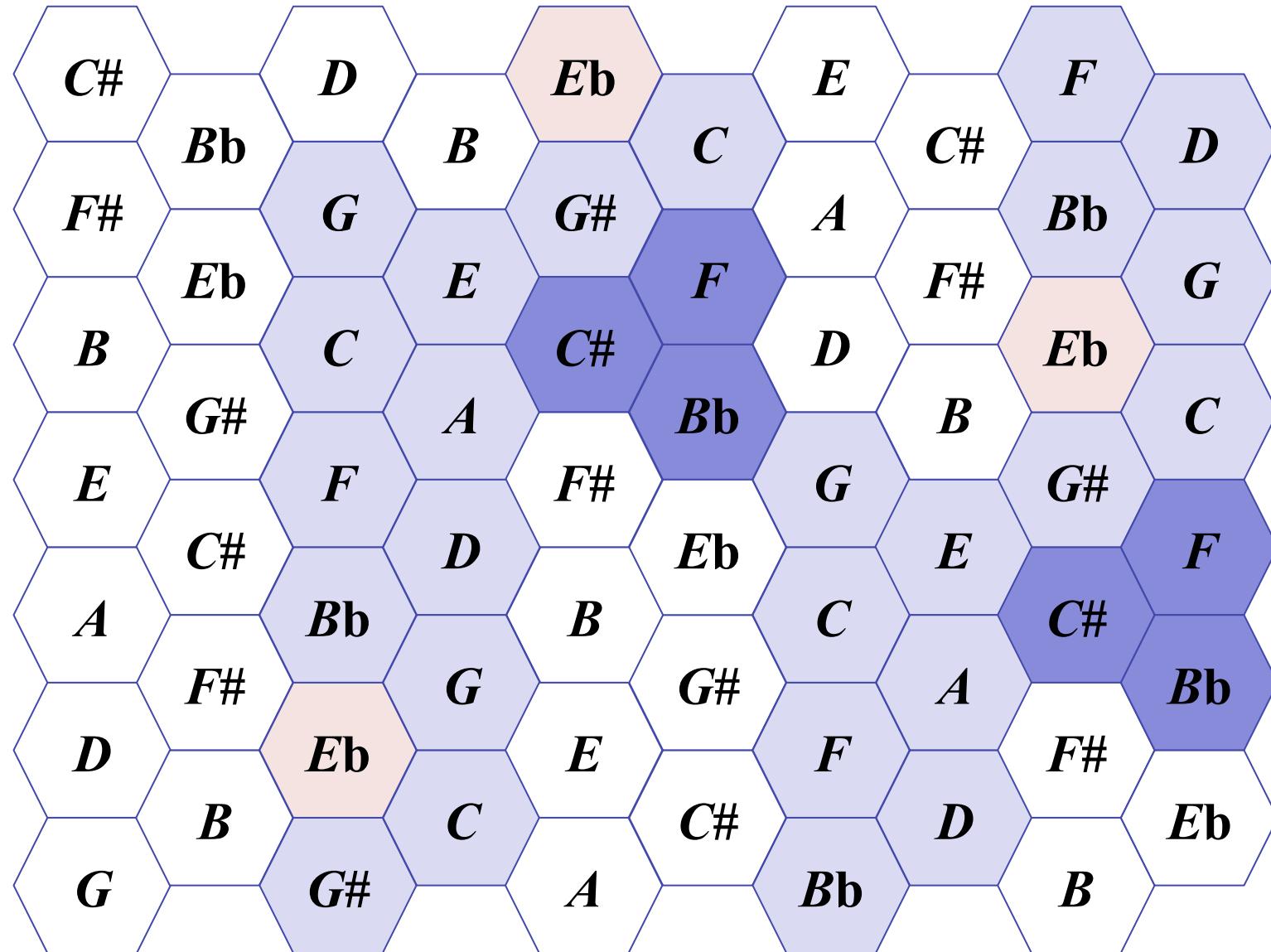
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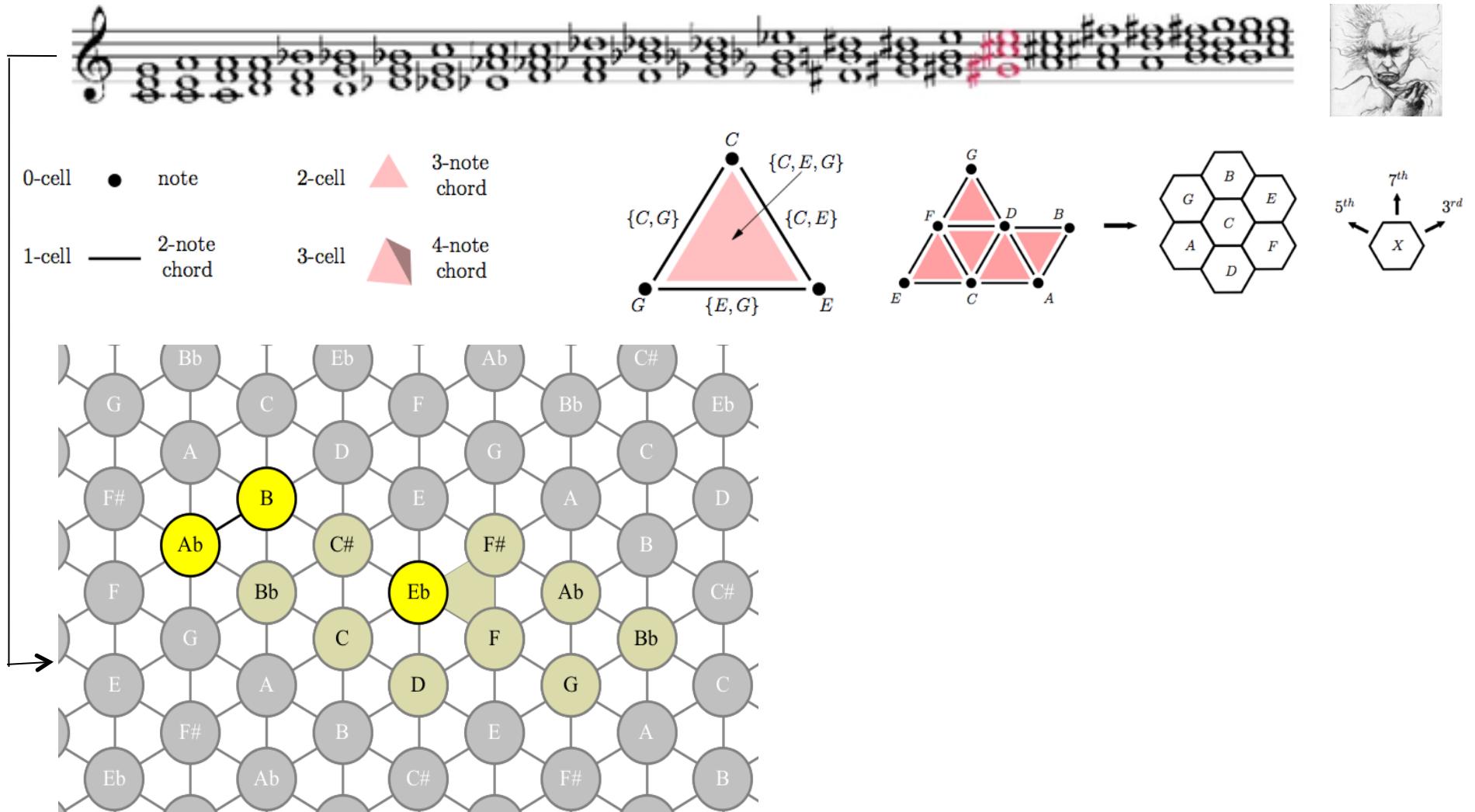
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# Extract of the 2<sup>nd</sup> movement of the Symphony No. 9 (L. van Beethoven)

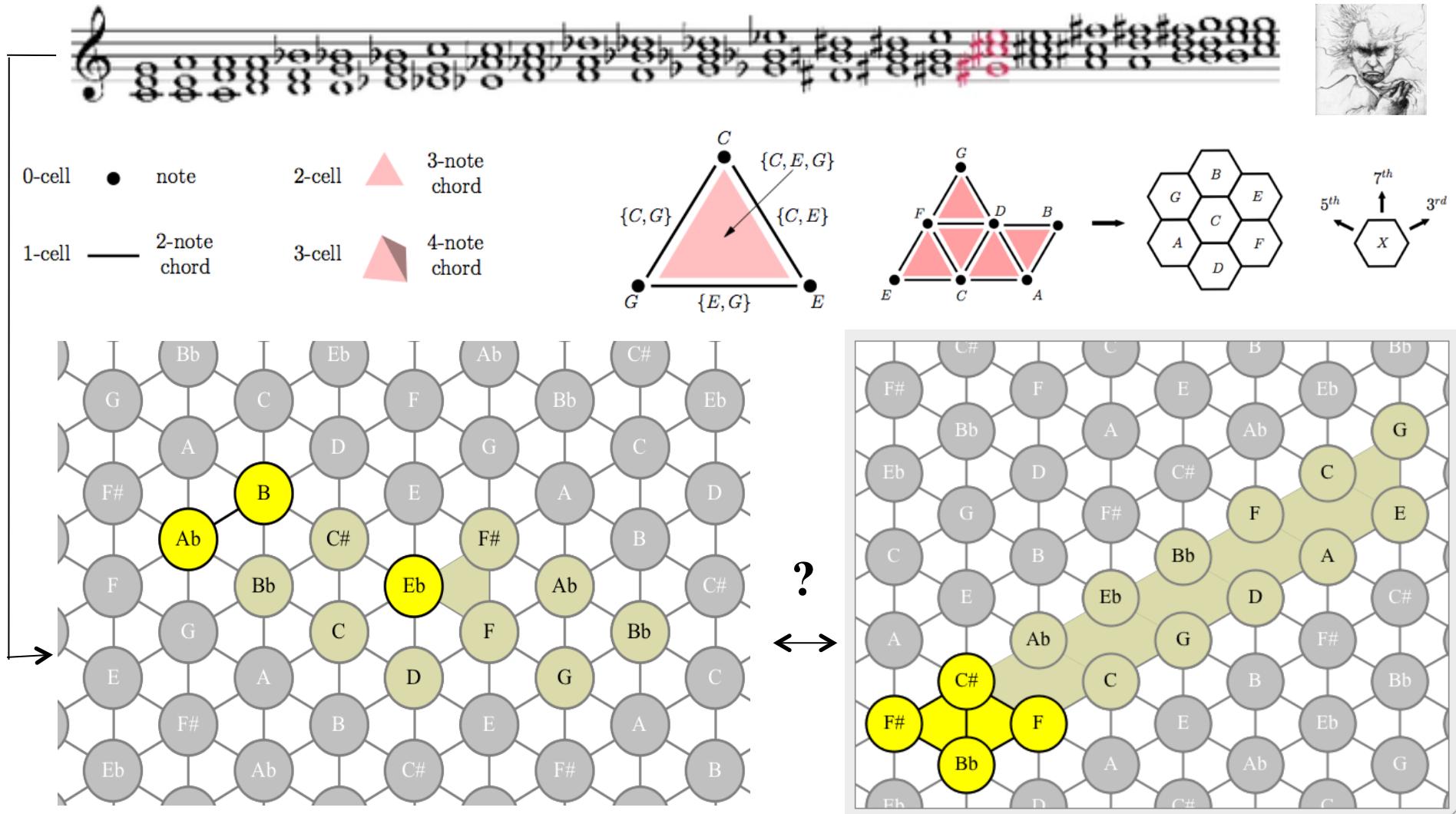


# Tonnetz et programmation spatiale



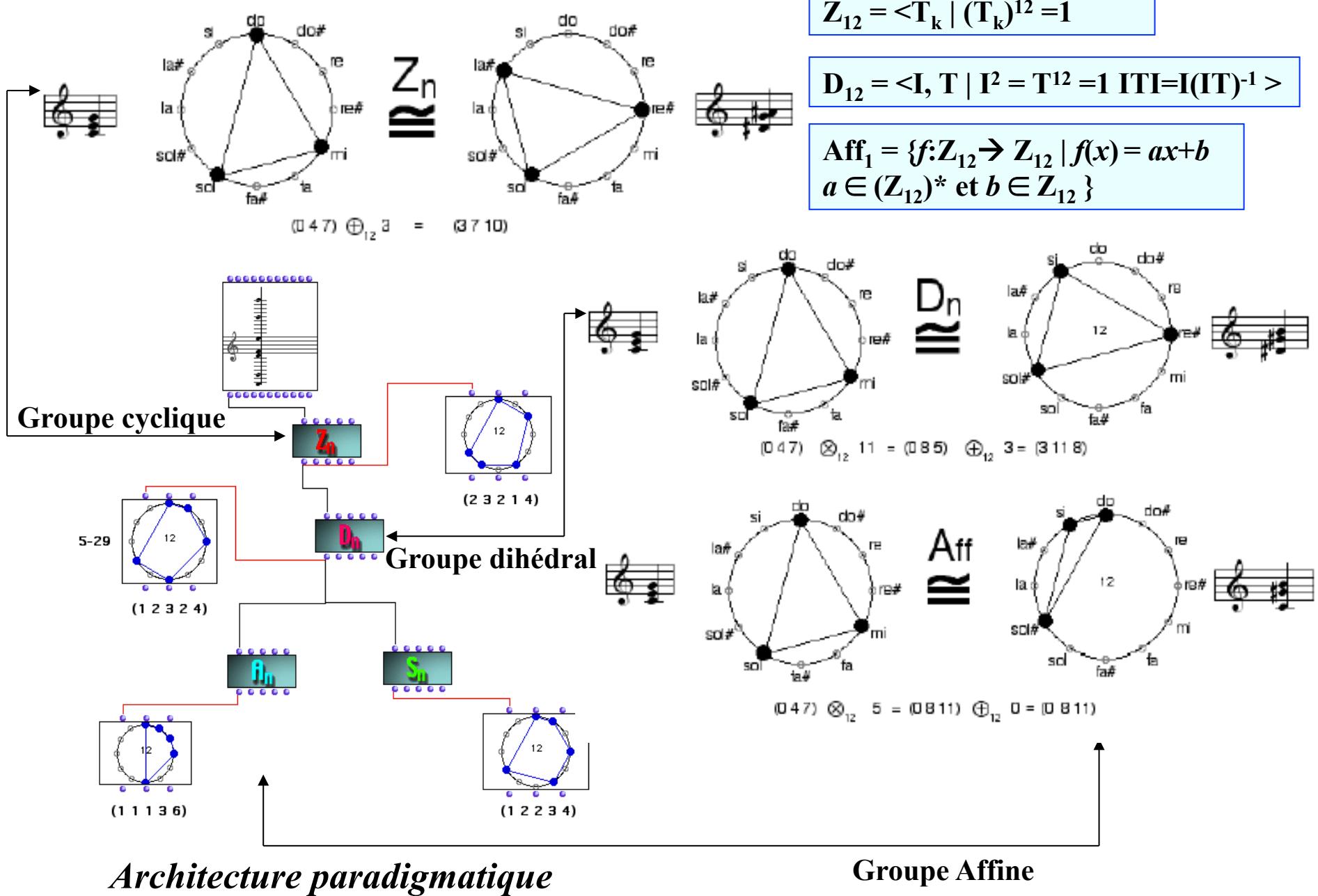
- L. Bigo, M. Andreatta, J.-L. Giavitto, O. Michel, A. Spicher, « Computation and Visualization of Musical Structures in Chord-based Simplicial Complexes », Fourth International Conference, MCM 2013, McGill University, Springer, LNCS.

# *Tonnetz et programmation spatiale*

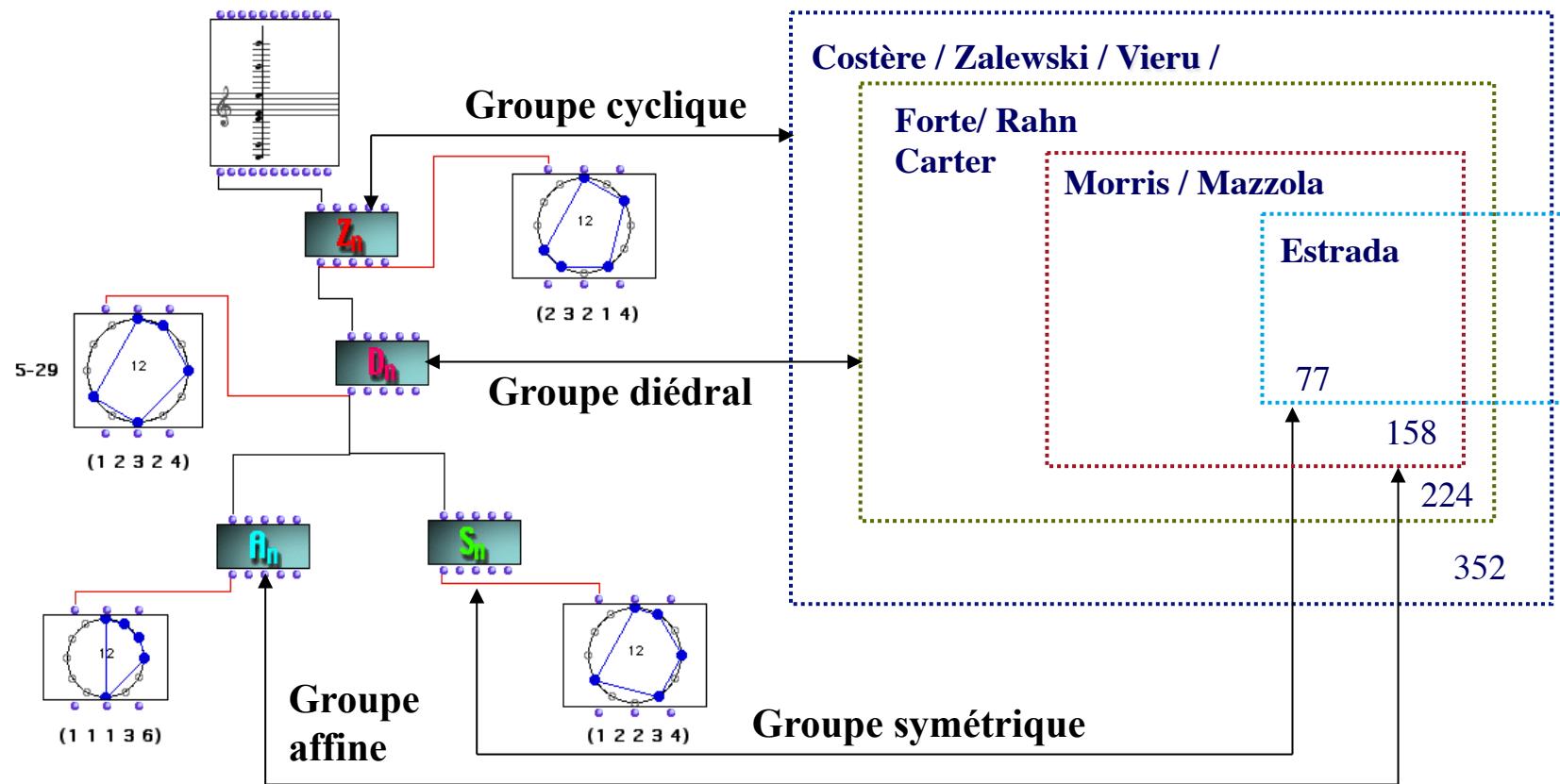


- L. Bigo, M. Andreatta, J.-L. Giavitto, O. Michel, A. Spicher, « Computation and Visualization of Musical Structures in Chord-based Simplicial Complexes », Fourth International Conference, MCM 2013, McGill University, Springer, LNCS.

# Classification paradigmatique des structures musicales



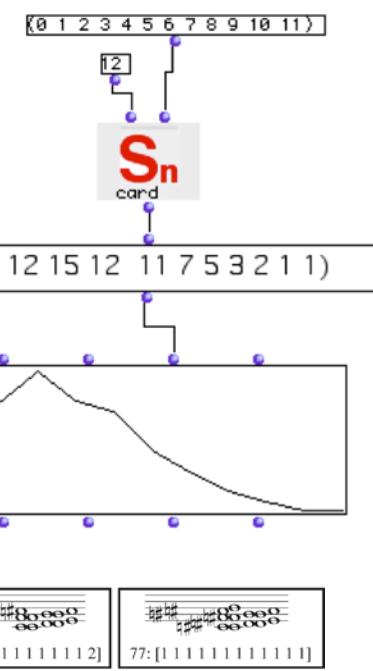
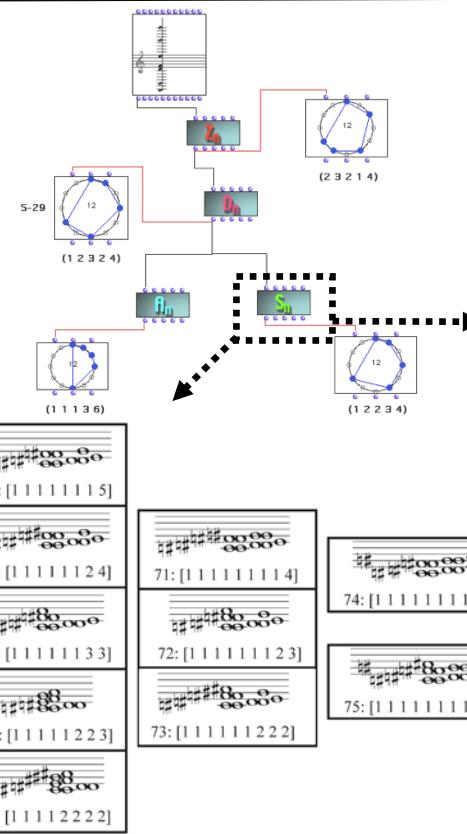
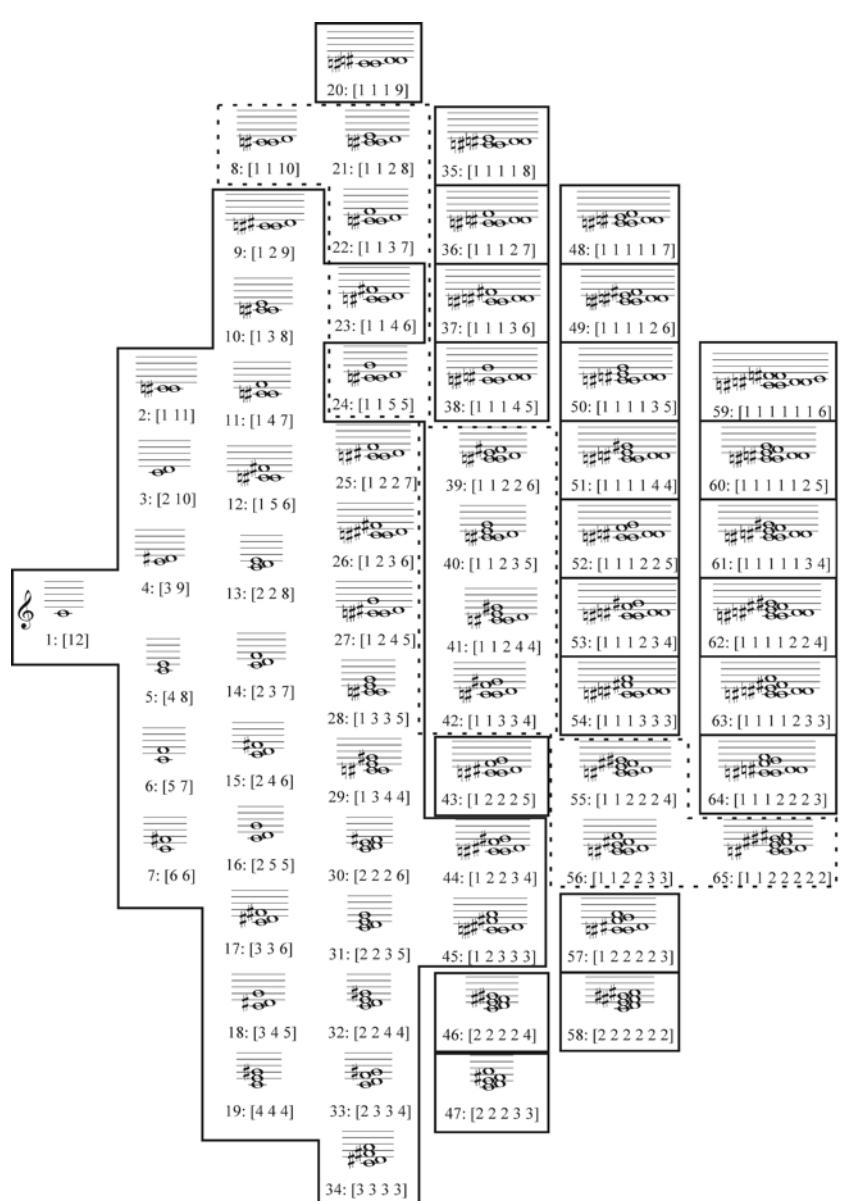
# Enumération d'orbites et catalogues d'accords



cardinalité

	1	2	3	4	5	6	7	8	9	10	11	12
$Z_n$	1	6	19	43	66	80	66	43	19	6	1	1
$D_n$	1	6	12	29	38	50	38	29	12	6	1	1
$A_n$	1	5	9	21	25	34	25	21	9	5	1	1
$S_n$	1	6	12	15	12	11	7	5	3	2	1	1

# Analyse musicale comme déplacement spatial



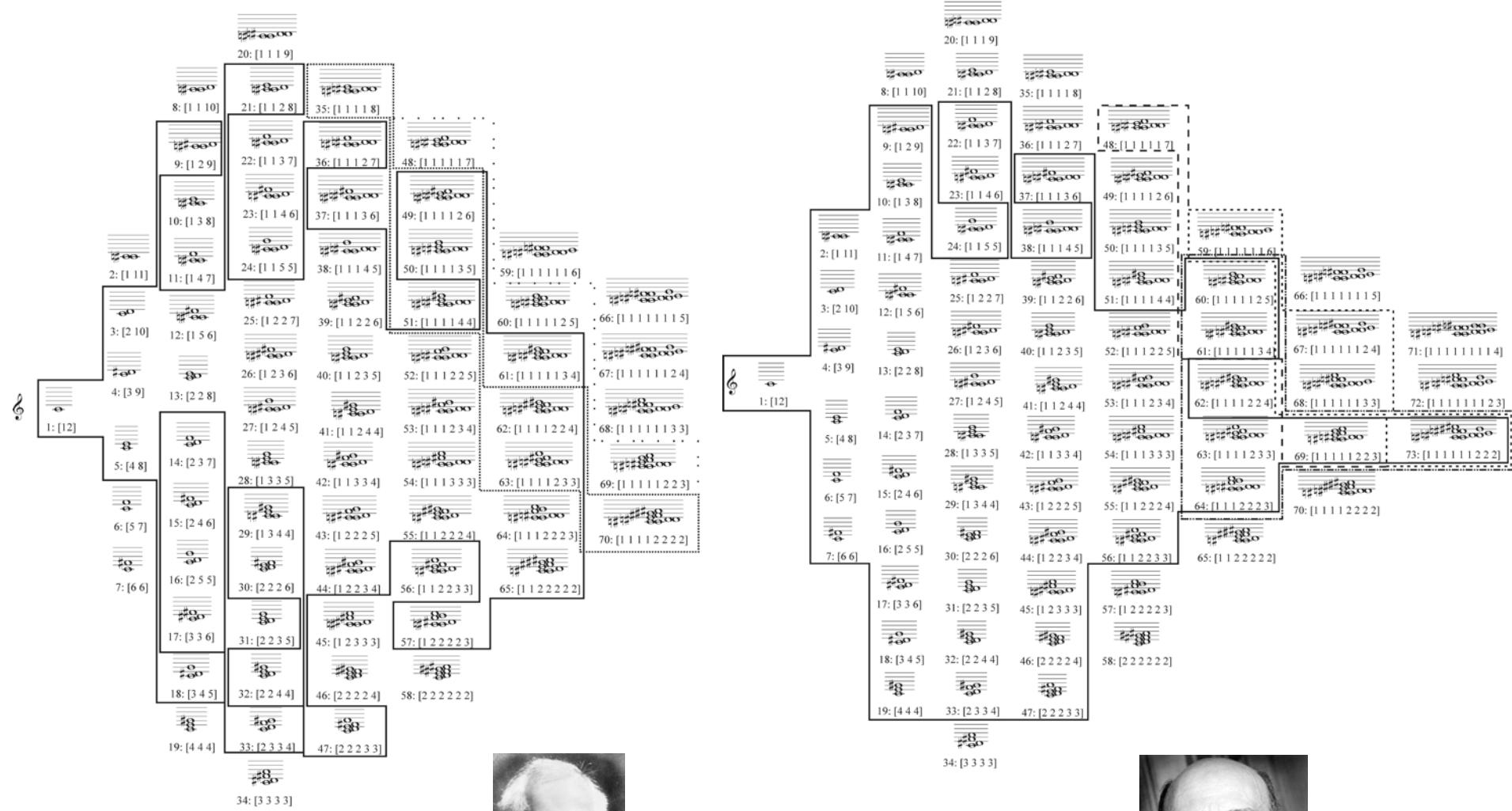
# L. Van Beethoven, Quatuor n° 17



**Julio Estrada**

# Le permutoèdre des « partitions » de Julio Estrada

# L'espace comme paramètre de style



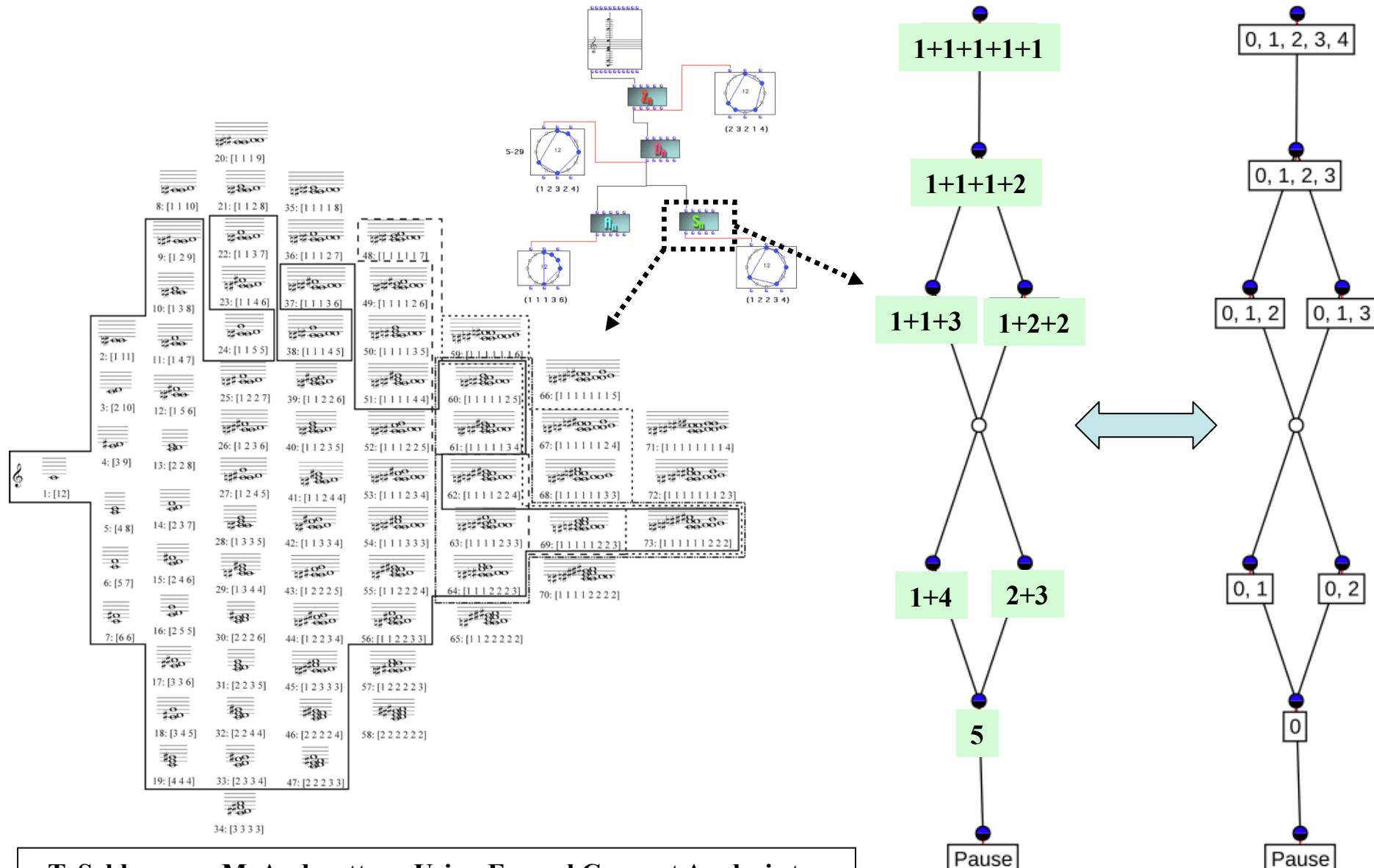
**B. Bartok, Quatuor n° 4  
(3<sup>e</sup> mouvement)**



**A. Schoenberg,  
*Six pièces op. 19***



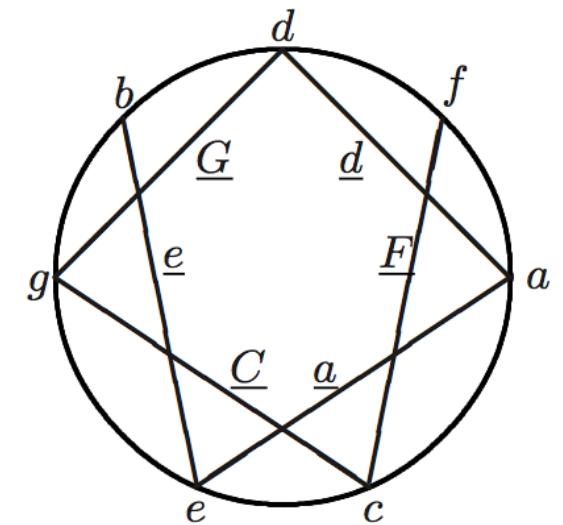
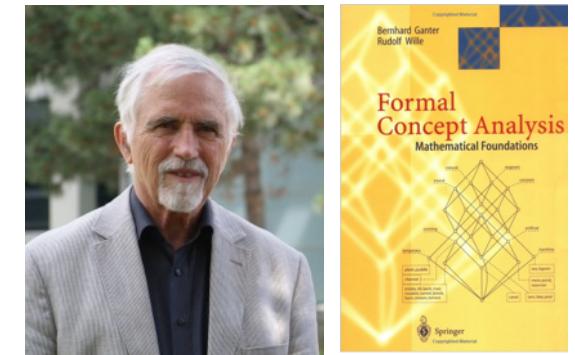
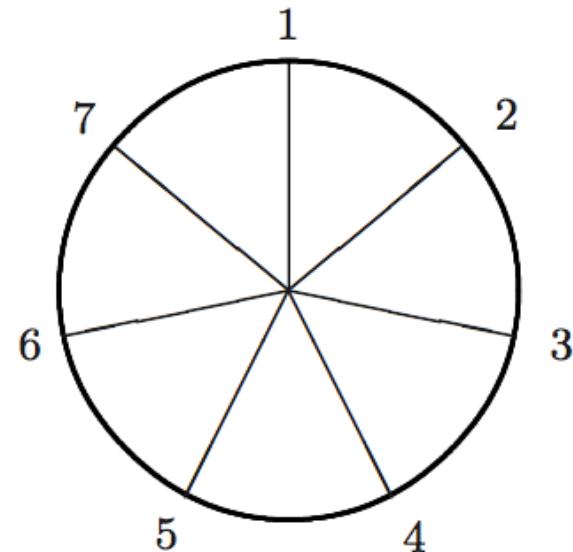
# Permutoèdre comme treillis de concepts formels



- T. Schlemmer, M. Andreatta, « Using Formal Concept Analysis to represent Chroma Systems », Fourth International Conference, MCM 2013, McGill University, Springer, LNCS.

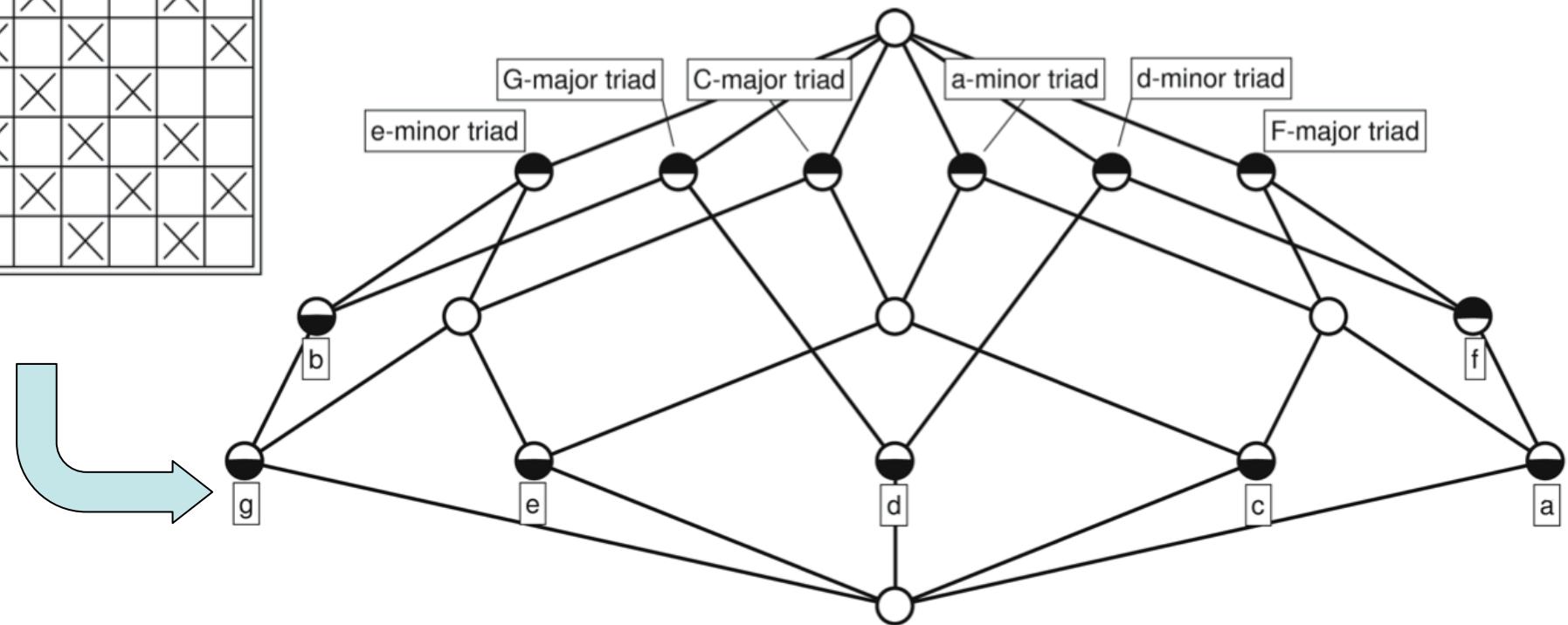
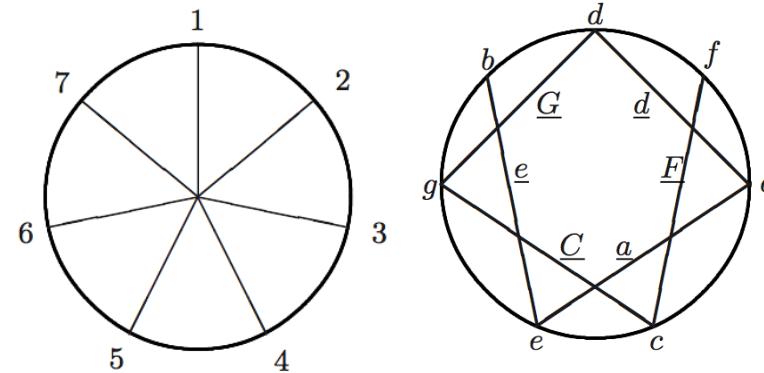
# Structures d'ordre en musique

		C-major triad	d-minor triad	e-minor triad	F-major triad	G-major triad	a-minor triad
c	X				X		X
d		X			X		
e	X		X				X
f		X	X	X			
g	X		X	X			
a		X	X	X			X
b			X		X		



# Un treillis des concepts pour la gamme diatonique

	C-major triad
	d-minor triad
	e-minor triad
	F-major triad
	G-major triad
	a-minor triad
c	X
d	X X
e	X X X
f	X X X
g	X X X
a	X X X
b	



# Des relations binaires au treillis des concepts

---

## 2. CONCEPT LATTICES

We start defining a *context* as a triple  $(G, M, I)$  where  $G$  and  $M$  are sets, and  $I$  is a binary relation between  $G$  and  $M$ ; the elements of  $G$  and  $M$  are called *objects* [Gegenstände] and *attributes* [Merkmale], respectively. If  $gIm$  for  $g \in G$  and  $m \in M$  we say: the object  $g$  has the attribute  $m$ . For  $A \subseteq G$  and  $B \subseteq M$  we define

$$\begin{aligned} A' &:= \{m \in M \mid gIm \text{ for all } g \in A\}, \\ B' &:= \{g \in G \mid gIm \text{ for all } m \in B\}. \end{aligned}$$

The mappings given by  $A \mapsto A'$  and  $B \mapsto B'$  are said to form a *Galois connection* between the power sets of  $G$  and  $M$ , i.e. they fulfill the following basic properties (cf. Birkhoff [3; pp. 122-125]).

PROPOSITION. For a context  $(G, M, I)$ :

- (1)  $A_1 \subseteq A_2$  implies  $A'_1 \supseteq A'_2$  for  $A_1, A_2 \subseteq G$ ,
- (1')  $B_1 \subseteq B_2$  implies  $B'_1 \supseteq B'_2$  for  $B_1, B_2 \subseteq M$ ,
- (2)  $A \subseteq A''$  and  $A' = A'''$  for  $A \subseteq G$ ,
- (2')  $B \subseteq B''$  and  $B' = B'''$  for  $B \subseteq M$ .

# Des relations binaires au treillis des concepts

Now, a concept [Begriff] of the context  $(G, M, I)$  may be defined as a pair  $(A, B)$  where  $A \subseteq G$ ,  $B \subseteq M$ ,  $A' = B$ , and  $B' = A$ ;  $A$  and  $B$  are called the extent and the intent of the concept  $(A, B)$ , respectively. The hierarchy of concepts is captured by the definition

$$(A_1, B_1) \leq (A_2, B_2) := A_1 \subseteq A_2 \quad (-B_1 \supseteq B_2)$$

for concepts  $(A_1, B_1)$  and  $(A_2, B_2)$  of  $(G, M, I)$ ;  $(A_1, B_1)$  is called the *subconcept* of  $(A_2, B_2)$ , and  $(A_2, B_2)$  is called the *superconcept* of  $(A_1, B_1)$ .

$$A' := \{m \in M \mid gIm \text{ for all } g \in A\}$$

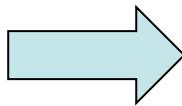
**A** = extension du concept  $(A, B)$   
**B** = intension du concept  $(A, B)$

## Attributes

	<i>B</i>					
<i>A</i>						
	X	X	X	X		
	X	X	X	X		
	X	X	X	X		

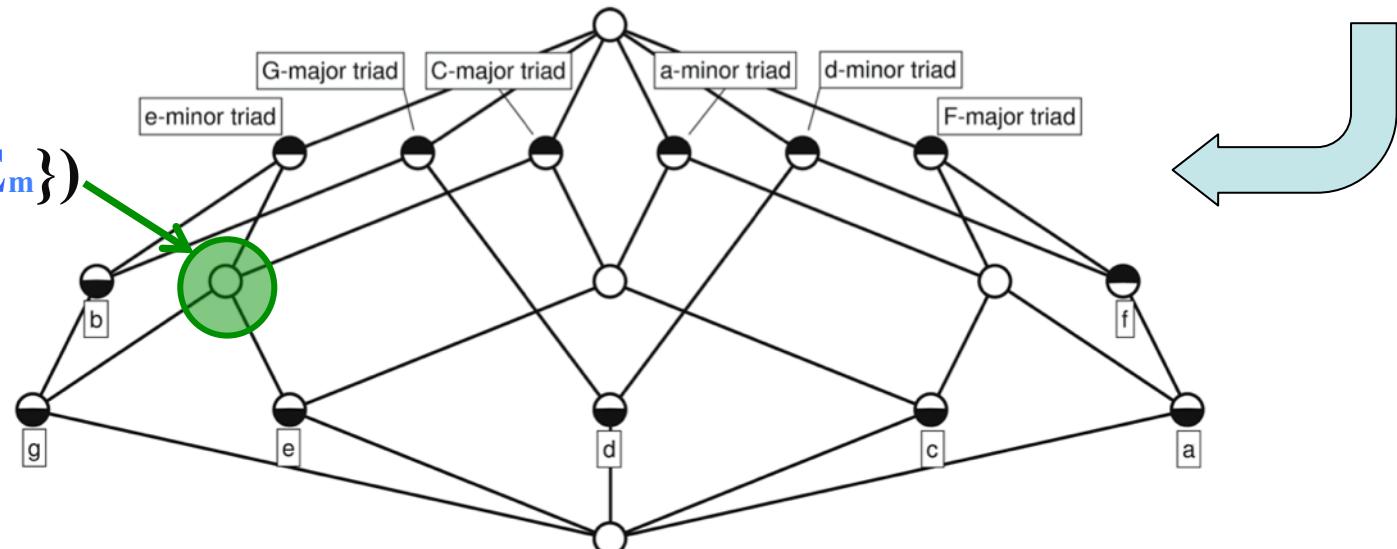
# Un treillis des concepts pour la gamme diatonique

		C-major triad	d-minor triad	e-minor triad	F-major triad	G-major triad	a-minor triad
c	X				X		X
d	X	X			X		
e	X	X	X				X
f	X	X	X				
g	X	X	X	X			
a	X	X	X				X
b		X	X	X			



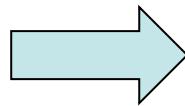
	C <sub>M</sub>	E <sub>m</sub>				
mi	X	X				
sol	X	X				

({*mi, sol*}, {C<sub>M</sub>, E<sub>m</sub>})



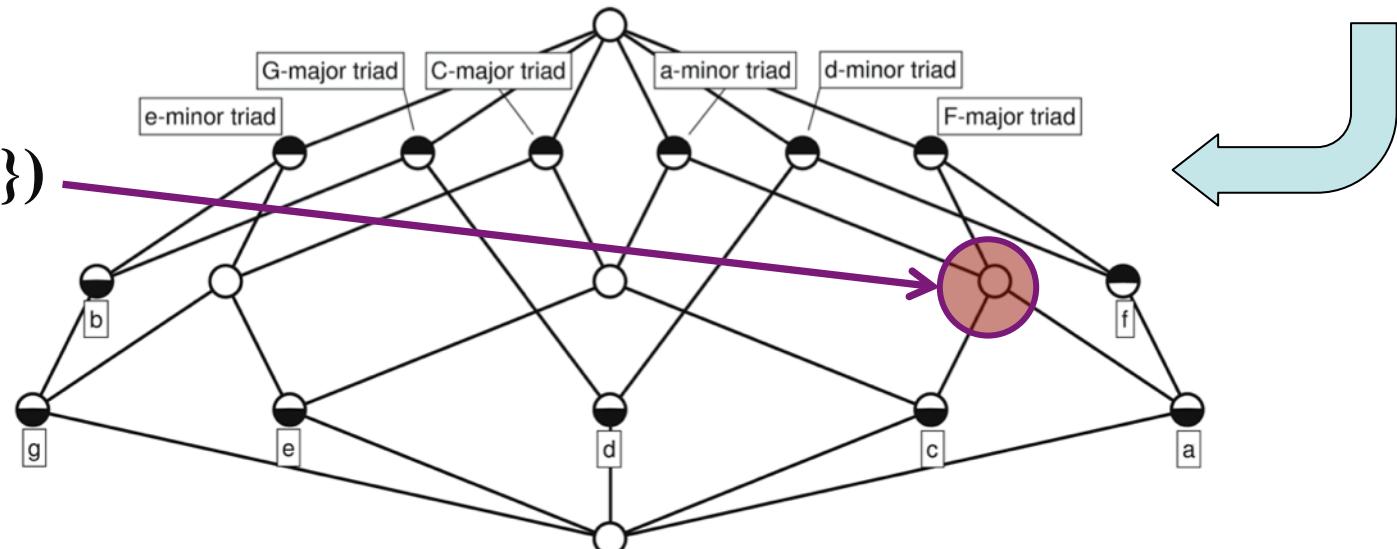
# Un treillis des concepts pour la gamme diatonique

		C-major triad	d-minor triad	e-minor triad	F-major triad	G-major triad	a-minor triad
c	X				X		X
d	X	X			X		X
e	X	X	X			X	X
f	X	X	X				
g	X	X	X				
a	X	X	X				
b		X	X	X			



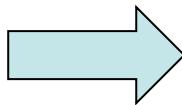
	C <sub>M</sub>	E <sub>m</sub>	A <sub>m</sub>	F <sub>M</sub>		
<i>mi</i>	X	X				
<i>sol</i>	X	X				
<i>la</i>			X	X		
<i>do</i>			X	X		

({*la, do*}, {F<sub>M</sub>, A<sub>m</sub>})



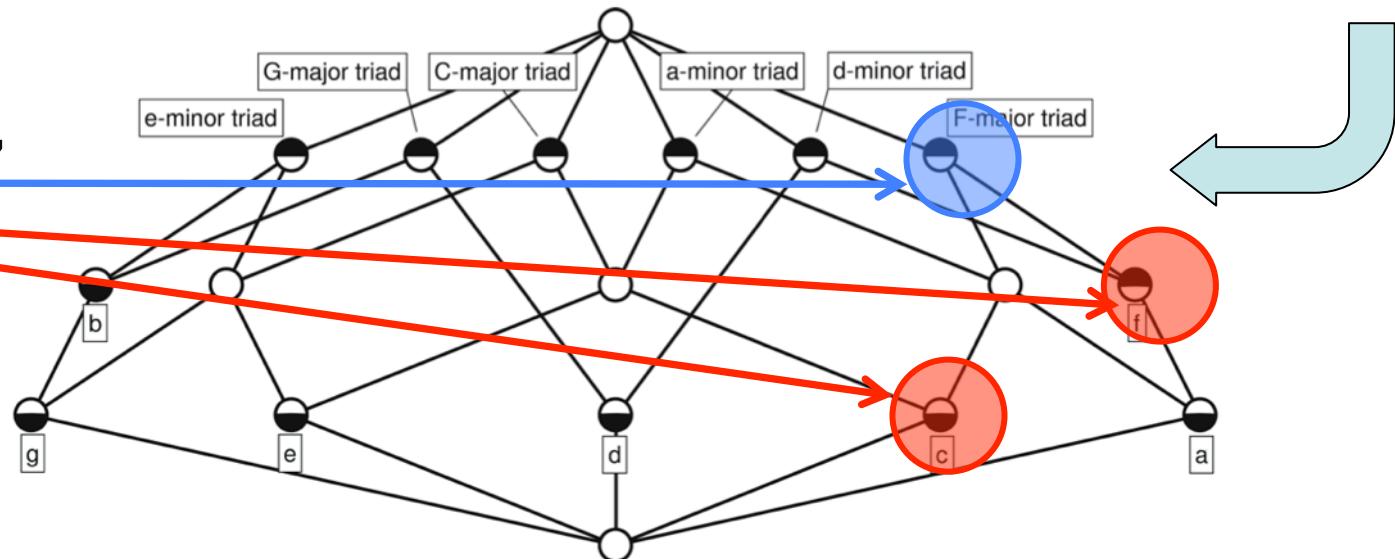
# Opérateur de clôture « »

	C-major triad	d-minor triad	e-minor triad	F-major triad	G-major triad	a-minor triad
c	X			X		X
d	X	X		X		X
e	X	X	X			X
f	X	X	X			
g	X	X	X			
a	X	X	X			X
b		X	X	X		



	C <sub>M</sub>	E <sub>m</sub>	A <sub>m</sub>	F <sub>M</sub>		
mi	X	X				
sol	X	X				
la			X	X		
do			X	X		
fa				X		

$$\{do, fa\}'' = \{F_M\}'$$



# Opérateur de clôture « ” » comme implication logique

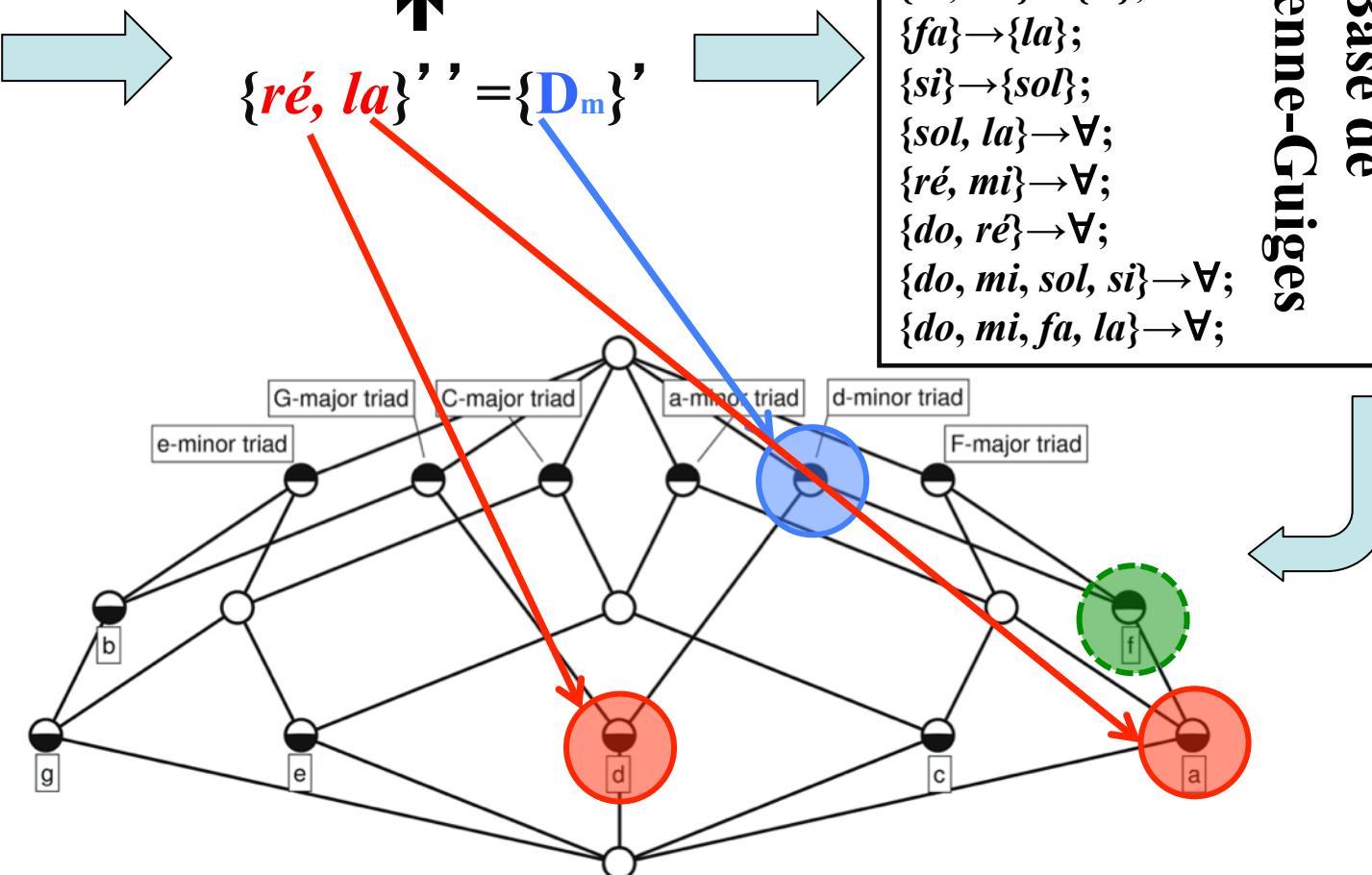
	C-major triad	d-minor triad	e-minor triad	F-major triad	G-major triad	a-minor triad
c	X			X		X
d	X	X			X	
e	X	X	X			X
f	X	X	X	X		
g	X	X	X	X	X	
a	X	X	X	X	X	X
b		X	X	X	X	X

$$\{ré, la\} \rightarrow \{fa\};$$

$$\{ré, la\}'' = \{D_m\}'$$

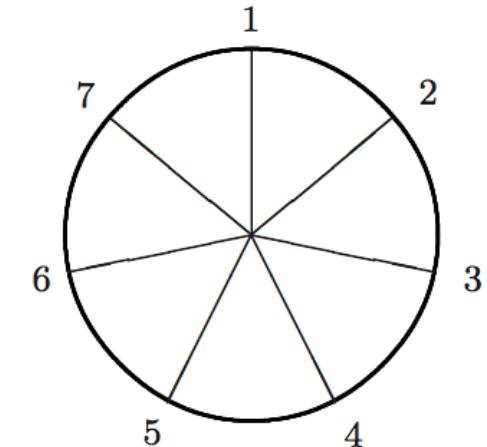
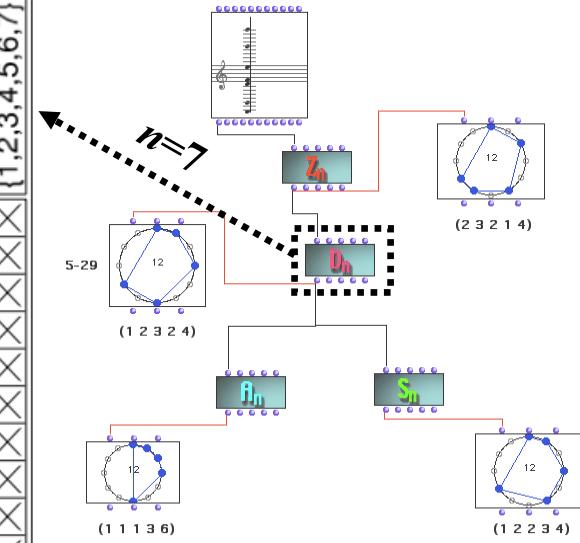
$\{ré, la\} \rightarrow \{fa\};$   
 $\{do, sol\} \rightarrow \{mi\};$   
 $\{mi, la\} \rightarrow \{do\};$   
 $\{ré, sol\} \rightarrow \{si\};$   
 $\{fa\} \rightarrow \{la\};$   
 $\{si\} \rightarrow \{sol\};$   
 $\{sol, la\} \rightarrow \forall;$   
 $\{ré, mi\} \rightarrow \forall;$   
 $\{do, ré\} \rightarrow \forall;$   
 $\{do, mi, sol, si\} \rightarrow \forall;$   
 $\{do, mi, fa, la\} \rightarrow \forall;$

Duquenne-Guigues  
Base de

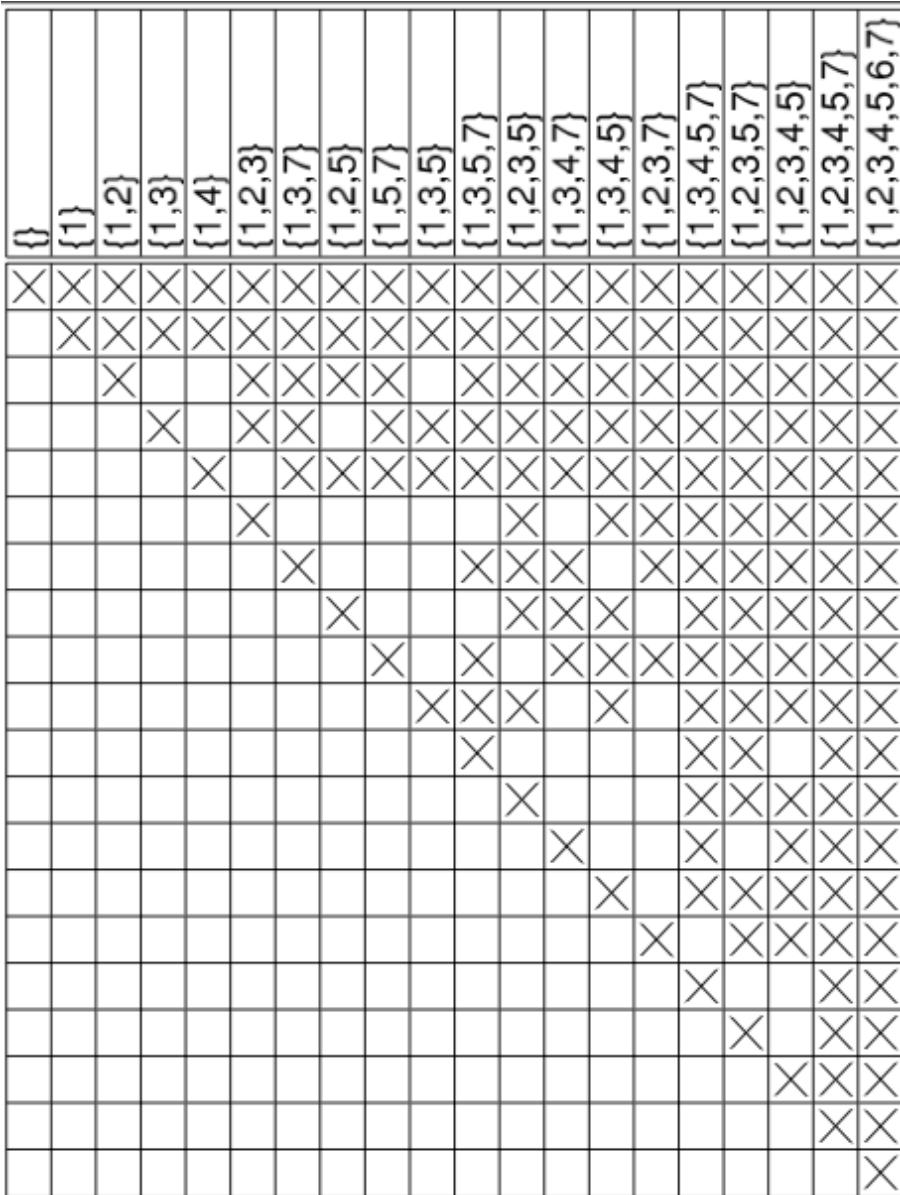


# FCA et classification paradigmique

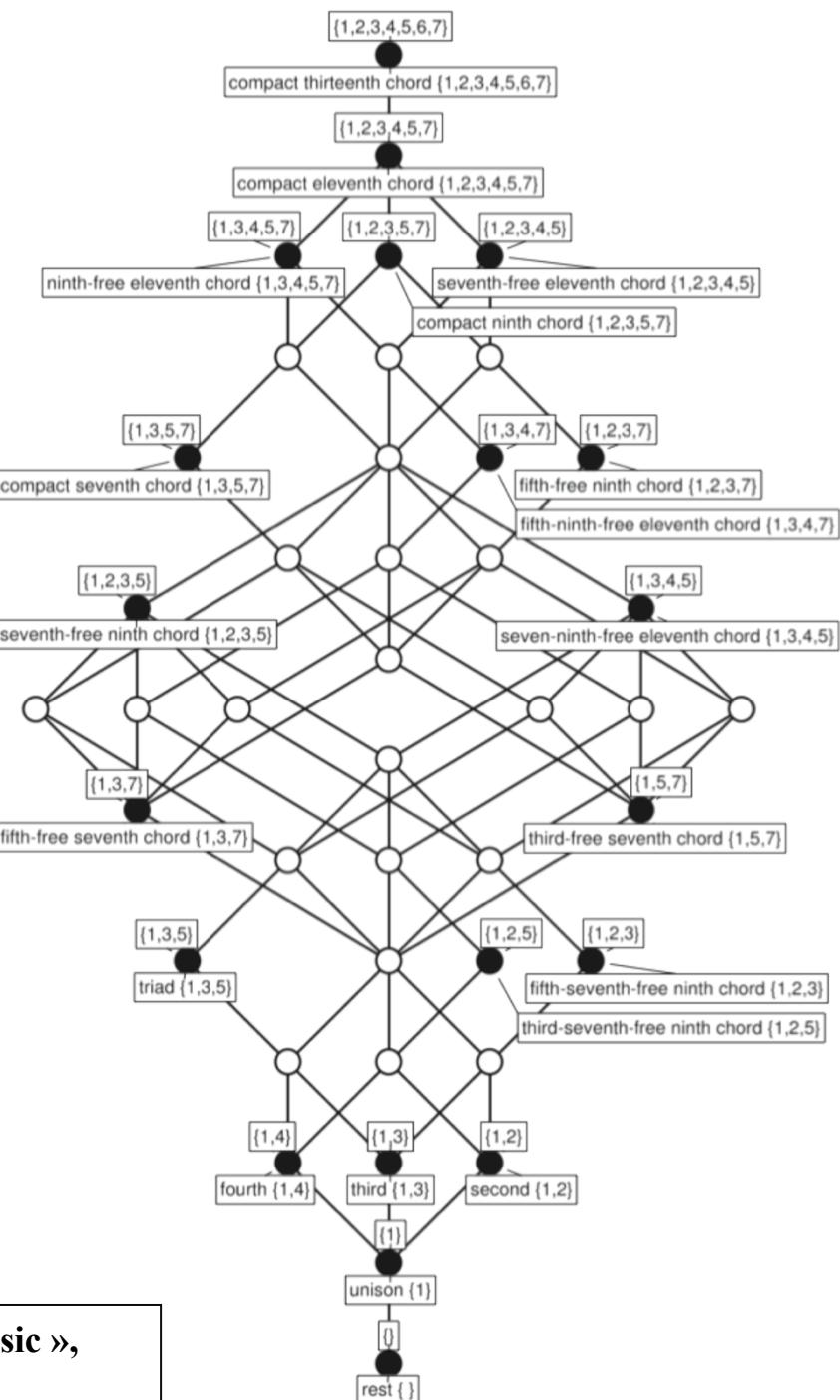
	{}	{1}	{1,2}	{1,3}	{1,4}	{1,2,3}	{1,3,7}	{1,2,5}	{1,5,7}	{1,3,5}	{1,3,5,7}	{1,2,3,5}	{1,3,4,7}	{1,3,4,5}	{1,2,3,7}	{1,3,4,5,7}	{1,2,3,5,7}	{1,2,3,4,5}	{1,2,3,4,5,7}	{1,2,3,4,5,6,7}
rest { }	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X
unison {1}		X																		
second {1,2}			X																	
third {1,3}				X																
fourth {1,4}					X															
fifth-seventh-free ninth chord {1,2,3}						X														
fifth-free seventh chord {1,3,7}							X													
third-seventh-free ninth chord {1,2,5}								X												
third-free seventh chord {1,5,7}									X											
triad {1,3,5}										X										
compact seventh chord {1,3,5,7}											X									
seventh-free ninth chord {1,2,3,5}												X								
fifth-ninth-free eleventh chord {1,3,4,7}												X								
seventh-ninth-free eleventh chord {1,3,4,5}													X							
fifth-free ninth chord {1,2,3,7}													X							
ninth-free eleventh chord {1,3,4,5,7}														X						
compact ninth chord {1,2,3,5,7}														X						
seventh-free eleventh chord {1,2,3,4,5}															X					
compact eleventh chord {1,2,3,4,5,7}																X				
compact thirteenth chord {1,2,3,4,5,6,7}																	X			



# Le treillis de concepts formels

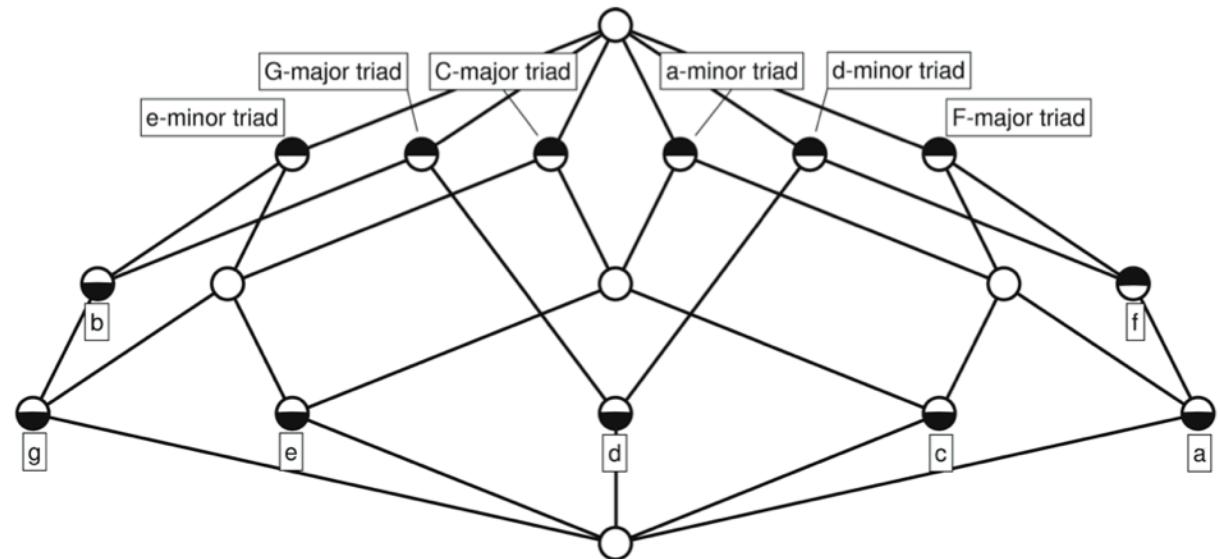


R. Wille & R. Wille-Henning, « Towards a Semantology of Music »,  
ICCS 2007, Springer, 2007]

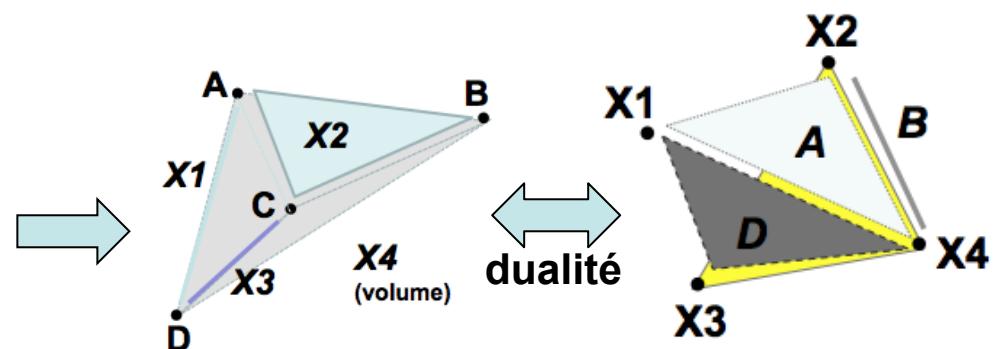


# Analyse des concepts formels et topologie

	C-major triad	d-minor triad	e-minor triad	F-major triad	G-major triad	a-minor triad
c	X		X	X		
d		X		X		
e	X	X				X
f	X	X	X			
g	X	X	X	X		
a	X	X	X	X	X	
b		X	X	X		



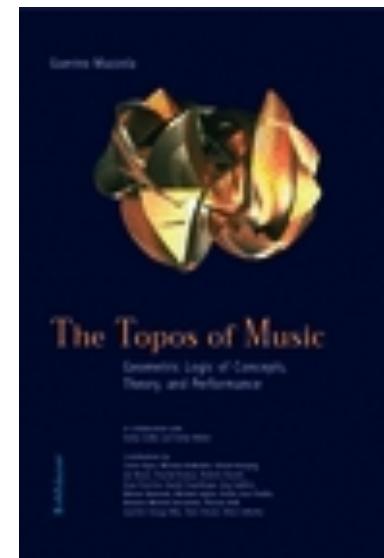
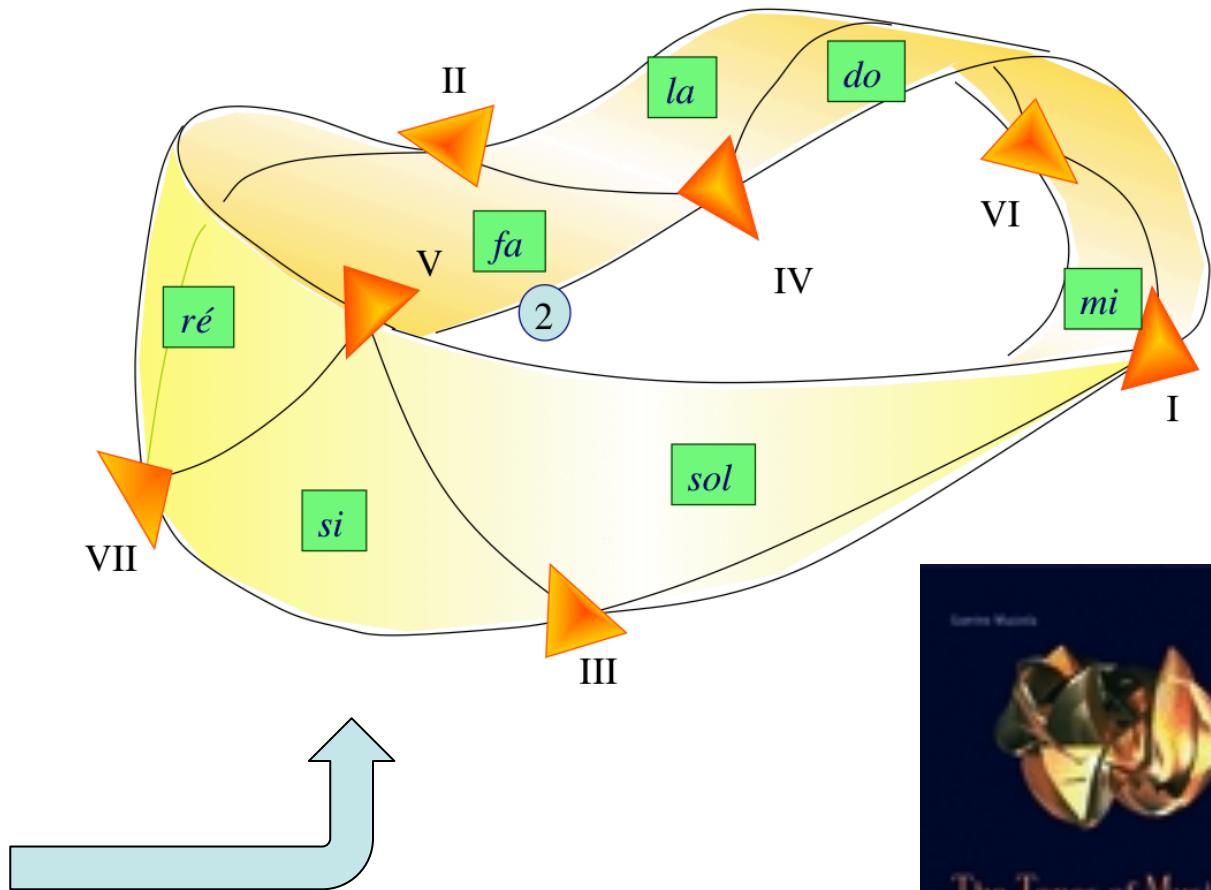
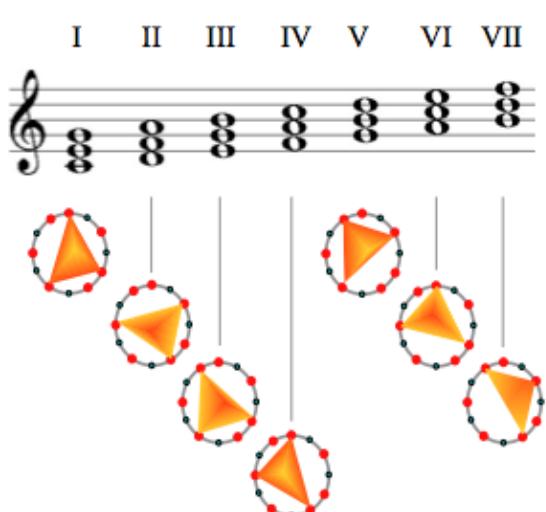
	A	B	C	D
X1	1	0	0	1
X2	1	1	1	0
X3	0	0	1	1
X4	1	1	1	1



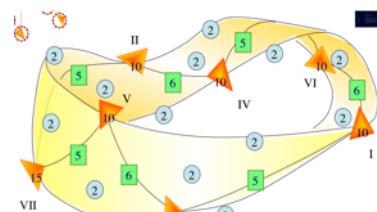
# Structure topologique du treillis diatonique

	C-major triad	d-minor triad	e-minor triad	F-major triad	G-major triad	a-minor triad
c	X		X			X
d	X	X		X		X
e	X	X				X
f	X	X	X			
g	X	X	X			
a	X	X	X	X		X
b		X	X			

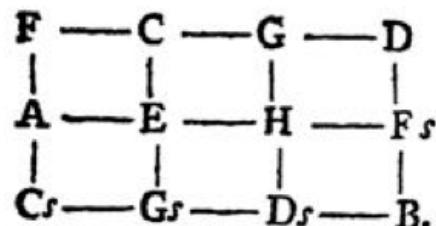
accord diminué



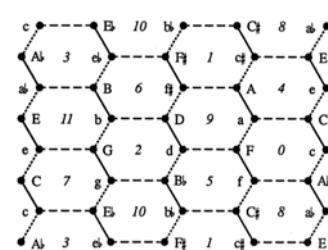
# Représentations spatiales en musique



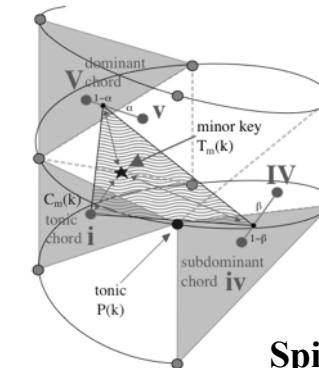
## Tonality strip [Mazzola]



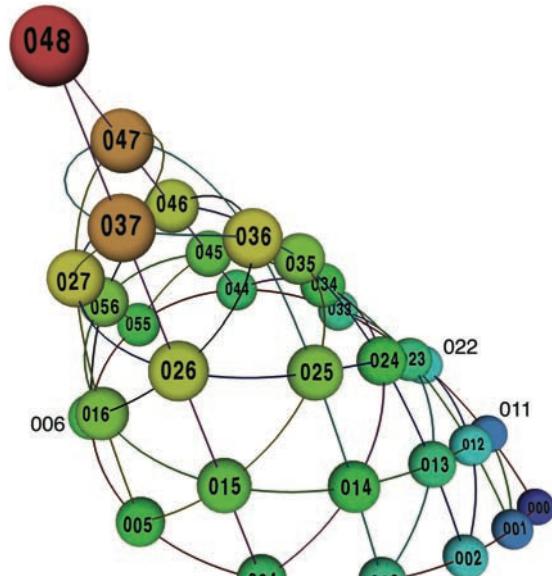
## Speculum Musicum [Euler]



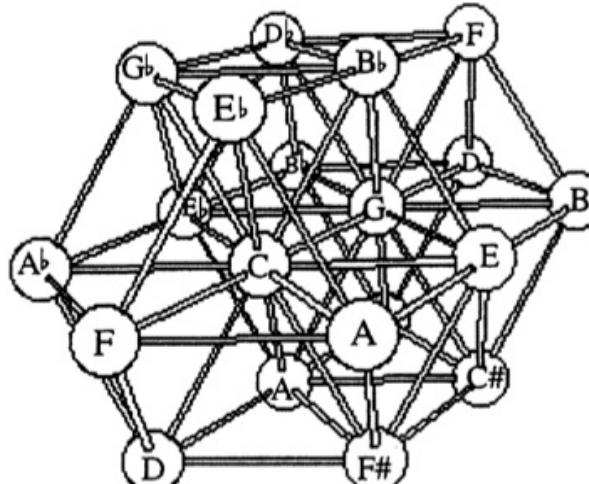
## Chicken Wire Torus [Douthett & Steinbach]



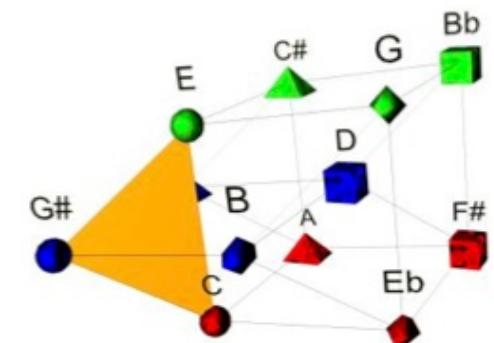
## Spiral Array [Chew]



## Orbifolds [Tymoczko]



## 3D Tonnetz [Gollin]



## Model Planet [Barouin]

# Les géométries d'un Prélude...

