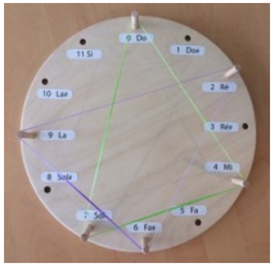


UNIVERSITÀ DEGLI STUDI DI TRENTO 25th November 2021
Dipartimento di Matematica

From music to maths and backwards: introducing algebra, topology and category theory into computational musicology

Moreno Andreatta

CNRS / IRMA / Université de Strasbourg
CNRS / IRCAM / Sorbonne Université
http://repmus.ircam.fr/moreno/smir

IRMA Institut de Recherche Mathématique Avancée · Université de Strasbourg · cnrs · ircam Centre Pompidou · Sorbonne Université

MATEMATICA, GEOMETRIA E MUSICA

Buone vibrazioni tra pentagramma, forme e formule

Giovedì 15 dicembre 2016, ore 17.45

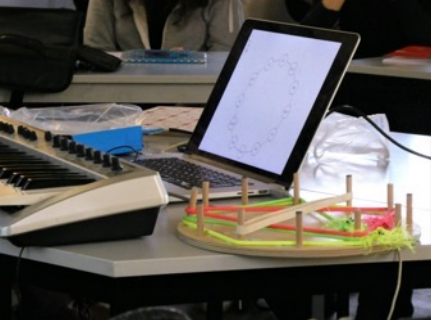
presso il

TEATRO COLOSSEO

Via Madama Cristina, 71 - Torino

Alberto Conte
Professore emerito Geometria superiore, Università di Torino
Moreno Andreatta
Direttore di ricerca al CNRS (Centre National de la Recherche Scientifique) presso l'IRCAM di Parigi e "matemusicista"
con la partecipazone di **Paolo Conte**, avvocato e musicista

➔ **https://www.giovediscienza.it/it/history/appuntamenti-31-edizione/337-matematica-geometria-e-musica**





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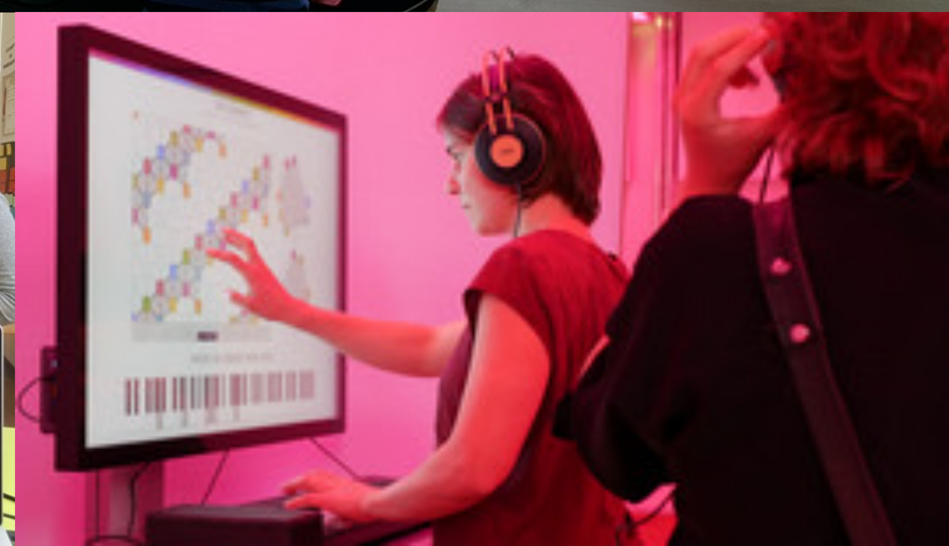

Workshops for High School students

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MISSION INTERDISCIPLINAIRE
POUR LES INITIATIVES
TRANSVERSES
ET INTERDISCIPLINAIRES

INVESTISSEMENTS D'AVENIR

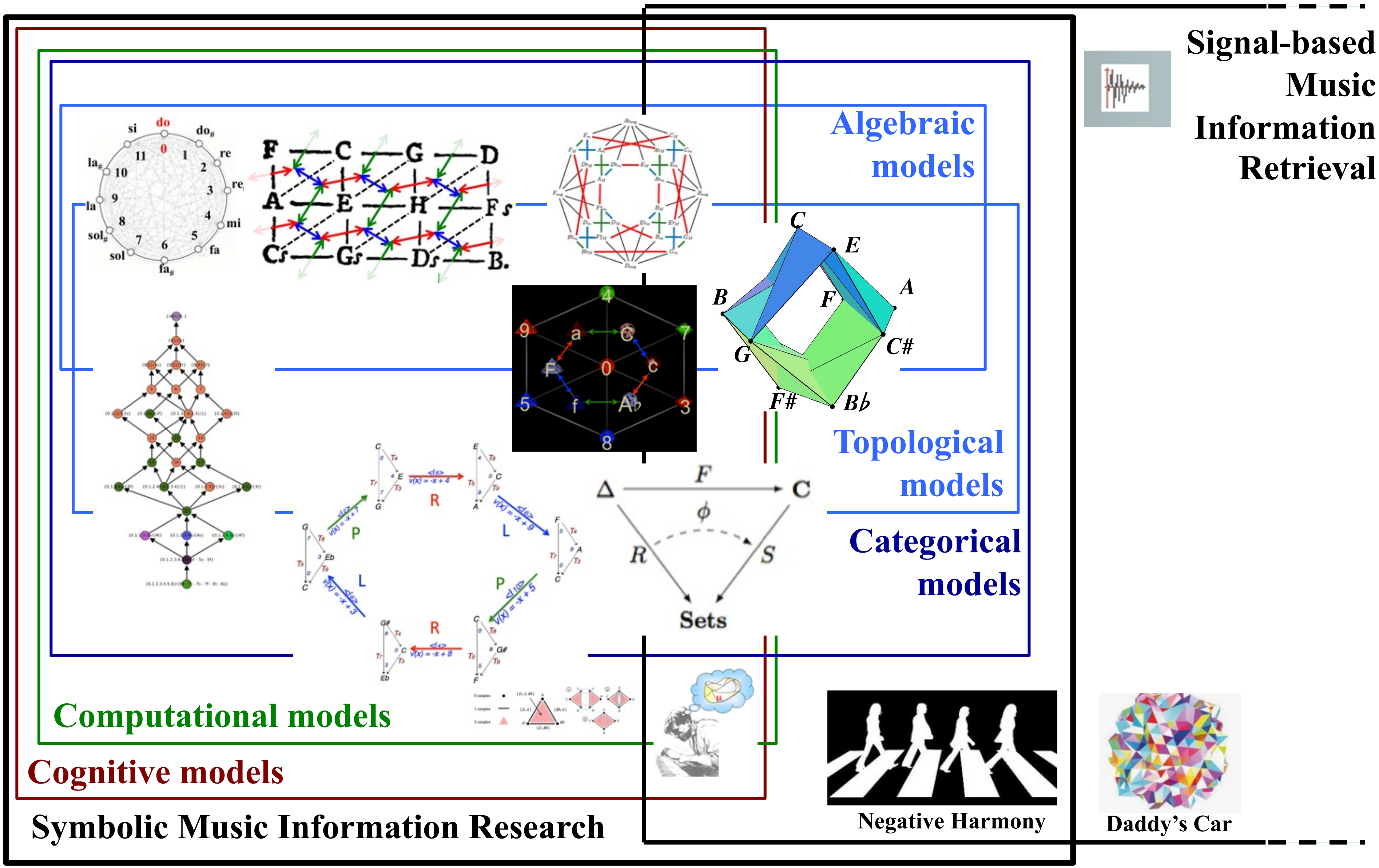
Saint-Brieuc (Bretagne)

(in the future…?)

? ?

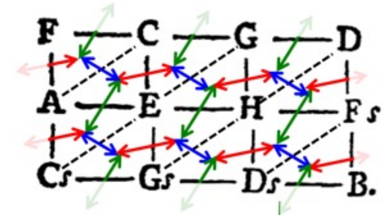
Centro Servizi Culturali
Santa Chiara

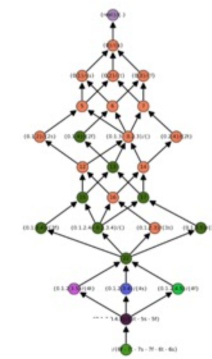
The SMIR Project: advanced maths for the working musicologist



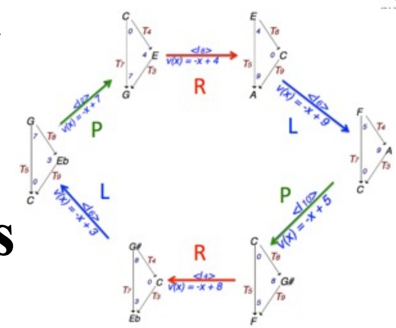
The SMIR Project: Structural Music Information Research

- **Generalized Tonnetze and Persistent Homology**
  - The Tonnetz as a simplicial complex
  - Algebraic classification of the twelve possible Tonnetze
  - Isotropic and anisotropic Tonnetze
  - Application to automatic stylistic classification

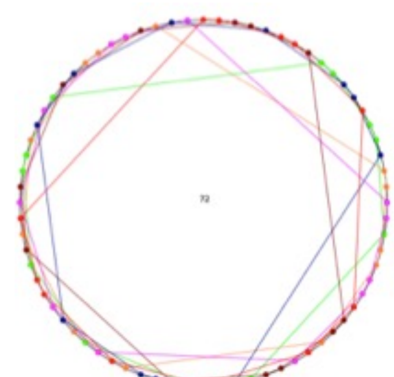




- **Formal Concept Analysis and Mathematical Morphology**
  - Lattice structure of formal concepts
  - Derivation operators (in FCA) and dilation/erosion in MM
  - Application to pattern recognition and extraction
- **Category theory and Transformational Theory**
  - From K-nets to PK-nets
  - Diatonicism



- **‘Mathemusical’ problems and open conjectures**
  - Tiling rhythmic canons and Fuglede Spectral Conjecture
  - Homometric musical structures
- **Philosophy, Epistemology and Cognitive Science**
  - Geometry-based Neo-structuralism in music analysis
  - Processes and techniques of mathemusical learning



➔ **Andreatta, M., « From music to mathematics and backwards: introducing algebra, topology and category theory into computational musicology », in M. Emmer and M. Abate (eds.), *Imagine Math 6* - Springer, 2018**

Structure of the two doctoral courses:

(36 hours)

First course:

- Moreno Andreatta: An overview of some research axes in **Mathemusical Research**
- Emmanuel Amiot: **Discrete Fourier Transform** in music analysis: from tilings to musical scales
- Greta Lanzarotto: **Tiling Canons** and Fuglede Spectral Conjecture

Second course:

- Franck Jedrzejewski: **Homometry** and neo-Riemannian Theory
- Thomas Noll: **Word theory** and its application to scales, modes, chords and rhythms
- Alexandre Popoff: **Category-Theory** formalization of transformational music analysis

Some examples of PhD in maths / music / computer science

- **Gonzalo Romero**, *Morphologie mathématique et analyse musicale*, PhD in **computer science** at **Sorbonne University** (supervised by Carlos Agon, in collaboration with Isabelle Bloch and Moreno Andreatta)
- **Riccardo Giblas,** Topic *to be defined,* PhD in **maths** in cotutelle agreement, **University of Padova** (L. Fiorot & Alberto Tonolo) / **Université de Strasbourg** (M. Andreatta), ongoing.
- **Victoria Callet**, *Modélisation topologique de structures et processus musicaux*, PhD in **maths**, Université de Strasbourg (supervised by Pierre Guillot and Moreno Andreatta, IRMA)
- **Matias Fernandez Rosales,** *Mathematical models in Computer-assisted composition,* PhD in composition and research, **HEAR/University of Strasbourg** (supervision: Daniel D'Adamo, Xavier Hascher, Moreno Andreatta)
- **Greta Lanzarotto**, *Fuglede Spectral Conjecture, Musical Tilings and Homometry*, PhD in **maths** in cotutelle agreement, **University of Pavia** (L. Pernazza) / **Université de Strasbourg** (M. Andreatta), ongoing.
- **Alessandro Ratoci**, Vers l'hybridation stylistique assistée par ordinateur, PhD in music **composition & research**, Sorbonne University / IRCAM (cosupervised with Laurent Cugny), ongoing
- **Sonia Cannas**, *Représentations géométriques et formalisations algébriques en musicologie computationnelle*, PhD in **maths** in cotutelle agreement, **University of Pavia** (L. Pernazza) / **Université de Strasbourg** (A. Papadopoulos & M. Andreatta), 2019.
- **Grégoire Genuys**, *Théorie de l'homométrie et musique*, PhD in **maths**, **Sorbonne University** / IRCAM (cosupervised with Jean-Paul Allouche), 2017.
- **Hélianthe Caure**, *Pavages en musique et conjectures ouvertes en mathématiques*, PhD in **computer science**, **Sorbonne University** (cosupervised with Jean-Paul Allouche), 2016.
- **Mattia Bergomi**, *Dynamical and topological tools for (modern) music analysis,* PhD in **maths** in a cotutelle agreement **Sorbonne University** / **University of Milan** (with Goffredo Haus, 2015).
- **Charles De Paiva**, *Systèmes complexes et informatique musicale*, thèse de doctorat, Programme Doctoral International « Modélisation des Systèmes Complexes », PhD in **musicology** in a cotutelle agreement, **Sorbonne University** / **UNICAMP**, Brésil, 2016.
- **Louis Bigo**, *Représentation symboliques musicales et calcul spatial*, PhD in **computer science**, **University of Paris Est Créteil** / **IRCAM**, 2013 (with Olivier Michel and Antoine Spicher)
- **Emmanuel Amiot**, *Modèles algébriques et algorithmiques pour la formalisation mathématique de structures musicales*, PhD in, **Sorbonne University** / **IRCAM**, 2010 (cosupervised with Carlos Agon) **computer science**
- **Yun-Kang Ahn**, *L'analyse musicale computationnelle*, PhD in **computer science**, **Sorbonne University** / **IRCAM**, 2009 (cosupervised with Carlos Agon)

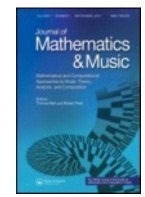
Maths & Music in Academic Research

Conferences of the SMCM:

- 2007 Technische Universität (Berlin, Allemagne)
- 2009 Yale University (New Haven, USA)
- 2011 IRCAM (Paris, France)
- 2013 McGill University (Canada)
- 2015 Queen Mary University (Londres)
- 2017 UNAM (Mexico City)
- 2019 Universidad Complutense de Madrid (Spain)
- 2022 Georgia State University (Atlanta, USA)

Official Journal and MC code (00A65: Mathematics and Music)

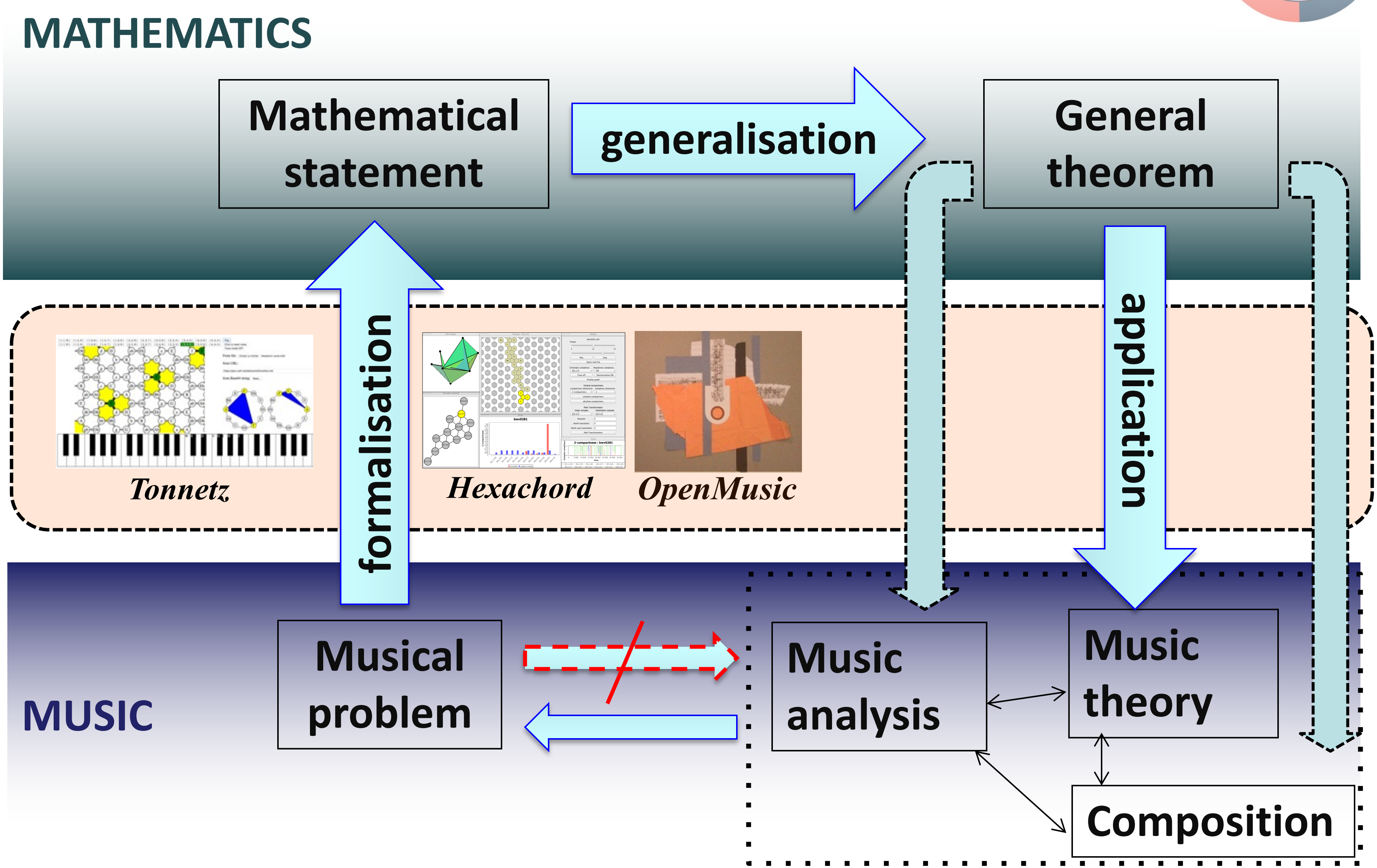
- *Journal of Mathematics and Music*, Taylor & Francis
  (Editors: Th. Fiore, C. Callender | Associate eds.: E. Amiot, J. Yust)



Books Series:

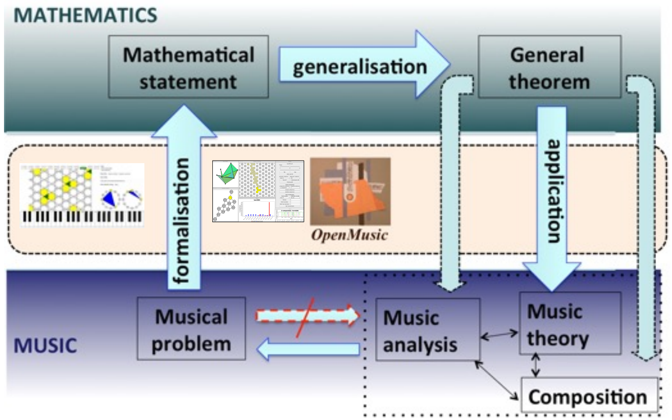
- *Computational Music Sciences* Series, Springer (G. Mazzola & M. Andreatta eds. – 12 books published (since 2009)
- Collection *Musique/Sciences*, Ircam-Delatour France (J.-M. Bardez & M. Andreatta dir. – 16 books published (since 2006)

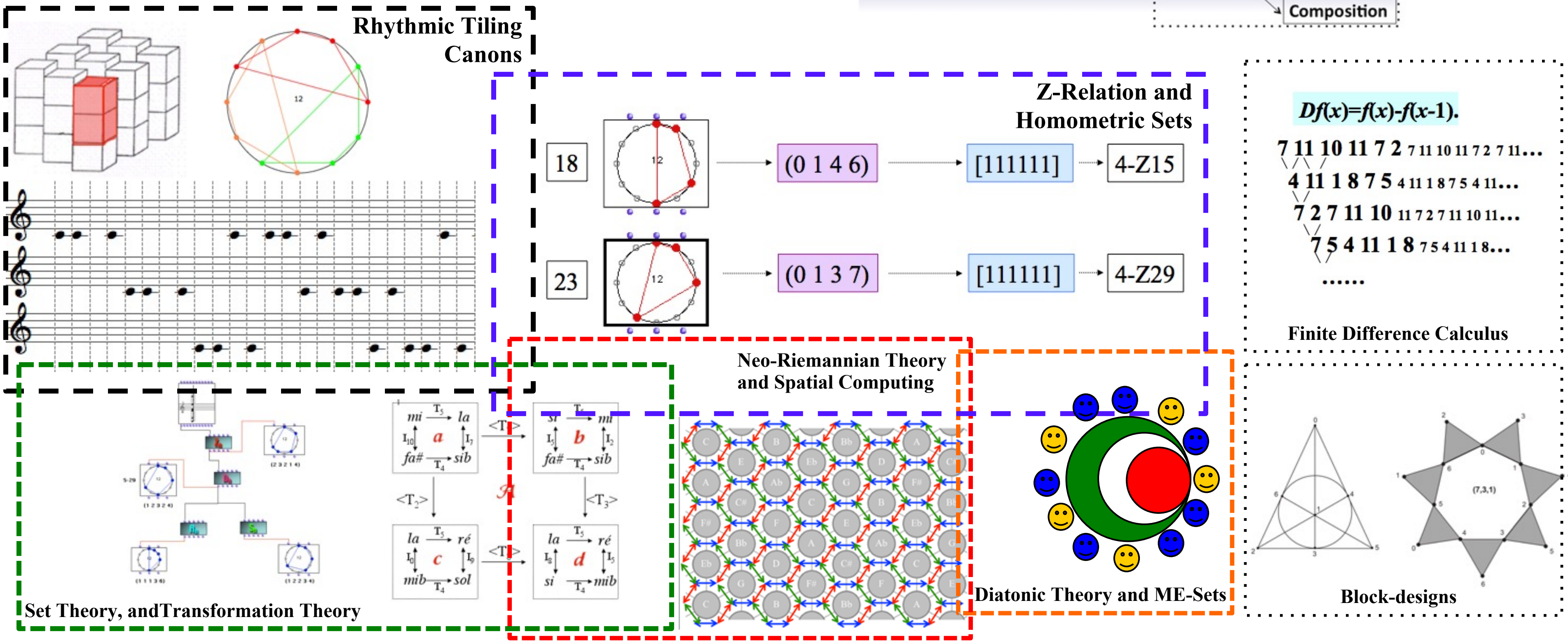
The double movement of a 'mathemusical' activity



Some musically-driven mathematical problems

- Tiling Rhythmic Canons
- *Z* relation and homometry
- Transformational Theory
- Music Analysis, Spatial Computing and FCA
- Diatonic Theory and Maximally-Even Sets
- Periodic sequences and Finite Difference Calculus
- Block-designs in composition





➔ M. Andreatta : *Mathematica est exercitium musicae*, Habilitation Thesis, IRMA University of Strasbourg, 2010

Periodic sequences and finite difference calculus

$$Df(x)=f(x)-f(x-1)$$

f = 7 11 10 11 7 2 7 11 10 11 7 2 7 11…

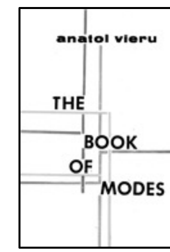
Df = 4 11 1 8 7 5 4 11 1 8 7 5 4 11…

$D^2 f$ = 7 2 7 11 10 11 7 2 7 11 10 11…

$D^3 f$ = 7 5 4 11 1 8 7 5 4 11 1 8…

$D^k f$ = ……

Anatol Vieru



V	0	3	8	7	11	0	11	10	6	9	0	9	1	2	9	8	4	3	6
VIII	0	0	0	0	3	3	7	2	0	0	0	6	3	3	3	4	8	0	0
IV	3	3	4	4	1	11	11	8	3	3	9	4	1	7	11	8	11	3	9
IX	0	0	0	0	0	3	6	[1]	3	3	3	3	9	0	3	6	[10]	6	6
IV	0	10	3	9	10	0	9	7	0	6	7	9	6	4	9	3	4	6	3

Anatol Vieru: *Zone d'oubli* pour alto (1973)

Reducible and reproducible sequences

f = 11 6 7 2 3 10 11 6 …

Df = 7 1 7 1 7 1 7 1…

D^2f = 6 6 6 6 6…

D^4f = 0 0 0

Reducible sequences:

$\exists\ k \geq 1$ **such that**

$D^k f = 0$

---

f = 7 11 10 11 7 2 7 11 …

Df = 4 11 1 8 7 5 4 11 1 …

D^2f = 7 2 7 11 10 11 7 2 7 …

D^3f = 7 5 4 11 1 8 7 5 4 11 1 8…

D^4f = 10 11 7 2 7 11 10 11 …

D^5f = 1 8 7 5 4 11 1 8 …

D^6f = 7 11 10 11 7 2 7 11 …

Reproducible sequences:

$\exists\ k \geq 1$ **such that**

$D^k f = f$

A decomposition property of any periodic sequence

$Df(x)=f(x)-f(x-1)$.

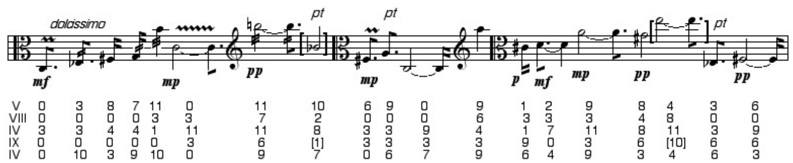
7 11 10 11 7 2 7 11 10 11 7 2 7 11…

4 11 1 8 7 5 4 11 1 8 7 5 4 11…

7 2 7 11 10 11 7 2 7 11 10 11…

7 5 4 11 1 8 7 5 4 11 1 8…

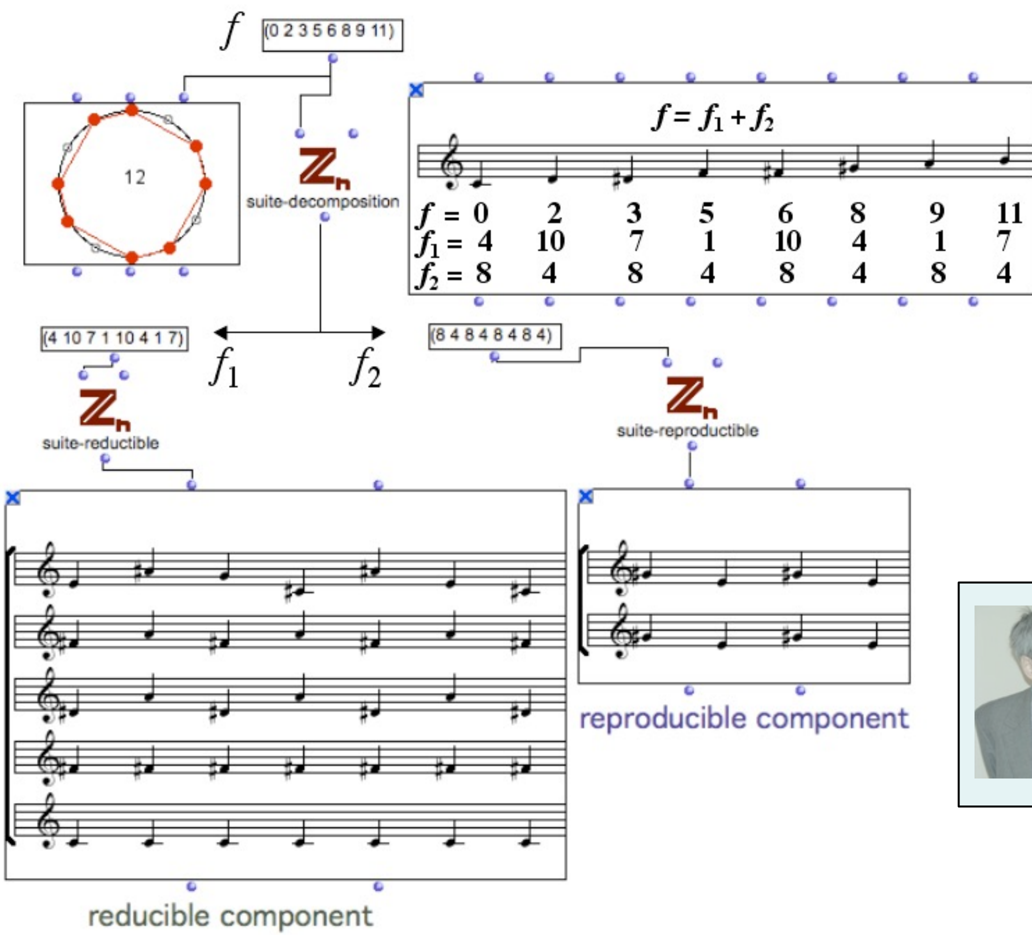
……



Anatol Vieru: *Zone d'oubli* for viola (1973)

Reducible sequences:
$\exists\ k \geq 1$ such that $D^k f = 0$

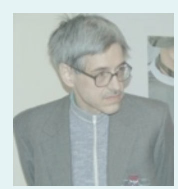
Reproducible sequences:
$\exists\ k \geq 1$ such that $D^k f = f$



- **Decomposition theorem**

(Vuza & Andreatta, *Tatra M.*, 2001)

Every periodic sequence f can be decomposed in a unique way as a sum $f_1 + f_2$ of a reducible sequence f_1 and a reproducible sequence f_2



D. Vuza, M. Andreatta (2001), « On some properties of periodic sequences in Anatol Vieru's modal theory », Tatra Mountains Mathematical Publications, Vol. 23, p. 1-15

A decomposition property of any periodic sequence

Theorem (Vuza & Andreatta, 2001): Any periodic sequence can be decomposed <u>in a unique way</u> as a sum of a *reducible* and a *reproducible* sequence.

Growing by additions and proliferation of values

A^3g = 8 4 8 10 11 1 5 1 2 10 2 4 5 7 11 7 8 4 …

A^2g = 8 4 2 1 2 4 8 1 8 4…

Ag = 8 10 11 1 2 4 5 7 8 10 …

g = 2 1 2 1 2 1 2 1

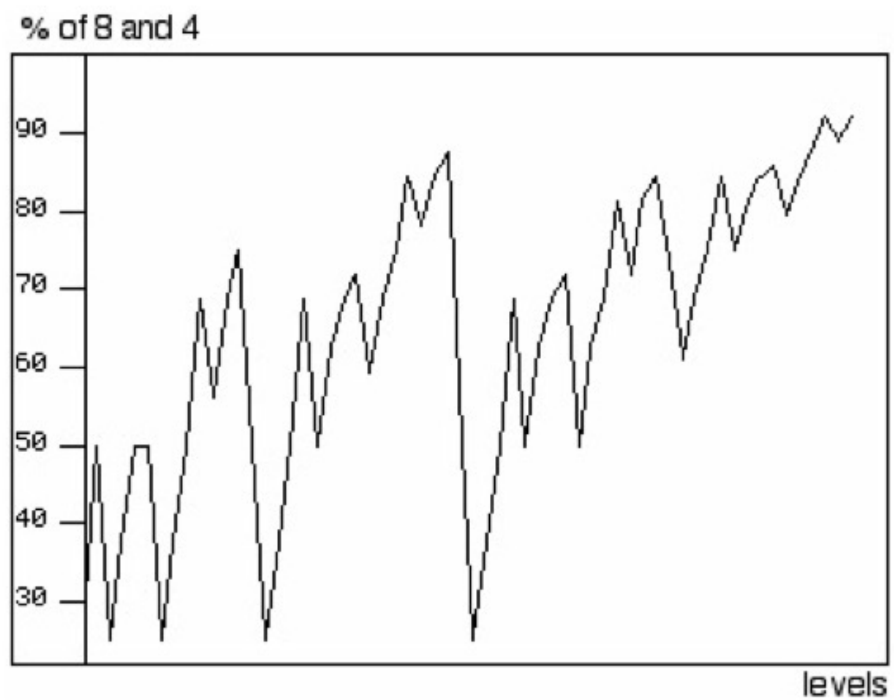


Fig. 11 Initial values equal to 8

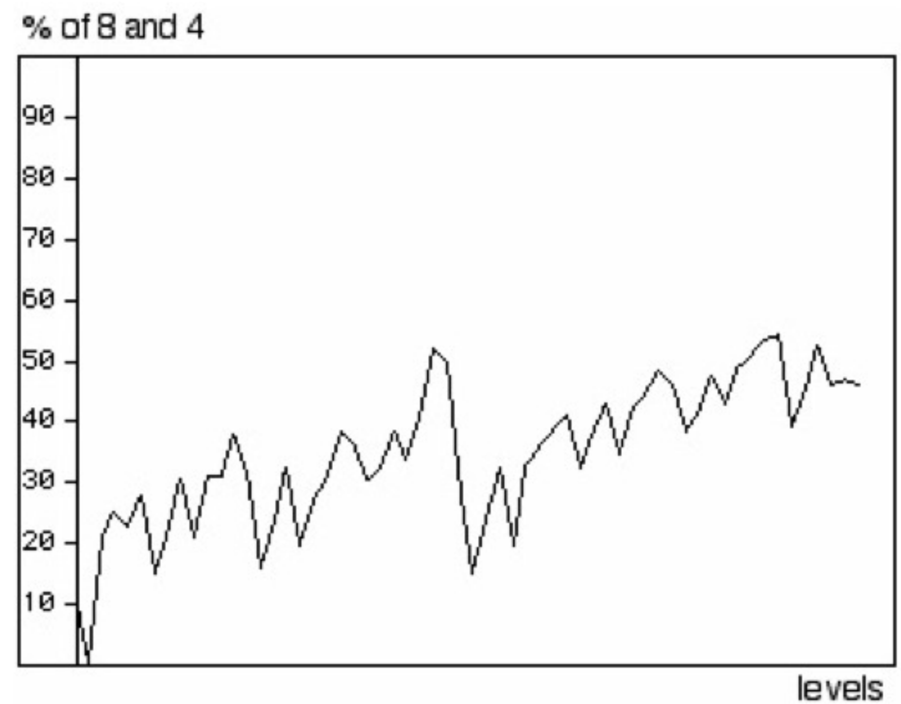


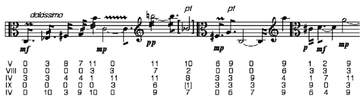
Fig. 15 Initial values equal to 4

Introduction

In his *Book of Modes*, the romanian composer Anatol Vieru studies periodic sequences taking values in $\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$ using finite difference calculus. From the musical point of view, each coefficient of the sequence may represent a pitch class or an interval, as well as a rhythmic beat.

Musical meaning

Starting from the constant sequence (6), corresponding to the triton interval, Vieru collects other periodic sequences by iteratively applying the finite sum operator. Then he decodes from each sequence a musical aspect, giving rise to a composition, *Zone d'Oubli*.



The mathematical environment

The periodic sequences

On the $\mathbb{Z}_m$-module $\mathbb{Z}_m^{\mathbb{N}}$, consider the shifting operator:

$$\theta(f)(i) := f(i+1) \quad \forall i \in \mathbb{N}.$$

A sequence $f \in \mathbb{Z}_m^{\mathbb{N}}$ is *periodic* if there exists $k \geq 1$ such that

$$f \in \ker(\theta^k - \mathrm{id}).$$

The minimal k satisfying this condition is the *period* of f. We work in the $\mathbb{Z}_m$ -(sub)module of periodic sequences

$$P_m = \bigcup_{k \geq 1} \ker(\theta^k - \mathrm{id}).$$

The operators

On the module P_m of periodic sequences, we consider two other operators:

- the *discrete derivative*
$$\Delta := \theta - \mathrm{id};$$
- given $c \in \mathbb{Z}_m$, a *finite sum operator* : for each $f \in P_m$,
$$\Sigma_c f(i) := \begin{cases} c \text{ if } i = 1 \\ f(n-1) + \Sigma_c f(n-1) \text{ if } i > 1. \end{cases}$$

We write just Σ to mean Σ_0; we say that $\Sigma^k f$ is the discrete *k-primitive* of f.

Some properties of the operators

- For every $c \in \mathbb{Z}_m$ and $f \in P_m$, $\Sigma_c f = \Sigma f + (c)$, where (c) is the constant sequence $(c, c, c, \dots)$.
- For every $c \in \mathbb{Z}_m$,
$$\Delta \circ \Sigma = \mathrm{id}.$$
- The period of Σf is a multiple of the period of f.

Decomposition in p-parts

If $m \in \mathbb{N}$, $m \geq 2$ and its factorization is:

$$m = \prod_{i=1}^{s} p_i^{n_i},$$

the group isomorphism

$$\mathbb{Z}/m\mathbb{Z} \to \bigoplus_{i=1}^{s} \mathbb{Z}/p_i^{n_i}\mathbb{Z}$$

gives rise to an isomorphism of abelian groups

$$P_m \to \bigoplus_{i=0}^{s} P_{p_i^{n_i}}$$
$$f \mapsto (f \bmod p_i^{n_i})_{0 \leq i \leq s}.$$

$f \bmod p_i^{n_i}$ is the p_i-part of f. We can study separately the p_i-parts, so we can restrict to work in $P_{p_i^{n_i}}$.

Δ-nilpotent and Δ-idempotent sequences

Definitions

- $f \in P_m$ is *Δ-nilpotent* (resp. *Δ-idempotent*) if there exists $k \geq 1$ such that $\Delta^k f = 0$ (resp. $\Delta^k f = f$); the minimal k such that this happens is said nilpotency (resp. idempotency) order of f.
- We denote by I_m^Δ the subset of P_m of Δ-idempotent sequences and by N_m^Δ the subset of Δ-nilpotent sequences.

Fitting Lemma

The *Fitting Lemma* gives the decomposition

$$P_m = N_m^\Delta \oplus I_m^\Delta$$

hence any $f \in P_m$ uniquely determines $f_I \in I_m^\Delta$ and $f_N \in N_m^\Delta$ such that

$$f = f_I + f_N.$$

The properties of Δ-nilpotent sequences

The period

Let $f \in P_{p^n}$ be a periodic sequence. Then:

- $f \in N_{p^n}^\Delta$ if and only if the period of f is p^m for $m \in \mathbb{N}$;
- if $f \in N_{p^n}^\Delta$ with period p^m and nilpotency order η, then $\eta \leq np^m$.

Sum of constants

Sequences in N_m^Δ are *finite sums* of discrete k-primitives of *constant sequences* ; furthermore, one shows that for any $c \in \mathbb{Z}_m$ and any $i \in \mathbb{N}$:

$$\Sigma^k(c)(i) \equiv_m c\binom{i}{k}.$$

Kummer's and Luca's generalized Theorems

We need to study the binomial coefficient modulo a power of a prime. The main classical tools in this setting are:

Kummer's Theorem

Given integers $i \geq k \geq 0$ and a prime number p, the p-adic valuation $\nu_p(\binom{i}{k})$ is equal to the number of carries when k is added to $i-k$ in base p.

This result allows us to find out if a binomial coefficient is 0 modulo a power of a prime.

Lucas's generalized Theorem

The generalization of Lucas's Theorem permits to compute explicitly a binomial coefficient modulo a power of a prime. Example: consider $41 = [1112]_3$ and $11 = [0102]_3$; lets us compute the residue class of $\binom{41}{11}$ modulo 9. One gets

$$\binom{[1112]_3}{[0102]_3} \equiv_9 \left\langle {11 \atop 01} \right\rangle \left\langle {1 \atop 1} \right\rangle^{-1} \left\langle {11 \atop 10} \right\rangle \left\langle {1 \atop 0} \right\rangle^{-1} \left\langle {12 \atop 02} \right\rangle \equiv_9 7.$$

Angled parentheses here denote a generalization of the binomial coefficient to the case when the numerator is smaller than the denominator.

New results

The period of primitives of constants

Let (c) be a non zero constant sequence in P_{p^n} with $c = p^l b$, $p \nmid b$. Let $s \in \mathbb{N}$ and $[a_k a_{k-1} \cdots a_1 a_0]_p$, $a_k \neq 0$, the representation of s in base p. Then the sequence $\Sigma^s(c)$ has period p^{n-l+k}.

Leading coefficient

Given $f \in N_{p^n}^\Delta$, there is a constant c among the constants in the decomposition of f which *definitively leads* the period of the primitives of f.

The sequence from Messiaen's second mode of limited transpositions

Vieru noticed that starting from a particular sequence and collecting its primitives, some values prolifer. This observation gave rise to some interest in understanding the motivation of this proliferation. The sequence is $f = (2,1,2,4,8,1,8,4)$, whose decomposition in Δ-nilpotent and Δ-idempotent part coincides with the decomposition in p-parts:

$$f_I = (2,1) \in \mathbb{Z}/3\mathbb{Z}, \quad f_N = (2,1,0,0,1,0,0) \in \mathbb{Z}/4\mathbb{Z}.$$

f_I is also Σ-idempotent, so it gives constant contribution to the primitives of f. On the other hand,

$$f_N = \Sigma^4(2) + \Sigma^3(3) + \Sigma^2(2) + \Sigma(3) + (2)$$

is the decomposition in primitives of constant sequences and $\Sigma^3(3)$ is the leading term for the period. Studying the behaviour of the primitives of the constant sequences (2) and (3), we found an algebraic explanation to the proliferation.

Riccardo Gilblas

Luisa Fiorot

Alberto Tonolo

Università degli Studi di Padova

Université de Strasbourg

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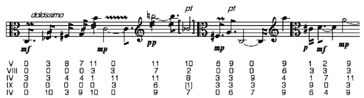

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$$\mathbb{Z}/m\mathbb{Z} \to \bigoplus_{i=1}^{s} \mathbb{Z}/p_i^{n_i}\mathbb{Z}$$

gives rise to an isomorphism of abelian groups

$$P_m \to \bigoplus_{i=0}^{s} P_{p_i^{n_i}}$$
$$f \mapsto (f \bmod p_i^{n_i})_{0\leq i\leq s}.$$

$f \bmod p_i^{n_i}$ is the p_i-part of f. We can study separately the p_i-parts, so we can restrict to work in $P_{p_i^{n_i}}$.

Δ-nilpotent and Δ-idempotent sequences

Definitions

- $f \in P_m$ is *Δ-nilpotent* (resp. *Δ-idempotent*) if there exists $k \geq 1$ such that $\Delta^k f = 0$ (resp. $\Delta^k f = f$); the minimal k such that this happens is said nilpotency (resp. idempotency) order of f.
- We denote by I_m^Δ the subset of P_m of Δ-idempotent sequences and by N_m^Δ the subset of Δ-nilpotent sequences.

Fitting Lemma

The *Fitting Lemma* gives the decomposition

$$P_m = N_m^\Delta \oplus I_m^\Delta$$

hence any $f \in P_m$ uniquely determines $f_I \in I_m^\Delta$ and $f_N \in N_m^\Delta$ such that

$$f = f_I + f_N.$$

The properties of Δ-nilpotent sequences

The period

Let $f \in P_{p^n}$ be a periodic sequence. Then:

- $f \in N_{p^n}^\Delta$ if and only if the period of f is p^m for $m \in \mathbb{N}$;
- if $f \in N_{p^n}^\Delta$ with period p^m and nilpotency order η, then $\eta \leq np^m$.

Sum of constants

Sequences in N_m^Δ are *finite sums* of discrete k-primitives of *constant sequences* ; furthermore, one shows that for any $c \in \mathbb{Z}_m$ and any $i \in \mathbb{N}$:

$$\Sigma^k(c)(i) \equiv_m c\binom{i}{k}.$$

Kummer's and Luca's generalized Theorems

We need to study the binomial coefficient modulo a power of a prime. The main classical tools in this setting are:

Kummer's Theorem

Given integers $i \geq k \geq 0$ and a prime number p, the p-adic valuation $\nu_p(\binom{i}{k})$ is equal to the number of carries when k is added to $i-k$ in base p.
This result allows us to find out if a binomial coefficient is 0 modulo a power of a prime.

Lucas's generalized Theorem

The generalization of Lucas's Theorem permits to compute explicitly a binomial coefficient modulo a power of a prime. Example: consider $41 = [1112]_3$ and $11 = [0102]_3$; lets us compute the residue of $\binom{41}{11}$ modulo 9. One gets

$$\binom{[1112]_3}{[0102]_3} \equiv_9 \left\langle {11 \atop 01} \right\rangle \left\langle {1 \atop 1} \right\rangle^{-1} \left\langle {11 \atop 10} \right\rangle \left\langle {\ldots \atop 0\ldots} \right\rangle \ldots \left\langle {12 \atop 02} \right\rangle \equiv_9 7.$$

Angled parentheses here denote a generalization of the binomial coefficient to the case when the numerator is smaller than the denominator.

New results

The period of primitives of constants

Let (c) be a non zero constant sequence in P_{p^n} with $c = p^l b$, $p \nmid b$. Let $s \in \mathbb{N}$ and $[a_k a_{k-1} \cdots a_1 a_0]_p$, $a_k \neq 0$, the representation of s in base p. Then the sequence $\Sigma^s(c)$ has period p^{n-l+k}.

Leading coefficient

Given $f \in N_{p^n}^\Delta$, there is a constant c among the constants in the decomposition of f which *definitively leads* the period of the primitives of f.

The sequence from Messiaen's second mode of limited transpositions

Vieru noticed that starting from a particular sequence and collecting its primitives, some values prolifer. This observation gave rise to some interest in understanding the motivation of this proliferation. The sequence is $f = (2, 1, 2, 4, 8, 1, 8, 4)$, whose decomposition in Δ-nilpotent and Δ-idempotent part coincides with the decomposition in p-parts:

$$f_I = (2, 1) \in \mathbb{Z}/3\mathbb{Z}, \quad f_N = (2, 1, 0, 0, 1, 0, 0) \in \mathbb{Z}/4\mathbb{Z}.$$

f_I is also Σ-idempotent, so it gives constant contribution to the primitives of f. On the other hand,

$$f_N = \Sigma^4(2) + \Sigma^3(3) + \Sigma^2(2) + \Sigma(3) + (2)$$

is the decomposition in primitives of constant sequences and $\Sigma^3(3)$ is the leading term for the period. Studying the behaviour of the primitives of the constant sequences (2) and (3), we found an algebraic explanation to the proliferation.

Riccardo Gilblas

Luisa Fiorot

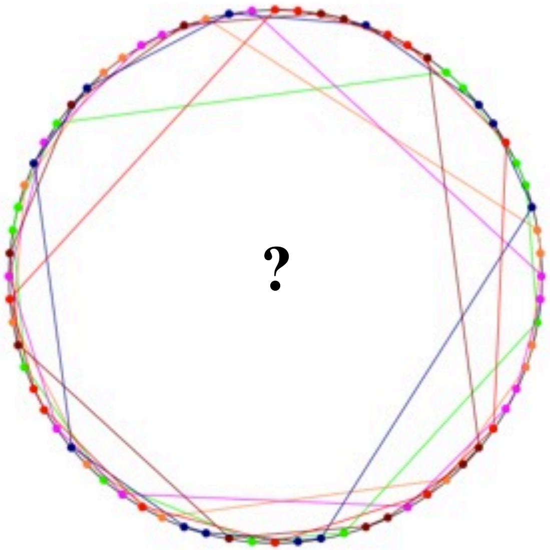
Alberto Tonolo

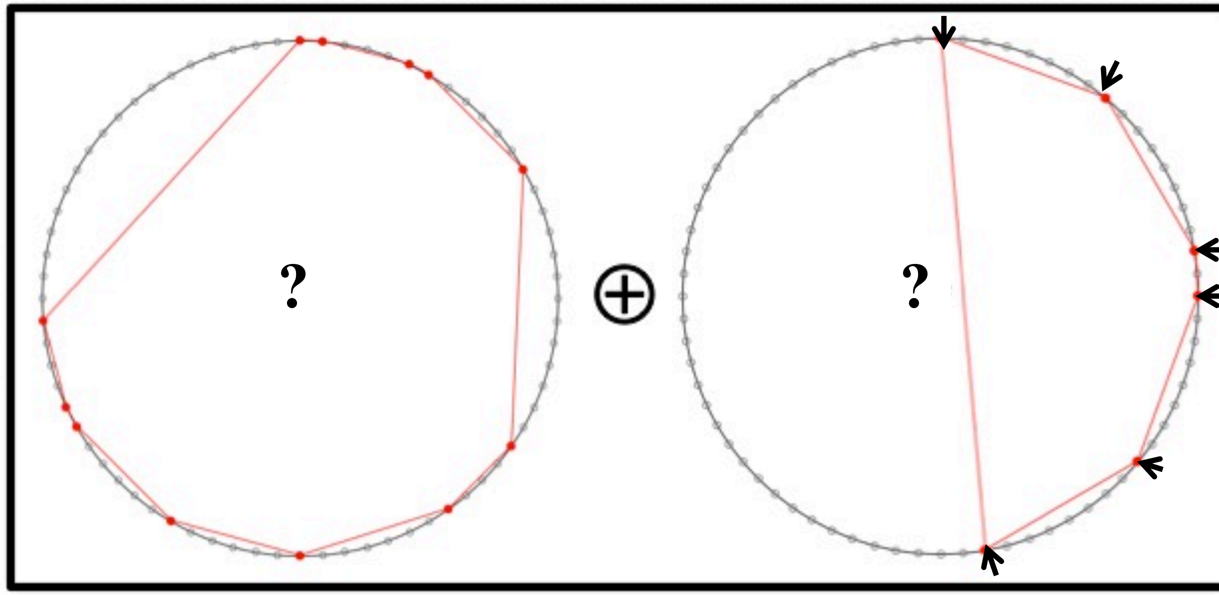
Università degli Studi di Padova

Université de Strasbourg

Aperiodic Rhythmic Tiling Canons (Vuza Canons)

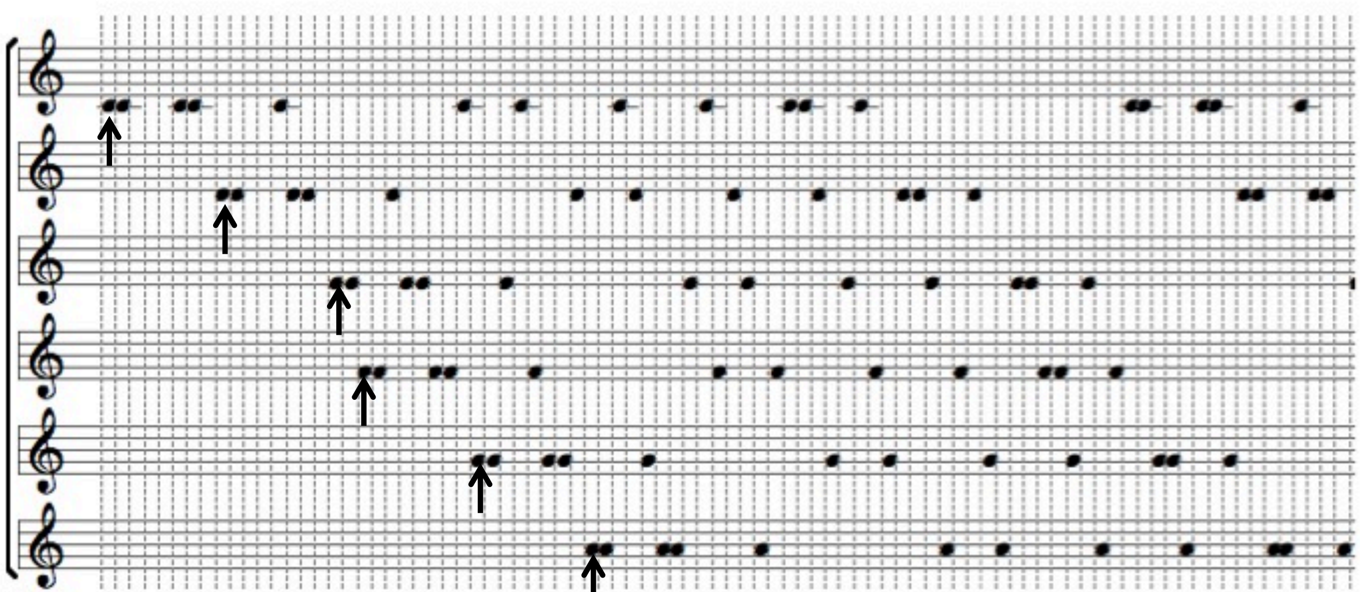
Which is the smallest factorisation of Z_n into two non-periodic subsets?





Anatol Vieru

Dan Vuza



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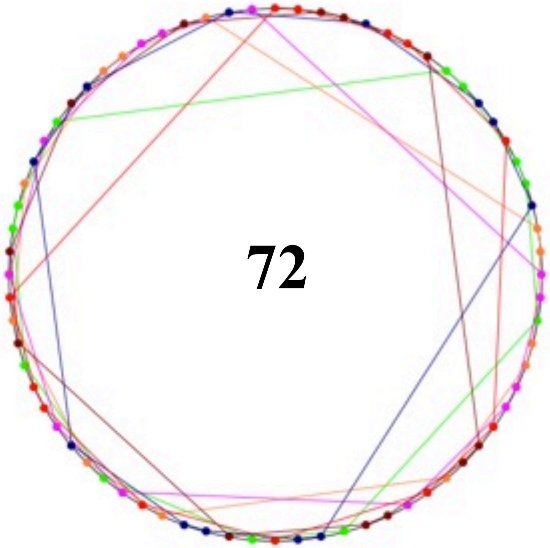

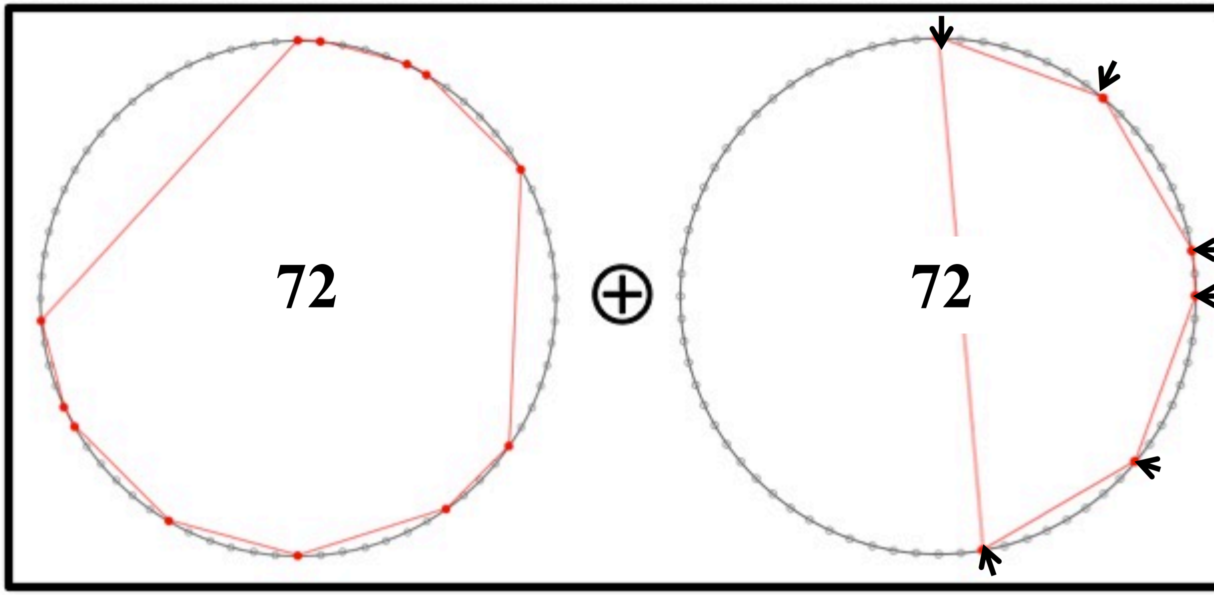

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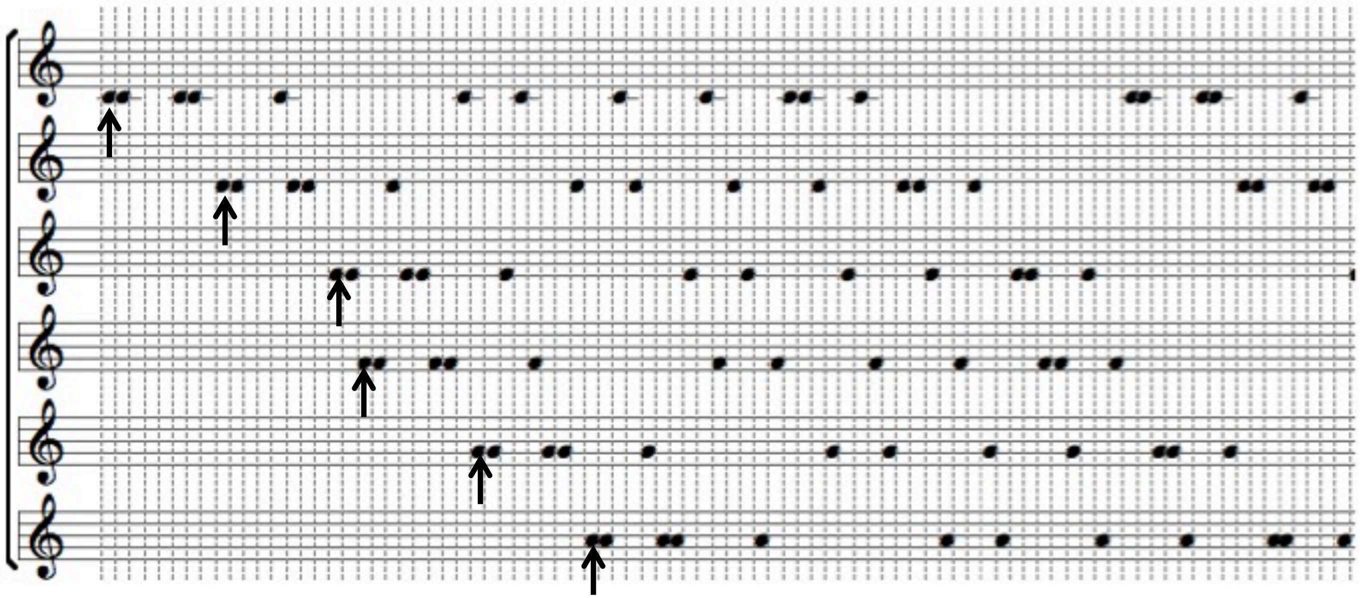


Vuza Canons and Fuglede Spectral Conjecture

A subset of the *n*-dimensional Euclidean space tiles by translation *iff* it is *spectral*.

(*J. Func. Anal.* 16, 1974)

➔**False in dim. $n \geq 3$**
(Tao, Kolountzakis, Matolcsi, Farkas and Mora)

➔ Open in dim. 1 et 2

DEFINITION 6 *A subset A of some vector space (say $\mathbb{R}^n$) is spectral iff it admits a Hilbert base of exponentials, i.e. if any map $f \in L^2(A)$ can be written*

$$f(x) = \sum f_k \exp(2i\pi \lambda_k . x)$$

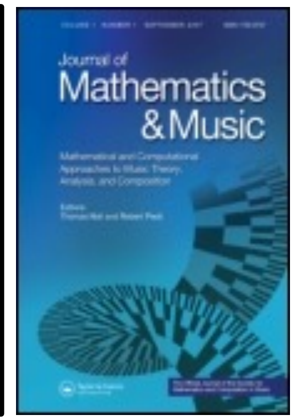
for some fixed family of vectors $(\lambda_k)_{k \in \mathbb{Z}}$ where the maps $e_k : x \mapsto \exp(2i\pi \lambda_k . x)$ are mutually orthogonal (i.e. $\int_A \overline{e_k} e_j = 0$ whenever $k \neq j$).

$\downarrow$ ***n*=1**

DEFINITION 8. *A subset $A \in \mathbb{Z}$ is* spectral *if there exists a spectrum $\Lambda \subset [0,1]$ (i.e., a subset with the same cardinality as A) such that $e^{2i\pi(\lambda_i - \lambda_j)}$ is a root of $A(X)$ for all distinct $\lambda_i, \lambda_j \in \Lambda$.*

Theorem (Amiot, 2009)

- *All non-Vuza canons are spectral.*
- *Fuglede Conjecture is true (or false) iff it is true (or false) for Vuza Canons*



Orders of non-Hajos groups ('Bad groups')

72
108 120 144 168 180
200 216 240 252 264 270 280 288
300 312 324 336 360 378 392 396
400 408 432 440 450 456 468 480
500 504 520 528 540 552 560 576 588 594
600 612 616 624 648 672 675 680 684 696
700 702 720 728 744 750 756 760 784 792
800 810 816 828 864 880 882 888…

(Sloane's sequence A102562)

M. Andreatta & C. Agon (eds), « Tiling Problems in Music », Special Issue of the *Journal of Mathematics and Music*, Vol. 3, Number 2, July 2009 (with contributions by E. Amiot, F. Jedrzejewski, M. Kolountzakis and M. Matolcsi)

Fuglede Spectral conjecture for convex domains is true (in all dimensions)

(Submitted on 28 Apr 2019)

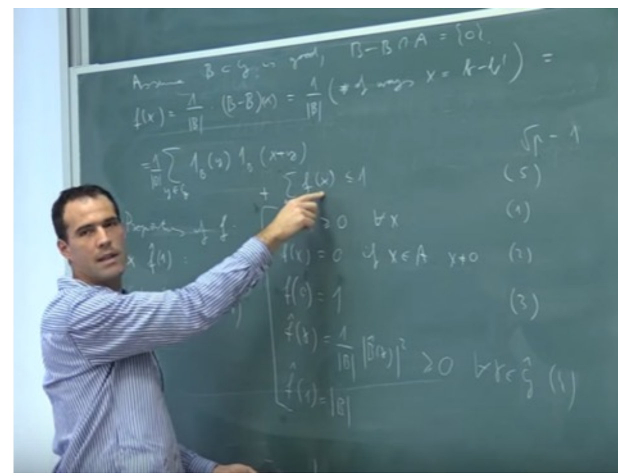
Let Ω be a convex body in $\mathbb{R}^d$. We say that Ω is spectral if the space $L^2(\Omega)$ has an orthogonal basis of exponential functions. There is a conjecture going back to Fuglede (1974) which states that Ω is spectral if and only if it can tile the space by translations. It has long been known that if a convex body Ω tiles then it must be a polytope, and it is also spectral. The converse, however, was proved only in dimensions $d \le 3$ and under the a priori assumption that Ω is a polytope.
In this paper we prove that for every dimension d, if a convex body $\Omega \subset \mathbb{R}^d$ is spectral then it must be a polytope, and it can tile the space by translations. The result thus settles Fuglede's conjecture for convex bodies in the affirmative. Our approach involves a construction from crystallographic diffraction theory, that allows us to establish a geometric "weak tiling" condition necessary for the spectrality of Ω.

Subjects: **Classical Analysis and ODEs (math.CA)**; Functional Analysis (math.FA); Metric Geometry (math.MG)

MSC classes: 42B10, 52B11, 52C07, 52C22

Cite as: arXiv:1904.12262 **[math.CA]**
(or arXiv:1904.12262v1 **[math.CA]** for this version)

Nir Lev (Bar-Ilan University, Tel-Aviv)



Mate Matolcsi (Rényi Institute, Budapest)

Vuza's Algorithm and other approaches (PhD Greta Lanzarotto)

An Integer Linear Programming Model for Tilings

Gennaro Auricchio [*1], Luca Ferrarini [†1] and Greta Lanzarotto [‡1,2]

[1]Department of Mathematics, University of Pavia
[2]IRMA, University of Strasbourg

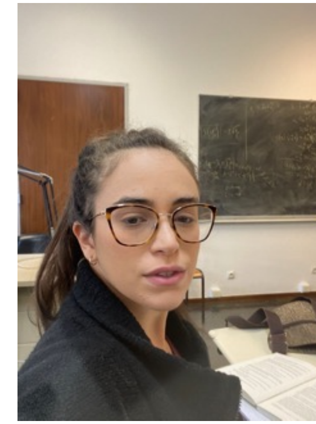
July 8, 2021

Abstract

In this paper, we propose an Integer Linear Model whose solutions are the aperiodic rhythms tiling with a given rhythm A.
We show how this model can be used to efficiently check the necessity of the Coven-Meyerowitz's $(T2)$ condition and also to define an iterative algorithm that finds all the possible tilings of the rhythm A.
To conclude, we run several experiments to validate the time efficiency of this model.

Keywords: Integer Programming, Mathematics and Music, Tiling Problems, Vuza Canons, $(T2)$ Conjecture

AMS: 90C10, 05B45



Greta Lanzarotto

Ludovico Pernazza

Université de Strasbourg

Vuza's Algorithm and other approaches (PhD Greta Lanzarotto)

An Integer Linear Programming Model for Tilings

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July 8, 2021

TO BE SUBMITTED

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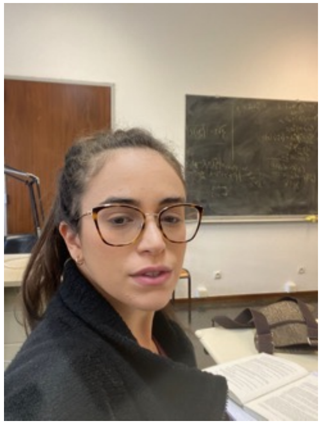


Greta Lanzarotto

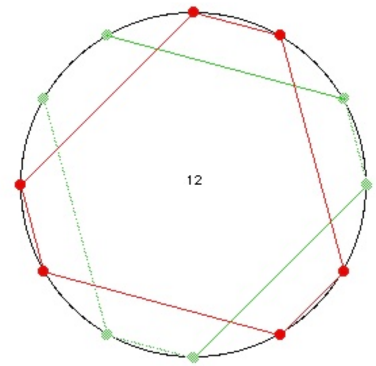
Ludovico Pernazza

Université de Strasbourg

Polynomial representation of tiling canons

$$\Delta_n = 1 + X + X^2 + \ldots + X^{n-1} = \prod \Phi_d(X) \qquad \begin{matrix} d \mid n \\ d \neq 1 \end{matrix}$$

$$\Phi_2(X) = 1 + X$$
$$\Phi_3(X) = 1 + X + X^2$$
$$\Phi_4(X) = 1 + X^2$$
$$\Phi_6(X) = 1 - X + X^2$$



$$\Delta_{12} = 1 + X + \ldots + X^{11} = \Phi_2 \times \Phi_3 \times \Phi_4 \times \Phi_6 \times \Phi_{12}$$

$$A(X) = \Phi_2 \times \Phi_3 \times \Phi_6 \times \Phi_{12} = 1 + X + X^4 + X^5 + X^8 + X^9$$

$$B(X) = \Phi_4 = 1 + X^2$$

$$B = \{0, 2\}$$

$$A = \{0, 1, 4, 5, 8, 9\}$$

Coven & Meyerowitz conditions

$$\Phi_2(X) = 1 + X$$
$$\Phi_3(X) = 1 + X + X^2$$

(T1) $A(1) = 6 = \Phi_2(1) \times \Phi_3(1) = 2 \times 3$

(T2) $\Phi_2 \mid A(X)$ et $\Phi_3 \mid A(X) \Rightarrow \Phi_{2\times 3} \mid A(X)$

E. Coven & A. Meyerowitz, "Tiling the integers with translates of one finite set", *J. Algebra*, 212, pp.161-174, 1999

C&M Conditions, tiling and spectrality

There is no loss of generality in restricting attention to translates of a finite set A of *nonnegative* integers. Then $A(x) = \sum_{a \in A} x^a$ is a polynomial such that $\#A = A(1)$. Let S_A be the set of prime powers s such that the s-th cyclotomic polynomial $\Phi_s(x)$ divides $A(x)$. Consider the following conditions on $A(x)$.

(T1) $A(1) = \prod_{s \in S_A} \Phi_s(1)$.

(T2) If $s_1, \ldots, s_m \in S_A$ are powers of distinct primes, then $\Phi_{s_1 \cdots s_m}(x)$ divides $A(x)$.

C&M (1999)

- T1 + T2 => tiling
- tiling => T1

Laba (2002)

- T1 + T2 => spectral
- T2 => spectral
- spectral => T1

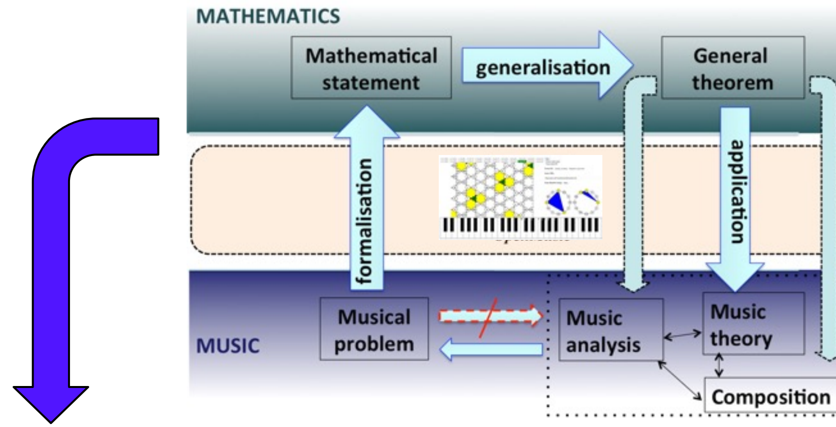
Amiot (2009)

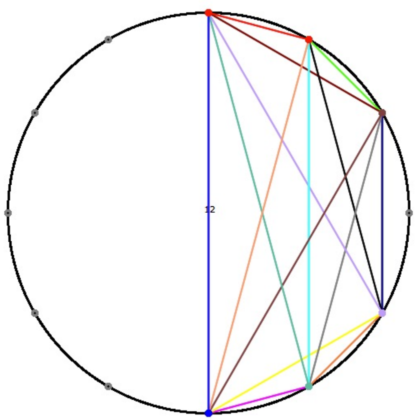
- Tiling of a Hajos group => T2.

Is condition T2 mandatory for tiling?

E. Coven & A. Meyerowitz, "Tiling the integers with translates of one finite set", *J. Algebra*, 212, pp.161-174, 1999

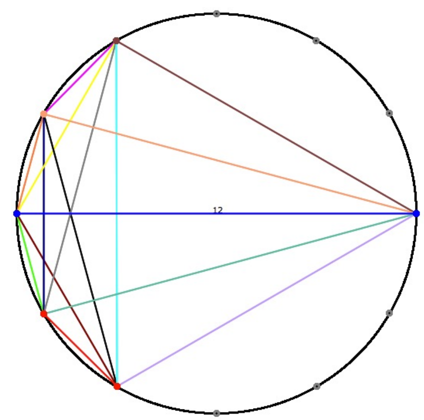
A historical example of "mathemusical" problem

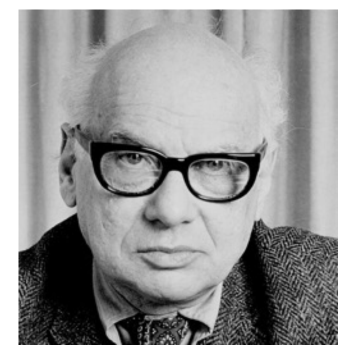




?

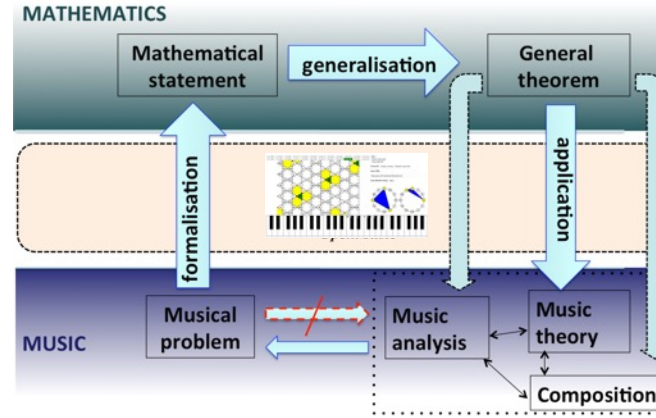
Hexachord Theorem

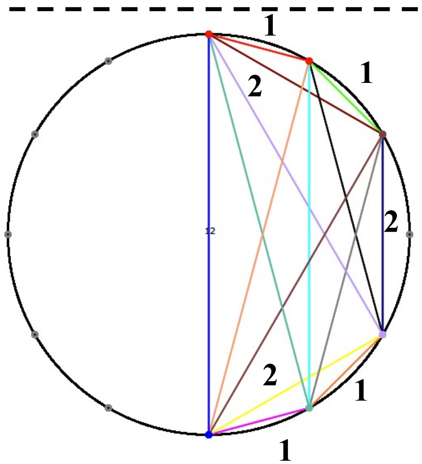




M. Babbitt

A historical example of "mathemusical" problem





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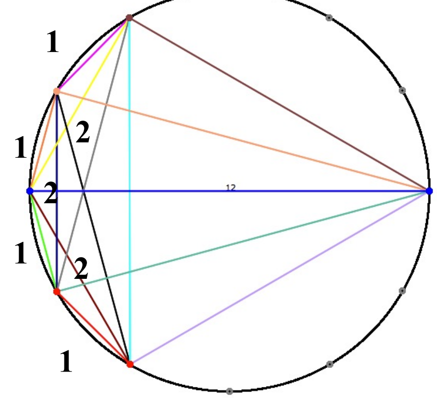
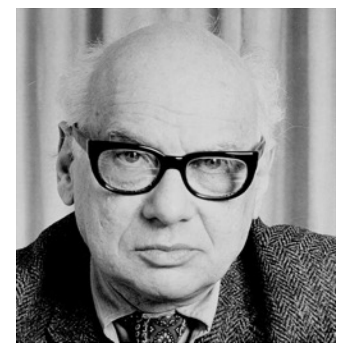
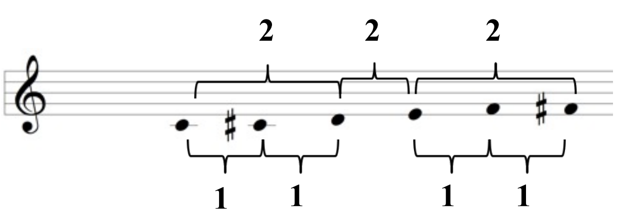
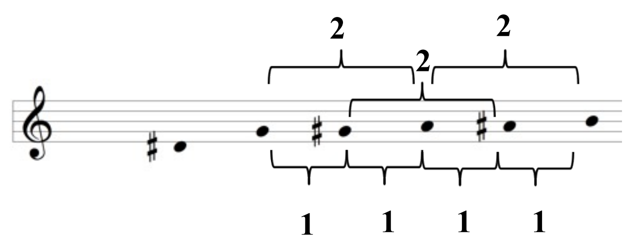
Hexachord Theorem





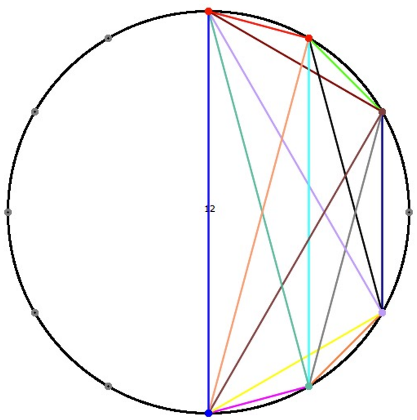
M. Babbitt





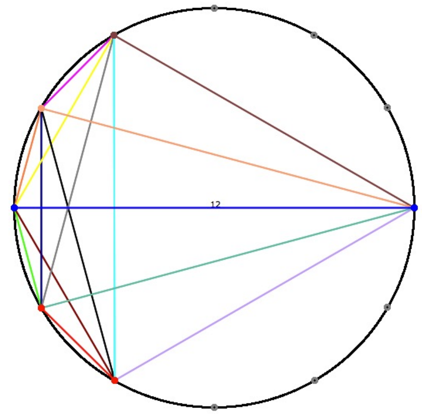
➔ https://guichaoua.gitlab.io/web-hexachord/hexachordTheorem

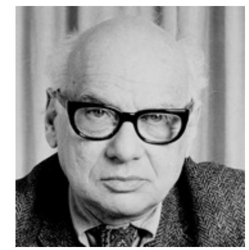
The shortest proof of Babbitt's Theorem?



$\approx$

Hexachord Theorem





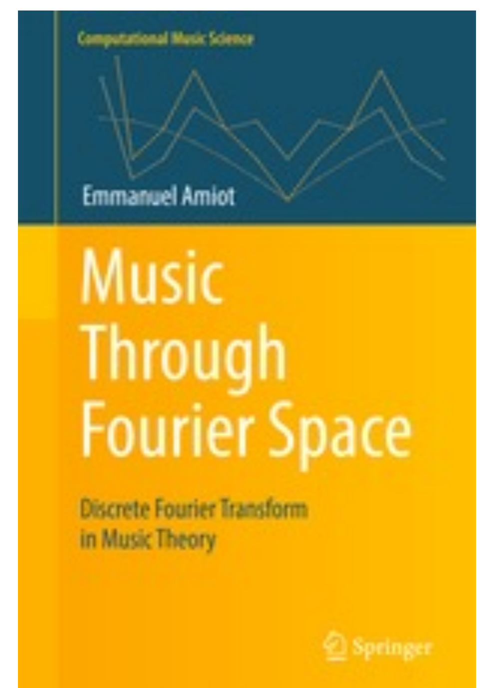
M. Babbitt

$$IC_A = [4, 3, 2, 3, 2, 1\ ] = [4, 3, 2, 3, 2, 1\ ] = IC_{A'}$$

$$IC_A(k) = (1_A \star 1_{-A})(k)$$

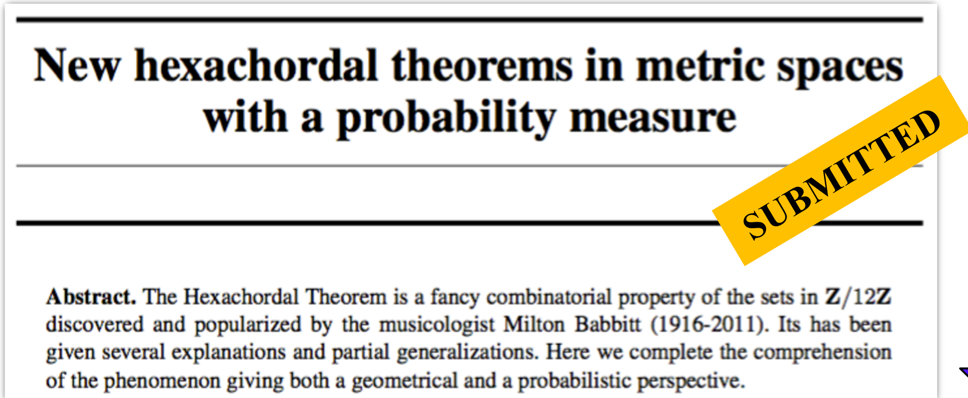
$$\mathcal{F}_A : t \mapsto \sum_{k \in A} e^{-2i\pi kt/c}$$

$$\forall \mathrm{k}\ \mathcal{F}(\mathrm{IC}_{\mathbb{Z}_c \setminus \mathrm{A}})(\mathrm{k}) = \mathcal{F}(\mathrm{IC}_\mathrm{A})(\mathrm{k})$$



E. Amiot : « Une preuve élégante du théorème de Babbitt par transformée de Fourier discrète », Quadrature, 61, 2006.

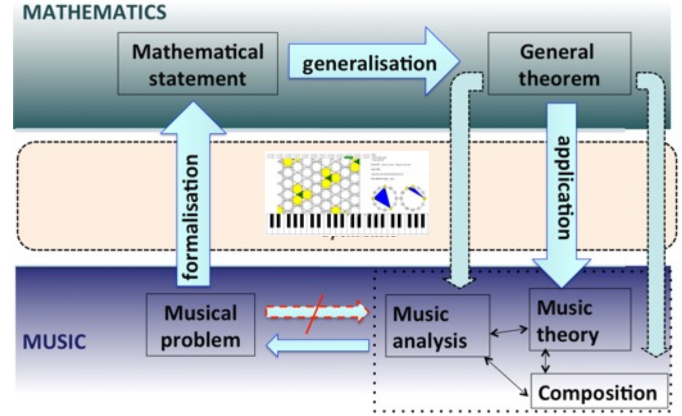
Some generalizations of Babbitt's Theorem



New hexachordal theorems in metric spaces with a probability measure

SUBMITTED

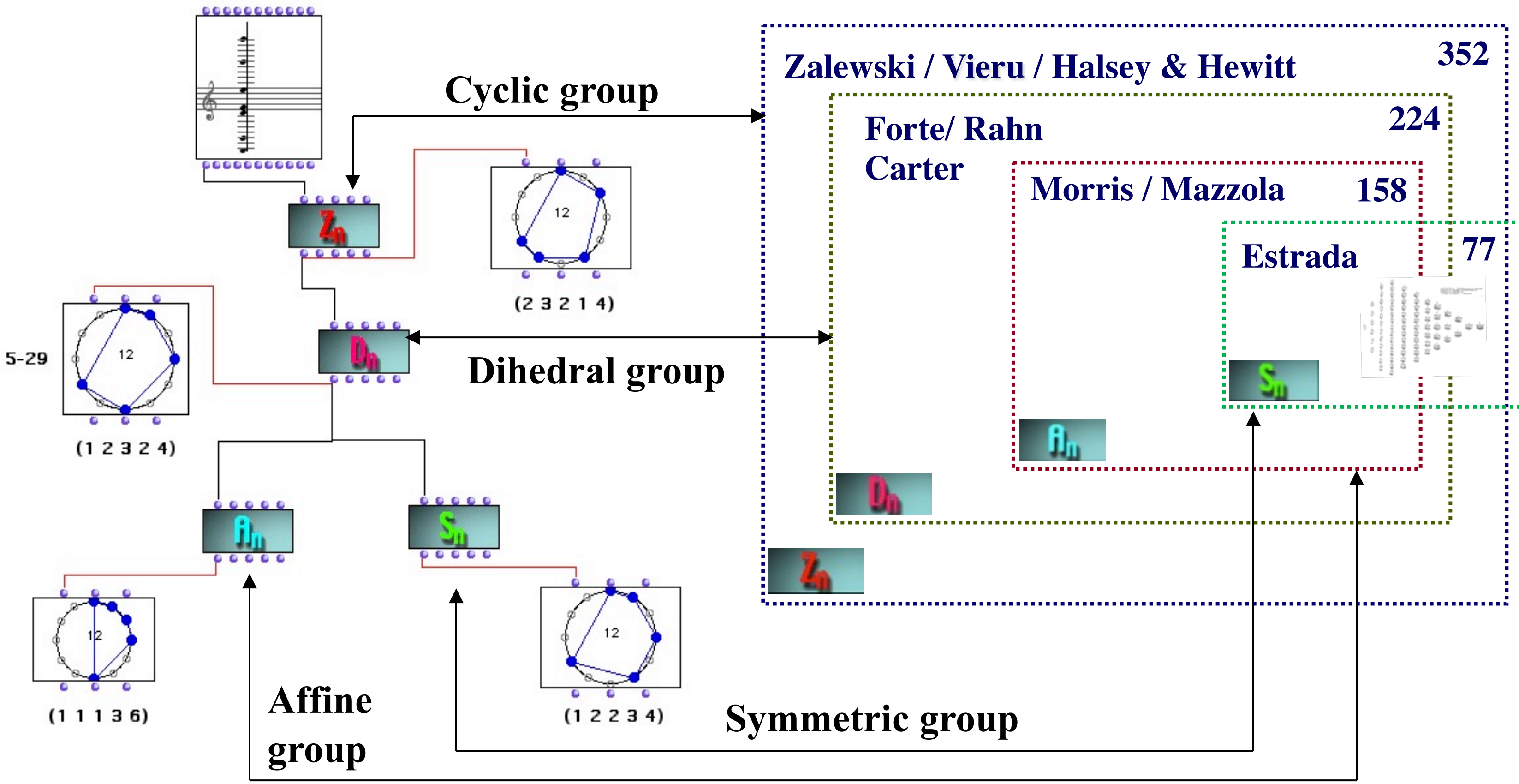
Abstract. The Hexachordal Theorem is a fancy combinatorial property of the sets in $\mathbf{Z}/12\mathbf{Z}$ discovered and popularized by the musicologist Milton Babbitt (1916-2011). Its has been given several explanations and partial generalizations. Here we complete the comprehension of the phenomenon giving both a geometrical and a probabilistic perspective.



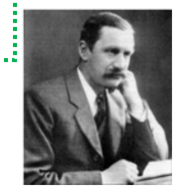
- The finite/compact spaces on which a group is transitively acting (homogeneous spaces).
  - Compact Lie groups
  - Spheres, torii, etc.
  - Cayley graphs of finite groups with generators (Hypercubes, symmetric group, etc.)
  - Other homogeneous graphs (Petersen graph, truncated icosahedron, etc.)

Nicolas Juillet

Group actions and the classification of musical structures

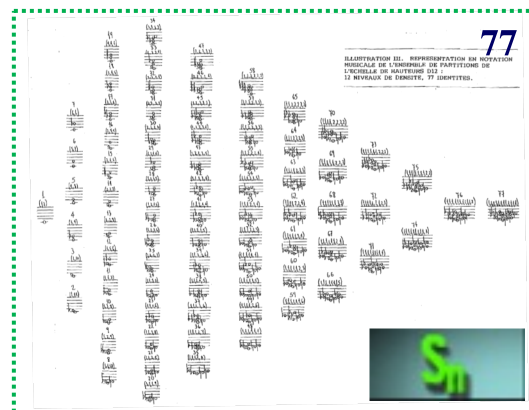



F. Klein

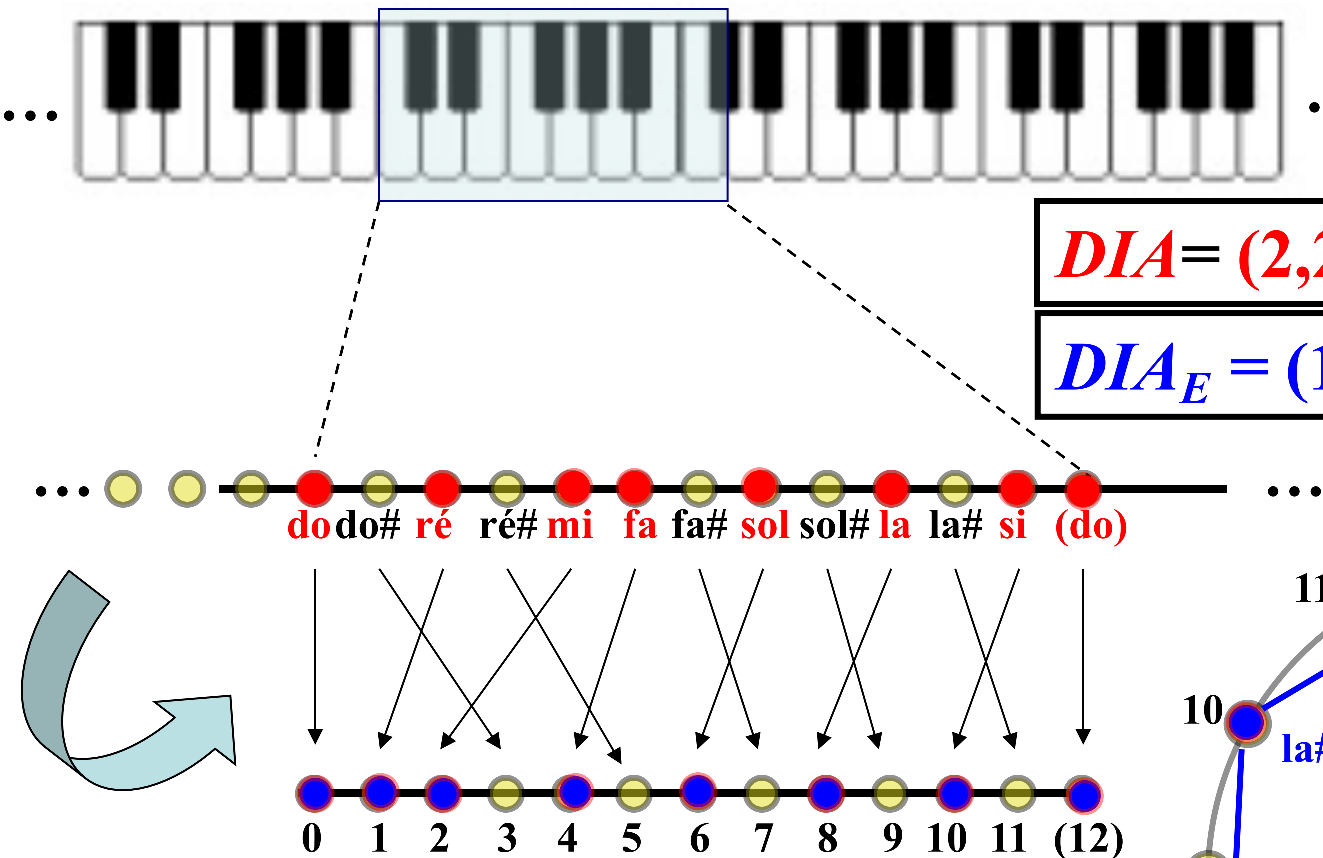

W. Burnside


G. Polya

	1	2	3	4	5	6	7	8	9	10	11	12
Z_n	1	6	19	43	66	80	66	43	19	6	1	1
D_n	1	6	12	29	38	50	38	29	12	6	1	1
A_n	1	5	9	21	25	34	25	21	9	5	1	1
S_n	*1*	*6*	*12*	*15*	*13*	*11*	*7*	*5*	*3*	*2*	*1*	*1*

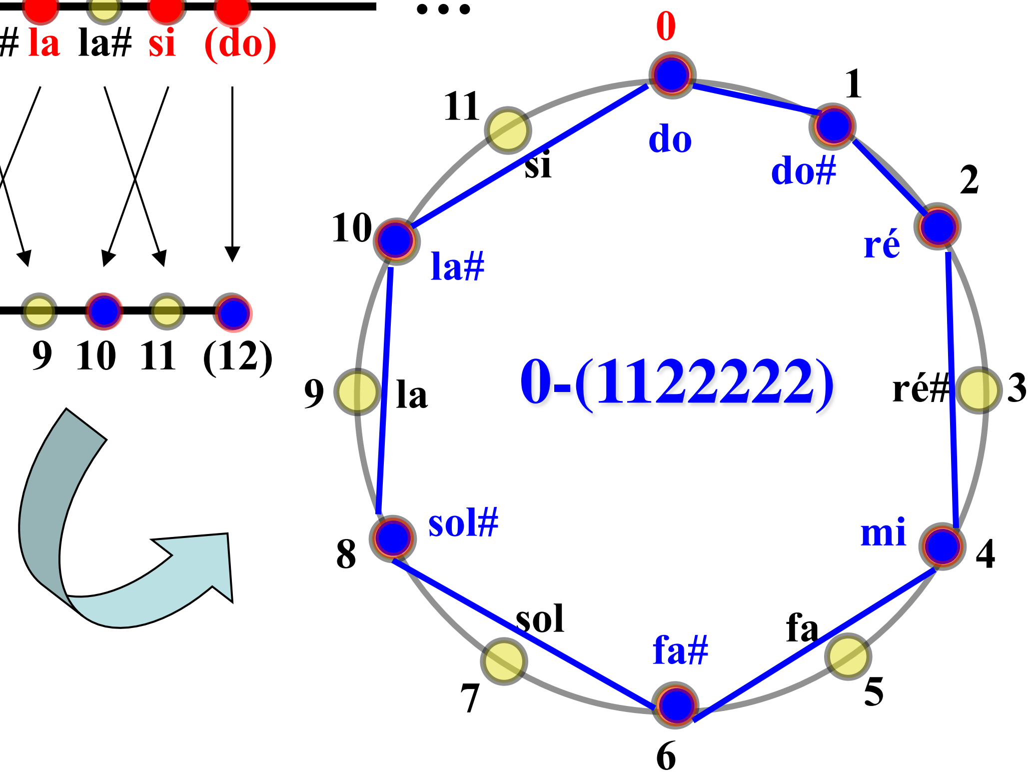


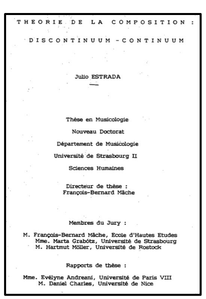
The partition lattice of musical structures

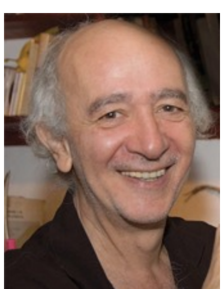


$DIA = (2,2,1,2,2,2,1)$

$DIA_E = (1,1,2,2,2,2,2)$

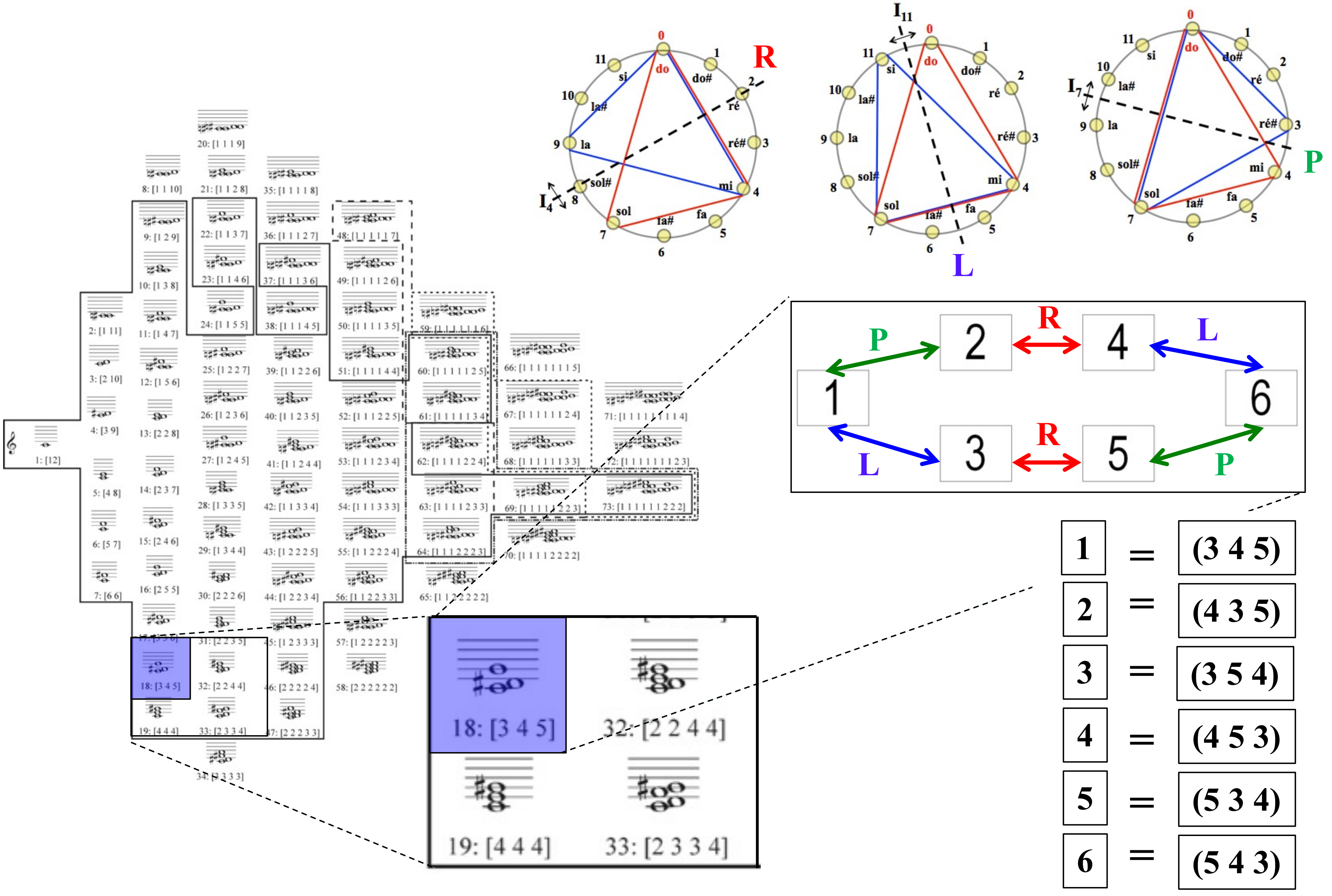




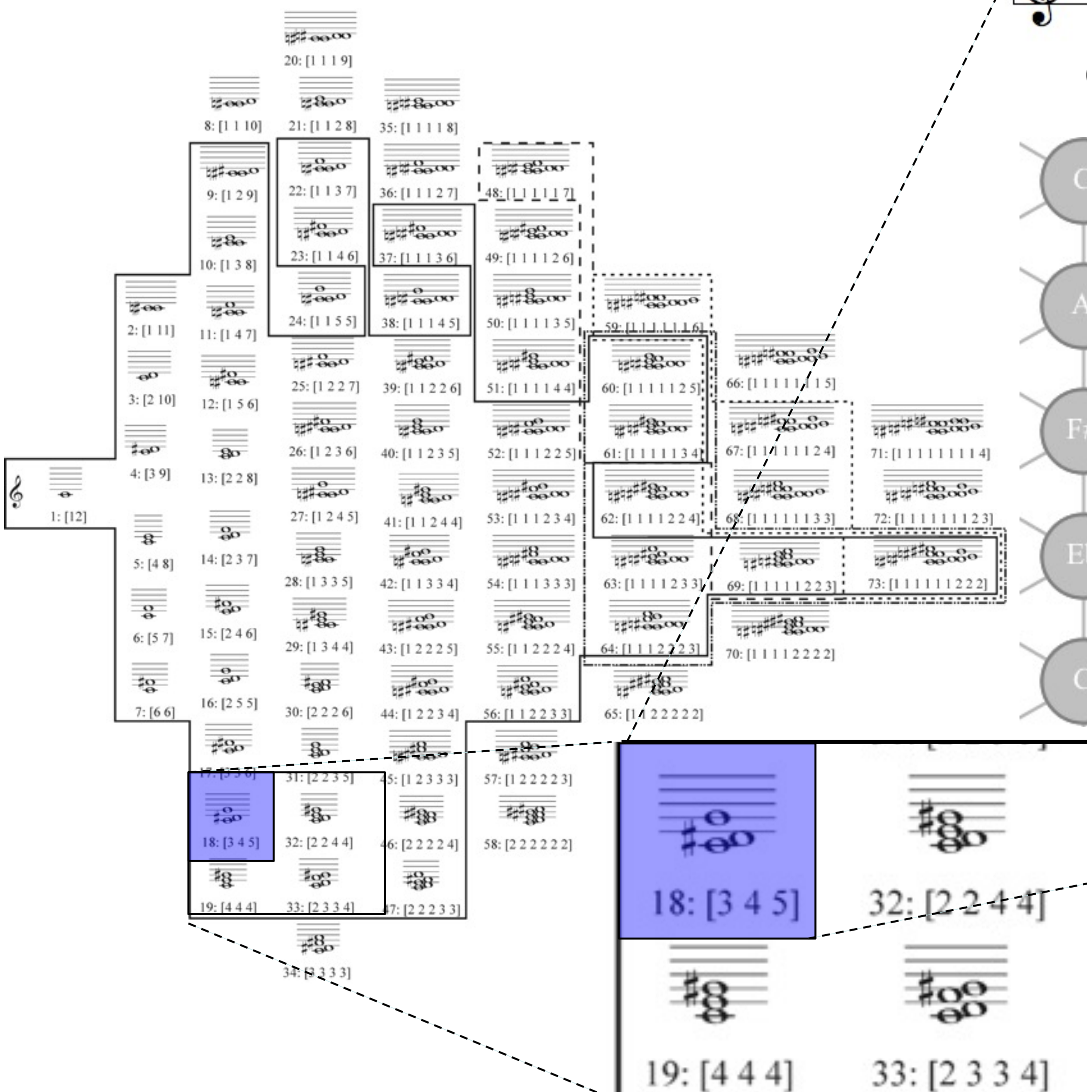


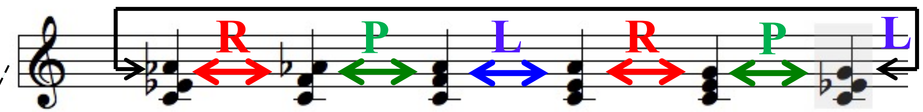
Julio Estrada, *Théorie de la composition : discontinuum – continuum,* université de Strasbourg II, 1994

Permutohedron and *Tonnetz*: a structural inclusion

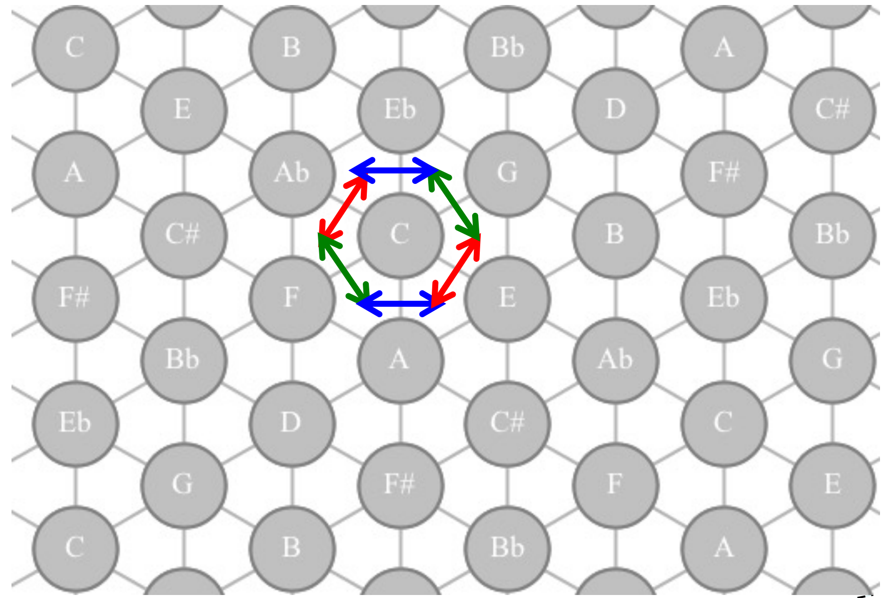


Permutohedron and *Tonnetz*: a structural inclusion





(3 5 4) (5 3 4) (5 4 3) (4 5 3) (4 3 5) (3 4 5)

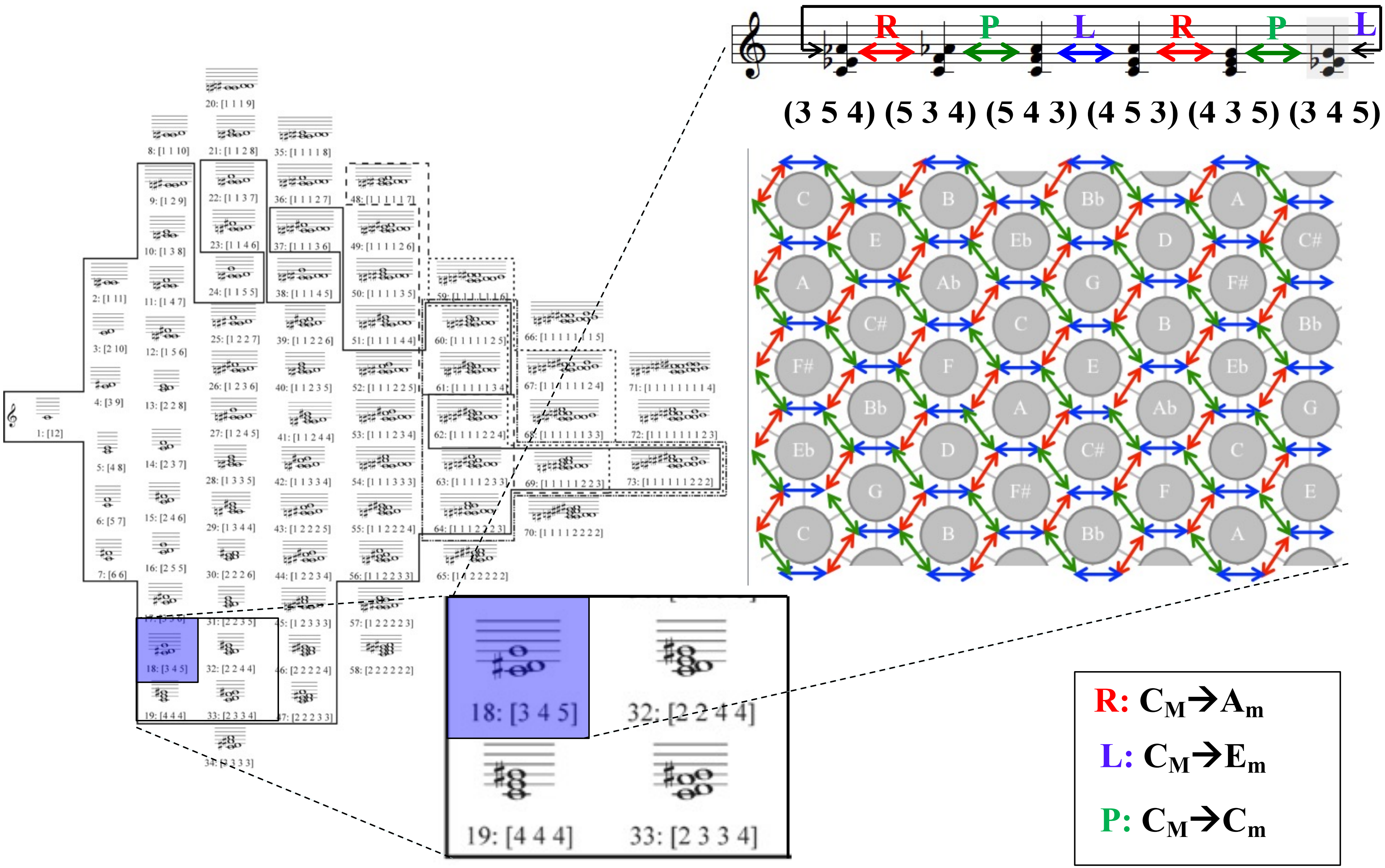


R: $C_{maj} \rightarrow A_{min}$

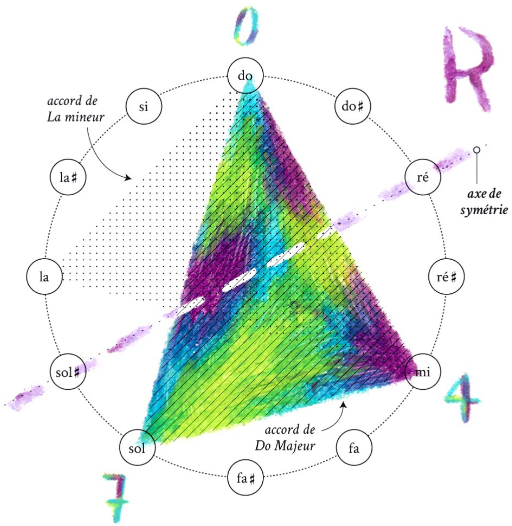
L: $C_{maj} \rightarrow E_{min}$

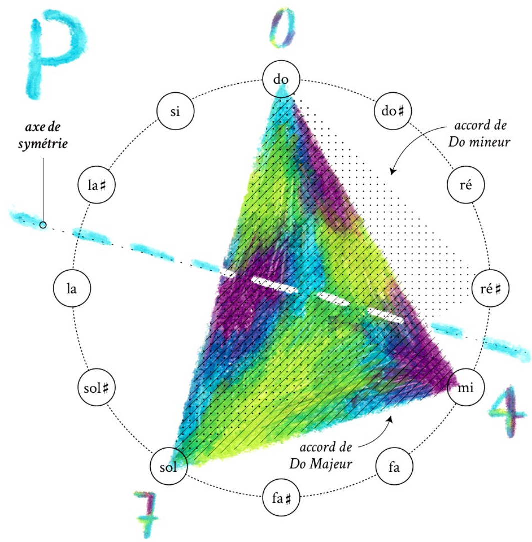
P: $C_{maj} \rightarrow C_{min}$

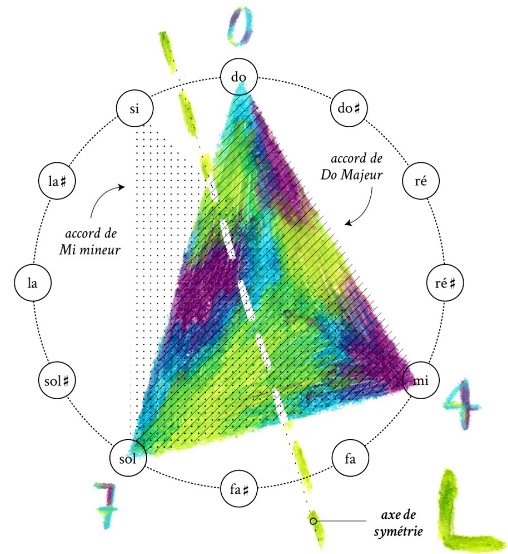
Permutohedron and *Tonnetz*: a structural inclusion



The three main major-minor symmetries







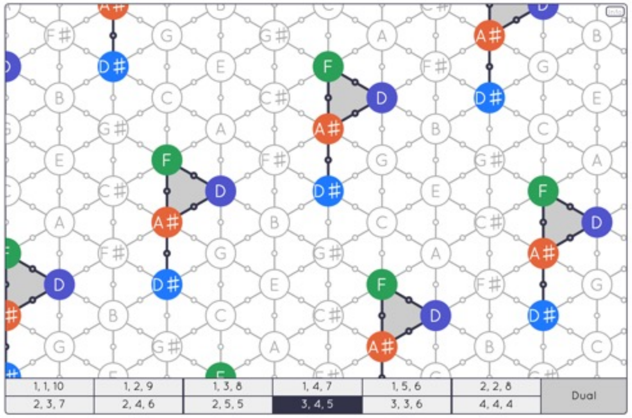
R as RELATIVE | **P as PARALLEL** | **L as LEADING-TONE (EXCHANGE)**

C major ↕ A minor

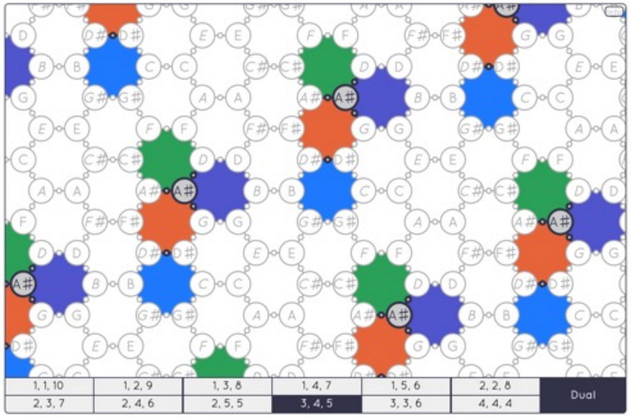
C major ↕ C minor

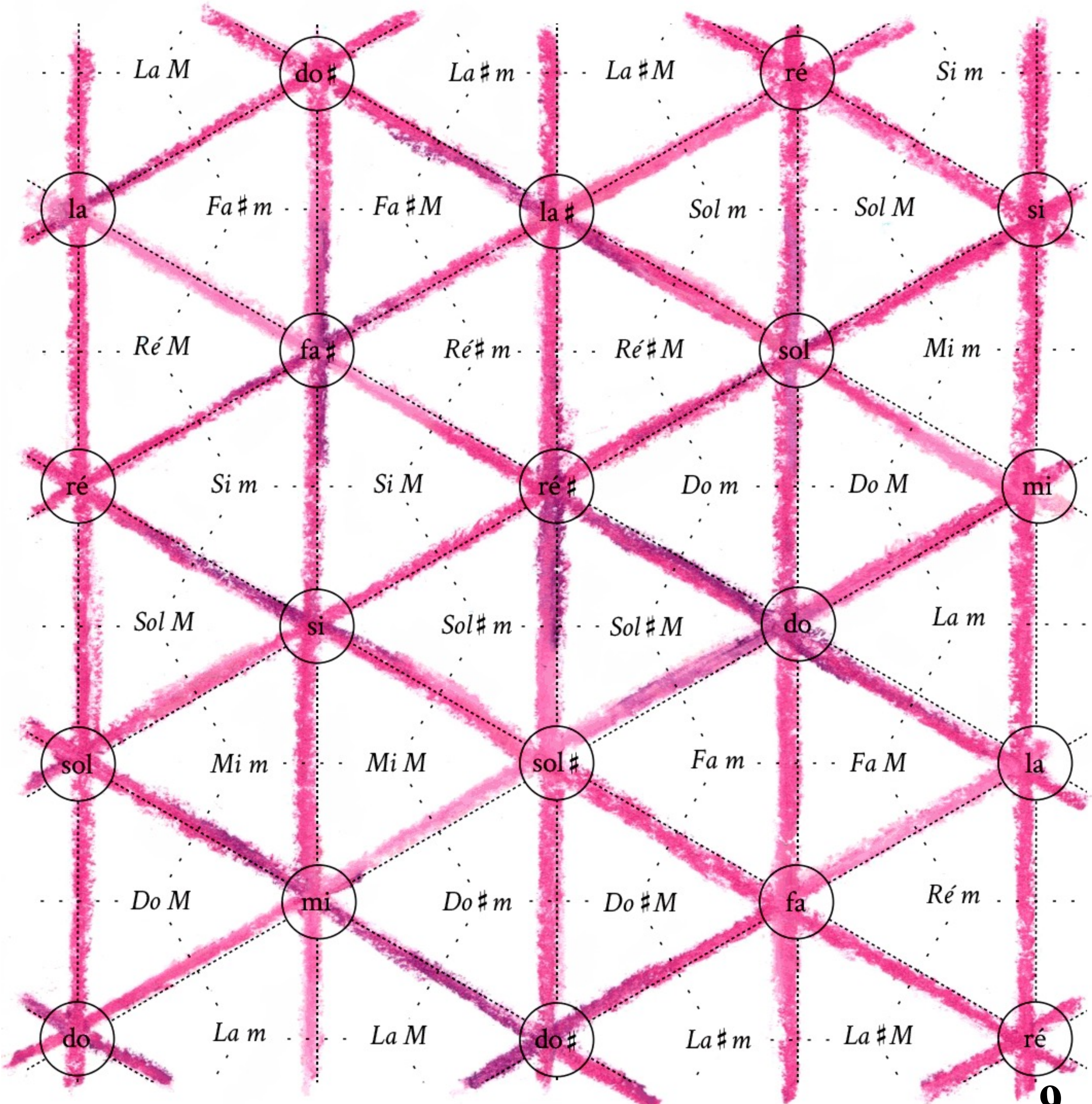
C major ↕ E minor

From the *Tonnetz* to the dual one



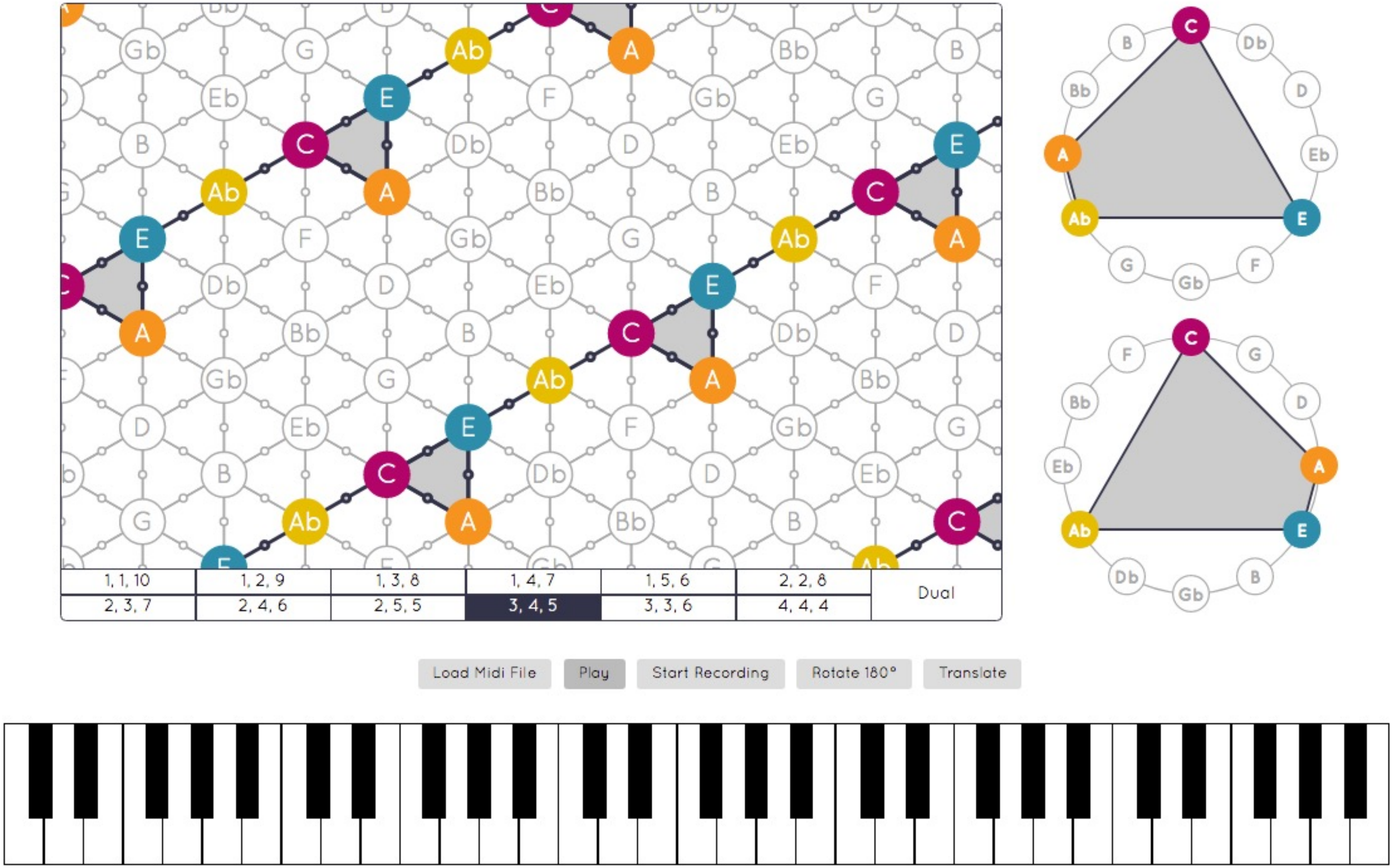
duality





THE TONNETZ

ONE KEY – MANY REPRESENTATIONS



➔ **www.morenoandreatta.com/software/**

la.la.lab
THE MATHS OF MUSIC

May. 17, 2019 to Dec. 2020

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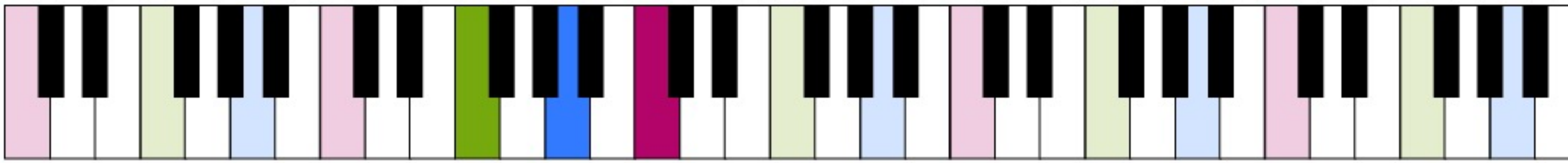
HEIDELBERG
LAUREATE FORUM
FOUNDATION

https://imaginary.org/



La.La.Lab brings the visitor to an interactive exploration and discovery of music from a mathematical perspective. The exhibition pivots over three axis:

- **Music theory.** Learning what tools build music, and how these tools are used to create art. Basic concepts and historical comments.

- **Current research.** The latest trends of research in the connection of maths and music. Artificial Intelligence, theoretical and new instruments, classification and composition tools.

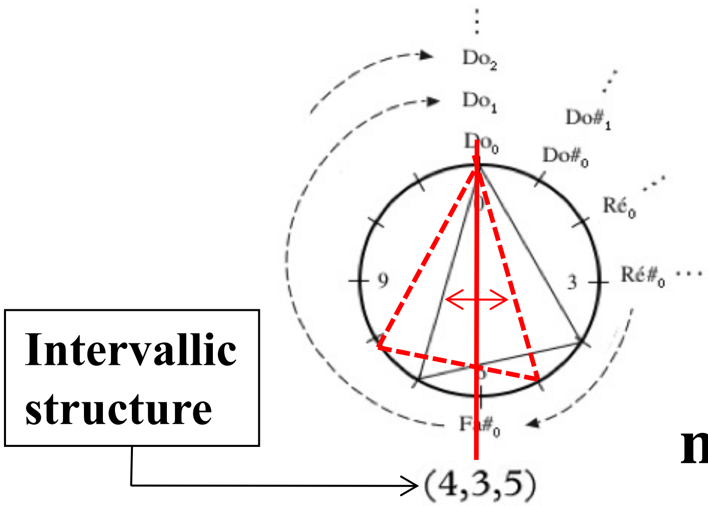
- **Art and entertainment.** A joyful display of artworks from artists and mathematicians in the field. Talks/concerts at scheduled events

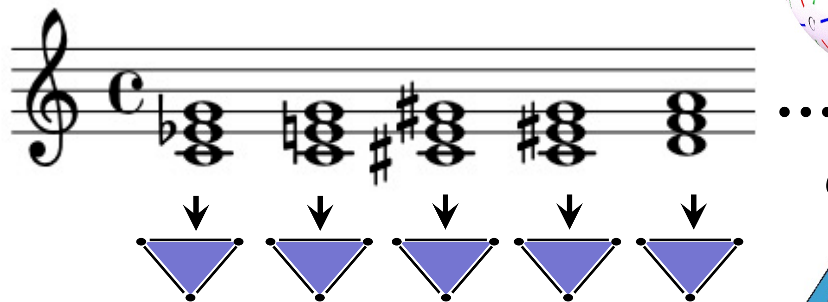
The Tonnetz web environment (developer: C. Guichaoua)

The *Tonnetz* as a simplicial complex

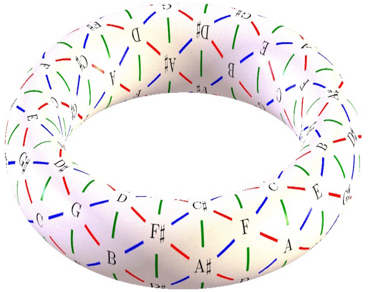
L. Bigo, *Représentation symboliques musicales et calcul spatial*, PhD, Ircam / LACL, 2013

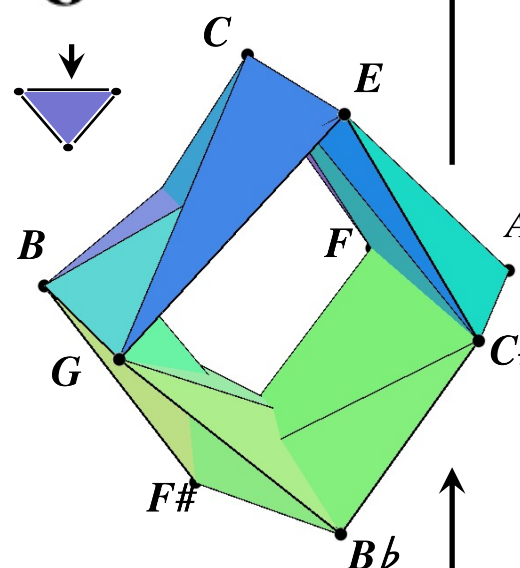
- Assembling chords related by some equivalence relation
  - Equivalence up to transposition/inversion:

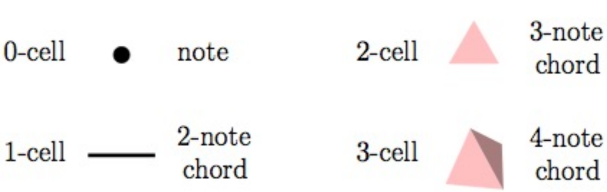


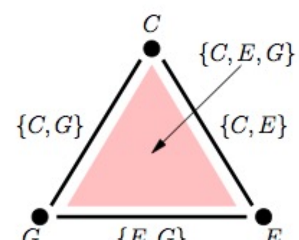


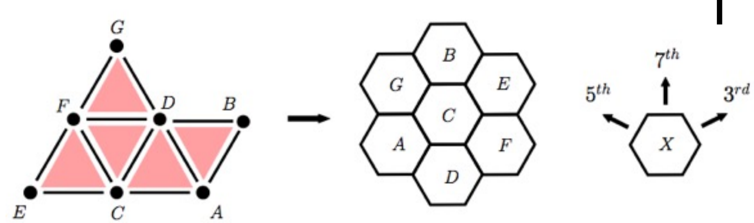
major/minor triads





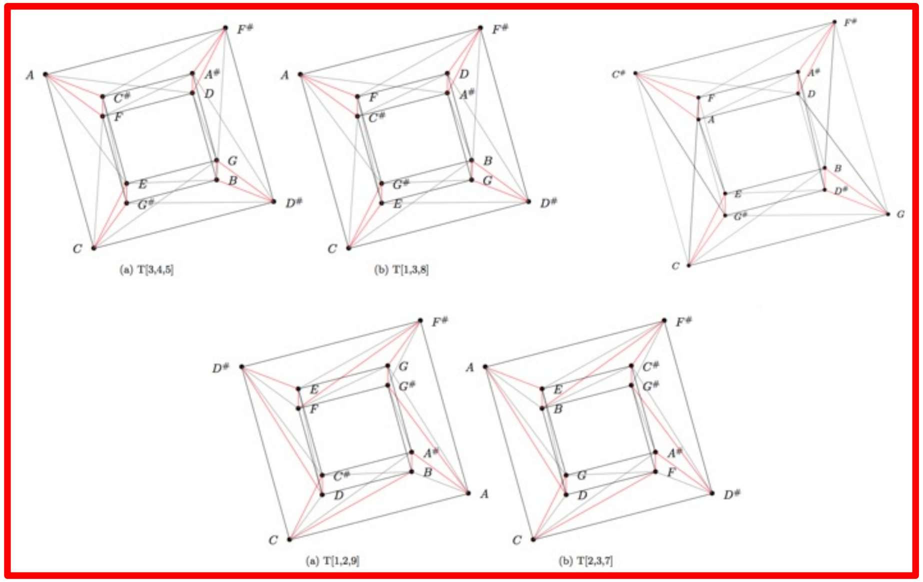






The *Tonnetze* as simplicial complexes

Tonnetz	Betti Numbers		
	β_0	β_1	β_2
$T[1,2,9]$	1	2	1
$T[1,3,8]$	1	2	1
$T[1,4,7]$	1	2	1
$T[2,3,7]$	1	2	1
$T[3,4,5]$	1	2	1
$T[1,1,10]$	1	1	0
$T[2,5,5]$	1	1	0
$T[2,2,8]$	2	2	0
$T[1,5,6]$	1	1	6
$T[2,4,6]$	2	2	6
$T[3,3,6]$	3	0	3
$T[4,4,4]$	4	0	0

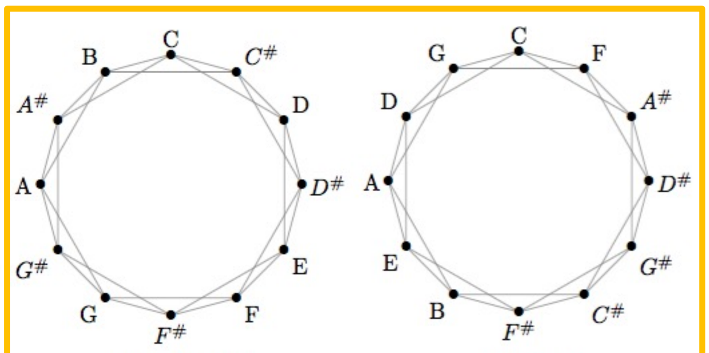


(a) T[3,4,5] (b) T[1,3,8]

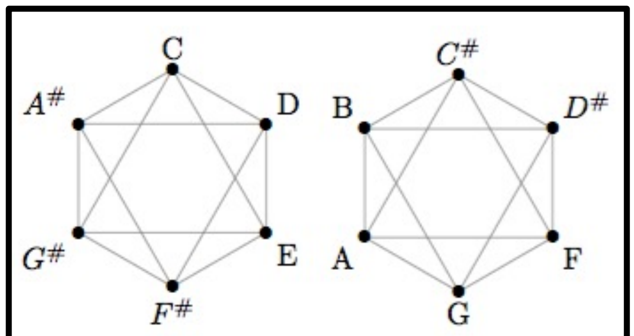
(a) T[1,2,9] (b) T[2,3,7]

L. Bigo

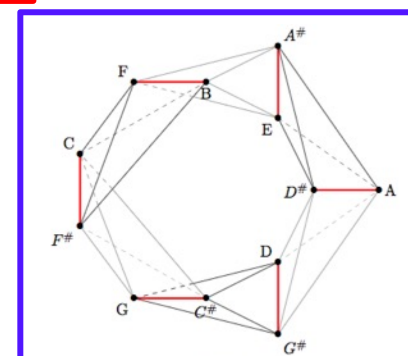
P. Lascabettes



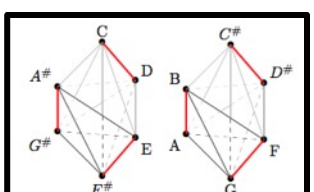
(a) $T[1,1,10]$ (b) $T[2,5,5]$



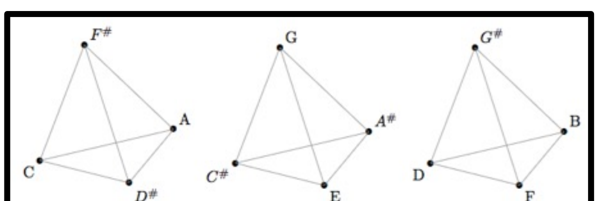
(c) $T[2,2,8]$



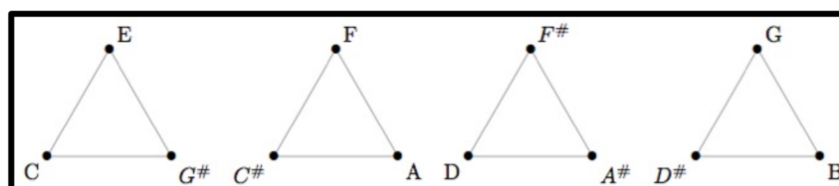
(a) $T[1,5,6]$



(b) $T[2,4,6]$

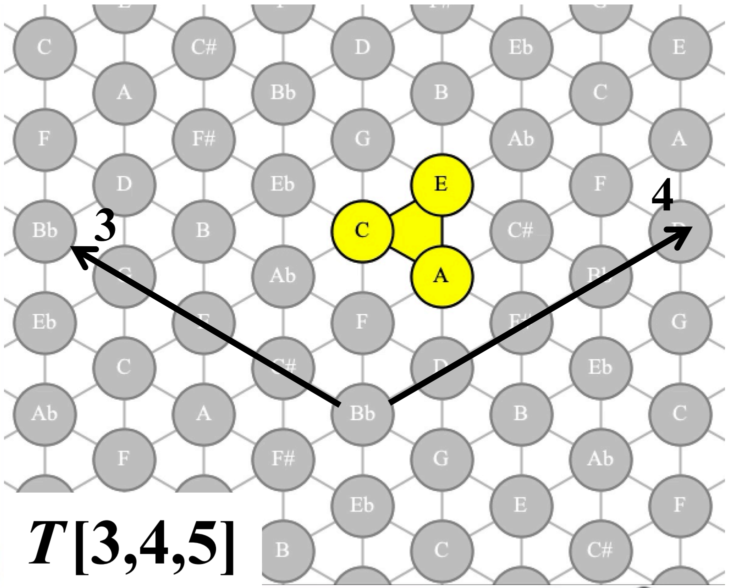


(a) $T[3,3,6]$

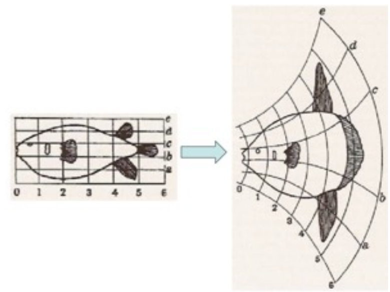


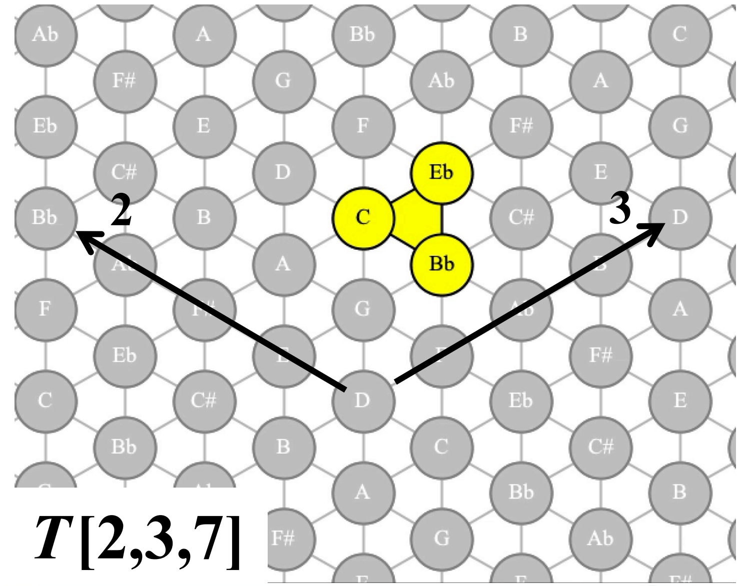
(b) T[4,4,4]

Musical style and space trajectories

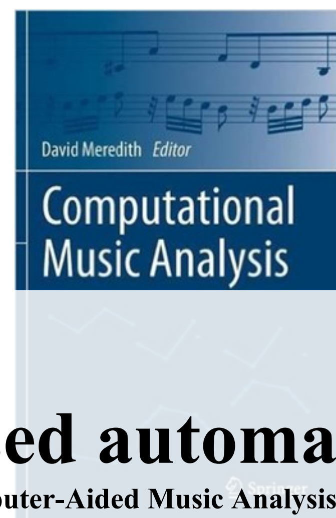


T[3,4,5]





T[2,3,7]

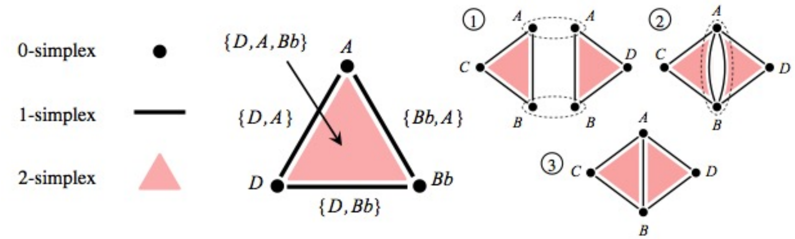


Towards a geometry-based automatic musical style analysis

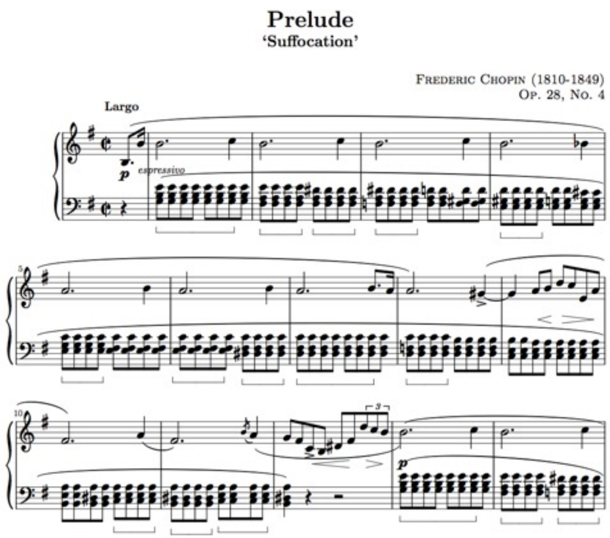
Bigo L., M. Andreatta (2015), Topological Structures in Computer-Aided Music Analysis, in D. Meredith (ed.), *Computational Music Analysis*, Springer

Towards a topological signature of a musical piece
A structural approach in Music Information Retrieval

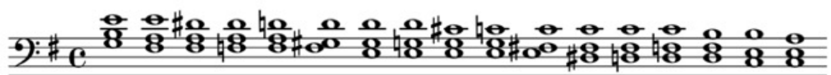
The simplices and their self-assembly



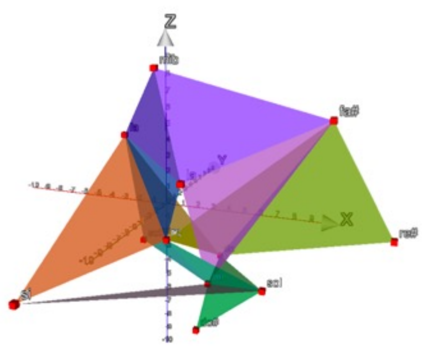
The score

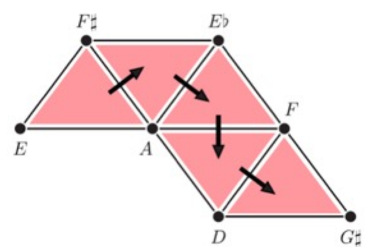


Score reduction



The simplicial complex generated by the piece



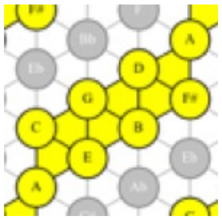


A specific trajectory in the complex

Topological signature?

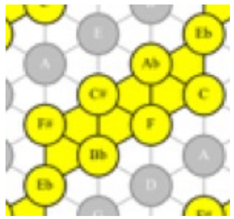
Towards an anisotropic *Tonnetz*

Mattia Bergomi

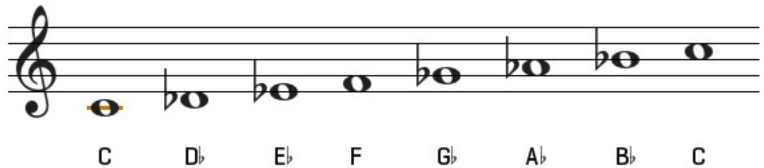


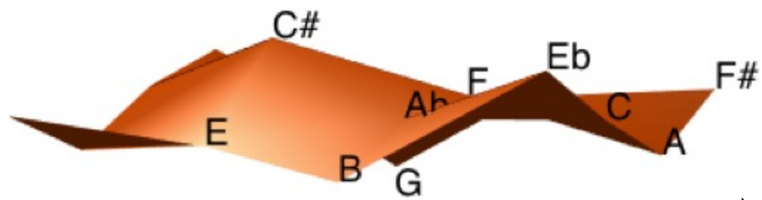
Ionian mode

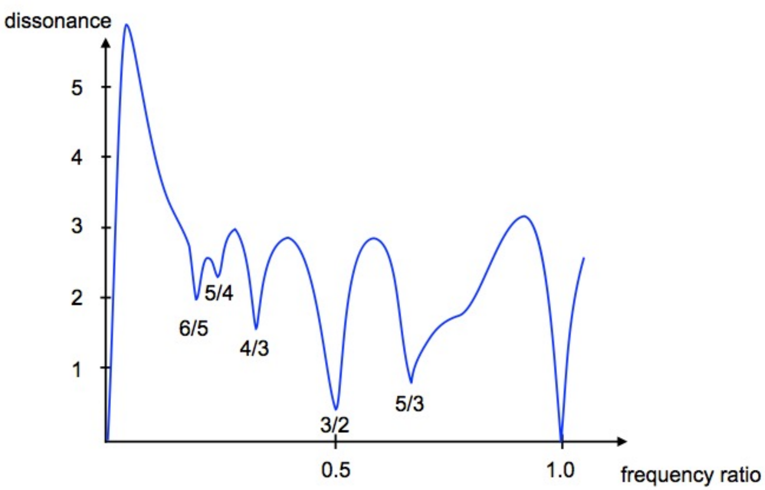
=

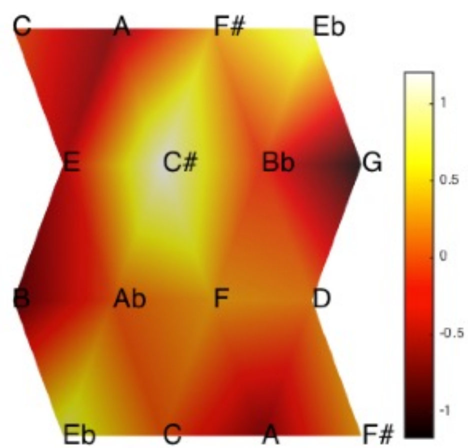


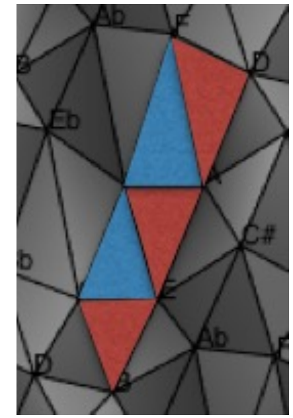
Locrian mode











Ionian mode

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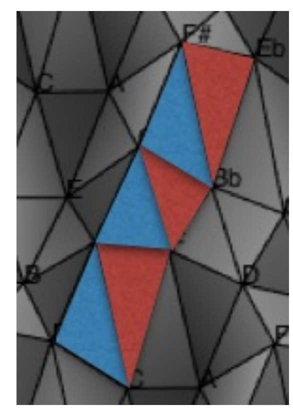


Locrian mode

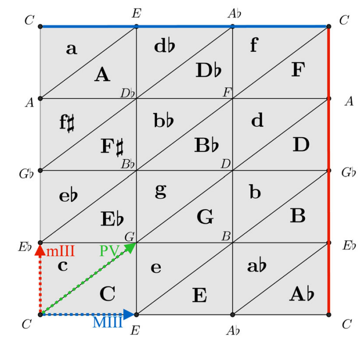
Persistent homology and music

Mattia Bergomi

Homological persistence in time series: an application to music classification

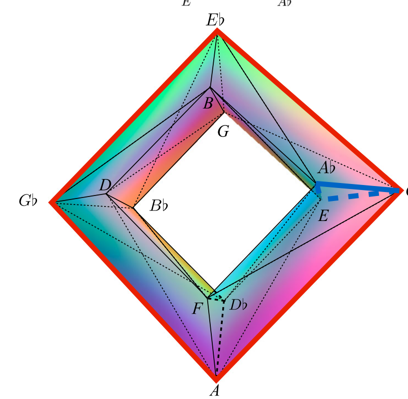
Mattia G. Bergomi [a*] and Adriano Baratè [b]

[a] *Veos Digital, Milan, Italy;* [b] *Music and Computer Science Laboratory, University of Milan, Milan, Italy*



(*Received 22 May 2019; accepted 2 June 2020*)

Meaningful low-dimensional representations of dynamical processes are essential to better understand the mechanisms underlying complex systems, from music composition to learning in both biological and artificial intelligence. We suggest to describe time-varying systems by considering the evolution of their geometrical and topological properties in time, by using a method based on persistent homology. In the static case, persistent homology allows one to provide a representation of a manifold paired with a continuous function as a collection of multisets of points and lines called persistence diagrams. The idea is to fingerprint the change of a variable-geometry space as a time series of persistence diagrams, and afterwards compare such time series by using dynamic time warping. As an application, we express some music features and their time dependency by updating the values of a function defined on a polyhedral surface, called the *Tonnetz*. Thereafter, we use this time-based representation to automatically classify three collections of compositions according to their style, and discuss the optimal time-granularity for the analysis of different musical genres.

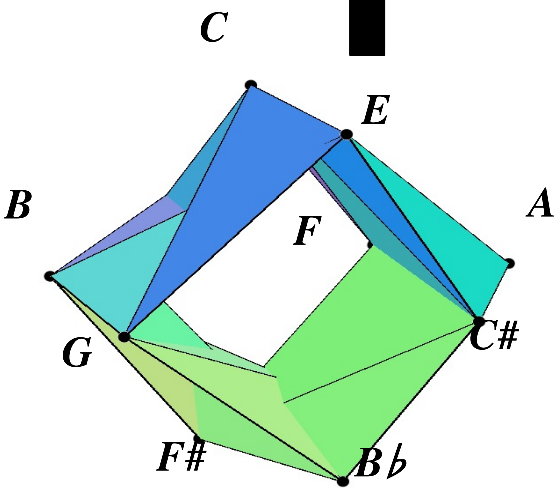


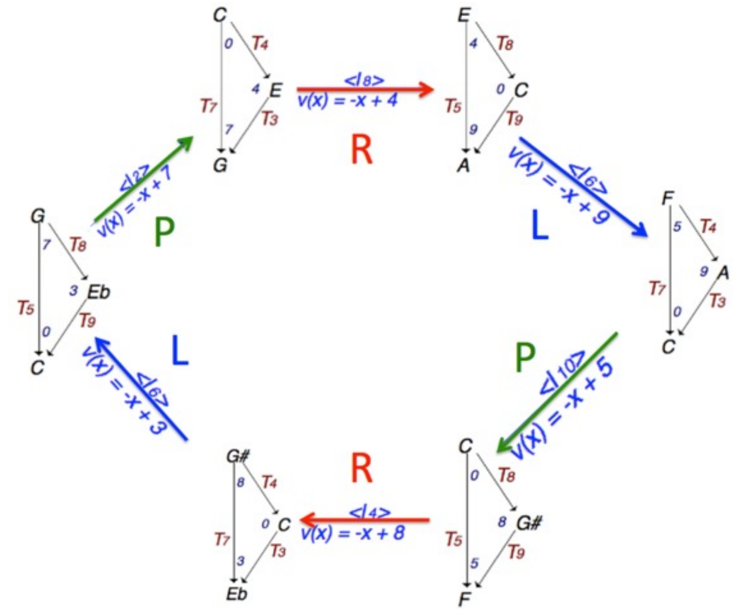
Keywords: *Tonnetz*; topology; time-series analysis; persistent homology; dynamic time warping; classification; style

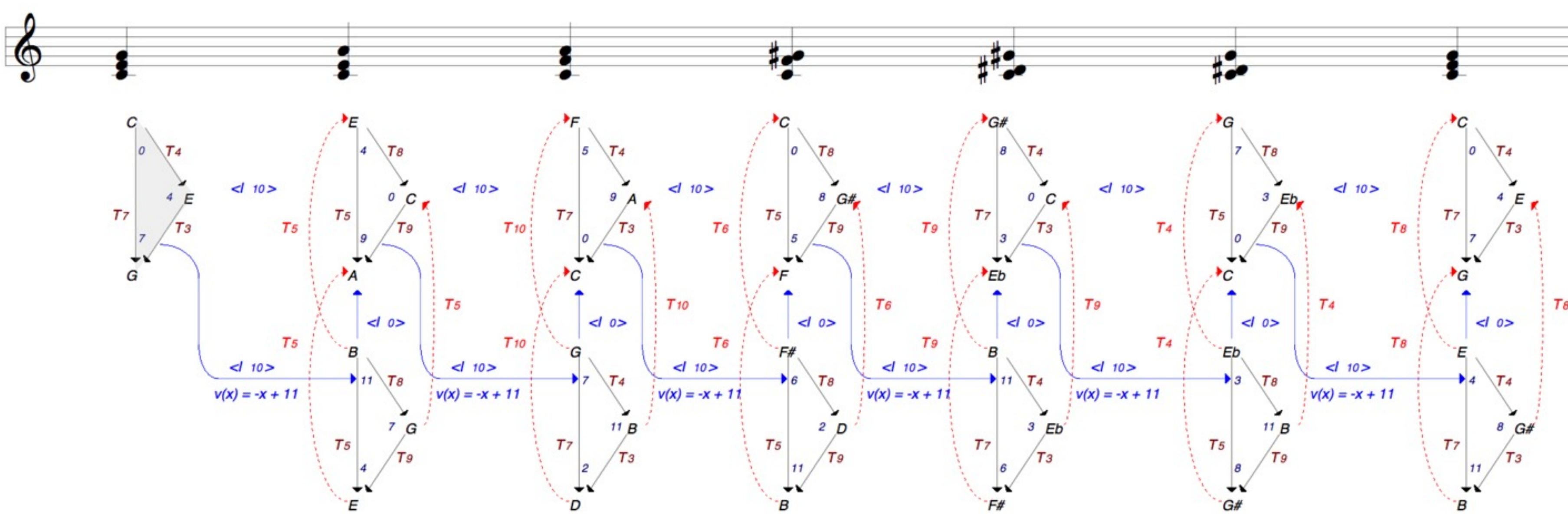
- **Mattia Bergomi, *Dynamical and topological tools for (modern) music analysis,* Sorbonne/LIM Milan, 2015.**
- **Mattia Bergomi, "Homological persistence in time series: an application to music classification", Journal of Mathematics and Music, Vol. 14, Nr. 2, pp. 204-221, 2020 (Special Issue on Geometry and Topology in Music; Guest Editors: M. Andreatta, E. Amiot, and J. Yust).**

➔ **Victoria Callet**, *Modélisation topologique de structures et processus musicaux*, ongoing PhD, Université de Strasbourg

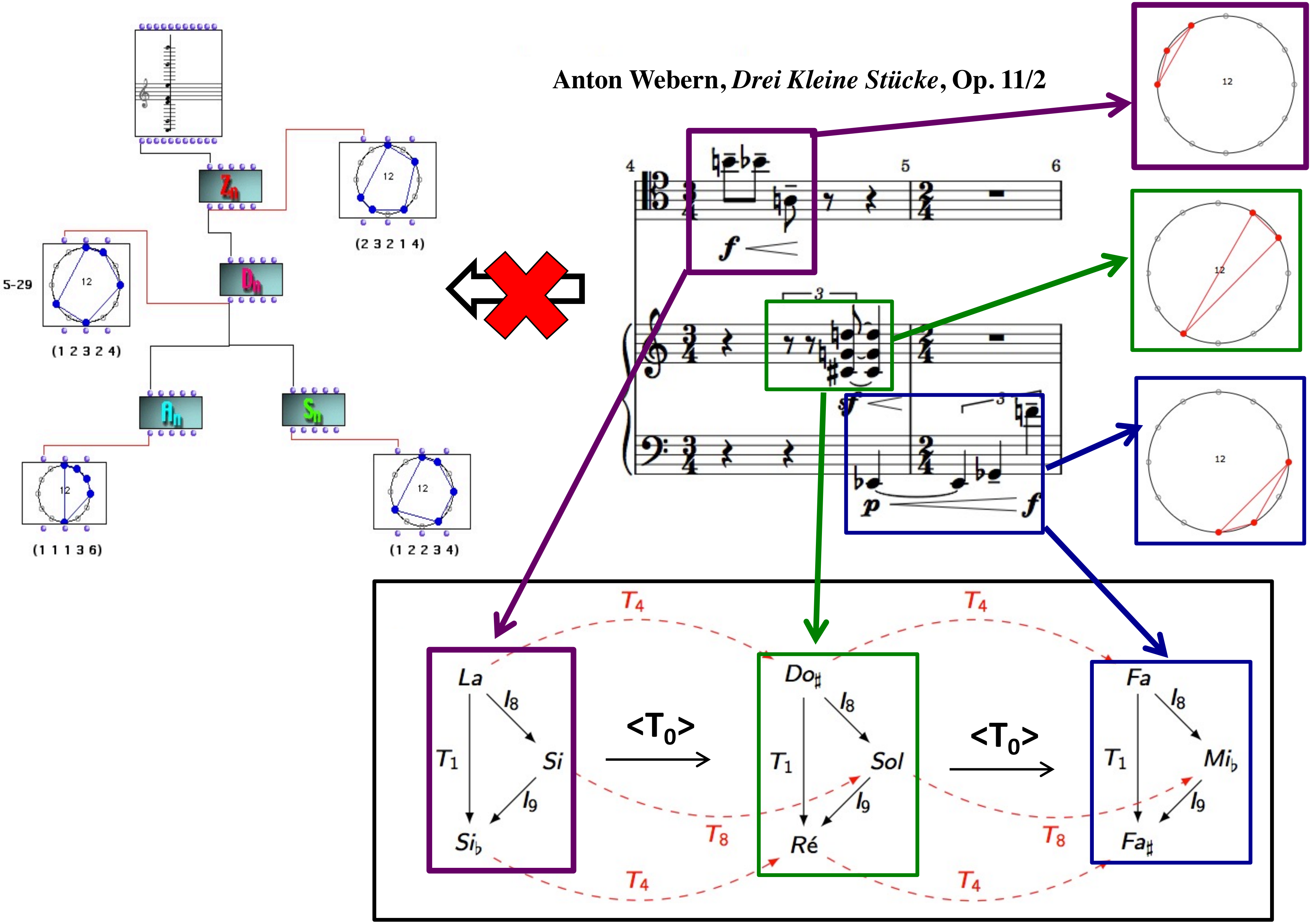
Mixing Algebra, Topology and Category Theory







Limitations of a paradigmatic action-based approach



A category-based approach of transformational analysis

$$
\begin{tikzcd}
C \arrow[r, "T_3"] \arrow[d, leftrightarrow, "I_2"'] & E\flat \arrow[d, "T_7M_7"] \\
D \arrow[r, "T_2"'] & E
\end{tikzcd}
\quad
\begin{tikzcd}
E\flat \arrow[r, "T_3"] \arrow[d, leftrightarrow, "I_2"'] & F\sharp \arrow[d, "T_7M_7"] \\
B \arrow[r, "T_2"'] & C\sharp
\end{tikzcd}
\quad
\begin{tikzcd}
F\sharp \arrow[r, "T_3"] \arrow[d, leftrightarrow, "I_2"'] & A \arrow[d, "T_7M_7"] \\
G\sharp \arrow[r, "T_2"'] & B\flat
\end{tikzcd}
\quad
\begin{tikzcd}
A \arrow[r, "T_3"] \arrow[d, leftrightarrow, "I_2"'] & C \arrow[d, "T_7M_7"] \\
F \arrow[r, "T_2"'] & G
\end{tikzcd}
$$

$$
\begin{tikzcd}
\square \arrow[rr, "F"] \arrow[dr, "R"'] & \arrow[d, phantom, "\phi"] & \square \arrow[dl, "S"] \\
& \square &
\end{tikzcd}
$$

Definition 1 *Let* $\mathbf{C}$ *be a category, and* S *a functor from* $\mathbf{C}$ *to the category* **Sets**. *Let* Δ *be a small category and* R *a functor from* Δ *to* **Sets**. *A PK-net of form* R *and of support* S *is a 4-tuple* (R, S, F, ϕ), *in which*

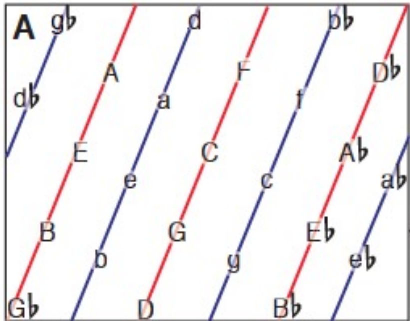
- F *is a functor from* Δ *to* $\mathbf{C}$,
- *and* ϕ *is a natural transformation from* R *to* SF.

The definition of a PK-net is summed up by the following diagram:

$$
\begin{tikzcd}
\Delta \arrow[rr, "F"] \arrow[dr, "R"'] & \arrow[d, phantom, "\phi"] & \mathbf{C} \arrow[dl, "S"] \\
& \mathbf{Sets} &
\end{tikzcd}
$$

Popoff A., M. Andreatta, A. Ehresmann, « A Categorical Generalization of Klumpenhouwer Networks », MCM 2015, Queen Mary University, Springer, p. 303-314

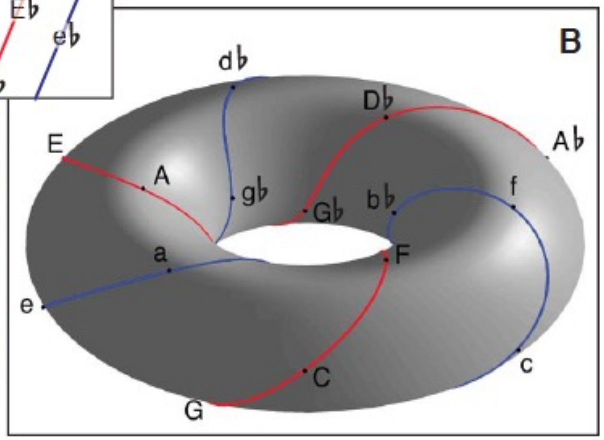
Neurosciences and *Mathemusical* Learning



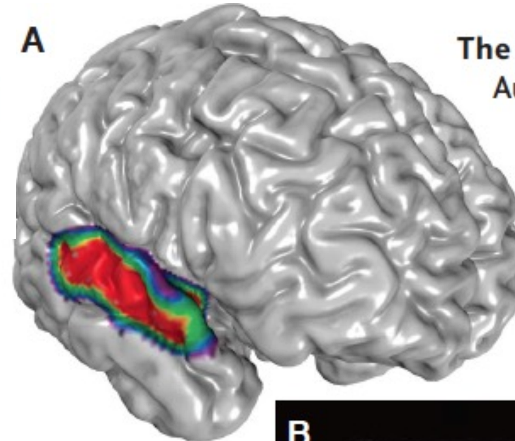
PERSPECTIVES: NEUROSCIENCE

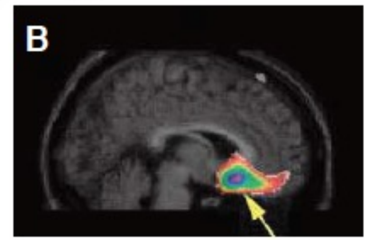
Mental Models and Musical Minds

Robert J. Zatorre and Carol L. Krumhansl

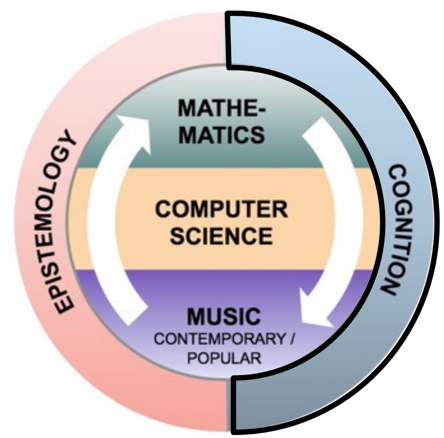


Mental key maps. (**A**) Unfolded version of the key map, with opposite edges to be considered matched. There is one circle of fifths for major keys (red) and one for minor keys (blue), each wrapping the torus three times. In this way, every major key is flanked by its relative minor on one side (for example, C major and a minor) and its parallel minor on the other (for example, C major and c minor). (**B**) Musical keys as points on the surface of a torus.

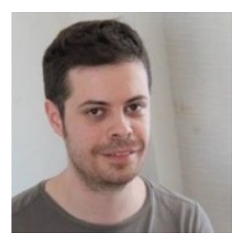




The sensation of music. (**A**) Auditory cortical areas in the superior temporal gyrus that respond to musical stimuli. Regions that are most strongly activated are shown in red. (**B**) Metabolic activity in the ventromedial region of the frontal lobe increases as a tonal stimulus becomes more consonant.



E. Bisesi



J.-L. Besada

C. Guichaoua

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THANK YOU FOR YOUR ATTENTION