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Embeddability of the combinohedron

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Abstract

Let $e_1,\dots,e_m$ be m different symbols, let $r_1 \geqslant \cdots \geqslant r_m$ be positive integers, and let $n=\sum_{i=1}^{m} r_i$. The *combinohedron*, denoted by $C(r_1,\dots,r_m)$, is the loopless graph whose vertices are the n-tuples in which the symbol e_i appears exactly r_i times, and where an edge joins two vertices if and only if one can be transformed into the other by interchanging two adjacent entries. The graph known as *permutohedron* is a particular case of the combinohedron. Here, we extend to the combinohedron some results on embeddability of the permutohedron.

MSC: 05C10; 05C45; 52C07

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1. Introduction

Let **N**, **Z**, and **R** denote the natural, the integer, and the real numbers, respectively. Let $m\in\mathbf{N}$ and $R=\{r_1,\dots,r_m\}\in\mathbf{N}^m$, with $r_i\in\mathbf{N}$, for $i=1,\dots,m$. Without loss of generality, we assume $r_1 \geqslant \cdots \geqslant r_m$. Finally, let $e_1,\dots,e_m$ be m different symbols. If $n=\sum_{i=1}^{m} r_i$, then the *combinohedron*, denoted by $C(r_1,\dots,r_m)$ or $C(R)$ for short, is the loopless graph whose vertex set $V(R)$ corresponds to all n-tuples in which the symbol e_i appears exactly r_i times, for $i=1,\dots,m$, and where an edge joins vertices $(u_1,\dots,u_n)\in V(R)$ and $(v_1,\dots,v_n)\in V(R)$ if and only if there exists an index k $(1\leqslant k<n)$ such that $u_k=v_{k+1}$, $u_{k+1}=v_k$, and $u_i=v_i$ for all $1\leqslant i\leqslant n$, $i\neq k,k+1$. Note that the graph known as the *permutohedron* (denoted by P_n) is a particular case of the combinohedron by setting $r_i=1$, for all i. There is a vast literature on the

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permutohedron [1,4,7–9]. In particular, Gaiha and Gupta [14] defined a permutohedron P_n as the convex hull of $\{(a_{\pi(1)},\ldots,a_{\pi(n)}) \mid \pi \text{ is a permutation of } (1,2,\ldots,n)\}$, where $a_1 < \cdots < a_n$ are integers. They proved that every vertex of P_n has $n-1$ adjacent vertices and gave a method for their determination. Further related results on the permutohedron can be found in [15–17,20,23].

As combinatorial objects, combinohedra are not only graphs, they are graded posets. More formally, the *multinomial lattice* $L(r_1,\ldots,r_m)$ consists of all words of length $n=\sum_{i=1}^m r_i$ on the alphabet $\{1,\ldots,m\}$ consisting of r_1 1's, r_2 2's, $\ldots, r_m$ m's. For words t and s, it is said that t *covers* s if s and t differ only on an adjacent pair, in numerical order in s, but in reverse order in t. This covering relation defines an order on $L(r_1,\ldots,r_m)$ in which $L(r_1,\ldots,r_m)$ becomes a lattice. The basic reference on multinomial lattices is a paper by Bennett and Birkhoff [3]. The *permutation lattice* results from the special case $L(1,\ldots,1)$. The following observation was kindly pointed out to us by one of the referees.

Remark 1.1. The multinomial lattice $L(r_1,\ldots,r_m)$ is isomorphic to the quotient of the usual lattice order of the Coxeter group of the permutations by the parabolic subgroup associated to the repeated elements (cf. for instance [18] for definitions).

The aim of this paper is to furnish some interesting explicit embeddings of the combinohedron $C(R)$ into the cubic and root lattices which are closely related with the *order dimension* of $L(R)$. Recall that the *order dimension* of a lattice L is defined as the minimal number of chains which admit an order-embedding of L into their direct product. One motivation for studying this dimension is that such a minimum set provides a compact representation of the relation, using relatively few bits (see [19] for further details). We shall show that $C(r_1,\ldots,r_m)$ can be embedded into the cubic lattice of dimension $\sum_{i=2}^m r_i$. From this, we can conclude that the order dimension of $L(r_1,\ldots,r_m)$ is less than or equal to $\sum_{i=2}^m r_i$. This is not surprising, as a stronger result due to Flath [13] states that the order dimension of $L(r_1,\ldots,r_m)$ is equal to $\sum_{i=2}^m r_i$. However, that proof uses notions of Ferrers dimension of incident structures as well as formal concept analysis and does not yield an explicit embedding.

The next section is devoted to the computation of a few invariants of $C(R)$, like its diameter, and the minimum and maximum degrees of its vertices. Then, in Section 3 we demonstrate how to embed this graph in the cubic and root lattices, well known for their connection with sphere close packing problems. For general terminology on graphs and lattices the reader is referred to [6,10], respectively.

2. Some facts

Let $N(R)=(r_1+\cdots+r_m)!/r_1!r_2!\cdots r_m!$ be the number of vertices in $V(R)$. Note that $N(R)=1$ if and only if $m=1$. Let $u=(u_1,\ldots,u_n)$ and $v=(v_1,\ldots,v_n)$ be vertices of $V(R)$. The *antipode* of u is $u^*=(u_1^*,\ldots,u_n^*)\in V(R)$, where $u_j^*=u_{n+1-j}$, for

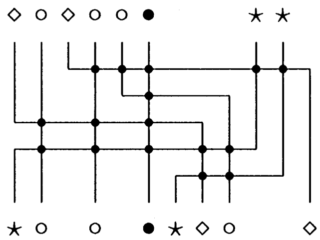
$j=1,\ldots,n$. Finally, let $d(u,v)$ be the number of edges in a minimum-length path from u to v. Barbut and Frey [2] described an algorithm to find $d(u,v)$ in case $R=\{1,\ldots,1\}$, which can be easily generalized for any set R of integers by means of the following procedure.

Step 1: Form a two-rowed array by writing u in the top row and v in the bottom row. Assume all symbols in the bottom row as non marked.

Step 2: For $i=1,\ldots,n$, draw a line to join u_i in the top row to v_j in the bottom row, whenever v_j is the most left non-marked symbol such that $v_j=u_i$ (no three lines may be concurrent, two lines may cross at most once, and no line can have ups and downs), and then mark v_j in the bottom row.

Step 3. Compute $d(u,v)$ as the number of crossings induced by the n lines drawn.

For example, if $u=(\diamondsuit,\circ,\diamondsuit,\circ,\circ,\bullet,\bigstar,\bigstar)$ and $v=(\bigstar,\circ,\circ,\bullet,\bigstar,\diamondsuit,\circ,\diamondsuit)$, then $d(u,v)=16$ arises by counting the number of crossings in the diagram given below.



The special vertex $(\underbrace{e_1,\ldots,e_1}_{r_1},\underbrace{e_2,\ldots,e_2}_{r_2},\ldots,\underbrace{e_m,\ldots,e_m}_{r_m})\in V(R)$ will be denoted by Θ. Also, the *diameter* of $C(R)$ is defined as $\varphi(R)=\max_{u,v\in V(R)}\{d(u,v)\}$. Notice that $\varphi(R)$ corresponds to the length of the maximal chain in the multinomial lattice $L(r_1,\ldots,r_m)$.

Proposition 2.1. *If* $m\geqslant 2$, *then* $\varphi(R)=d(\Theta,\Theta^*)=\sum_{1\leqslant i<j\leqslant m} r_ir_j$.

Proof. The proposition follows from an easy induction on m, and because for the vertex Θ and its antipode Θ^* the above procedure yields a maximum number of crossings. $\square$

The degree of a vertex $v\in V(R)$ is the number of edges incident to v. Let $\underline{\delta}(R)$ and $\bar{\delta}(R)$ denote the minimum and maximum degrees of vertices in $V(R)$, respectively.

Proposition 2.2. (a) $\underline{\delta}(R)=m-1$, *and*

$$\text{(b)}\quad \bar{\delta}(R)=\begin{cases} n-1 & \textit{if } 2r_1\leqslant n,\\ 2(n-r_1) & \textit{otherwise.}\end{cases}$$

Proof. (a) Trivially, a vertex with minimum number of neighbors $(m-1)$ is Θ.

(b) If $2r_1 \leqslant n$ then, in order to get a vertex with $n-1$ neighbors, just intercalate, from right to left, the r_1 symbols e_1 in the sequence (recall $r_1 \geqslant \cdots \geqslant r_m$)

$$(\underbrace{\underbrace{e_2,\ldots,e_m}_{m-1},\ldots,\underbrace{e_2,\ldots,e_m}_{m-1}}_{(m-1)r_m},\underbrace{\underbrace{e_2,\ldots,e_{m-1}}_{m-2},\ldots,\underbrace{e_2,\ldots,e_{m-1}}_{m-2}}_{(m-2)(r_{m-1}-r_m)},\ldots,\underbrace{\underbrace{e_2,e_3}_{2},\ldots,\underbrace{e_2,e_3}_{2}}_{2(r_3-r_4)},\underbrace{e_2,\ldots,e_2}_{r_2-r_3}).$$

If $2r_1 > n$ then $n-(2r_1-n) = 2(n-r_1)$ is clearly the degree of vertex

$$(\underbrace{e_1,e_2,\ldots,e_1,e_2}_{2r_2},\underbrace{e_1,e_3,\ldots,e_1,e_3}_{2r_3},\ldots,\underbrace{e_1,e_m,\ldots,e_1,e_m}_{2r_m},\underbrace{e_1,\ldots,e_1}_{2r_1-n}). \qquad \square$$

3. Embeddings

For given $a_1,\ldots,a_d \in \mathbf{N}$, the *cubic slice* $[\![a_1,\ldots,a_d]\!]$ is the graph whose vertices are the points in the cubic lattice $\mathbf{Z}^d = \{(x_1,\ldots,x_d)\colon x_i \in \mathbf{Z}\}$, satisfying $0 \leqslant x_i \leqslant a_i$, for $i=1,\ldots,d$, and two vertices are joined by an edge if and only if the Euclidean distance between them is equal to one. If, additionally, $\gamma \in \mathbf{N}$ is given, then the *root slice* $\langle\!\langle a_1,\ldots,a_d\rangle\!\rangle_\gamma$ is defined as the graph whose vertices are the points in the root lattice $\mathbf{A}^d = \{(x_1,\ldots,x_d) \in \mathbf{Z}^d\colon x_1+\cdots+x_d = \gamma\}$, satisfying $0 \leqslant x_i \leqslant a_i$, for $i=1,\ldots,d$, and an edge joins two vertices if and only if the Euclidean distance between them is equal to $\sqrt{2}$. Note that if γ is large enough then $\langle\!\langle a_1,\ldots,a_d\rangle\!\rangle_\gamma$ is the empty graph.

We say that a graph H is *embeddable* in a graph G, if H is isomorphic to some subgraph of G. Barbut and Frey [2] have shown that P_n (that is, $C(1,\ldots,1)$) is embeddable in $[\![1,2,\ldots,n-1]\!]$, for $n \geqslant 2$. Hereafter, we extend this result by proving that, if $m \geqslant 2$, one can find $t = n-r_1$ positive integers $a_1,\ldots,a_t$ such that $C(r_1,\ldots,r_m)$ is embeddable in the cubic slice $[\![a_1,\ldots,a_t]\!]$ as well as in the root slice $\langle\!\langle a_1,\ldots,a_t,\varphi(R)\rangle\!\rangle_{\varphi(R)}$, where $\varphi(R)$ is the diameter of $C(r_1,\ldots,r_m)$. Moreover, devoting particular attention to $C(r,\underbrace{1,\ldots,1}_{w})$ at the end of this section, we find that this combinohedron is embeddable in the root slice $\langle\!\langle rw,\underbrace{n-1,\ldots,n-1}_{w}\rangle\!\rangle_{w(r+n-1)/2}$. In all cases, our proofs provide the corresponding embeddings.

Theorem 3.1. *Let $r_1 \geqslant \cdots \geqslant r_m$ be positive integers with $m \geqslant 2$. Then the combinohedron $C(r_1,\ldots,r_m)$ is embeddable in the cubic slice*

$$\left[\!\!\left[\underbrace{r_1,\ldots,r_1}_{r_2},\underbrace{r_1+r_2,\ldots,r_1+r_2}_{r_3},\ldots,\underbrace{\sum_{i=1}^{m-1} r_i,\ldots,\sum_{i=1}^{m-1} r_i}_{r_m}\right]\!\!\right].$$

Let us introduce some notation before proving Theorem 3.1. For $i=1,\ldots,m$, distinguish the r_i replicas of the symbol e_i by $e_i^1,\ldots,e_i^{r_i}$. Let $n=\sum_{i=1}^m r_i$. Let $u=(u_1,\ldots,u_n)$ be a vertex of $C(r_1,\ldots,r_m)$, and, for $j=1,\ldots,r_i$, let $I_u(e_i^j)$ be the nonnegative integer ψ

such that $u_{\psi+1}=e_i^j$. In words, ψ is the number of symbols that, in u, are "at the left" of replica e_i^j. Notice that $0 \leqslant \psi \leqslant n-1$. Moreover, let $I'_u(e_i^j)$ be the number of symbols *not equal* to e_i^j that, in u, are "at the left" of replica e_i^j. Without loss of generality, assume $I_u(e_i^j) < I_u(e_i^k)$ and $I'_u(e_i^j) < I'_u(e_i^k)$ whenever $j<k$. Also, denote by $u\backslash e_i$ the $(n-r_i)$-tuple formed by deleting from u all the replicas of e_i.

Let $t=n-r_1$, and denote by $p(u)=(p_1^u,\dots,p_t^u)$ the point in $[\![a_1,\dots,a_t]\!]$ assigned to u in the proof. If v is also a vertex of $C(r_1,\dots,r_m)$, then $|p(u)-p(v)|=(|p_1^u-p_1^v|,\dots,|p_t^u-p_t^v|)$, and $\sigma(p(u),p(v))$ denotes the Euclidean distance between points $p(u)$ and $p(v)$. Let w_i be the binary t-vector with a digit 1 in the ith entry, and zeros elsewhere. Thus, $\sigma(p(u),p(v))=1$ if and only if $|p(u)-p(v)|=w_i$ for some i $(1\leqslant i\leqslant t)$. Finally, for vectors $p=(p_1,\dots,p_t)$ and $q=(q_1,\dots,q_{t'})$, their concatenation $(p_1,\dots,p_t,q_1,\dots,q_{t'})$ will be denoted as $p\oplus q$.

Proof of Theorem 3.1. A recursive, explicit assignment of each vertex u of $C(r_1,\dots,r_m)$ to a point of the cubic slice $[\![a_1,\dots,a_t]\!]$ will be proposed. It will then be enough to prove: (i) no two distinct vertices of $C(r_1,\dots,r_m)$ are assigned to the same point of the cubic slice, and (ii) if u and v are adjacent vertices in $C(r_1,\dots,r_m)$, then $\sigma(p(u),p(v))=1$. We proceed by induction on m.

Starting with $m=2$, assign to each vertex u of $C(r_1,r_2)$ the point $p(u)=(p_1^u,\dots,p_{r_2}^u)$, with $p_j^u=I'_u(e_2^j)$, for $j=1,\dots,r_2$. As $p(u)=p(v)$ for vertices u and v of $C(r_1,r_2)$ implies $u=v$, then each vertex of $C(r_1,r_2)$ has been assigned to a distinct point. Moreover, since $0\leqslant p_j^u\leqslant r_1$, for $j=1,\dots,r_2$, then $p(u)\in[\![\underbrace{r_1,\dots,r_1}_{r_2}]\!]$, and (i) follows.

Now, consider vertices u and v of $C(r_1,r_2)$ adjacent to one another, where v can be obtained from u by interchanging e_1^i and e_2^j, for some $i\in\{1,\dots,r_1\}$ and $j\in\{1,\dots,r_2\}$, i.e., $I_u(e_1^i)=I_v(e_2^j)$, and $I_u(e_2^j)=I_v(e_1^i)$. Without loss of generality, assume $I_u(e_1^i)+1=I_v(e_1^i)$. Since $I'_u(e_2^s)=I'_v(e_2^s)$, for $s=1,\dots,j-1,\ j+1,\dots,r_2$, and $I'_u(e_2^j)=I'_v(e_2^j)+1$, then $|p(u)-p(v)|=w_j$. Hence, $\sigma(p(u),p(v))=1$, and (ii) follows.

Assuming that the theorem holds for $m=k$, we deal with the case $m=k+1$. Assign to each vertex u of $C(r_1,\dots,r_{k+1})$ the point $p(u)=p(u\backslash e_{k+1})\oplus(I_u(e_{k+1}^1),I'_u(e_{k+1}^2),\dots,I'_u(e_{k+1}^{r_{k+1}}))$. Then, by an argument as that for case $m=2$, any two vertices of $C(r_1,\dots,r_{k+1})$ have been assigned to distinct points of the cubic slice

$$\left[\!\!\left[\underbrace{r_1,\dots,r_1}_{r_2},\underbrace{r_1+r_2,\dots,r_1+r_2}_{r_3},\dots,\underbrace{\sum_{i=1}^{k}r_i,\dots,\sum_{i=1}^{k}r_i}_{r_{k+1}}\right]\!\!\right].$$

Let u and v be adjacent vertices of $C(r_1,\dots,r_{k+1})$, where v can be obtained from u by interchanging e_f^ℓ and e_g^j, for some $\ell\in\{1,\dots,r_f\}$ and $j\in\{1,\dots,r_g\}$. Assume, without loss of generality, that $\xi=I_u(e_f^\ell)=I_v(e_g^j)$, $\xi+1=I_u(e_g^j)=I_v(e_f^\ell)$, and $1\leqslant f<g\leqslant k+1$. Let us consider two cases. If $g\leqslant k$ then $I_u(e_{k+1}^i)=I_v(e_{k+1}^i)$ for each $i\in\{1,\dots,r_{k+1}\}$, and the induction hypothesis yields $\sigma(p(u\backslash e_{k+1}),\ p(v\backslash e_{k+1}))=1$; hence $\sigma(p(u),p(v))=1$, and (ii) holds.

On the other hand, if $g=k+1$ then $\xi=I_u(e_f^\ell)=I_v(e_{k+1}^j)$ and $\xi+1=I_u(e_{k+1}^j)=I_v(e_f^\ell)$. Since $I'_u(e_{k+1}^s)=I'_v(e_{k+1}^s)$, for $s=1,\dots,j-1,j+1,\dots,r_{k+1}$ and $I'_u(e_{k+1}^j)=\vartheta$, $I'_v(e_{k+1}^j)=\vartheta-1$ for some $\vartheta\in\{1,\dots,\sum_{i=1}^k r_i\}$, then $|p(u)-p(v)|=w_i$ for some $i\in\{0,\dots,\sum_{j=1}^k r_j\}$, hence (ii) is proved, and the theorem follows. $\square$

Note that the sum of all the lenghts a_i, in the cubic slice proposed in Theorem 3.1, is equal to $\sum_{1\leqslant i<j\leqslant m} r_ir_j$ $(=\varphi(R)$, the length of the maximal chain in $L(r_1,\dots,r_n))$. Also note that the entire slice might not be needed. This theorem is illustrated in Fig. 1 where, for the sake of clarity, some labels have been omitted.

Now, turning to root lattices notice first that, for any given $\gamma\in\mathbf{Z}$, the expression

$$\mathbf{A}^{\zeta-1}=\{(x_1,\dots,x_\zeta)\in\mathbf{Z}^\zeta\colon x_1+\cdots+x_\zeta=\gamma\}$$

uses ζ coordinates to define a $(\zeta-1)$-dimensional root lattice: $\mathbf{A}^{\zeta-1}$ lies in the $(\zeta-1)$-dimensional hyperplane $\sum x_i=\gamma$ in $\mathbf{Z}^\zeta$. Of course, $\mathbf{A}^1$ and $\mathbf{A}^2$ are equivalent, respectively, to $\mathbf{Z}$ and to the familiar hexagonal lattice, a triangulation of $\mathbf{R}^2$, and are so called because the corresponding Voronoi cells are hexagons. Furthermore, $\mathbf{A}^3$ is equivalent to the face-centered cubic lattice (or *fcc*), portrayed in chemistry textbooks, and found in the structure of some crystals as well as in pyramidal arrangements of cannon balls.

Proposition 3.2. *Let $R=\{r_1,\dots,r_m\}$. Then $C(R)$ is embeddable in the root slice*

$$\langle\!\langle a_1,\dots,a_t,\varphi(R)\rangle\!\rangle_{\varphi(R)} = \left\langle\!\!\left\langle \underbrace{r_1,\dots,r_1}_{r_2},\underbrace{r_1+r_2,\dots,r_1+r_2}_{r_3},\dots,\underbrace{\sum_{i=1}^{m-1} r_i,\dots,\sum_{i=1}^{m-1} r_i}_{r_m},\ \varphi(R)\right\rangle\!\!\right\rangle_{\varphi(R)}.$$

Proof. Recall $t=n-r_1$. Assign vertex u of $C(R)$ to the $(t+1)$-dimensional point $(z_1,\dots,z_t,\varphi(R)-\sum_{i=1}^t z_i)$, where $(z_1,\dots,z_t)$ is precisely the point assigned to vertex u in the embedding provided by the proof of Theorem 3.1. From Proposition 2.1, $\varphi(R)=\sum_{1\leqslant i<j\leqslant m} r_ir_j$, which can be expressed as $\varphi(R)=\sum_{k=1}^t a_k$, and slightly extending the arguments in the proof of Theorem 3.1, it is easy to see that: (i) any two distinct vertices of $C(R)$ are assigned to different points of $\langle\!\langle a_1,\dots,a_t,\varphi(R)\rangle\!\rangle_{\varphi(R)}$, and (ii) if u and v are adjacent vertices of $C(R)$, then the distance between their corresponding points in $\langle\!\langle a_1,\dots,a_t,\varphi(R)\rangle\!\rangle_{\varphi(R)}$ is equal to $\sqrt{2}$. $\square$

Our next result proposes an alternate embedding of $C(r_1,\dots,r_m)$ in the root lattice, in the case $r_i=1$, for $i=2,\dots,m$.

Theorem 3.3. Let $r,w\in\mathbf{N}$. *Then the combinohedron $\mathscr{C}=C(r,\underbrace{1,\dots,1}_{w})$ is embeddable in the root slice $\langle\!\langle rw,\underbrace{n-1,\dots,n-1}_{w}\rangle\!\rangle_{w(r+n-1)/2}$.*

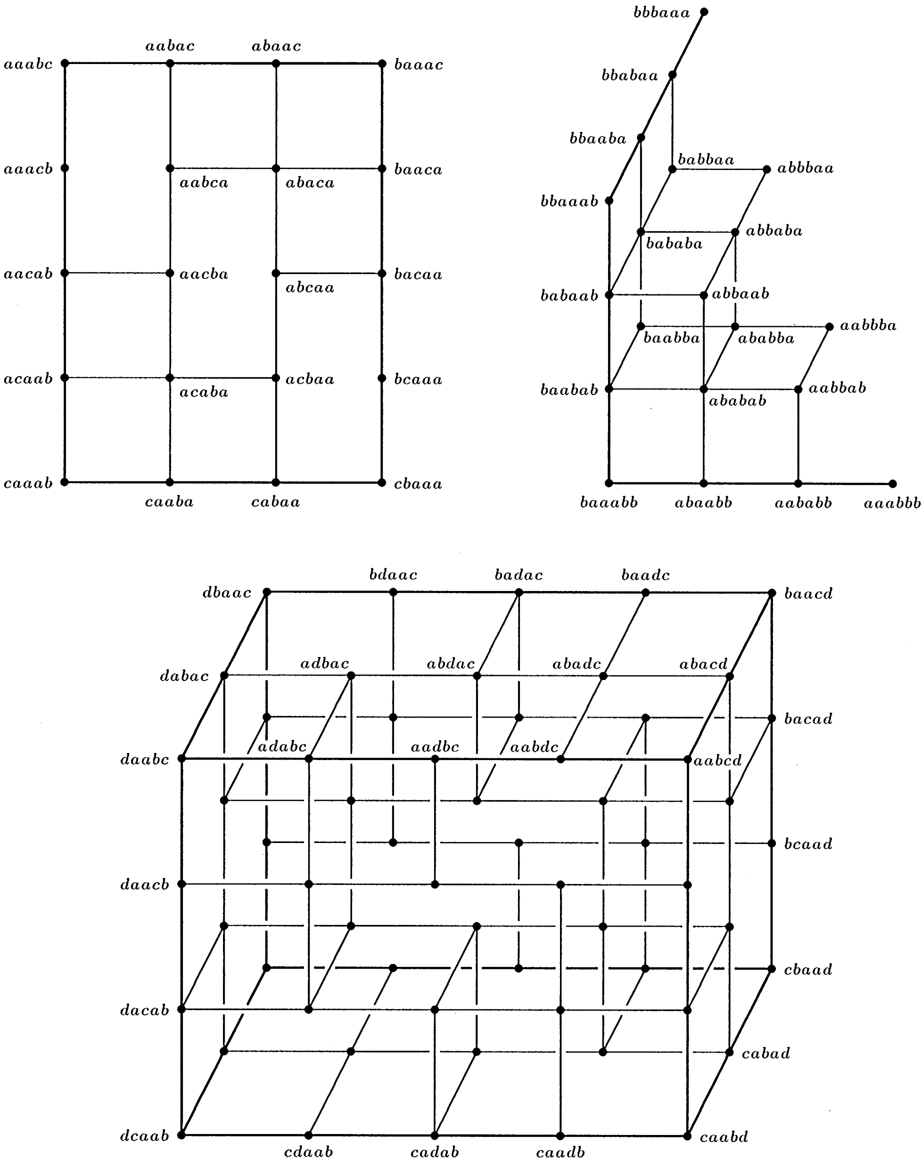


Fig. 1. Embeddings of $C(3,1,1)$, $C(3,3)$, and $C(2,1,1,1)$, in $[\![3,4]\!]$, $[\![3,3,3]\!]$, and $[\![2,3,4]\!]$, respectively.

Proof. Recall the notation used in the proof of Theorem 3.1. Assign to each vertex $u=(u_1,\ldots,u_n)$ of $\mathscr{C}$, the point $z(u)=(z_1,\ldots,z_m)$, where $z_i=I_u(e_i^1)$, for $i=2,\ldots,m$, and $z_1=\sum_{j=1}^{r}(I_u(e_1^j)+1-j)$.

Clearly, z_1 can also be expressed as $(r-r^2)/2+\sum_{j=1}^{r}I_u(e_1^j)$; hence $\sum_{i=1}^{m}z_i=n(n-1)/2+(r-r^2)/2$ which, after some algebraic manipulation, leads to $\sum_{i=1}^{m}z_i=$

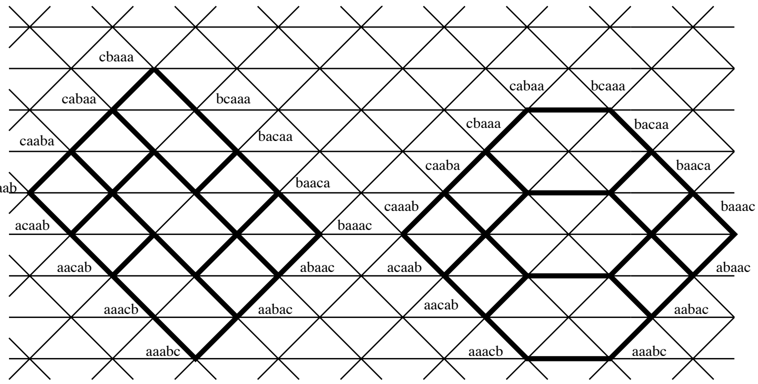


Fig. 2. The hexagonal lattice. Embedding of $C(3,1,1)$ in $\langle\!\langle 3,4,7\rangle\!\rangle_7$ and $\langle\!\langle 6,4,4\rangle\!\rangle_7$.

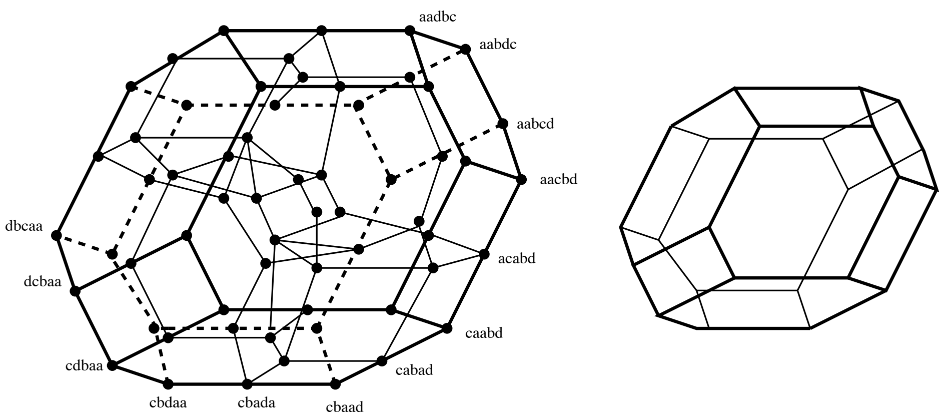


Fig. 3. At left, embedding of $C(2,1,1,1)$ in $\langle\!\langle 6,4,4,4\rangle\!\rangle_9$. At right, the convex hull arising from this embedding.

$w(n+r-1)/2$. Moreover, since $0 \leqslant z_1 \leqslant rw$, and $0 \leqslant z_i \leqslant n-1$, for $i=2,\dots,m$, we have $z(u) \in \langle\!\langle rw, \underbrace{n-1,\dots,n-1}_{w}\rangle\!\rangle_{w(r+n-1)/2}$.

Following an argument similar to that used in the proof of Theorem 3.1, we arrive at the following conclusions: (i) no two distinct vertices of $\mathscr{C}$ are assigned to the same point, and (ii) if u and v are adjacent vertices in $\mathscr{C}$, then the distance between $z(u)$ and $z(v)$ is equal to $\sqrt{2}$. Hence the theorem. $\square$

Fig. 2 shows embeddings of $C(3,1,1)$ in $\langle\!\langle 3,4,7\rangle\!\rangle_7$ and $\langle\!\langle 6,4,4\rangle\!\rangle_7$, as provided by Proposition 3.2 and Theorem 3.3, respectively (central labels have been omitted). Theorem 3.3 enables us to embed $C(2,1,1,1)$ in $\langle\!\langle 6,4,4,4\rangle\!\rangle_9$, as is shown in Fig. 3, left (most labels omitted). The reader is advised to contrast these embeddings with those in Fig. 1.

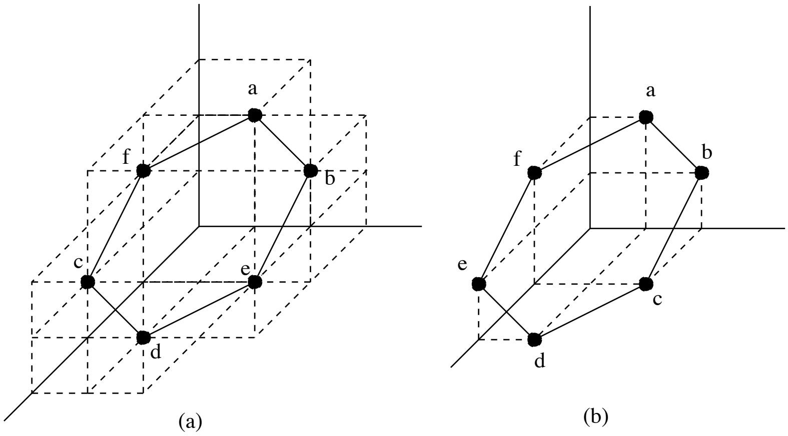


Fig. 4. (a) Embedding of $\prod_2$ and (b) embedding of $C(1,1,1)$ as proposed in Theorem 3.3.

As a polytope, the permutahedron $\prod_{d-1} \subseteq \mathbf{R}^d$ is defined as the convex hull of all vectors that are obtained by permuting the coordinates of the vector $(1,\ldots,d)$; see [11, Section 2.9, Fig. 3] and [22]. Note that $\prod_{d-1}$ is the special case when $a_i = i$ of P_d as defined by Gaiha and Gupta [14, see Section 1].

The vertices of $\prod_{d-1}$ can be identified with the permutations of the symmetric group $\mathscr{S}_d$ (by associating with $(x_1,\ldots,x_d)$ the permutation that maps $x_i \to i$) in such a way that two vertices of $\prod_{d-1}$ are connected by an edge if and only if the corresponding permutations differ by an adjacent transposition. In other words, $G(\prod_{d-1})$ is isomorphic to $C(\underbrace{1,\ldots,1}_{d})$, where $G(\prod_{d-1})$ denotes the graph defined on the set of vertices of $\prod_{d-1}$, with vertices u and v in $\prod_{d-1}$ being adjacent in $G(\prod_{d-1})$ if $[u,v]$ is a one-dimensional face of $\prod_{d-1}$.

Notice that for the case $r = 1$, the embedding of Theorem 3.3 assigns to a permutation σ of $(1,\ldots,n)$ the point $(\sigma^{-1}(1)-1, \sigma^{-1}(2)-1, \ldots, \sigma^{-1}(n)-1)$ which is just a translation of $\prod_n$.

Let us illustrate this with the case $n = 3$. The points of $\prod_2$ are: $a = (1,2,3)$, $b = (1,3,2)$, $c = (3,1,2)$, $d = (3,2,1)$, $e = (2,3,1)$ and $f = (2,1,3)$, see Fig. 4(a). The corresponding points, computed as in Theorem 3.3 are: $a = (0,1,2)$, $b = (0,2,1)$, $c = (2,0,1)$, $d = (2,1,0)$, $e = (1,2,0)$ and $f = (1,0,2)$, see Fig. 4(b).

A well-known property of $\prod_{d-1}$ is that it tiles $\mathbf{R}^{d-1}$, that is, there exists a polyhedral subdivision of $\mathbf{R}^{d-1}$ all of whose maximal cells are translates of $\prod_{d-1}$. This comes from the fact that $\prod_{d-1}$ is a *zonotope* with the *unimodularity property* (see [5,21] for further details).

We conjecture here that, for any combinohedron of the form $C(r,\underbrace{1,\ldots,1}_{w})$, the polyhedron defined by the convex hull of the set of points $z(u)$ as computed in the proof of Theorem 3.3 has the property to tile $\mathbf{R}^{m-1}$; see Fig. 5 for the case

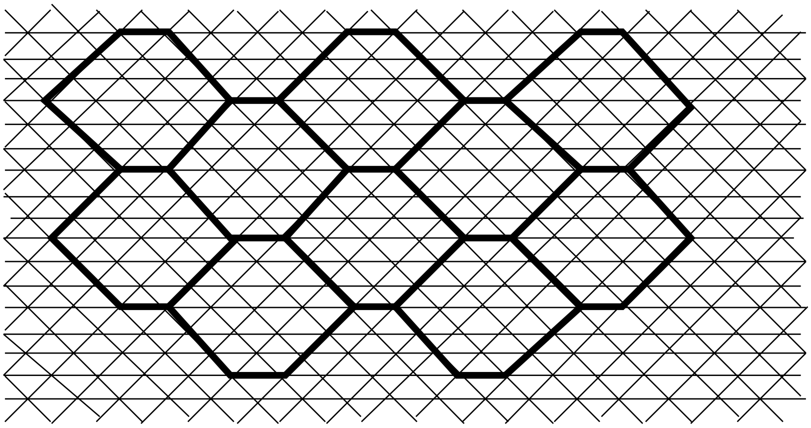


Fig. 5. Tiling of $\mathbf{R}^2$ with $C(3,1,1)$ shown in $\langle\!\langle 6,4,4 \rangle\!\rangle_7$.

$C(3,1,1)$, and remark that the convex hull of $C(2,1,1,1)$ is exactly $\prod_3$ and therefore it tiles $\mathbf{R}^3$.

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