THESE DE DOCTORAT DE L'UNIVERSITE PIERRE ET MARIE CURIE

Ecole doctorale ED 386: Sciences Mathématiques de Paris Centre

Doctorat réalisé à

l'Institut de Mathématiques de Jussieu - Paris Rive Gauche

en cotutelle avec

l'Institut de Recherche et Coordination en Acoustique/Musique

pour obtenir le grade de

DOCTEUR de l'UNIVERSITE PIERRE ET MARIE CURIE

Doctorat présenté et soutenu publiquement par

Grégoire GENUYS

le 23 octobre 2017

Sujet de la thèse :

Non-commutative homometric musical structures and chord distances in geometric pitch spaces

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Abstract

This work is the culmination of a doctoral thesis which merges two fields: mathematics and music. The full content is written in english, except for the introduction which is written in french, since the doctoral thesis has been done in a french university. We study two main topics: homometry and the distances between musical chords (a chord is a set of at least three notes heard simultaneously), each topic corresponding to a part of this work. At first homometry is a way to characterize a set from the differences between its elements. Two sets are *homometric* if they have the same set of differences. In a musical point of view homometry characterizes a melody, which is a set of notes played successively, from the musical intervals (the interval between two notes are the mathematical difference between these two notes) that compose it: two melodies are homometric if they have the same set of intervals (the same *interval content*). Concerning the distances between chords, the idea is to define a mathematical way to measure how far/how near two chords are one from the other. These two parts are independant and can be considered separately.

The first part deals with the problem of homometry. We begin with a presentation of the general concept of homometry and we focus on the Z-relation, which concerns homometry between musical notes ([2], [3], [10], [13]). It can be seen as an homometry between sets whose elements belong to $\mathbb{Z}/12\mathbb{Z}$ (the set of the integers modulo 12), which is mathematically a *commutative group*. The main purpose of our work is to consider homometry in non-commutative groups, which is a new and challenging problem. For this purpose we define an adequate framework for a general non-commutative group, and more specially in a semi-direct product. We chose to focus on semi-direct products because we have two interesting candidates that are often studied in musical theory: the dihedral group and the time-spans group. For each one we study homometry in details and we consider in subsection 2.3 a central notion that we call a *lift*. The lift is a way to switch from homometry in a semi-direct product to the homometry in one of the groups that form this semi-direct product. For instance in the case of the dihedral group, where we interpret sets as chord sequences, it allows us to build homometric chord sequences from homometric melodies formed by the roots [1] of the previous chords (Thm. 2.5, Cor. 2.2, Thm. 2.6). In the time-spans

[1] A chord being composed of at least three notes, the root of the chord is the note that gives its name to the chord. For example, the C-major chord is composed of the notes {C,E,G}, C being the root of the chord.

group we interpret sets as musical rhythms [2]. One of the main results we prove (Thm. 3.4, Thm. 3.5) is that in some cases, we can always find homometric lifts. At the end of this part, we generalize the process we used with these two groups in a general semi-direct product.

The second part deals with the quesion of distances between chords. We have two main objectives: studying the topological properties of the chord space defined by Tymoczko ([30]), and building from this space a way to measure distances between chords that do not have the same number of notes. First we present existing distances ([21], [26], [31]) using various approaches such as graphical ones (via the *Tonnetz*) or other ones based on the interval content. Then we define the chord space of Tymoczko and we prove some of its properties: in particular we show that it is a metric space. Consequently we can define a distance between chords that have the same number of notes (i.e. that lie in the same chord space of Tymoczko). We then focus on a way to measure distances between chords that do not have the same number of notes. For this purpose we define the space that contains all the chords (the mathematical union of the chord spaces of Tymoczko) and we define, using the Hausdorff distance, a distance d on this space (subsection 7.2). Finally we propose in section 8 a musical application of a modified version (for musical reasons) of d with two Bach's chorals. The idea is to draw a graph (called *graph of distances*) that shows the values of the distances between consecutive chords of a musical piece, and then to study the graphical properties of this graph. We show that the relevant information we can obtain come more from local patterns than from isolated values of the distance.

[2] Claude Abromont ([1]) defines a musical rhythm as *“ the result of the organisation of the durations, the timbres or the accents of a musical phrase ̈*

Remerciements/Acknowledgments

Je souhaite tout d'abord remercier mes encadrants Jean-Paul Allouche et Moreno Andreatta de m'avoir donné la chance de réaliser cette thèse et de m'avoir accompagné dans mon travail.

J'adresse de très chaleureux remerciements à Andrée Ehresmann et Alexandre Popoff pour leur collaboration et leur aide plus que précieuses tout au long de ma thèse : à Andrée tout particulièrement pour l'aspect catégorielle de ma recherche ainsi que pour sa présence efficace et régulière, et à Alexandre pour son apport important sur le sujet de l'homométrie, aussi bien concernant la facette théorique que computationnelle.

De chaleureux remerciements aussi à l'équipe de l'Ircam, en particulier à Léopold Crestel et Philippe Esling pour leur apport algorithmique qui m'a aidé lors de l'application musicale des distances d'accords, à Pierre Talbot et Clément Poncelet pour leurs partages et leur cohabitation, et à Sylvie Benoit pour ses échanges, son aide et sa cohabitation.

Je souhaite remercier vivement Maxime, aussi bien pour l'aide concrète qu'il m'a apportée que pour sa présence, ses idées, sa curiosité et son écoute tout au long de ma recherche. De même un grand merci à Guillemette pour sa présence, ses relectures et son soutien. Merci à Jill pour ses corrections, ses apports et ses échanges. Une grande reconnaissance enfin pour ma famille et pour mes l'ensemble de mes amis avec lesquels j'ai échangé à propos de mon travail.

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Introduction

Le travail qui suit est l'aboutissemement d'une thèse à la rencontre de deux grands chemins : celui des mathématiques et celui de la musique. Plutôt que “ rencontre entre chemins ”, nous devrions plus justement dire que ce travail appartient à un sentier à part entière issu de ces chemins. En effet les deux disciplines que sont les mathématiques et la musique ont une longue histoire en commun, qui remonte à Euler, à Pythagore et même au-delà. Les mathématiques ont été largement utilisées pour formaliser des notions musicales (d'un point de vue acoustique et théorique) et inversement la musique a fait éclore de nouveaux problèmes mathématiques. Au cours du dernier siècle beaucoup de compositeurs ont utilisé systématiquement des approches mathématiques pour élaborer leurs créations. Nous pouvons penser à Boulez qui a été influencé par les matématiques ou Xenakis qui a employé la théorie des ensembles et des probabilités pour construire son univers sonore. Réaliser une thèse dans une telle branche nécessite donc un juste équilibre entre un contenu pertinent musicalement et élaboré mathématiquement. Nous avons essayé ici d'avoir une approche multiple afin de nous engager sur plusieurs fronts. Nous trouverons ainsi des résultats théoriques en mathématiques et des applications musicales. De telles applications nécessitent des outils informatiques propres à la modélisation et l'analyse musicale. La création et le développement de ces outils constituent l'axe principal de la recherche scientifique au sein de l'équipe Représentations Musicales de l'IRCAM, aussi cela a été une très belle opportunité pour nous de réaliser cette thèse dans cette équipe. Nous avons choisi de rédiger ce texte en anglais pour la raison suivante : la communauté des chercheurs et compositeurs gravitant autour des problématiques mathématiques/musique est assez réduite, et l'ensemble des travaux proposés dans cette branche est majoritairement écrit en anglais. Aussi, dans un souci de diffusion au sein de cette communauté, nous avons jugé préférable de suivre cette tendance.

Deux grandes notions sont abordées ici : celle d'homométrie et celle de distance entre les accords musicaux (un accord tant une superposition d'au moins trois notes jouées simultanment), chacune constituant une partie de la thèse. L'homométrie permet de caractériser un ensemble à partir des valeurs des différen- ces entre ses éléments : deux ensembles sont homométriques s'ils possèdent le même ensemble de différences. D'un point de vue musical cela revient à caractériser par exemple une mélodie, c'est-a-dire un ensemble de notes de musique jouées successivement, à partir des intervalles musicaux (l'intervalle entre deux notes est l'écart – la différence mathématique – entre ces notes) qui composent cette mélodie. Deux mélodies sont homométriques si elles possèdent le même ensemble d'intervalles. La notion de distance entre les accords consiste quant à elle à créer un outil permettant d'évaluer la proximité entre deux accords quelconques (deux ensembles de notes). Ces deux grandes notions ont donc des similarités dans le sens où elles comparent des ensembles de notes entre eux. De plus, comme nous le verrons, le contenu intervallique (correspondant à l'ensemble des différences) à la base de l'homométrie peut être utilisé comme mesure de distance entre les accords. Toutefois les deux parties de notre travail sont largement indépendantes quant à leur contenu et peuvent être considérées séparément. Nous présentons maintenant ce qui motive les recherches que nous avons effectuées pour ces deux sujets.

Beaucoup de chercheurs se sont penchés sur le problème de l'homométrie, y compris dans d'autres branches que la recherche musicale. L'homométrie provient d'ailleurs, comme nous le verrons, de la cristallographie, et donc n'avait au départ aucun lien établi avec la musique. La plupart des travaux abordant le lien entre homométrie et musique concerne la Z-relation ([2], [3], [10], [13]). Deux ensembles de notes sont dit en Z-relation s'ils ont le même ensemble d'intervalles (donc les ensembles sont homométriques) et ne sont pas reliés par transposition ou inversion. Un ensemble de notes pouvant être vu comme un ensemble d'entiers modulo 12 (il y a douze tons chromatiques dans les gammes occidentales), la Z-relation est en fait une homométrie dans l'ensemble des entiers modulo 12 (qu'on généralisera aux entiers modulo n). Cet ensemble est mathématiquement un *groupe commutatif* (ou abélien), ce qui signifie que pour deux nombres a et b, $a \times b = b \times a$. Il existe aussi des études dans d'autres groupes, eux aussi commutatifs, comme les groupe des entiers relatifs. Ce qui motive notre travail est un aspect tout à fait nouveau qui n'a à notre connaissance jamais été étudié, même s'il est évoqué rapidement dans [20], à savoir l'homométrie dans des groupes non-commutatifs. Dans ce cas la précédente équation n'est plus vérifiée, et nous pouvons dès lors distinguer deux multiplications : une multiplication à droite et une multiplication à gauche. Cela conduit à deux comportements mathématiques différents pour l'homométrie. Nous avons choisi, pour cette première approche, un type spécial de groupes : les produits semi-directs, car ils sont la plupart du temps non-abéliens, et nous permettent d'avoir une approche assez générale de l'homométrie non-commutative. Rappelons qu'un produit semi-direct correspond au produit ensembliste de deux groupes, muni d'une loi de multiplication permettant de lui donner une structure de groupe. Mathématiquement notre ambition est donc satisfaite dans le sens où ce sujet ouvre de nouvelles voies d'exploration. D'autre part, pour une étude plus concrète nous avons déjà deux candidats (deux produits semi-directs) qui sont régulièrement étudiés en théorie de la musique, à savoir le groupe diédral (groupe des symétries d'un polygone régulier) et le groupe des time-spans, introduit par Lewin dans [16] (Lewin (1933 – 2003) a été un compositeur, critique et théoricien d'une grande importance dans la recherche musicale du XXe siècle. Il a notamment développé l'analyse transformationnelle, qui est une approche algébrique de la musique). Ainsi notre étude apportera aussi l'aspect musical recherché.

La première partie, qui traite ce sujet, est construite de la manière suivante. Nous présentons d'abord l'homométrie de manière générale et la Z-relation, puis nous expliquons ce que signifie l'homométrie dans un groupe non-commutatif et plus particluièrement dans un produit semi-direct. Cela nécessite de définir, comme nous l'avons vu, deux homométries : l'une pour la multiplication à droite et l'autre pour la multiplication à gauche. Elles seront appelées homométrie pour l'action à droite et homométrie pour l'action à gauche. Nous étudions ensuite dans le détail l'homométrie (pour l'action à droite ou à gauche) dans le groupe diédral, que nous interprétons comme une homométrie entre des ensembles composés d'accords musicaux (des triades majeures et mineures[3]). Cela nous amène à définir dans le paragraphe 2.3 une notion centrale : le lift. Celui-ci permet de créer un lien entre une homométrie non-commutative dans un produit semi-direct, et l'homométrie dans l'un des groupes formant ce produit semi-direct. Dans le cas du groupe diédral, cela nous amène à considérer les liens entre une homométrie avec des ensembles de triades majeures et mineures et des ensembles en Z-relation dont les éléments sont les toniques (c'est-à-dire les notes fondamentales) de ces triades. Un des résultats fondamentaux que nous montrons (Thm. 2.5, Cor. 2.2, Thm. 2.6) est que dans certains cas, on peut toujours obtenir des ensembles d'accords homométriques pour l'action à droite dans le groupe diédral, à partir d'ensembles de notes homométriques dans le groupe des entiers modulo n. Dans notre terminologie, nous dirons que l'on peut "lifter" à droite dans le groupe diédral des ensembles en Z-relation. D'un point de vue plus musical, nous pouvons obtenir un enchainement d'accords homométriques à droite, dont les ensembles formés par les toniques de ces accords sont en Z-relation.

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[3] Les triades sont des accords composés de trois sons : la fondamentale (celle qui donne son nom l'accord), la tierce, et la quinte. Seule la tierce permet de distinguer un accord mineur d'un accord majeur de mme fondamentale. Par exemple, l'accord de Do mineur est constitué des notes {Do, Mi bémol, Sol} tandis que l'accord de Do majeur est constitué des notes {Do, Mi, Sol}.

Nous étudions ensuite l'homométrie dans le groupe des time-spans, que nous voyons comme une homométrie entre des rythmes [4]. Nous regarderons plus précisément un sous-groupe du groupe des time-spans, plus adapté à l'étude des rythmes musicaux. Les résultats obtenus (Thm. 3.4, Thm. 3.5) sont relativement similaires à ceux obtenus pour le groupe diédral. Le procédé que nous utilisons pour étudier ces deux groupes est en fait presque le même – nous le généralisons d'ailleurs dans la section 4 à tout produit semi-direct ; il consiste en la décomposition d'un ensemble selon les projections sur les groupes formant le produit semi-direct. Il repose aussi sur l'utilisation de l'outil efficace qu'est la transformée de Fourier discrète, qui permet de traduire des propriétés musicales (homométrie, convolution, transposition) en des opérations algébriques simples.

En ce qui concerne la question des distances entre accords musicaux, il existe déjà des études sur ce sujet, utilisant différentes approches ([21], [26], [31]). Il n'y a en effet, *a priori*, pas de manière "naturelle" – dans le sens d'instinctive – de définir une distance entre des accords. L'approche que nous adoptons ici a été introduite par Tymoczko ([30]). Celui-ci construit un espace d'accords ayant un nombre de notes fixé, basé sur le principe du *voice-leading* [5] : deux accords sont proches s'ils sont reliés par une "petite" conduite de voix. Par exemple l'accord de Fa majeur (Fa-La-Do) est proche de l'accord de La mineur (La-Do-Mi) car ils sont reliés par un petit déplacement : le La reste La (aucun déplacement), le Do reste Do (aucun déplacement) et le Fa va vers le Mi (déplacement de 1 demi-ton [6]). En totalité ces deux accords sont donc distants de 1 demi-ton, ainsi ils seront proches dans l'espace de Tymoczko. Notre objectif est double : d'une part nous souhaitons étudier les aspects topologiques de cet espace (qui possède une structure mathématique intéressante) et donner certaines démonstrations de ses propriétés mathématiques, d'autre part nous voulons définir à partir de cette construction une mesure de distances entre des accords n'ayant pas le même nombre de notes. Ceci n'avait pas été fait dans ce cadre, même s'il existe toutefois

[4] Claude Abromont ([1]) définit le rythme musical comme *“ le résultat de l'organisation des durées, des timbres ou des accents successifs dans une phrase musicale ¨*

[5] Le *voice-leading* (conduite de voix) correspond aux mouvements indépendants des voix internes à un accord pour former un autre accord.

[6] Dans la musique occidentale, le demi-ton est le plus petit intervalle entre deux notes. Il correspond á l'écart entre une touche blanche et une touche noire sur un piano.

des distances d'autres natures qui le permettent, comme nous le verrons. Le fait de proposer une telle mesure est intéressant d'un point de vue musical dans le sens où cela nous permet de comparer entre eux des accords de toutes sortes : des triades (3 sons), des accords de septième (4 sons) ou plus, ou des accords non-classifiés avec un nombre de notes quelconque.

La deuxième partie, qui traite ce sujet, débute par une brève revue de quelques mesures de distances déjà existantes, impliquant notamment des graphes (*Tonnetz*) ou bien le contenu intervallique (mesure **Angle**). Suit une étude topologique de l'espace d'accords défini par Tymoczko ([31]), qui est mathématiquement un orbifold que l'on peut munir d'une métrique (distance) le rendant complet. L'espace de Tymoczko contient des accords ayant un nombre de notes fixé. Aussi pour atteindre notre deuxième objectif nous définissons l'espace total des accords comme la réunion des espaces de Tymoczko pour toutes les cardinalités : nous obtenons mathématiquement une réunion dénombrable d'espaces métriques, qui est un espace contenant tous les accords possibles, toutes cardinalités confondues. La question est alors de savoir s'il est possible de munir cet espace total d'une distance ayant un comportement raisonnable, c'est-a-dire respectant certaines propriétés attendues. Parmi ces propriétés, nous souhaitons par exemple que la distance entre deux accords ayant le même nombre de notes (mettons m) dans cet espace total soit égal à la distance entre ces accords dans l'espace naturel dans lequel ils vivent, à savoir l'espace de Tymoczko des accords à m notes. Nous souhaitons aussi que la topologie sur l'espace total corresponde à la topologie provenant de l'union des espaces de Tymoczko (signifiant qu'un ouvert sur l'espace total est réunion dénombrable d'ouverts sur les espaces de Tymoczko). Nous proposons dans le paragraphe 7.2 une mesure qui vérifie ces propriétés et utilise notamment la distance de Hausdorff permettant de calculer la distance entre des ensembles. Cette mesure est alors modifiée (paragraphe 7.3) légèrement pour des raisons musicales, et ensuite mise à l'épreuve lors d'une application musicale concrète avec deux chorals de Bach dans la section 8. L'idée est de produire, pour un choral (et pour un morceau de musique quelconque en général), ce que nous appelons un *graphique de distances* qui montre les valeurs des distances entre les accords successifs du choral au cours du temps. Cela permet une étude graphique du comportement harmonique de la pièce. Il ressort de ces graphiques que l'information pertinente ne semble pas résider dans les valeurs ponctuelles/isolées des distances, mais davantage dans des motifs (ensemble de distances consécutives) qui encodent certaines informations musicales intéressantes.

Afin de mettre en lumière ce qui est de l'ordre de notre contribution personnelle dans ce travail, nous donnons d'ores et déjà quelques détails : les preuves explicitement rédigées sont réalisées par l'auteur (sauf lorsque l'inverse est mentionné, quand nous pensons que la preuve clarifie le discours) ; la généralisation aux groupoides dans la section 4 (qui est basée toutefois sur un premier travail de John Mandereau ([18]) ainsi que la formalisation de l'homométrie dans les produits semi-directs (incluant l'ensemble des résultats dans le groupe diédral et le groupe des time-spans, notamment le théorème Thm. 2.5 et ses corollaires) est nouvelle. Dans la deuxième partie, nous donnons des preuves explicites à certains résultats présentés par Tymoczko (Thm. 6.1, Cor. 6.1). L'ensemble des résultats qui suivent sur les distances dans la section 7 (complétude, distances dans l'espace total des accords et applications musicales) est un apport personnel.

Nous vous souhaitons une bonne lecture !

Part I

Homometry in Some Semi-direct Products

Presentation

Homometry theory appeared in the 1930s and concerned at first crystallography. The question was to determine the structure of a crystal from an observation of it with X-rays: it was possible to measure the magnitude of the Fourier transform of this strucure, but not the phase of it. Thus the problem was to know if it was yet possible to find the complete structure of the crystal. This problem had applications in various fields, such as music: the question was then to characterize a set of notes (a chord, a melody) from the intervals that compose this set. In this context the concept of homometry is generally called *Z-relation*, and mathematically it requires to study the commutative group of the integers modulo n ($\mathbb{Z}/n\mathbb{Z}$).

The main objective of this part is to study homometry in a *non-commutative* framework. There are very few works that approach this problem, and as far as we know there is no work giving a detailed study with concrete groups. Mathematically, semi-direct products are good candidates since they are generally non-commutative, that is why we will focus on two specific semi-direct products that are well-known from the mathemusical community: the dihedral group and the time-spans group (introduced by Lewin in [16]).

In this part we first explain what is homometry, with a special attention given to the famous case of the integers modulo n and to the General Interval Systems (GIS). We propose a generalization in a more general context, namely when we use groupoids instead of groups. In the second section we study non-commutative homometry in the dihedral group – interpreted musically as homometry between sets of chords – and in the third section homometry in the time-spans group – interpreted musically as homometry between rhythms. Finally we give some results that generalize these two studies with general semi-direct products.

1 General Framework for Homometry in non-Commutative Groups

In this part we explain the mathematical concept of homometry. We first focus on homometry in $\mathbb{Z}_n$ (called *Z-relation* – cf. the work of A. Forte [9]) and more generally in a commutative group with the concept of Generalized Interval System (GIS) introduced by D. Lewin ([16]). Then we discuss the non-commutative case and we define homometry for the right and for the left actions of a non-commutative group on itself. As we said we will only study homometry in special non-commutative groups in this work, namely semi-direct products. That is why we recall the definition of a semi-direct product and clarify the notion of GIS in it. At the end of this part we generalize the concepts of GIS and homometry to groupoids, which is a more formal structure than groups. This last study uses category theory.

1.1 Presentation of the Concept of Homometry

At first the concept of homometry consisted in characterizing a spatial distribution of atoms with different weights (according to their size). The aim was to find a set of points in $\mathbb{R}^n$ for a given set of distances between these points.

In more mathematical terms, if $(G, +)$ is a commutative group and A a finite multiset (the fact that we consider multisets is equivalent to the attribution of integer weights to each element) of elements in G, we define the *multiset of differences of* A as

$$\Delta A := \{a - b \mid a, b \in A\}.$$

Remark that even if A is a set and not a multiset, ΔA is generally a multiset. The former question is then to reconstruct a multiset A from its multiset of differences ΔA.

We said that two multisets A and B are *homometric* if they have the same multiset of differences i.e. if

$$\Delta A = \Delta B.$$

Clearly translation and inversion are two mathematical operations that preserve homometry. More precisely, if we write $T_g A = A + \{g\}$ the translation of A by $g \in G$ and $IA = -A$ the inversion of A, we have

$$\begin{cases} \Delta(T_g A) = \Delta A & \text{(invariance under translation)} \\ \Delta(IA) = \Delta A & \text{(invariance under inversion).} \end{cases}$$

Hence we conclude that it is not possible to have unicity when reconstructing a set from its multisets of differences. That is why we have to be more accurate in our question: is it possible to reconstruct a multiset A from ΔA modulo translations and inversions. It leads us to the following definition.

Definition 1.1. *Let A and B be two multisets in G. We say that A and B are* non trivially *homometric if $\Delta A = \Delta B$ and if A and B are not linked by translation or inversion.*

We give an example of non-trivial homometric sets in $\mathbb{Z}$, drawn from [15].

Example 1.1. In $\mathbb{Z}$, the sets

$$A = \{0, 1, 3, 8, 9, 11, 12, 13, 15\}$$

and

$$B = \{0, 1, 3, 4, 5, 7, 12, 13, 15\}$$

are non-trivially homometric. In fact they have a specific property which is that if we define $U = \{6, 7, 9\}$ and $V = \{-6, 2, 6\}$ we have

$$A = U + V,\ B = U - V.$$

In general for two sets U and V in G, the sets $U + V$ and $U - V$ are always homometric (they can obviously be trivially homometric), as explained in [15]. It can be a way to build homometric sets in general. We also mention this property since we will use later the idea of decomposing homometric sets in subsets that have specific relationships.

When we study homometry in music, we are interested in the case $G = \mathbb{Z}/n\mathbb{Z}$. This is presented in the next subsection. From now on we will systematically use the notation $\mathbb{Z}_n$ to designate $\mathbb{Z}/n\mathbb{Z}$.

1.2 Homometry in $\mathbb{Z}_n$

When we study homometry in the fields of musical analysis or musical composition, the group we consider is $\mathbb{Z}_{12}$ and more generally $\mathbb{Z}_n$, for some $n \in \mathbb{N}$. In that case the denomination that we find in the literature is *Z-relation*. There are many works about Z-relation, written by mathematicians or musicians, with various approaches such as group theory, measure theory, distribution theory and Fourier transform, matricial or polynomial notations, etc. Here we present only some special features concerning Z-relation, that we will use later.

We first describe $\mathbb{Z}_n$ as the musical set of pitch classes, then we define and we give some important and general properties of Z-relation. Finally we focus on the discrete Fourier transform of *pitch classes* sets (p.c. sets) and its links with Z-relation.

$\mathbb{Z}_n$ as the Musical Set of Pitch Classes

In order to study atonal music which used completely new points of view concerning composition, some musicologists developed what they called the *set theory*. In this theory we do not consider notes anymore but equivalence classes of notes (called *pitch classes*) modulo two equivalence relations: octave transposition and enharmony. Consequently we identify for instance all the C of an instrument, and we identify also $B^\sharp$, C and $D^{\flat\flat}$. It is coherent since atonal music does not consider the tonal aspect of notes but only their pitch. We obtain finally 12 pitch classes, one for each note of the chromatic scale, that are refered to as numbers between 0 and 11 with the following convention: C is assigned the pitch class 0. Thus $C^\sharp = D^\flat$ is assigned 1 and B is assigned 11, and the twelve notes can be seen as elements of the group $\mathbb{Z}_{12}$. This group is often represented as a musical clock, as in Fig. 1.

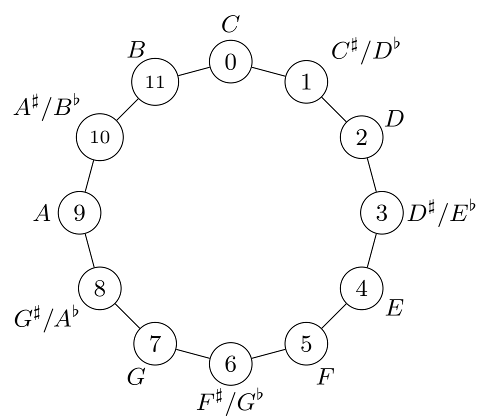


Fig. 1. $\mathbb{Z}_{12}$ as the musical clock.

With this formalism we can easily calculate the interval between two notes n_1 and n_2: it is equal to $n_2 - n_1 \bmod 12$, and it corresponds to the number of semitones (the number of notes on the keyboard) between these two notes. For instance the interval between C and G is $7-0=7$ (a fifth). Now if we consider a finite subset M_1 of notes – i.e. a subset in $\mathbb{Z}_{12}$ – we can calculate all the intervals between elements of M_1. We obtain the multiset of differences (or multiset of intervals)

$$\Delta M_1 = \{n_i - n_j \bmod 12 \mid n_i, n_j \in M_1\}.$$

Following Def. 1.1, we say that two sets of notes are *non-trivially homometric* if they have the same multiset of intervals and are not linked by translation or inversion. In fact two such sets are also said to be *Z-related*, as explained in the next paragraph.

This is for the case $\mathbb{Z}_{12}$, but quite systematically mathematicians study the general space $\mathbb{Z}_n$, for some $n \in \mathbb{N}$. It allows the musicians to work also with microtonal music (i.e. when we divide an octave in more than twelve notes), considering for instance $n = 24$ notes (quarter tones).

Z-relation

The concept of Z-relation was discussed by Hanson ([12]) and mainly by Allen Forte ([9]). It relies on two functions introduced by David Lewin in [16], namely the *interval function* and the *interval vector* of sets in $\mathbb{Z}_n$.

Definition 1.2. *The* interval function *between two sets A and B in $\mathbb{Z}_n$ is defined, for all $k \in \mathbb{Z}_n$, by*

$$\boldsymbol{ifunc}(A,B)(k) := \sharp\{(a,b) \in A \times B \mid b - a = k\}. \tag{1}$$

The interval vector *of A is*

$$\boldsymbol{iv}(A)(k) := \boldsymbol{ifunc}(A,A)(k) = \sharp\{(a,a') \in A \times A \mid a' - a = k\}. \tag{2}$$

Here $\sharp$ designates the cardinality of a set. As the interval vector is symmetric ($\mathbf{iv}(A)(k) = \mathbf{iv}(A)(12-k)$), we can use in practice the *interval content* $\mathbf{ic}(A)$ which is the set of first digits of the interval vector $(\mathbf{iv}(A)(1), ..., \mathbf{iv}(A)(n/2))$, except that the last one is divided by 2 if n is even. We give an example drawn from [13]. The interval vector of the set $A = \{0,1,3,4,7,9\}$ is

$$\mathbf{iv}(A) = (6,2,2,4,3,2,4,2,3,4,2,2)$$

and the interval content is

$$\mathbf{ic}(A) = (2,2,4,3,2,2).$$

Notice that in the interval vector we mention $\mathbf{iv}(A)(0) = 6$ which corresponds to the cardinality of A, but it not compulsory. We do not mention it in the interval content.

We can now define the Z-relation.

Definition 1.3. *Two sets A and B in $\mathbb{Z}_n$ are* Z-related *(or in Z-relation) if they have the same interval content: $\boldsymbol{ic}(A) = \boldsymbol{ic}(B)$.*

In fact we obtain via the interval content, or the interval vector, another expression of the multiset of differences/intervals ΔA, and it is easy to see that A and B are Z-related if and only if they are homometric. Hence translations and inversions are operations that preserve Z-relation, so we will naturally call non trivially Z-related sets, two sets that have the same interval content and are not linked by translation or inversion. We give a well-known and important example.

Example 1.2. The two sets $\{0, 1, 4, 6\}$ and $\{0, 1, 3, 7\}$ are non-trivially Z-related. Musically they correspond to $\{C, D^\flat, E, G^\flat\}$ and $\{C, D^\flat, E^\flat, G\}$. They have the same interval content: $(1, 1, 1, 1, 1, 1)$. We see from the interval content an interesting property for these sets: they contain all the intervals (they are often called "all interval tetrachords").

Example 1.3. A major triad and a minor triad are *trivially* Z-related since they have the same interval content $(0, 0, 1, 1, 1, 0)$, but are linked by inversion (and eventually transposition).

There are interesting theorems that we will not present here (a set and its complement are Z-related, the hexachord theorem, etc.). For more information the reader can refer to the article of F. Jedrzejewski and T. Johnson ([13]: general theorems, pumping lemma, enumeration of Z-related sets, group action for Z-relation), to J.S. Goyette's thesis ([10]) or to the master thesis of G. Lachaussée ([15]: group theory approach) for instance. There are also results that concern more our personal topic, in the work of E. Amiot ([2], [3]: Fourier transform for homometry) and the work of Mandereau and al. ([19], [20]:using the GIS).

In the next paragraph we use the discrete Fourier transform to characterize homometry in $\mathbb{Z}_n$.

Discrete Fourier Transform and Homometry

The results we present in this paragraph come mainly from the work of Amiot ([2],[3]). He proves the convenience of the discrete Fourier transform when using the functions **ifunc** and **iv** for subsets in $\mathbb{Z}_n$. Let us do a recap.

Let A be a subset in $\mathbb{Z}_n$, we define the *characteristic function* of A as

$$\mathbb{1}_A(t) = \begin{cases} 1, & \text{if } t \in A \\ 0, & \text{if } t \neq A. \end{cases}$$

Then for another subset B in $\mathbb{Z}_n$, **ifunc** appears as the convolution product of the characteristic functions of $-A$ and B:

$$\begin{aligned} \mathbf{ifunc}(A,B)(t) &= \sum_{k \in \mathbb{Z}_n} \mathbb{1}_A(k)\mathbb{1}_B(t+k) \\ &= \sum_{k \in \mathbb{Z}_n} \mathbb{1}_{-A}(k)\mathbb{1}_B(t-k) \\ &= \mathbb{1}_{-A} \star \mathbb{1}_B(t). \end{aligned}$$

Indeed $\mathbb{1}_A(k)\mathbb{1}_B(t+k) = 0$ except for $k \in A$ and $t+k \in B$ i.e. when there exists $(a,b) \in A \times B$ such that $a - b = t$.

Definition 1.4. *Following Lewin and Amiot, we define the Discrete Fourier Transform (DFT) of a subset $A \in \mathbb{Z}_n$ as the Fourier transform of its characteristic function, i.e. as a map from $\mathbb{Z}_n$ to $\mathbb{C}$ – an element of $\mathbb{C}^{\mathbb{Z}_n}$ – defined as follows:*

$$\mathcal{F}_A = \mathcal{F}(\mathbb{1}_A) : t \mapsto \sum_{k \in A} e^{-2i\pi kt/n}.$$

The values $\mathcal{F}_A(t)$, $t \in \mathbb{Z}_n$, are the Fourier coefficients.

Remark that$\mathbb{1}_A$ is a map from $\mathbb{Z}_n$ to $\mathbb{C}$, thus its Discrete Fourier Transform (DFT) is well defined for t mod n as $\mathcal{F}_A(t+n) = \mathcal{F}_A(t)$.

There are various interesting properties of the DFT of a characteristic function, but in the present work we will only mention some of them. For more details please refer to [2]. First $\mathcal{F}_A$ characterizes the subset A by the following identity (inverse Fourier transform):

$$\mathbb{1}_A(t) = \frac{1}{n} \sum_{k \in \mathbb{Z}_n} e^{+2i\pi kt/n} \mathcal{F}_A(k).$$

It means that $\mathcal{F}_A$ contains the same information as A, which allows us to switch from the DFT to the characteristic function.

Secondly we have some useful formulas. Here T_pA designates the translation of A by p, IA the inversion of A, and $\lambda_m A$ the multiplication of a set A by m, i.e. if $A = \{a_1, ..., a_N\}$, $\lambda_m A = \{ma_1, ..., ma_N\}$.

Proposition 1.1. *Let A be a subset in $\mathbb{Z}_n$ and $(p,t) \in \mathbb{Z}_n^2$. We have*

- $\mathcal{F}_{T_pA}(t) = e^{-2i\pi pt/n}\mathcal{F}_A(t)$;
- $\mathcal{F}_{I_0A}(t) = \overline{\mathcal{F}_A(t)}$;
- $\mathcal{F}_{\lambda_m A}(t) = \mathcal{F}_A(mt)$.

Proof. This result is not our contribution but we give the proof in order to accustom the reader to the use of the Fourier transform. Let $t \in \mathbb{Z}_n$. For the first point of the proposition we have

$$\mathcal{F}_{T_pA}(t) = \sum_{k \in T_pA} e^{-2i\pi kt/n} = \sum_{k-p \in A} e^{-2i\pi(k+p)t/n} = e^{-2i\pi pt/n}\mathcal{F}_A(t).$$

For the second point:

$$\mathcal{F}_{I_0A}(t) = \sum_{k \in I_0A} e^{-2i\pi kt/n} = \sum_{k \in A} e^{2i\pi kt/n} = \overline{\mathcal{F}_A(t)}.$$

For the last point:

$$\mathcal{F}_{\lambda_m A}(t) = \sum_{k \in A} e^{-2i\pi(mk)t/n} = \sum_{k \in A} e^{2i\pi k(mt)/n} = \mathcal{F}_A(mt).$$

□

Then if we apply the Fourier transform to **ifunc**(A, B) i.e. to the convolution product, we obtain for $t \in \mathbb{Z}_n$

$$\mathcal{F}(\mathbf{ifunc}(A,B))(t) = \overline{\mathcal{F}_A(t)}\mathcal{F}_B(t) \tag{3}$$

which is a very convenient formulation. As **iv**(A)=**ifunc**(A, A) we deduce from Eq. 3 that $\mathcal{F}(\mathbf{iv}(A)) = |\mathcal{F}_A|^2$ and using the fact that $|\mathcal{F}_A|$ holds all the information about **iv**(A) by inverse Fourier transform, we get the well-known characterization of homometry in $\mathbb{Z}_n$.

Proposition 1.2. *Let A and B be two subsets of $\mathbb{Z}_n$:*

$$A \text{ and } B \text{ are homometric} \iff |\mathcal{F}_A| = |\mathcal{F}_B|. \tag{4}$$

Hence as Amiot says in [3]: "The question of finding all (or at least some) homometric sets boils down to finding the *phase* of the Fourier coefficients, since their *magnitude* is common to all homometric [sets]. Hence it is often called the *Phase Retrieval Problem* in the literature." The Phase Retrieval Problem is a hard and famous problem in the world of homometry. This problem is raised in various works and also in a more general context called k-homometry. Here are some references about this question: the article of J. Mandereau and al. ([19]), the works of Rosenblatt ([24]), Jaming and Kolountzakis ([14]), Amiot ([3]), Pebody...

In the rest of this part we will use systematically the discrete Fourier transform to characterize non-commutative homometry and the above-mentioned properties will be useful. We will present later another way to study homometry as well, which is based on characteristic polynomials. This is explained in the paragraph 2.6.

Now we forget for a while homometry in $\mathbb{Z}_n$ and the Fourier transform. We will focus on the concept of GIS and define homometry in a non-commutative group, since our work concerns these groups.

1.3 GIS – Homometry for the Left and for the Right Actions

The notion of *Generalized Interval System* (GIS) was introduced by D. Lewin in [16]. It formalizes the concept of *interval* between two elements in an abstract musical space. As homometry is a characterization of a set by its interval content, a GIS is then a convenient structure to define homometry in abstract spaces (for instance in other spaces than groups). This is well described in [20] where is given an abstract formulation of homometry in a formal musical space S, by transferring Haar measure on a locally compact topological group onto S. In this section we use a similar approach: we recall the definition of a GIS, we use it to describe "right and left homometry" in a non-commutative group and particularly in a semi-direct product. This presentation is less general than in [20] but it gives the exact framework for the future explorations in the dihedral group and the time-spans group. Besides in section 4 we propose a further generalization of GIS to groupoids (GIGS).

Presentation of a GIS

We begin with the definition of a Generalized Interval System (GIS) given by Lewin in [16].

Definition 1.5. *A* Generalized Interval System (GIS) *is a triple* $(S, G, \boldsymbol{int})$*, where* S *is a set called* space *of the GIS,* G *a group called* interval group *of the GIS, and* $\boldsymbol{int}$*:* $S \times S \longrightarrow G$ *a map such that:*

(A) For every r, s, t *in* $S : \boldsymbol{int}(r, s)\boldsymbol{int}(s, t) = \boldsymbol{int}(r, t)$.

(B) For every s *in* S, i *in* $G,$ *there is a unique* t *in* S *such that* $\boldsymbol{int}(s, t) = i$.

As mentioned in [34], properties (A) and (B) are equivalent to defining a simply transitive right action of group G on S, such that for every s, t in S: $s.\mathbf{int}(s, t) = t$.

In every GIS, the musical parameter space S and the interval group G have the same cardinality. More precisely, condition (B) implies that the application

$$\begin{aligned} \mathbf{label} : S &\longrightarrow G \\ t &\longmapsto \mathbf{int}(s, t) \end{aligned}$$

is bijective. As said in [20], this bijection can be used for two purposes. The first possibility is using the interval group G itself as the space S. In this case, the group action that defines the GIS is right translation, i.e. for every s, t in G

$$\mathbf{int}(s, t) = s^{-1}t. \tag{5}$$

As a consequence, every group defines a canonical GIS associated with it via this group action. To avoid confusion that may arise from this identification of the interval group G and the space S, elements of the space will be called points, elements of the interval group will be called intervals.

The second possibility is using **label** for transferring some additional structure of the interval group G onto S. We know ([20]) that any locally compact Hausdorff topological group has a right Haar measure [7], which is uniquely defined up to a multiplicative constant. As far as we are concerned we will consider a group and a space with the discrete topology, the associated Haar measure is then simply the cardinality function. Writing $\mathcal{F}\text{in}(S)$ the set of finite subsets of S, we can thus define the interval function and the interval vector.

Definition 1.6. *The* interval function *between* A *and* B *in* $\mathcal{F}in(S)$ *is*

$$\begin{aligned} \boldsymbol{ifunc}(A, B) : G &\longrightarrow \mathbb{R}_+ \\ g &\longmapsto \sharp(B \cap Ag). \end{aligned}$$

[7] we do not give any detail of this construction since it is well explained in [20] and we will not use it later.

The interval vector *of $A \in \mathcal{F}in(S)$ is*

$$\boldsymbol{iv}(A) : G \longrightarrow \mathbb{R}_+$$
$$g \longmapsto \sharp(A \cap Ag).$$

Since $B \cap Ag = \{a \in A \mid \exists b \in B, \mathbf{int}(a,b) = g\}$, this definition is a straightforward generalization of the definition of interval vector in $\mathbb{Z}_n$. We can now define homometry in the context of a GIS.

Definition 1.7. *We say that two sets A and B in $\mathcal{F}in(S)$ are* homometric *if $\boldsymbol{iv}(A) = \boldsymbol{iv}(B)$.*

Notice that for the right action of a group on itself, when we define $\mathbf{int}(s,t) = s^{-1}t$, we favour the left translation over the right translation. Indeed for r in G we have

$$\mathbf{int}(rs, rt) = (rs)^{-1}(rt) = s^{-1}r^{-1}rt = \mathbf{int}(s,t)$$

but *a priori*

$$\mathbf{int}(sr, tr) = (sr)^{-1}(tr) = r^{-1}s^{-1}tr \neq \mathbf{int}(s,t).$$

Then the above definition of interval vector is invariant under left translation only. If we consider the action by left translation and we define

$$\widetilde{\mathbf{int}}(s,t) = ts^{-1}$$

then the interval vector is invariant under right translation only. If G is a commutative group, the two definitions are obviously equal and the situation is simpler. However, our main interest in the present work is precisely the non-commutative case.

Right and Left Homometry in a non-Commutative GIS

In a non-abelian group, we can then introduce two distinct definitions for the functions **ifunc** and **iv**, since there are two different definitions of interval, depending on if we consider right or left action. It leads to two different GIS. From now on we will write ${}^r\mathbf{int}$ for right interval and ${}^l\mathbf{int}$ for left interval. Then when we use the group itself as the space of the GIS we obtain ${}^r\mathbf{int}(s,t) = s^{-1}t$ and ${}^l\mathbf{int}(s,t) = ts^{-1}$. This way we define the right interval function (the one we defined in Def. 1.6) and the left interval function.

Definition 1.8. *The* right interval function *between* $A, B \in \mathcal{F}in(S)$ *is*

$$\begin{aligned} {}^{r}\boldsymbol{ifunc}(A,B) : G &\longrightarrow \mathbb{R}_{+} \\ g &\longmapsto \sharp(B \cap Ag). \end{aligned}$$

The left interval function *between* $A, B \in \mathcal{F}in(S)$ *is*

$$\begin{aligned} {}^{l}\boldsymbol{ifunc}(A,B) : G &\longrightarrow \mathbb{R}_{+} \\ g &\longmapsto \sharp(B \cap gA). \end{aligned}$$

We define similarly the right interval vector and the left interval vector, and finally homometry for the right action and homometry for the left action.

Definition 1.9. *We say that two sets A and B in $\mathcal{F}in(S)$ are* homometric for the right action *(resp.* homometric for the left action*) if* ${}^{r}\boldsymbol{iv}(A) = {}^{r}\boldsymbol{iv}(B)$ *(resp.* ${}^{l}\boldsymbol{iv}(A) = {}^{l}\boldsymbol{iv}(B)$*).*

In the rest of this work, we will often say *right-homometric* to mean *homometric for the right action*, and similarly for the left action.

As we saw, for the action of a group on itself, ${}^{r}\mathbf{int}$ is invariant under left translation. Hence ${}^{r}\mathbf{ifunc}$ and ${}^{r}\mathbf{iv}$ are invariant under left translation: in other words left translation preserves right homometry. Similarly ${}^{l}\mathbf{int}$ is invariant under right translation, hence right translation preserves left homometry. Then when two sets A and B are right (resp. left) translated one from the other we will say that A and B are *trivially* left (resp. right) homometric.

We saw that inversion is also an interval preserving operation for homometry in $\mathbb{Z}_n$. What about inversion in a GIS formed by a group acting on itself? We just recall that for $A \in \mathcal{F}\text{in}(G)$, the inversion IA is given by

$$IA = A^{-1} = \{a^{-1} \mid a \in A\}.$$

If we look at the right interval between s^{-1} and t^{-1}, we remark that

$${}^{r}\mathbf{int}(s^{-1}, t^{-1}) = st^{-1} = (ts^{-1})^{-1} = ({}^{l}\mathbf{int}(s,t))^{-1}. \tag{6}$$

Consequently I is hardly an interval preserving operation, that is why we will keep the denomination *trivial* for translations in general. However Eq. (6) leads us to an important proposition which is a personal contribution.

Proposition 1.3. *Let $A, B \in \mathcal{F}in(G)$. A is non-trivially right- (resp. left-) homometric with B if and only if IA is non-trivially left- (resp. right-) homometric with IB.*

Proof. 'A and B are right-homometric' means ${}^r\text{iv}(A) ={}^r \text{iv}(B)$. If we fix an element g in G we have

$$\begin{aligned}\sharp(A \cap Ag) &= \{a \in A \mid \exists a' \in A, {}^r\,\mathbf{int}(a, a') = g^{-1}\} \text{ (we use } \mathbf{iv}(A)(g) = \mathbf{iv}(A)(g^{-1})) \\ &= \{a \in A \mid \exists a' \in A, {}^r\,\mathbf{int}(a, a')^{-1} = g\} \\ &= \{a \in A \mid \exists a' \in A, {}^l\,\mathbf{int}(a^{-1}, a'^{-1}) = g\} \text{ (we use Eq. (6))} \\ &= \sharp(IA \cap gIA).\end{aligned}$$

Then we conlude ${}^l\text{iv}(IA)(g) ={}^l \text{iv}(IB)(g)$. It is true for all g in G then we conclude that IA is left-homometric with IB. It works similarly if A and B are left-homometric.

Besides, if A and B are right translated, there exists g in G such that $Ag = B$. We conclude immediately that $g^{-1}IA = IB$ i.e. IA and IB are left translated one from the other. It works similarly if A and B are left translated one from the other. Then we conclude the result concerning the triviality. □

This proposition is important, as we will see later when studying the dihedral group. It gives a simple way to switch from right-homometric sets to left-homometric sets, and produces interesting results such as: when there is no other interval preserving operation than translation, there is the same number of non trivial right-homometric sets than non trivial left-homometric sets.

As we said before we will focus in this work on the cases where G is a semi-direct product acting on itself. In the following paragraph we recall what is a semi-direct product and we give the formulas of right and left intervals.

Semi-direct Products: a Special non-Commutative Case

Let $(Z, +)$ and $(H, .)$ be two groups and $\phi : H \longrightarrow \mathcal{A}ut(Z)$ (for a definition of $\mathcal{A}ut(Z)$ please refer to Def. 2.3).

Definition 1.10. *The semi-direct product $G := Z \rtimes_\phi H$ of Z by H is the cartesian product between Z and H with the following multiplication law:*

$$(z_1, h_1)(z_2, h_2) = (z_1 + \phi(h_1)(z_2), h_1 h_2). \tag{7}$$

With this definition of multiplication, G is a group. The neutral is $(0_Z, 1_H)$, and the inverse of an element (z, h) is $(\phi(h^{-1})z^{-1}, h^{-1})$.

In general such a semi-direct G is not commutative even if the groups Z and H are commutative. Following the previous subsection we can then consider the two GISs associated to the right (resp. to the left) action of G on itself by right (resp. left) multiplication. If (z_1, h_1) acts on the left on (z_2, h_2) then we have the formula just above (Eq. 7), and if (z_1, h_1) acts on the right on (z_2, h_2) then we have

$$(z_2, h_2)(z_1, h_1) = (z_2 + \phi(h_2)(z_1), h_2 h_1).$$

Concerning the intervals we obtain

$$\begin{aligned} {}^r\mathbf{int}((z_1, h_1), (z_2, h_2)) &= (\phi(h_1^{-1})(z_1^{-1} + z_2), h_1^{-1} h_2), \\ {}^l\mathbf{int}((z_1, h_1), (z_2, h_2)) &= (z_2 + \phi(h_2 h_1^{-1})(z_1^{-1}), h_2 h_1^{-1}). \end{aligned}$$

We are now able to compute ${}^r\mathbf{iv}(A)$ and ${}^l\mathbf{iv}(A)$ for A in $\mathcal{F}\text{in}(G)$, following the previous definitions, and we can define right and left homometry in the context of semi-direct products.

Remark 1.1. Considering $\phi : H \longrightarrow \mathcal{A}ut(Z)$ is like considering a group action of H on Z with the action application

$$\begin{aligned} Z \times H &\longrightarrow Z \\ (z, h) &\longmapsto \phi(h)(z). \end{aligned}$$

With the multiplicative law given in Eq. 7 we can then turn the cartesian product $Z \times H$ into the semi-direct product $Z \rtimes_\phi H$. Thus it is possible to consider homometry at different levels: homometry in H as the acting group on Z (in the GIS $(Z, H, \mathbf{int})$ as we did in the previous subsection), homometry in Z as a group acting on itself, and homometry in the semi-direct product G as a group acting on itself. All these homometries can be considered for the right or for the left actions.

Consequently an interesting question is: What are the links between homometry in $G = Z \rtimes_\phi H$ and homometries in Z and H? The appraoching results concerning the dihedral group and the time-spans group will give some answers to this question, more precisely to the question of the links between homometry in G and homometry in Z.

In practice Z and G will always be chosen commutative, and ϕ will be the simple multiplication $\phi(h)(z) = hz = zh$. In that case the semi-direct product is simpy written $Z \rtimes H$ and the multiplicative law is

$$(z_1, h_1)(z_2, h_2) = (z_1 + h_1 z_2, h_1 h_2) \tag{8}$$

and the intervals are

$$^r\mathbf{int}((z_1,h_1),(z_2,h_2)) = (h_1^{-1}(z_2 - z_1), h_1^{-1}h_2), \tag{9}$$

$$^l\mathbf{int}((z_1,h_1),(z_2,h_2)) = (z_2 - h_2h_1^{-1}z_1, h_2h_1^{-1}). \tag{10}$$

The inverse of an element (z,h) is $(z,h)^{-1} = (h^{-1}z^{-1}, h^{-1})$.

We will study in the next sections two semi-direct products: the dihedral group $(\mathbb{Z}_n,+) \rtimes (\mathbb{Z}_2,.)$ and the time-spans group $(\mathbb{R}_+,+) \rtimes (\mathbb{R}_+^*,.)$. We will use the equations Eq. 9 to calculate explicitly the intervals. We generalize before the concept of GIS to groupoids, which is a more general structure than groups.

1.4 Generalization to Groupoids

In this subsection we generalize the framework proposed in [20] when G is a groupoid. A groupoid is a category in which every morphism is invertible, at some points it can be seen as a generalization of the notion of group. To define homometry in this abstract context we have to describe the action of a groupoid on a set S, which is the aim of the first part of this subsection, and to define the concept of interval. It leads us to the second part where we generalize the notion of GIS. Following Mandereau ([18]), we call such generalization a Generalized Interval Groupoid System. Then using some properties of measure theory, fibre spaces and the work of Seda ([27]) we explain how it is possible to build a G-invariant measure on S from measures on fibres, and to define the interval function and the interval vector.

Groupoid Action on a Set

First of all we recall the definition of a groupoid. There are different ways to define it, we choose the categorical point of view. In this perspective a groupoid is a category in which every morphism is an isomorphism.

Definition 1.11. *A groupoid $\mathcal{G}$ is given by:*

- *A set G_0 of* objects*;*
- *For each pair of objects x and y in G_0, a set $G(x,y)$ of* morphisms *from x to y (written $f : x \to y$);*
- *For every object x, a designated element id_x of $G(x,x)$;*

- *For each triple of objects x, y and z, a function of composition*

$$\begin{aligned} G(y,z) \times G(x,y) &\longrightarrow G(x,z) \\ (g,f) &\longmapsto gf; \end{aligned}$$

- *For each pair of objects x and y, a function* inverse

$$\begin{aligned} inv : G(x,y) &\longrightarrow G(y,x) \\ f &\longmapsto f^{-1}. \end{aligned}$$

Besides, for any $f : x \to y, g : y \to z$, and $h : z \to w$, we need

- $fid_x = f$ *and* $id_y f = f$;
- $(hg)f = h(gf)$;
- $ff^{-1} = id_y$ *and* $f^{-1}f = id_x$.

If f belongs to $G(x,y)$, x is called the source *and y is called the* target *of f. The set of morphisms is often written G. We note $\alpha : G \longrightarrow G_0$ and $\beta : G \longrightarrow G_0$, respectively the source function and the target function of a morphism.*

A groupoid action can be seen as a functor $F : G \longrightarrow \mathcal{S}et$ where $\mathcal{S}et$ is the category of sets. As a consequence for each object $x \in G_0$, $F(x)$ represents a set and we can define $S = \bigsqcup_{x \in G_0} F(x)$, the disjoint union of all the sets $F(x)$ for x in G_0. Thus we have a right groupoid action on S given by

$$\begin{aligned} G * S &\longrightarrow S \\ (g,s) &\longmapsto sg := F(g)(s), \end{aligned}$$

where $G * S = \{(g,s) \in G \times S \mid \alpha(g) = x \text{ and } s \in F(x)\}$. Following the work of C. Ehresmann ([6]), we will suppose also that for every $g \in G$, there is $s \in S$ such that sg is defined (which is equivalent to suppose that $F(x) \neq \emptyset$ for every object $x \in G_0$), and for every $s \in S$, there is $g \in G$ such that sg is defined.

Consequently we can define the application

$$\begin{aligned} \pi : S &\longrightarrow G_0 \\ s &\longmapsto x, \text{ such that } se \text{ is defined,} \end{aligned}$$

which is surjective thanks to our hypothesis and satisfies $\pi(sg) = \beta(g)$.

Remark 1.2. We saw that a group action on a group can be represented as a semi-direct product. In our context (G is a groupoid) the space $G * S$ can be

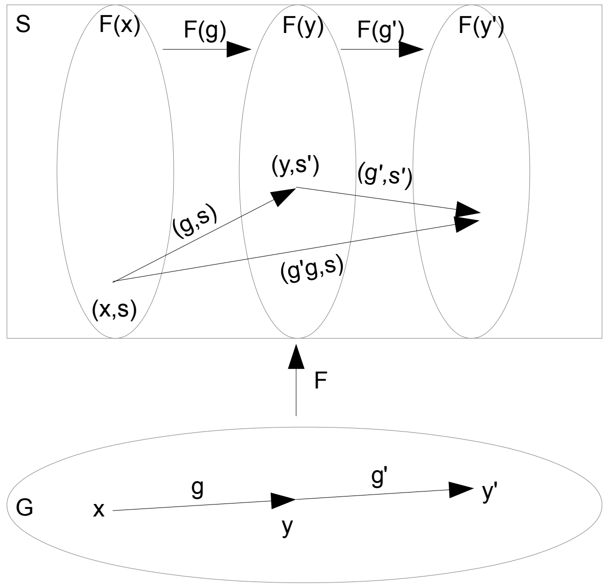


Fig. 2. The groupoid G acting on the set S.

defined similarly with a *partial* – it is not always defined – composition law between two elements given by

$$(g', s')(g, s) = (g'g, s) \text{ if } s' = gs. \tag{11}$$

In fact the space $G * S$ is a groupoid. An object is a pair (x, s) where $x \in G_0$ and $s \in F(x)$, and a morphism is a pair (g, s) where $g \in G(x, y)$ and $s \in F(x)$. Such a morphism is an isomorphism since G is a groupoid. The composition of morphisms is given by (11). This is represented on Fig. 2.

Remark 1.3. It is not mandatory to suppose that G is a groupoid to define our action. Indeed it can be done with a general category. However we will use later the fact that the morphisms are invertible.

The action can have additional properties. It is said to be:

- **transitive**: when for every pair $(s, s') \in S^2$, there exists $g \in G$ such that $sg = s'$. In that case the groupoid G is also *transitive* in the sense of category theory – meaning that for all x and y in G_0, we have $G(x, y) \neq \emptyset$.

- **free**: when for every pair $(g, g') \in G^2$ and $s \in S : sg = sg' \Longrightarrow g = g'$.

- **simply transitive**: when it is both free and transitive. For every pair $(s, s') \in S^2$, there exists a unique $g \in G$ such that $sg = s'$.

Remark 1.4. Defining a simply transitive action of G on S is equivalent to defining a representable functor $F : G \longrightarrow \mathcal{S}et$, where G is viewed as a category with one object A and invertible morphisms. Recall that a (covariant) representable functor is a functor naturally isomorphic to the $\mathcal{H}om(A, -)$ functor.

From the definition of the action and what we did before about group action, a natural definition for an *interval function* between two element s and s' in S would be

$$\mathbf{int}(s, s') = \{g \in G \mid sg = s'\} \tag{12}$$

implying that $\mathbf{int}(s, s')$ is a subset in G. A fundamental property that we may expect concerns the composition of intervals:

$$\mathbf{int}(s, s')\mathbf{int}(s', s'') = \mathbf{int}(s, s'') \tag{13}$$

for $(s, s', s'') \in S^3$. Recall that it is the property (A) in the definition of a GIS (cf. Def. 1.5). We saw that together with the property (B) it implied the simple transitivity of the group action. If we define **int** as in (12) and we do not impose any constraint on the groupoid action, Eq. (13) is not defined *a priori*. Indeed we can imagine a situation where $\mathbf{int}(s, s')$ and $\mathbf{int}(s', s'')$ are both equal to the empty set but $\mathbf{int}(s, s'')$ is not empty. In that case the product $\mathbf{int}(s, s')\mathbf{int}(s', s'')$ is not well defined and we can hardly give a correct meaning to Eq. (13). However notice that if for instance $\mathbf{int}(s', s'')$ and $\mathbf{int}(s, s'')$ are not empty, then $\mathbf{int}(s, s')$ is not empty since we just have to compose $g \in \mathbf{int}(s', s'')$ and g'^{-1} for $g' \in \mathbf{int}(s, s'')$ to obtain $g'^{-1}g \in \mathbf{int}(s, s')$.

If we suppose that the groupoid action is *transitive*, then $\mathbf{int}(s, s')$ is not the empty set for every s and s' in S, and defining

$$\mathbf{int}(r, s)\mathbf{int}(s, t) := \{gg' \mid (g, g') \in \mathbf{int}(r, s) \times \mathbf{int}(s, t)\} \tag{14}$$

formula (13) is meaningful but not true *a priori*. In fact we have in that case

$$\mathbf{int}(s, s')\mathbf{int}(s', s'') \subset \mathbf{int}(s, s'').$$

Then the question is: Is the converse inclusion true? The following proposition gives a affirmative answer.

Proposition 1.4. *Let G be a groupoid acting* transitively *on a non empty set S. With the above definitions for the interval function – Eq. (12) and (14) – we have*

$$\text{(i) } \mathbf{int}(r,s) \text{ is in bijection with } \mathbf{int}(t,u);$$
$$\text{(ii) } \mathbf{int}(r,s)\mathbf{int}(s,t) = \mathbf{int}(r,t).$$

Proof. (i) Let $h_{rt} \in \mathbf{int}(r,t)$ and $h_{su} \in \mathbf{int}(s,u)$ (such elements exist since the action is transitive). The following application is a bijection

$$\begin{aligned} \mathbf{int}(r,s) &\longrightarrow \mathbf{int}(t,u) \\ g &\longmapsto h_{rt}^{-1} g h_{su}. \end{aligned}$$

(ii) We saw that $\mathbf{int}(r,s)\mathbf{int}(s,t) \subseteq \mathbf{int}(r,t)$. For the other inclusion we consider $g \in \mathbf{int}(r,t)$ and $h \in \mathbf{int}(s,t)$. We have

$$g = (gh^{-1})h \in \mathbf{int}(r,s)\mathbf{int}(s,t)$$

thus we are done. □

Remark that we used the fact that G is a groupoid since we considered invertible morphims in the proof. This proposition gives an interesting result if we consider homometry in groupoids: if two finite sets are homometric, they have the same cardinality.

In the next paragraph we generalize the concept of GIS for groupoids. We will consider *simply transitive* actions since it allows us to define an equivalent definition of GIS. As simple transitivity implies transitivity all the previous equations are thus valid. Nevertheless we wanted, with this short study of transitive action, to show that it may be possible to work with more general hypothesis.

Generalized Interval Groupoid System (GIGS): Generalization of GIS

We define now a generalization of GIS for groupoids. This was done by Mandereau in [18].

Definition 1.12. *A* Generalized Interval Groupoid System (GIGS) *is a tuple $(S, G, \pi, \boldsymbol{int})$ where S is a non empty set called the* space *of the GIGS, G is a groupoid called* interval groupoid, *$\pi : S \longrightarrow G_0$ and $\boldsymbol{int} : S \times S \longrightarrow G$, such* *that:*

(i) π *is surjective;*

(ii) $\forall (s,t) \in S^2$*:* $\alpha(\boldsymbol{int}(s,t)) = \pi(s)$ *and* $\beta(\boldsymbol{int}(s,t)) = \pi(t)$*;*

(iii) $\forall (r,s,t) \in S^3$*:* $\boldsymbol{int}(r,s)\boldsymbol{int}(s,t) = \boldsymbol{int}(r,t)$*;*

(iv) $\forall s \in S, \forall g \in G(\pi(s),.)$*: there is a unique* $t \in S$ *such that* $\boldsymbol{int}(s,t) = g$*.*

Lemma 1.1. *The definition of a GIGS* $(S, G, \pi, \mathbf{int})$ *is equivalent to the definition of a simply transitive right action of a groupoid* G *on the non empty set* S*. More precisely:*

(1) If we are given a GIGS $(S, G, \pi, \mathbf{int})$ *then the action is defined by* $s.\mathbf{int}(s,t) = t$ *with* $s, t \in S$;

(2) If we have a simply transitive right action, then π *and* $\mathbf{int}$ *are defined as in the previous part, and* $(S, G, \pi, \mathbf{int})$ *is a GIGS.*

Proof. (1) We suppose that $(S, G, \pi, \mathbf{int})$ is a GIGS. π is surjective, then

$$S = \bigsqcup_{x \in G_0} \pi^{-1}(x).$$

We define the functor $F : G \longrightarrow \mathcal{S}et$, such that $F(x) = \pi^{-1}(x)$ for $x \in G_0$ (hence $S = \bigsqcup_{x \in G_0} F(x)$), and for $s \in S$ and $g \in G(\pi(s),.)$ we define

$$F(g)(s) := t$$

if $\mathbf{int}(s,t) = g$ (it is well defined thanks to (iv)). We can easily check the usual properties of a functor:

- thanks to (iii), $F(g') \circ F(g)(s) = F(gg')(s)$ when it's well defined;
- still with (iii), we obtain $\mathbf{int}(s,s)\mathbf{int}(s,s) = \mathbf{int}(s,s)$, so

$$\mathbf{int}(s,s) = \mathbf{int}(s,s)\mathbf{int}(s,s)^{-1}$$

  hence $\mathbf{int}(s,s)$ is the identity element of the object $\pi(s)$ (we use here (ii));
- $s.\mathbf{int}(s,s) = s$, in other words $F(e)(s) = s$ if e is the identity element of $\pi(s)$.

The functor defines a simply transitive action on S. (2) We suppose now that we have a functor $F : G \longrightarrow Set$ defining a simply transitive right action of G on $S = \bigsqcup_{x \in G_0} F(x)$. Then we define π and $\mathbf{int}$ as in the first part, and we see that (i), (ii) and (iii) are verified. For (iv), let $(s,g) \in S \times G(\pi(s),.)$ and $t = sg$.

We suppose that there exists $t' = sg$. Since the action is simply transitive, there exists a unique $g' \in G$ such that $tg' = t'$. As a consequence

$$sg = t' = tg' = (sg)g' = s(gg')$$

and in particular $sg = s(gg') = t'$. As the action is simply transitive we must have

$$g = gg' \Longrightarrow g' = g^{-1}g$$

then g' is a unit and $t = t'$. Then (iv) is verified and $(S, G, \pi, \mathbf{int})$ is a GIGS. $\square$

The condition (iv) of Def. 1.12 allows us to define the following bijection, for a fixed $s \in S$:

$$\begin{aligned} \mathbf{label} : S &\longrightarrow G(\pi(s), .) \\ t &\longmapsto \mathbf{int}(s, t). \end{aligned}$$

Then two comments are in order. We can not use the groupoid itself as the space of the GIGS (as we did in section 1.3 with groups) since they are not in bijection. For the same reason we can not transport a measure from G to S easily and consequently define an interval vector. For this purpose we will use, in the following paragraph, the work of Seda concerning measures on fibre spaces ([27]).

G-invariant Measures on S – Defining ifunc and iv

In [27] Seda describes how it is possible to build a measure on a fibre space. In this paragraph we will explain this procedure in the particular case of a GIGS. The aim is to define a measure on S that is invariant under the action of G. We assume that G is a Hausdorff and locally compact topological groupoid – meaning that G and the set of objects G_0 are Hausdorff and locally compact topological spaces and that all the structure maps (source, target, identity, composition, inverse) are continuous maps – and S is a locally compact topological space.

The global idea is not to propose an complete study about measures on fibre spaces (in particular we will not give all the details for the choice of Baire measures instead of Borel measures), it is more to give an example of such a construction and to explain the general process.

Definition 1.13. *Let $x \in G_0$, the subspace $S_x = \pi^{-1}(x)$ is called the* fibre of S over x.

Given $g \in G(x,y)$, let ϕ_g be the induced map $S_x \longrightarrow S_y$ defined by

$$\phi_g(s) = sg.$$

As G is a groupoid, g is invertible and then ϕ_g is an homeomorphism. Then all the fibres S_x, for $x \in G_0$, are in bijection.

Let us consider the simple case of the product $X \times Y$ with a measure μ on X, a measure ν on Y and $m = \mu \times \nu$ their product. Then every fibre $S_x \simeq \{x\} \times Y$ supports a copy μ_x of ν and the map $x \longmapsto \mu_x$ is a disintegration of m. It says that

$$m = \int_{x \in G_0} \mu_x \,\mathrm{d}\mu(x)$$

i.e. m is the integral of the family $\{\mu_x\}$ with respect to μ. This example leads Seda to propose generally that in studying measures m on S one might consider measures which are the integral with respect to a measure μ on X of a family of measures $\{\mu_x \mid x \in X\}$. Besides, it seems natural to seek measures that are invariant under the action of G on S.

Always following Seda, we will consider Baire measures i.e. measures defined on the σ-algebra of Baire sets (denoted by $B(S)$) generated by the compact G_δ sets, which have the form $\cap_{i\in\mathbb{N}} U_i$, each U_i being an open set of S. Baire measures give finite measure on every compact Baire set. Recall that a continuous function with compact support is always Baire measurable.

Definition 1.14. *For each $x \in G_0$ let μ_x be a Baire measure defined on S_x. The indexed familly of measures $\{\mu_x \mid x \in G_0\}$ is* non trivial *if each μ_x is non trivial, and it will be called* G-invariant *if each ϕ_g is measurable and measure preserving. More exactly if for every x and y in G_0, $g \in G(x,y)$ and $A \in B(S_x)$:*

$$\mu_x(A) = \mu_y(\phi_g(A)) = \mu_y(A.g). \tag{15}$$

In that case we may write $\mu_y = \phi_g(\mu_x)$, for μ_y is the image of μ_x under ϕ_g.

We would like to define a measure on the space S. First we give an interesting and easy construction of a G-invariant familly of measures on S. Consider a measure μ_z on a given fibre S_z, invariant under the action of the group $G(z,z)$. For each $x \in G_0$, let choose $r_x \in G(z,x)$ (it is possible since the action is simply transitive) and define $\mu_x := \phi_{r_x}(\mu_z)$. Let $g \in G(x,y)$ and A be a Baire subset of S_x. Let $h = r_x g r_y^{-1}$, it belongs to $G(z,z)$, then $\mu_z(B) = \mu_z(Bh)$ for every B

Baire subset of S_z, because μ_z is $G(z,z)$-invariant. We have

$$\begin{aligned}\mu_x(A) &= \mu_z(A.r_x^{-1})\\ &= \mu_z(A.r_x^{-1}h)\\ &= \mu_z(A.r_x^{-1}r_x g r_y^{-1})\\ &= \mu_z(A.g r_y^{-1})\\ &= \mu_y(A.g),\end{aligned}$$

thus the familly is G-invariant. With similar arguments we can prove easily that the definition of μ_x does not depend on the choice of $r_x \in G(z,x)$. In fact Seda says that the construction of any non-trivial G-invariant family can be achieved with this procedure.

We assume that such a $G(z,z)$-invariant measure exists on some S_z. The next step is to define a global measure on S. For this purpose we consider each measure μ_x as a Baire measure defined on S, with support contained in S_x, in accordance with the relation $\mu_x(A) := \mu_x(A \cap S_x)$ for A a Baire set in S.

Definition 1.15. *Let $\{\mu_x\}$ be a family of G-invariant Baire measures and μ be a Baire measure on G_0. We will say that the familly is* μ-integrable *if the function $M : G_0 \longrightarrow \mathbb{R}_+^*$ defined by $M(x) := \mu_x(A)$ is μ-measurable for every Baire set A in S and the set function m defined on S by*

$$m(A) = \int_{x \in G_0} \mu_x(A)\, \mathrm{d}\mu(x) \tag{16}$$

is a Baire measure.

We have then the following theorem which is an equivalent of the theorem of Tonelli.

Theorem 1.1. *For a Baire measure m on S the following statements are equivalent:*

a) m is G-invariant;

b) for any non-negative real valued continuous function f with compact support;

(i) $\displaystyle\int_S f dm = \int_{G_0}\int_{S_x} f d\mu_x d\mu$;

(ii) $\displaystyle\int_{S_x} (f \circ \phi_g) d\mu_x = \int_{S_y} f d\mu_y$ *for all* $(x,y) \in G_0^2, g \in G(x,y)$.

Then if there exists such G-invariant measure we can define the interval function and the interval vector.

Definition 1.16. *Let $A, B \in B(S)$, and m a Baire G-invariant measure on S, associated to the μ-integrable familly $\{\mu_x\}$. The* interval function *between A and B is*

$$\begin{aligned}\mathbf{ifunc}(A,B) : G &\longrightarrow \mathbb{R}_+ \\ g &\longmapsto m(B \cap Ag),\end{aligned}$$

where $Ag = \{ag \mid a \in A, \pi(a) = \alpha(g)\}$, $\pi(a) = \alpha(g)$ and

$$B \cap Ag = \{a \in A, \exists b \in B, \boldsymbol{int}(a,b) = g\}.$$

We can remark that if $g \in G(x,y)$, we have $Ag \subset S_y$ and then

$$\boldsymbol{ifunc}(A,B)(g) = m(B \cap Ag) = \mu_y(B \cap Ag).$$

The interval vector *of A is*

$$\begin{aligned}\mathbf{iv}(A) : G &\longrightarrow \mathbb{R}_+ \\ g &\longmapsto m(A \cap Ag).\end{aligned}$$

Remark 1.5. Those functions correspond to the right action of G on S. We can also consider the left action and define ${}^l\mathbf{ifunc}$ and ${}^r\mathbf{iv}$.

We can now define the concept of homometry in the context of groupoid action.

Definition 1.17. *Two sets $A, B \in B(S)$ are homometric for the right action (resp. for the left action) if ${}^r\boldsymbol{iv}(A) =^r \boldsymbol{iv}(B)$ (resp. ${}^l\boldsymbol{iv}(A) =^l \boldsymbol{iv}(B)$).*

Then the question would be: What are the properties verified by such homometric sets? It would be useful in practice to write $m(A) = \int_S \mathbb{1}_A dm$ in order to explain the interval function as a convolution product. But $\mathbb{1}_A$ is not Baire measurable *a priori*. In the present work we will not go any further in this direction. What we propose in the next paragraph – after the remark – is a generalization of the notion of interval in a semi-direct product with the action of a groupoid on a groupoid.

Remark 1.6. A group can be seen as a groupoid with one object z. Conversely there is a non-canonical equivalence between a given transitive groupoid G and a group. We fix an object $z \in G_0$ and for all object $x \in G_0$ a morphism $r_x \in G(z,x)$ (there exists such a morphism since the groupoid is transitive). Then we can associate to each morphism $g \in G(x,y)$ the morphism $f_g \in G(z,z)$ defined by

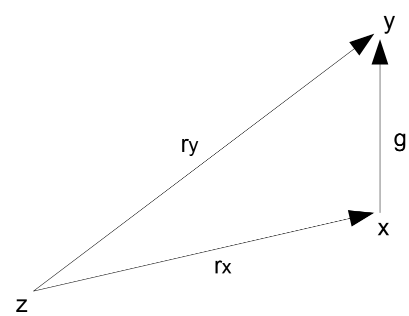


Fig. 3. Defining $f_g = r_y^{-1} g r_x$.

$f_g = r_y^{-1} g r_x$ (cf. Fig. 3). Then G is equivalent to the groupoid with the only element z and the morphisms f_g, i.e. is equivalent to a group.

We see that the (non-canonical) way we built a measure on a fibre space from a measure on one fibre (following Seda) was implicitly based on this equivalence. Indeed we fixed an object z in a fibre with a invariant measure and we chose morphisms r_x to build a measure on the whole space. In fact we used the equivalence between the groupoid G and the non-canonical associated groupoid with one element z and the morphisms f_g.

Groupoid Acting on a Groupoid

We saw in Rmk. 1.2 that the action of a groupoid G on a set S can be seen as a semi-direct product. Here we study another generalization of semi-direct products when the space S is a groupoid itself. We have to define more precisely the notion of groupoid action on a groupoid. For this purpose we use the work of Charles Ehresmann ([7]).

We begin with a functor $F : G \longrightarrow \mathcal{C}at$ where $\mathcal{C}at$ stands for the category of categories. The groupoid G acts on the groupoid H whose space of objects H_0 is the sum $\sum_{x \in G_0} F(x)$ of the sets $F(x)$ for x in G_0 (or the disjoint union of the sets $F(x)$).

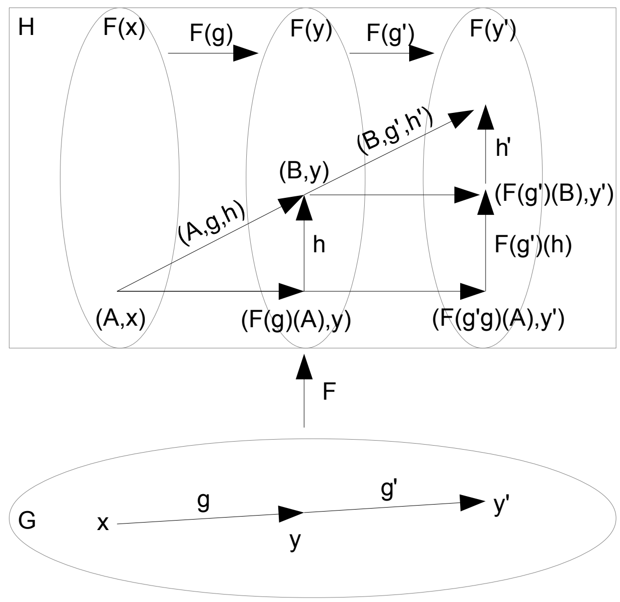


Fig. 4. The groupoid G acting on the groupoid H

To understand the action we describe the category $G * H$ that we obtain, similarly to the category $G * S$ of Rmk. 1.2. The objects are pairs (A, x) where $A \in F(x)$. The morphisms are

$$(A, g, h) : (A, x) \to (B, y),$$

where $g \in G(x, y)$ and $h : F(g)(A) \to B \in F(y)$. The composition of two morphisms (A, g, h) and (B, g', h') is given by

$$(B, g', h')(A, g, h) = (A, g'g, h'F(g')(h)) \tag{17}$$

if and only if $g'g$ exists and $h : F(g)(A) \to B$. This is reprensented on Fig. 4.

Hence there are two types of morphisms: the "vertical" morphisms (like $(F(g)(A), id_y, h) \approx h$) and the "horizontal" morphisms (like $(A, g, id) \approx (A, g)$). They generate all the morphisms since we have

$$(A, g, h) = (F(g)(A), id_y, h)(A, g, id).$$

In fact this construction does not require G to be a groupoid. If G is a category the definition of the action is still valid. However when G is a groupoid we have some simplifications since a morphism (A, g, h) can be represented by (g, h). Indeed if we are given a pair (g, h), then $g \in G(x, y)$ and the morphism h sends some object (A', y) on another object (B, y). G is a groupoid so g is invertible and $F(g)$ is also invertible (the inverse is $F(g^{-1})$). A is thus uniquely defined as $F(g)^{-1}(A')$ and $(g, h) \approx (A, g, h)$.

Then Eq. 17 is equivalent to

$$(g', h')(g, h) = (g'g, h'F(g')(h)). \tag{18}$$

This last formula is precisely the same formulation than the multiplicative law in a general semi-direct product (cf. Eq. 7). That is why with this composition law $G * H$ is a generalization of the notion of semi-direct product.

We will then naturally call *right interval* in $G * H$ between two elements (g_1, h_1) and (g_2, h_2) the set

$$^r\mathbf{int}((g_1, h_1), (g_2, h_2)) = \{(g, h) \in G * H \mid (g_1, h_1)(g, h) = (g_2, h_2)\}$$

and similarly for the left interval. There are obvious conditions for the existence of elements in such an interval. For instance in the above formula we need $g_1 g$ to be defined and $h : F(g)(A) \to A_1$ where A is the source of (g, h) and A_1 the source of (g_1, h_1).

It would be interesting to study the properties of this interval function, the conditions to verify Eq. 13 and other expected behaviours. As before we will not explore any further this direction. This idea is more to give a possible beginning for works on this topic.

Partial Conclusion

In this first section we presented the general concept of homometry, the Z-relation with p.c. sets in $\mathbb{Z}_n$ and its characterization with the discrete Fourier transform. We used then the GIS to define non-commutative homometry for the action of a group on a set, on more precisely for the action of a non-commutative group on itself, with the special case of semi-direct products. Finally we proposed a generalization of the definition of homometry given in [20] with locally compact groups acting on a set, to the case of groupoid. It led to a categorical study on groupoid action (a groupoid action on itself being seen as a generalization of semi-direct products) and finally to a definition of homometry in this context. In the next section we explore non-commutative homometry in the dihedral group.

2 Homometry in the Dihedral Group

The dihedral group D_n of order $2n$ is the group of symmetries of a regular polygon with n sides. Mathematically it is also the non-commutative semi-direct product of $(\mathbb{Z}_n, +)$ by $(\mathbb{Z}_2, .)$ in the sense of subsection 1.3. There are several reasons to study this group. As $\mathbb{Z}_n$ and $\mathbb{Z}_2$ are quite "simple" groups (in the sense that they are classical groups for mathematicians), $\mathbb{Z}_n \rtimes \mathbb{Z}_2$ is a good first candidate for us. It is also interesting because it has a deep link with very classical groups in music theory, namely the neo-Riemannian T/I and PLR-groups. More precisely these two groups are isomorphic to the dihedral group of order 24. We will show that D_{12} is also in bijection with the set S of major and minor triads. Finally, following Rmk. 1.1, we may find interesting relationships between homometry in D_n and homometry in $\mathbb{Z}_n$, the latter being already studied in the first section.

In this section we first define the dihedral group, we give some of its properties and we consider homometry in the GIS obtained by right and left actions of the dihedral group on itself. We give a musical interpretation in the case $n = 12$, building a bridge with the neo-Riemannian groups. Then we give a characterization of homometry for the right and the left actions using the discrete Fourier transform, and important specificities of homometry for the right action. We present a complete enumeration of homometric sets in the dihedral group for small values of n and small cardinalities of sets. Finally we introduce characteristic polynomials which is a convenient way to manipulate homometric sets in practice.

2.1 D_n as a Semi-direct Product

We begin with a presentation of the dihedral group of order $2n$ with various definitions, and we give some examples of homometric sets. Then we study the question of trivial homometry: What does *trivial* mean for homometry in D_n? The result is that we did not find any other interval preserving operation except for the translations.

The Dihedral Group D_n: Presentation and Homometry

A regular polygon with n sides has $2n$ different symmetries: n rotational symmetries and n reflection symmetries. The associated rotations and reflections make up the dihedral group D_n which then contains $2n$ elements. Usually we consider $n > 3$. If n is odd, each axis of symmetry connects the midpoint of one side to the opposite vertex. If n is even, there are $n/2$ axes of symmetry connecting the midpoints of opposite sides and $n/2$ axes of symmetry connecting opposite vertices. In either case, there are n axes of symmetry and $2n$ elements in the symmetry group. Reflecting in one axis of symmetry followed by reflecting in another axis of symmetry produces a rotation through twice the angle between the axes.

As we said, the dihedral group can be described as a semi-direct product.

Definition 2.1. *The dihedral group D_n of order $2n$ is the semi-direct product $(\mathbb{Z}_n, +) \rtimes (\mathbb{Z}_2, \times)$, where $\mathbb{Z}_2 = \{\pm 1\}$. Its elements are then pairs (k, ϵ) where $k \in \mathbb{Z}_n$ and $\epsilon = \pm 1$. The multiplicative law is given by*

$$(k, \epsilon)(l, \eta) = (k + \epsilon l, \epsilon\eta). \tag{19}$$

The neutral is $(0, 1)$. The inverse of (k, ϵ) is $(k, \epsilon)^{-1} = (-k\epsilon, \epsilon)$. It gives

$$(k, 1)^{-1} = (-k, 1) \text{ and } (k, -1)^{-1} = (k, -1).$$

Notice that D_2 is the cyclic group of order 2 and D_4 is the Klein four-group (with four elements). Both of them are commutative groups. However for $n > 3$, D_n is a *non-commutative* group, thus it acts on itself by right or left multiplication. If we look at (l, η) acting on (k, ϵ) on the right we obtain Eq. 19, and if we look at the left action we obtain

$$(l, \eta)(k, \epsilon) = (l + \eta k, \eta\epsilon). \tag{20}$$

We can then define the left interval between two pairs (k_1, ϵ_1) and (k_2, ϵ_2) in D_n as the unique element (l, η) in D_n such that $(l, \eta)(k_1, \epsilon_1) = (k_2, \epsilon_2)$, and the right interval as the unique element such that $(k_1, \epsilon_1)(l, \eta) = (k_2, \epsilon_2)$. Using Eq. 19 we get

$$
\begin{aligned}
{}^{l}\mathbf{int}((k_1, \epsilon_1), (k_2, \epsilon_2)) &= (k_2 - \epsilon_2\epsilon_1 k_1, \epsilon_1\epsilon_2),\\
{}^{r}\mathbf{int}((k_1, \epsilon_1), (k_2, \epsilon_2)) &= ((k_2 - k_1)\epsilon_1, \epsilon_1\epsilon_2).
\end{aligned}
$$

Let us give somes examples in D_8.

Example 2.1. - If we consider $(1, -1)$ and $(2, 1)$ in D_8 then

$$
\begin{aligned}
{}^{l}\mathbf{int}((1, -1), (2, 1)) &= (3, -1),\\
{}^{r}\mathbf{int}((1, -1), (2, 1)) &= (7, -1).
\end{aligned}
$$

- If we consider $(1,-1)$ and $(2,1)$ in D_8 then

$$^{l}\mathbf{int}((3,-1),(1,-1)) = (6,1),$$
$$^{r}\mathbf{int}((3,-1),(1,-1)) = (2,1).$$

Finally we define the right and the left interval vectors of a set A in D_n as described in the first chapter, and we say that two sets are right or left-homometric if they have the same interval vector. Here are two examples of homometric sets in D_{16} with their interval vector.

Example 2.2. - In D_{16}, the set $\{(0,1),(8,1),(0,-1),(4,-1)\}$ and the set $\{(0,1),(12,1),(0,-1),(8,-1)\}$ are left-homometric. Their interval vector is

$$^{l}\mathbf{iv} : [(0{,}1),(8,1),(0,-1),(4,-1),(8,1),(0,1),(8,-1),(12,-1),(0,-1),$$
$$(8,-1),(0,1),(4,1),(4,-1),(12,-1),(12,1),(0,1)].$$

- The sets $\{(0,1),(8,1),(1,-1),(5,-1)\}$ and $\{(0,1),(4,1),(1,-1),(9,-1)\}$ are right-homometric in D_{16}. Their interval vector is

$$^{r}\mathbf{iv} : [(0{,}1),(8,1),(1,-1),(5,-1),(8,1),(0,1),(9,-1),(13,-1),(1,-1),$$
$$(9,-1),(0,1),(12,1),(5,-1),(13,-1),(4,1),(0,1)].$$

We can also give an alternative definition of the dihedral group with generators and relations.

Definition 2.2. *The dihedral group has the presentation*

$$D_n = \langle r, c \mid r^n = 1, c^2 = 1, crc = r^{-1}\rangle. \tag{21}$$

It means that D_n is generated by r (rotation, ord(r)=n) and c (reflection, ord(c) = 2). It contains $2n$ elements: the n rotations $1, r, r^2, ..., r^{n-1}$ and the n reflections $c, cr, ..., cr^{n-1}$. Besides, they are linked by the relation $cr = r^{-1}c$.

In our context rotations correspond to translations (or transposition in music) and reflections correspond to inversions. It is explained in the following remark that gives the correspondences between Def. 2.1 and Def. 2.2.

Remark 2.1. To see the link between the two definitions of the dihedral group we remark that $(1,1)^n = (n,1) = (0,1) = id$, $(0,-1)^2 = (0,1) = id$ and

$$(0,-1)(1,1)(0,-1) = (-1,1) = (1,1)^{-1}.$$

Identifying the rotation r to the pair $(1,1)$ and the reflection c to the pair $(0,-1)$ we obtain the relations of Def. 2.2. Thus D_n is generated by $(1,1)$ and $(0,-1)$.

We will now talk about trivial and non-trivial homometry in the dihedral group.

Trivial and non Trivial Homometry in D_n

We know from the general study of the previous section that any left translation of a set in D_n preserves the right intervals and any right translation preserves the left intervals. For instance if we translate by $(2,1)$ the above mentioned set $\{(0,1),(8,1),(0,-1),(4,-1)\}$ in D_{16} on the left (i.e. we multiply on the the left every element of this set by $(2,1)$) we obtain the set $\{(2,1),(10,1),(2,-1),(6,-1)\}$ that is trivially right-homometric to the first one.

It is natural to search other operations in D_n that preserve right or left intervals. Recall that in $\mathbb{Z}_n$ we considered sets modulo translations and inversions. So we will first check if the inversion in D_n is interval preserving or not. In fact it is not. For instance the set $\{(0,1),(2,-1),(3,1)\}$ in D_{12} has the following interval vectors:

$$^{r}\mathbf{iv} : [(0,1),(2,-1),(3,1),(2,-1),(0,1),(11,-1),(9,1),(11,11),(0,1)],$$
$$^{l}\mathbf{iv} : [(0,1),(2,-1),(3,1),(2,-1),(0,1),(5,-1),(9,1),(5,-1),(0,1)].$$

The inverse $\{(0,1),(2,-1),(-3,1)\}$ has different interval vectors:

$$^{r}\mathbf{iv} : [(0,1),(2,-1),(9,1),(2,-1),(0,1),(5,-1),(3,1),(5,-1),(0,1)],$$
$$^{l}\mathbf{iv} : [(0,1),(2,-1),(9,1),(2,-1),(0,1),(11,-1),(3,1),(11,-1),(0,1)].$$

Obviously it can still exist other interval preserving operations. That is why we will study the group of automorphisms of the dihedral group, and look for interval preserving operations inside this group. We just recall some definitions.

Definition 2.3. *Let G be a group. An* automorphism *of G is a endomorphism (morphism from G to itself) which is an isomorphism (i.e. which admits an inverse). The set of automorphisms of G is written $\mathcal{A}ut(G)$, it is a group for the composition of morphisms.*

$\mathcal{A}ut(G)$ contains two types of morphisms, called inner *and* outer *automorphisms. The inner automorphisms are the conjugations by the elements of G (for each $a \in G$, conjugation by a is the operation $\phi_a : G \longrightarrow G, \phi_a(g) = a^{-1}ga$), and the outer automorphisms are the other automorphisms.*

To understand the automorphism group $\mathcal{A}ut(D_n)$ we will use Def. 2.2 of the dihedral group with presentation and relations.

Proposition 2.1. *The group $\mathcal{A}ut(D_n)$ of automorphisms of the dihedral group D_n is isomorphic to $\mathbb{Z}_n \rtimes \mathbb{Z}_n^\star$, where $\mathbb{Z}_n^\star$ is the group of invertible elements of $\mathbb{Z}_n$. Then it has $\phi(n)n$ elements, where $\phi(n)$ is the number of $k \in \{1, ..., n-1\}$ coprime with n.*

Proof. This result is not our contribution but we give the proof to describe the actions of the automorphisms, it will be useful later. Let π be an automorphism of D_n. To understand π we have to evaluate it on the generators of D_n, i.e. on r and c.

- r generates a cyclic subgroup of order n, then $\pi(r)$ must generate a cyclic subgroup of order n in D_n. $\mathbb{Z}_n$ is the only subgroup of order n, it is generated by the rotation r. Then $\pi(r) = r^k$ for a certain k that is coprime with n (i.e $k \in \mathbb{Z}_n^\star$).
- c generates a cyclic subgroup of order 2 in D_n then it must be the same for $\pi(c)$. There can be a cyclic subset of order 2 generated by a rotation but we have already counted it before and π is an automorphism, then $\pi(c)$ must be a reflection and $\pi(c) = cr^l$ for some $l \in \mathbb{Z}_n$.

Then $|\mathcal{A}ut(D_n)| < \phi(n)n$, but here we have exactly $\phi(n)n$ automorphisms then we have all of them.

An automorphism π is then denoted by a pair $\pi = (l, k) \in \mathbb{Z}_n \times \mathbb{Z}_n^\star$. On a rotation r^p it acts by $\pi(r^p) = r^{kp}$ and on a reflection cr^q it acts by

$$\pi(cr^q) = \pi(c)\pi(r^q) = cr^l r^{kq} = cr^{kq+l}.$$

Let us look at the action of the composition of two automorphisms $\pi = (l, k)$ and $\pi' = (l', k')$:

- on a rotation r^p: $(l, k) \circ (l', k')(r^p) = (l, k)(r^{k'p}) = r^{kk'p}$;
- on a reflection cr^q: $(l, k) \circ (l', k')(cr^q) = (l, k)(cr^{k'q+l'}) = cr^{kk'p+kl'+l}$.

Then the composition $(l, k) \circ (l', k')$ acts similarly to $(l + kl', kk')$, which is the definition of the multiplication $(l, k)(l', k')$ in the semi-direct product $\mathbb{Z}_n \rtimes \mathbb{Z}_n^\star$. □

Remark 2.2. In the previous proposition we described all the automorphims of the dihedral group: the inner and the outer automorphisms.

In the sense of Def. 2.1, a pair $(l,k) \in \mathbb{Z}_n \rtimes \mathbb{Z}_n^\star$ acts on $(p,1)$ (equivalent to a rotation) by $(l,k).(p,1) = (kp,1)$ and on $(q,-1)$ (equivalent to a reflection) by $(kq+l,-1)$. With this formulation we can now look at the interval preserving operations in $\mathcal{A}ut(D_n)$. The following proposition shows that there is not such operation.

Proposition 2.2. *There is no element of $\mathcal{A}ut(D_n)$ that preserves intervals in D_n except for the identity $(0,1)$.*

Proof. It is not complicated but a bit long if we want to do all the cases. We will do only one case and leave the others to the reader. Let us look, for instance, at the right interval between $(p,1)$ and $(q,1)$

$$^r\mathbf{int}((p,1),(q,1)) = (q-p,1).$$

If we apply (l,k) we get $(l,k).(p,1) = (kp,1)$ and $(l,k).(q,1) = (kq,1)$. Then

$$^r\mathbf{int}((kp,1),(kq,1) = (k(q-p),1)$$

and if (l,k) is interval preserving we must have $k=1$.

If $k=1$ we can look at

$$^r\mathbf{int}((p',-1),(q',1)) = (p'-q',-1)$$

and after the left action of $(l,1)$ we get

$$^r\mathbf{int}((p'+l,-1),(q',-1) = (p'+l-q',-1).$$

If $(l,1)$ is interval preserving we must have $l=0$. It works the same for left intervals. □

We did not find any other preserving interval operation, that is why we keep the following definition for non-trivial homometry.

Definition 2.4. *We say that two sets in D_n are* non-trivially *right (resp. left) homometric if there are right (resp. left) homometric and not linked by a left (resp. right) translation.*

The homometric pairs given in the previous paragraph (Ex. 2.2) are non-trivial homometric sets. When we do not specify, 'homometric sets' means 'non-trivial homometric sets'.

Now we will describe the (right and left) actions of the dihedral group of order 24 on itself as the actions of the two neo-Riemannian groups on the set S of major and minor triads.

2.2 Link with the T/I-group and the PLR-group when $n = 12$

The T/I-group and the PLR-group (also called the neo-Riemannian groups) are very famous groups in music theory, and have the specificity to act simply transitively on the set S of major and minor triads, so we can form GISs and then study homometry. Popoff proved in [23] that the actions of the neo-Riemannian groups on S can be interpreted as the right and the left actions of the dihedral group D_{12} on itself. Hence homometric sets in the dihedral group are isomorphic to homometric sets in the T/I-group or in the PLR-group in the case $n = 12$. The aim of this subsection is to explain these identifications.

We first define the set S of major and minor triads, then we focus on the two neo-Riemannian groups and their actions on S.

The Set S of Major and Minor Triads as D_{12}

As it is written in [5], "Triadic harmony has been in use for hundreds of years and is still used everyday in popular music". It is the base of occidental harmonic music.

A *triad* consists in three simultaneous played notes. A *major* triad consists of a *root* note, a second note 4 semitones above the root, and a third note 7 semitones above the root. For example the C-major triad consists of $\{C, E, G\}$ which is $\{0, 4, 7\}$.

A *minor* triad consists of a *root* note, a second note 3 semitones above the root, and a third note 7 semitones above the root. For example the C-minor triad consists of $\{0, 3, 7\} = \{C, E^\flat, G\}$.

Altogether, the major and minor triads form the set S of *consonant* triads, which are called consonant because of their smooth sound. A consonant triad is named after its root, for instance the C-major triad is $\{C, E, G\}$, the F-minor triad is $\{5, 8, 0\} = \{F, A^\flat, C\}$. Musicians commonly denote major triads by upper-case letters and minor triads by lower-case letters. We will often use this notation when there is no ambiguity.

The 24 elements belonging to the set S are presented on Fig. 1, drawn from [5].

In this table angular brackets denote ordered sets. Since we are speaking of simultaneously sounding notes, it is not necessary to insist on a particular order-

Table 1. The set S of major and minor triads.

Major triads	Minor triads
$C = \langle 0,4,7\rangle$	$\langle 0,8,5\rangle = f$
$C^\sharp = D^\flat = \langle 1,5,8\rangle$	$\langle 1,9,6\rangle = f^\sharp = g^\flat$
$D = \langle 2,6,9\rangle$	$\langle 2,10,7\rangle = g$
$D^\sharp = E^\flat = \langle 3,7,10\rangle$	$\langle 3,11,8\rangle = g^\sharp = a^\flat$
$E = \langle 4,8,11\rangle$	$\langle 4,0,9\rangle = a$
$F = \langle 5,9,0\rangle$	$\langle 5,1,10\rangle = a^\sharp = b^\flat$
$F^\sharp = G^\flat = \langle 6,10,1\rangle$	$\langle 6,2,11\rangle = b$
$G = \langle 7,11,2\rangle$	$\langle 7,3,0\rangle = c$
$G^\sharp = A^\flat = \langle 8,0,3\rangle$	$\langle 8,4,1\rangle = c^\sharp = d^\flat$
$A = \langle 9,1,4\rangle$	$\langle 9,5,2\rangle = d$
$A^\sharp = B^\flat = \langle 10,2,5\rangle$	$\langle 10,6,3\rangle = d^\sharp = e^\flat$
$B = \langle 11,3,6\rangle$	$\langle 11,7,4\rangle = e$

ing of the elements within the brackets. However the mathematical artifice of an ordering will simplify the discussion of the PLR-group that we are approaching.

When we represent $\mathbb{Z}_{12}$ as a musical clock, we picture triads by a chord polygon, as in Fig. 5 below. Besides when we will refer to some triad in S, we will write it s_t, s standing for the root of the triad (element of $\mathbb{Z}_{12}$) and $t = M$ or $t = m$ standing for major (M) or minor (m).

We will now build a bijection χ between the set S of major and minor triads and the dihedral group $D_{12} = \mathbb{Z}_{12} \rtimes \mathbb{Z}_2$. This identification is not canonical and we will have to distinguish it from the future identifications with the T/I and the PLR-groups. The pair $(s,+1) \in D_{12}$ will be considered as the major triad whose base root is $s \in \mathbb{Z}_{12}$ i.e. to the chord s_M, and the pair $(s,-1) \in D_{12}$ will be considered as the minor triad whose base root is $s \in \mathbb{Z}_{12}$ i.e. to s_m. In other terms $\chi((s,1)) = s_M$ and $\chi((s,-1)) = s_m$. For instance $(0,1)$, the identity element, corresponds to C, while $(0,-1)$ corresponds to c; $(8,-1)$ corresponds to $a^\flat$, etc. The set $\{c, D^\flat, E^\flat, e, a^\flat\}$ of the introduction corresponds thus to $\{(0,-1),(1,1),(3,1),(4,-1),(8,-1)\}$.

With the bijection χ we can associate to the action of the diedral group D_{12} on itslef by (right/left) multiplication, the action of D_{12} on the set S. For instance a pair $(p,1)$ acts on a triad s_M by

$$(p,1)s_M = \chi((p,1)(s,1)) = \chi((p+s,1)) = (p+s)_M.$$

We will see that these actions correspond in fact to the actions of the neo-Riemannian groups on S. In the next paragraph we focus on the T/I-group.

The T/I-group and the Left Action of D_{12}

The T/I-group is the group generated by the transpositions (i.e. the translations) and the inversions. There are 12 transpositions and 12 inversions, hence the T/I-group contains 24 elements. It acts simply transitively on $\mathbb{Z}_{12}$. The transposition by p semitones ($p \in \mathbb{Z}_{12}$) is generally written

$$\begin{aligned} T_p : \mathbb{Z}_{12} &\longrightarrow \mathbb{Z}_{12} \\ x &\mapsto x + p, \end{aligned}$$

and the inversion I_q ($q \in \mathbb{Z}_{12}$) is equal to $T_q \circ I_0$:

$$\begin{aligned} I_q : \mathbb{Z}_{12} &\longrightarrow \mathbb{Z}_{12} \\ x &\mapsto -x + q. \end{aligned}$$

Graphically, if we see $\mathbb{Z}_{12}$ as the musical clock, T_1 corresponds to clockwise rotation by 1/12 of a turn, while inversion I_0 corresponds to a reflection of the clock about the $0 - 6$ axis.

Besides, we have the following relations between transpositions and inversions:

$$T_p \circ T_q = T_{p+q \text{ mod } 12}, \tag{22}$$

$$T_p \circ I_q = I_{p+q \text{ mod } 12}, \tag{23}$$

$$I_p \circ T_q = I_{p-q \text{ mod } 12}, \tag{24}$$

$$I_p \circ I_q = T_{p-q \text{ mod } 12}. \tag{25}$$

In particular $T_1^{12} = id$ (where $T_1^{12} := \overbrace{T_1 \circ ... \circ T_1}^{\times 12}$), $I_0^2 = id$ and $I_0 \circ T_1 \circ I_0 = T_{-1}$, that are exactly the relations of Eq. (21) with $r = T_1$ and $c = I_0$. Hence the T/I-group is in fact a dihedral group of order 24.

The T/I-group acts also on S. Actually we can easily define an action on a triad from an action on pitch classes. For a triad s_t in S it is given by

$$T_p(s_t) = (p + s)_t \tag{26}$$

and

$$I_0(s_M) = (5 - k)_m, \tag{27}$$

$$I_0(s_m) = (5 - k)_M. \tag{28}$$

We give some examples of this action.

Example 2.3. - the triad C-major is 0_M. If we use the transposition by 1 semitone we obtain $T_1(0_M) = 1_M = D^\flat$.

- If we use the inversion by 0 semitone on the triad C-major we obtain $I_0(0_M) = (5, m) = f$. It is illustrated on Fig. 5.

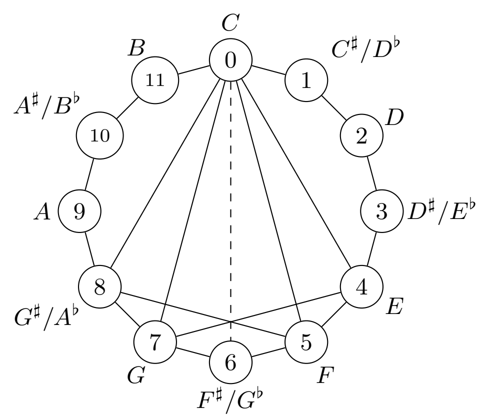


Fig. 5. I_0 acting on C-major gives F-minor. We see here the representation of triads as chord polygons.

Moreover this action (it is easy to check) is *simply transitive*. As we said in section 1.3 we have then a GIS $(S, T/I, \mathbf{int}_{T/I})$ and consequently elements in the T/I-group can be seen as *intervals* between triads.

Definition 2.5. *Given two chords s and s' in S, we call* interval in the T/I-group *between s and s' – written* $\boldsymbol{int}_{T/I}(s, s')$ *– the unique transformation sending s to s'.*

Example 2.4. - The interval between C-major and $D^\flat$-major is T_1 (cf. the previous example).

- The interval between C-minor and $E^\flat$-major is $\mathbf{int}_{T/I}(c, E^\flat) = I_{10}$.

Hence we can associate to a set of triads (a subset of S) an interval vector. For instance the subset $A = \{c, C, E^\flat\}$ contains: $\mathbf{int}_{T/I}(c, C) = I_7$, $\mathbf{int}_{T/I}(c, E^\flat) = I_{10}$ and $\mathbf{int}_{T/I}(C, E^\flat) = T_3$. We have also to count the inverted intervals:

$\mathbf{int}_{T/I}(C,c) = I_7$, $\mathbf{int}_{T/I}(E^\flat,c) = I_{10}$ and $\mathbf{int}_{T/I}(E^\flat,C) = T_9$ (recall that $T_p^{-1} = T_{-p}$ and $I_q^{-1} = I_q$). We obtain the interval vector

$$\mathbf{iv}_{T/I}(A) = (T_3, T_9, I_7, I_7, I_{10}, I_{10}).$$

We give below an example of homometric sets (same interval vector).

Example 2.5. The sets $A = \{c, D^\flat, E^\flat, e, a^\flat\}$ and $B = \{c, E^\flat, e, F, a^\flat\}$ are homometric for the action of the T/I-group on S. Figure 6 displays the intervals between triads of these sets.

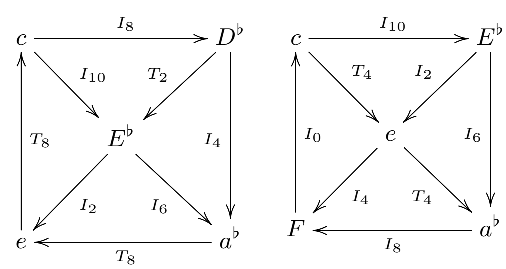


Fig. 6. Intervals in the T/I-group for the two sets $A = \{c, D^\flat, E^\flat, e, a^\flat\}$ (on the left) and $B = \{c, E^\flat, e, F, a^\flat\}$ (on the right). These sets are homometric in the T/I-group.

The two sets have the same interval content: $(T_2, T_4, T_4, T_8, I_0, I_2, I_4, I_6, I_8, I_{10})$. Note that the intervals are not all written on the graph for readibility. For example we obtain $\mathbf{int}_{T/I}(a^\flat, c) = T_4$ by

$$\mathbf{int}_{T/I}(E^\flat, c) \circ \mathbf{int}_{T/I}(a^\flat, E^\flat) = I_{10} \circ I_6 = T_4.$$ [8]

Besides it is an interval content and not an interval vector meaning that we do not write the inverted interval i.e $\mathbf{int}_{T/I}(E^\flat, c)$ for $\mathbf{int}_{T/I}(c, E^\flat)$ since we obtain them by simple inversion as we said before.

Following [23], we will now explain how we can see the action of the T/I-group on the set S as the left action of the dihedral group D_{12} on S and consequently, thanks to the bijection χ, as the left action of D_{12} on itself.

Consider the left action of $(p,1)$ on a chord s_t. We have

$$(p,1)s_t = \chi((p,1)(s,t)) = \chi((p+s,t)) = (p+s)_t,$$

[8] Notice that we do not write $\mathbf{int}_{T/I}(a^\flat, E^\flat) \circ \mathbf{int}_{T/I}(E^\flat, c)$ contrary to the definition of GIS. It is not important, since it depends only on the order we choose for the action. We made this choice to respect the traditional equations 25.

hence the elements $(p, 1)$ correspond to the translations T_p (cf. Eq. 26).

Consider the left action of $(5, -1)$. On a major chord it gives

$$(5, -1)s_M = \chi((5, -1)(s, 1)) = \chi((5 - s, -1)) = (5 - s)_m$$

and on a minor chord

$$(5, -1)s_m = \chi((5, -1)(s, -1)) = \chi((5 - s, 1)) = (5 - s)_M,$$

which is equivalent to the inversion I_0 (cf. Eq. 27). Owing to the group operation, the other group elements $(p, -1)$ are associated to the inversion transformations I_{p-5}.

For instance $T_7(C)$ is computed in D_{12} by $(7, 1)(0, 1) = (7, 1)$ which correponds to G and $I_2(C)$ is computed by $(7, -1)(0, 1) = (7, -1)$ which correponds to g.

With this new point of view we can picture the equivalent of Fig. 6 in the dihedral group. It is represented on Fig. 7, where we show the left intervals in D_{12} between the triads of the two sets A and B.

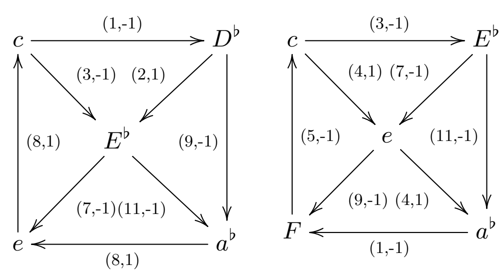


Fig. 7. Left intervals in D_{12} for $A = \{c, D^\flat, E^\flat, e, a^\flat\}$ and $B = \{c, E^\flat, e, F, a^\flat\}$.

We will use again this example because it has a specificity: it is also homometric for the action of the PLR-group.

The PLR-group and the Right Action of D_{12}

The PLR-group is generated by the transformations *parallel* (P), *leading tone exchange* (L) and *relative* (R). This group acts naturally on S but not on $\mathbb{Z}_n$, contrary to the T/I-group. For $s \in S$, $P(s)$ is the triad of opposite type as s with the first and third notes switched; $L(s)$ is the triad of opposite type as s

with the second and third notes switched; $R(s)$ is the triad of opposite type as s with the first and second notes switched.

We give some examples. Here the fact we use angular brackets is important because we have to take into account the order of the notes.

Example 2.6. - $P\langle 0,4,7\rangle = \langle 7,3,0\rangle$ i.e. $P(C) = c$.
- $L\langle 0,4,7\rangle = \langle 11,7,4\rangle$ i.e. $L(C) = e$.
- $R\langle 0,4,7\rangle = \langle 4,0,9\rangle$ i.e. $R(C) = a$.

We recognize in P, L and R classical transformations of triads. P changes the type of the triad but the root remains the same, R gives the relative tone. These transformations have the particularity to be *involutive*: if we compose them with themselves we get the identity transformation ($P \circ P = L \circ L = R \circ R = id$).

We give below the general formulation of the action of P on a triad s_t in S:

$$\begin{aligned} P : s_M &\longmapsto s_m \\ s_m &\longmapsto s_M; \end{aligned}$$

the action of L:

$$\begin{aligned} L : s_M &\longmapsto (s+4)_m \\ s_m &\longmapsto (s+8)_M; \end{aligned}$$

and the action of R:

$$\begin{aligned} R : s_M &\longmapsto (s+9)_m \\ s_m &\longmapsto (s+3)_M. \end{aligned}$$

In order to build the GIS associated to the action of the PLR-group on S we have to show that the action is simply transitive. This is proved in [5] where we find the following results.

Theorem 2.1. *The PLR-group is generated by L and R and is dihedral of order 24. Besides, its action on S is simply transitive.*

Proof. Refer to [5]. Notice that P can be expressed as a combination of the L and R transformations with $P = R(LR)^3$. □

Hence we get a function *interval* and a GIS $(S, PLR, \mathbf{int}_{PLR})$.

Example 2.7. - The interval between c and $D^\flat$ is LRL. Indeed $L(c) = A^\flat$, $R(A^\flat) = f$, and $L(f) = D^\flat$, i.e. $LRL(c) = D^\flat$.

- The interval between $D^\flat$ and $E^\flat$ is $\mathbf{int}_{PLR}(D^\flat, E^\flat) = PRLR$. Indeed $R(D^\flat) = b^\flat, L(b^\flat) = G^\flat, R(G^\flat) = e^\flat, P(e^\flat) = E^\flat$.

On fig. 8 we show Douthett and Steinbach's graph giving a 2D representation of the action of P, L and R on S. It is very convenient when we want to find the interval in the PLR-group between two triads. We can find graphically the results of Ex. 2.7 just above in a quicker way.

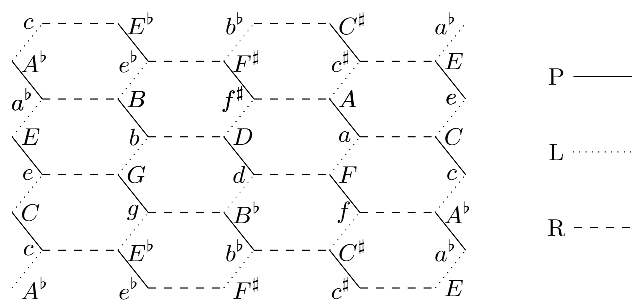


Fig. 8. Douthett and Steinbach's Graph

Then we define the interval vector in the PLR-group. Interestingly the two sets A and B of Ex. 2.5 are homometric also for the action of the PLR-group.

Example 2.8. The sets $A = \{c, D^\flat, E^\flat, e, a^\flat\}$ and $B = \{c, E^\flat, e, F, a^\flat\}$ in S are homometric for the action of the PLR-group. Figure 9 displays the intervals between triads of these sets.

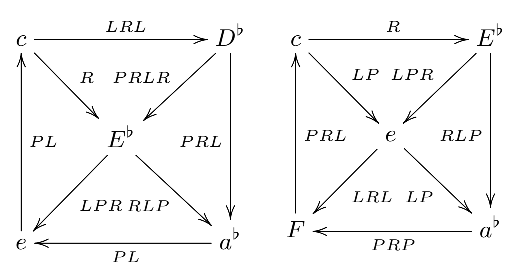


Fig. 9. Intervals in the PLR-group for the two sets $\{c, D^\flat, E^\flat, e, a^\flat\}$ (on the left) and $\{c, E^\flat, e, F, a^\flat\}$ (on the right). These sets are homometric in the PLR-group.

For both the interval content is

$$\mathbf{ic}_{PLR} = (R, PL, PL, PL, LRL, LRP, RLP, LPR, PRP, PRLR).$$

We explain now how we can see the action of the PLR-group on S as the right action of D_{12} on S. This is always drawn from [23].

Consider for example the right action of $(4, -1)$ on a major chord s_m. We have

$$s_M(4, -1) = \chi((s, 1)(4, -1)) = \chi((s + 4, -1)) = (s + 4)_m.$$

Consider now the right action of the same group element $(4, -1)$ on a minor chord s_m. We have

$$s_m(4, -1) = \chi((s, -1)(4, -1)) = \chi((s - 4, 1)) = (s - 4)_M.$$

We thus find that the right action of $(4, -1)$ is equivalent to the action of L. Similarly we can check that the action of P is associated with the right translation by the element $(0, -1)$ and that the action of R is associated to the right translation by $(9, -1)$.

For instance $P(C)$ is computed by $(0, 1)(0, -1) = (0, -1)$ which is c, $L(d^\flat)$ is $(1, -1)(4, -1) = (9, 1)$ which is A and $R(F)$ is $(4, 1)(9, -1) = (1, -1)$ which is $d^\flat$.

With this new point of view we can picture the equivalent of Fig. 9 in the dihedral group. It is represented on Fig. 10, where we show the right intervals in D_{12} between the triads of the two sets A and B.

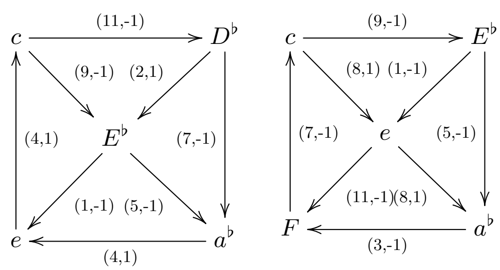


Fig. 10. Right intervals in D_{12} for $A = \{c, D^\flat, E^\flat, e, a^\flat\}$ and $B = \{c, E^\flat, e, F, a^\flat\}$.

In fact the two neo-Riemannian groups are dual in the sense that "each is the centralizer of the other in the symmetric group of the set S of major and minor triads” ([5]). As a conclusion Tab. 2 gives the correspondences between the T/I-group and the PLR-group with D_{12}.

Table 2. Correspondences between the neo-Riemannian groups and D_{12}.

$T/I \sim D_{12}$		$PLR \sim D_{12}$	
T_p	$(p, 1)$	P	$(0, -1)$
I_p	$(p + 5, -1)$	L	$(4, -1)$
		R	$(9, -1)$

We deduce a simple invariance property from these identifications. Homometry for the action of the T/I-group is invariant under the action of the T/I-group and conversely homometry for the action of the PLR-group is invariant under the action of the T/I-group. This gives for instance the following result: changing the nature (major $\longleftrightarrow$ minor) of all the triads of a set is equivalent to translate this set on the right by $(0, -1)$ (i.e. the action of P) hence it leads to a trivial homometric set for the action of the T/I-group. On the contrary two such sets can be non-trivially homometric for the action of the PLR-group.

Besides using the inversion we know from Prop. 1.3 that we can switch from homometry in the T/I-group to homometry in the PLR-group and conversely. We recall this general feature in the dihedral group later.

Finally we saw that the sets A and B of Ex. 2.5 and 2.7 are homometric for the two actions. It raises a new question: How can we find homometric sets for both left and right actions in the dihedral group? At some point we will deal with this question in the next subsection.

All we saw in this subsection concerned the case $n = 12$. This is the favored context for musical applications and we showed that there are relevant links with the actions of the neo-Riemannian groups. From now we will study the general case.

2.3 Homometry in D_n Using iv and ifunc

After some notations, we present in this part the equations of right and left homometry in the dihedral group, and we give some properties that lead us to define a central concept in this work: the *lift*. Using subsection 1.2, we then give the equivalent equations in the Fourier space and give some main results of this work concerning lifts with homometry for the right action. Finally we give some applications and illustrations of these results, using for instance the work of Goyette ([10]) or discussing some musical aspects. At the end we propose a table that summarizes the differences between right and left homometry.

Notations. We will work alternatively on subsets in D_n and subsets in $\mathbb{Z}_n$. To avoid confusions we will adopt the following convention: a subset in D_n will be written with the typography '$\mathcal{A}$' while a subset in $\mathbb{Z}_n$ will be written 'A'.

As we described D_n, a set $\mathcal{A} \in D_n$ is the union of two subsets corresponding respectively to the part with pairs having $+1$ as second component (major chords), and the part with pairs having -1 as second component (minor chords). For instance in D_{12} the set

$$\mathcal{A} = \{c, D^\flat, E^\flat, e, a^\flat\} = \{(0,-1),(1,1),(3,1),(4,-1),(8,-1)\}$$

of the introduction, is the union of the major chords $\{D^\flat, E^\flat\} = \{(1,1),(3,1)\}$ and the minor chords $\{c, e, a^\flat\} = \{(0,-1),(4,-1),(8,-1)\}$. We will call A_+ (resp. A_-) the set of the base roots in $\mathbb{Z}_n$ of the major chords (resp. minor chords) of A. In our case we have $A_+ = \{1,3\} \subset \mathbb{Z}_{12}$ and $A_- = \{0,4,8\} \subset \mathbb{Z}_{12}$.

More formally we have the following definitions.

Definition 2.6. *Let $\pi_1 : D_n = \mathbb{Z}_n \rtimes \mathbb{Z}_2 \longrightarrow \mathbb{Z}_n$ be the first projection and $\pi_2 : D_n = \mathbb{Z}_n \rtimes \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2$ be the second projection. For $\mathcal{A} \subset D_n$, we call $\mathcal{A}_+$ (resp. $\mathcal{A}_-$) the subset of $\mathcal{A}$ defined by $\mathcal{A}_+ = \{a \in \mathcal{A} \mid \pi_2(a) = +1\}$ (resp. $\mathcal{A}_- = \{a \in \mathcal{A} \mid \pi_2(a) = -1\}$). Hence $\mathcal{A} = \mathcal{A}_+ \cup \mathcal{A}_-$.*

Definition 2.7. *Let $\mathcal{A} \subset D_n$. We call $A_+ = \pi_1(\mathcal{A}_+)$ and $A_- = \pi_1(\mathcal{A}_-)$. It implies that A_+ and A_- are subsets of $\mathbb{Z}_n$, and that $\pi_1(\mathcal{A}) = A_+ \cup A_-$.*

Example 2.9. - If $\mathcal{B} = \{(0,1),(9,-1)\}$ in D_{10}, we have: $\mathcal{B}_+ = \{(0,1)\}$ and $\mathcal{B}_- = \{(9,-1)\}$, thus $B_+ = \{0\}$ and $B_- = \{9\}$.

- With the example just above $\mathcal{A} = \{(0,-1),(1,1),(3,1),(4,-1),(8,-1)\}$ in D_{12}, we have

$$\mathcal{A}_+ = \{(1,1),(3,1)\} \text{ and } \mathcal{A}_- = \{(0,-1),(4,-1),(8,-1)\}$$

so we obtain: $A_+ = \{1,3\}$ and $A_- = \{0,4,8\}$.

If there is no ambiguity, for a set $\mathcal{A} \in D_n$ we will write $A := \pi_1(\mathcal{A})$, so that $A = A_+ \cup A_-$.

We give now the characterization of right and left homometry in the dihedral group.

Characterization of Homometry

The purpose of the following theorem is to give a characterization of homometry in D_n using **iv** and **ifunc** and the above-mentioned notations.

Theorem 2.2. *Two sets $\mathcal{A}$ and $\mathcal{B}$ in D_n are homometric for the right action if and only if the two following equations hold:*

$$\begin{cases} \boldsymbol{iv}(A_+) + \boldsymbol{iv}(A_-) = \boldsymbol{iv}(B_+) + \boldsymbol{iv}(B_-) \\ \boldsymbol{ifunc}(A_+, A_-) = \boldsymbol{ifunc}(B_+, B_-). \end{cases} \tag{29}$$

Two sets $\mathcal{A}$ and $\mathcal{B}$ in D_n are homometric for the left action if and only if the two following equations hold:

$$\begin{cases} \boldsymbol{iv}(A_+) + \boldsymbol{iv}(A_-) = \boldsymbol{iv}(B_+) + \boldsymbol{iv}(B_-) \\ \boldsymbol{ifunc}(I_0A_+, A_-) = \boldsymbol{ifunc}(I_0B_+, B_-). \end{cases} \tag{30}$$

Proof. : Let $\mathcal{A}, \mathcal{B}$ be two right-homometric sets in D_n. Let us recall that

$$^r\mathbf{int}((k_1, \epsilon_1), (k_2, \epsilon_2)) = ((k_2 - k_1)\epsilon_1, \epsilon_1\epsilon_2) \tag{31}$$

for $(k_1, \epsilon_1), (k_2, \epsilon_2)$ in $\mathcal{A}$. Then we have two cases, which give the two equations Eq. 29 and 30: $\epsilon_2/\epsilon_1 = 1$ or $\epsilon_2/\epsilon_1 = -1$.

If $\epsilon_2/\epsilon_1 = 1$ **i.e.** $\epsilon_1 = \epsilon_2$. Either $\epsilon_1 = 1 = \epsilon_2$ then

$$^r\mathbf{int}((k_1, \epsilon_1), (k_2, \epsilon_2)) = (k_2 - k_1, 1)$$

for k_1, k_2 in A_+. To obtain all the intervals of that type we have to calculate $\mathbf{iv}(A_+)$. Or $\epsilon_1 = -1 = \epsilon_2$ then

$$^r\mathbf{int}((k_1, \epsilon_1), (k_2, \epsilon_2)) = (k_1 - k_2, 1)$$

for k_1, k_2 in A_-. To obtain all the intervals of that type we have to calculate $\mathbf{iv}(A_-)$. Then we must have

$$\mathbf{iv}(A_+) + \mathbf{iv}(A_-) = \mathbf{iv}(B_+) + \mathbf{iv}(B_-).$$

If $\epsilon_2/\epsilon_1 = -1$ **i.e.** $\epsilon_1 = -\epsilon_2$. Either $\epsilon_1 = 1, \epsilon_2 = -1$ then

$$^r\mathbf{int}((k_1, \epsilon_1), (k_2, \epsilon_2)) = (k_2 - k_1, -1)$$

for $k_1 \in A_+, k_2 \in A_-$. To obtain all the intervals of that type we have to calculate $\mathbf{ifunc}(A_+, A_-)$. Or $\epsilon_1 = -1, \epsilon_2 = 1$ then

$$^r\mathbf{int}((k_1, \epsilon_1), (k_2, \epsilon_2)) = (k_1 - k_2, 1)$$

for $k_1 \in A_-, k_2 \in A_+$. To obtain all the intervals of that type we have to calculate again $\mathbf{ifunc}(A_+, A_-)$. Then we must have

$$\mathbf{ifunc}(A_+, A_-) = \mathbf{ifunc}(B_+, B_-).$$

This gives the two equations of Thm. 2.2. Reciprocally if two sets verify equations (29) and (30), then they have the same right interval content. It works exactly the same with the left intervals. The difference comes from the fact that when $\epsilon_2/\epsilon_1 = -1$, we have

$$^l\mathbf{int}((k_1, \epsilon_1), (k_2, \epsilon_2)) = (k_1 + k_2, -1)$$

so we have to calculate $\mathbf{ifunc}(I_0A_+, A_-)$. □

We can notice that for both left and right homometry the first equation is the same, but the second ones show a big difference: for the left homometry it is symmetric between A_+ and A_-, and between B_+ and B_-, whereas it is not for the right homometry. It is due to the fact that

$$\mathbf{ifunc}(I_0A_+, A_-) = \mathbf{ifunc}(I_0A_-, A_+).$$

We give now the triviality conditions for homometric sets in D_n. It has no immediate application but it will be useful soon.

Proposition 2.3. *Two sets $\mathcal{A}$ and $\mathcal{B}$ in D_n are:*

- *trivially right-homometric if and only if there exists $p \in \mathbb{Z}_n$ such that*

$$(T_pA_+ = B_+ \text{ and } T_pA_- = B_-) \text{ or } (I_pA_+ = B_- \text{ and } I_pA_- = B_+); \quad (32)$$

- *trivially left-homometric if and only if there exists $p \in \mathbb{Z}_n$ such that*

$$(T_pA_+ = B_+ \text{ and } T_{-p}A_- = B_-) \text{ or } (T_pA_+ = B_- \text{ and } T_{-p}A_- = B_+). \quad (33)$$

Proof. We study each case, beginning with right homometry.

If $\mathcal{A}$ and $\mathcal{B}$ are trivially right-homometric we have two cases: $(p,1)\mathcal{A} = \mathcal{B}$ or $(p,-1)\mathcal{A} = \mathcal{B}$.

- If we have $(p,1)\mathcal{A} = \mathcal{B}$, reminding that $(p,1)(a,1) = (p+a,1)$ and that $(p,1)(a,-1) = (p+a,-1)$ we have

$$\begin{aligned}(p,1)\mathcal{A} = \mathcal{B} &\Longleftrightarrow (p,1)\mathcal{A}_+ = \mathcal{B}_+ \text{ and } (p,1)\mathcal{A}_- = \mathcal{B}_- \\ &\Longleftrightarrow T_pA_+ = B_+ \text{ and } T_pA_- = B_-.\end{aligned}$$

- If we have $(p,-1)\mathcal{A} = \mathcal{B}$, reminding that $(p,-1)(a,1) = (p-a,-1)$ and that $(p,-1)(a,-1) = (p-a,1)$ we have

$$\begin{aligned}(p,-1)\mathcal{A} = \mathcal{B} &\Longleftrightarrow (p,-1)\mathcal{A}_+ = \mathcal{B}_- \text{ and } (p,-1)\mathcal{A}_- = \mathcal{B}_+ \\ &\Longleftrightarrow I_pA_+ = B_- \text{ and } I_pA_- = B_+.\end{aligned}$$

If $\mathcal{A}$ and $\mathcal{B}$ are trivially left-homometric we have two cases: $\mathcal{A}(p,1) = \mathcal{B}$ or $\mathcal{A}(p,-1). = \mathcal{B}$.

- If we have $\mathcal{A}(p,1) = \mathcal{B}$, reminding that $(a,1)(p,1) = (a+p,1)$ and that $(a,-1)(p,1) = (a-p,-1)$ we have

$$\begin{aligned}\mathcal{A}(p,1) = \mathcal{B} &\Longleftrightarrow \mathcal{A}_+(p,1) = \mathcal{B}_+ \text{ and } \mathcal{A}_-(p,1) = \mathcal{B}_- \\ &\Longleftrightarrow T_pA_+ = B_+ \text{ and } T_{-p}A_- = B_-.\end{aligned}$$

- If we have $\mathcal{A}(p,-1) = \mathcal{B}$, reminding that $(a,1)(p,-1) = (a+p,-1)$ and that $(a,-1)(p,-1) = (a-p,1)$ we have

$$\begin{aligned}\mathcal{A}(p,-1) = \mathcal{B} &\Longleftrightarrow \mathcal{A}_+(p,-1) = \mathcal{B}_- \text{ and } \mathcal{A}_-(p,-1) = \mathcal{B}_+ \\ &\Longleftrightarrow T_pA_+ = B_- \text{ and } T_{-p}A_- = B_+.\end{aligned}$$

It covers all the cases. □

Let us denote by I the inversion operator in D_n. As $(k,1)^{-1} = (-k,1)$ and $(k,-1)^{-1} = (k,-1)$ for $k \in \mathbb{Z}_n$, it is easy to calculate $I(\mathcal{A})$ for a set $\mathcal{A}$ in D_n: we just have to take the inverse of A_+ and keep A_- unchanged. For example for $\mathcal{A} = \{(0,-1),(1,1),(3,1),(4,-1),(8,-1)\} \in D_{12}$, we obtain

$$I(\mathcal{A}) = \{(0,-1),(11,1),(9,1),(4,-1),(8,-1)\}.$$

We talked in Prop. 1.3 about the possibility to switch from right-homometric sets to left-homometric sets using the inversion. The following theorem is an equivalent of this result with the dihedral group.

Theorem 2.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be two sets in D_n. $\mathcal{A}$ and $\mathcal{B}$ are non-trivially right-homometric if and only if $I(\mathcal{A})$ and $I(\mathcal{B})$ are non-trivially left-homometric.*

Proof. It is just the application of Prop. 1.3 in the context of the dihedral group and the result of Prop. 2.2. □

Corollary 2.1. *For all $n \in \mathbb{N}$, the number of right-homometric sets in D_n is equal to the number of left-homometric sets. Besides, we can deduce all the left-homometric sets from the right-homometric sets, and reciprocally, with the inversion I.*

This result is useful when we deal with the problem of enumeration of homometric sets in D_n (which is an open problem as in $\mathbb{Z}_n$, cf. [13]) since we only have to do the calculation for right (or left) homometric sets and not both of them. It also shows a kind of symmetry between left and right homometry, but in fact they work very differently concerning a specific point given in the following proposition, which is simple but important for the issues we consider after.

Proposition 2.4. *If $\mathcal{A}$ and $\mathcal{B}$ are right-homometric in D_n, then their first projections $A = \pi_1(\mathcal{A})$ and $B = \pi_1(\mathcal{B})$ are homometric in $\mathbb{Z}_n$. Besides, if the homometry is trivial in D_n, the homometry is also trivial between the projections in $\mathbb{Z}_n$.*

Proof. If $\mathcal{A}$ and $\mathcal{B}$ are right-homometric in D_n, we have

$$\begin{aligned}\mathbf{iv}(A) &= \mathbf{iv}(A_+) + \mathbf{iv}(A_-) + \mathbf{ifunc}(A_+, A_-) + \mathbf{ifunc}(A_-, A_+)\\ &= \mathbf{iv}(B_+) + \mathbf{iv}(B_-) + \mathbf{ifunc}(B_+, B_-) + \mathbf{ifunc}(B_-, B_+)\\ &= \mathbf{iv}(B).\end{aligned}$$

Then A and B are homometric in $\mathbb{Z}_n$.

If $\mathcal{A}$ and $\mathcal{B}$ are trivially right-homometric we have several cases as usual (cf. Prop. 2.3). If

$$T_pA_+ = B_+ \text{ and } T_pA_- = B_-$$

we obtain

$$T_p(A_+ \cup A_-) = B_+ \cup B_- \Longrightarrow T_pA = B,$$

then A and B are trivially homometric in $\mathbb{Z}_n$. For the second case $(p, -1)\mathcal{A} = \mathcal{B}$ we obtain $I_pA = B$ then A and B are also trivially homometric in $\mathbb{Z}_n$. □

Remark 2.3. In other words, Prop. 2.4 says that right homometry in D_n "implies" homometry in $\mathbb{Z}_n$. However left homometry does not, the following pair in D_{10} gives a counterexample:

$$\{(0,1),(1,-1),(2,1),(5,-1),(7,-1)\}$$
$$\&\{(0,1),(1,-1),(6,1),(7,-1),(8,1)\}.$$

These sets are left-homometric but $\{0,1,2,5,7\}$ and $\{0,1,6,7,8\}$ are not homometric in $\mathbb{Z}_{10}$. Moreover if two sets are trivially left-homometric their projections are not trivially homometric *a priori*. For instance in D_{12} the sets

$$\mathcal{A} = \{(0,-1),(1,1),(5,1),(8,-1)\}$$
$$\mathcal{B} = \mathcal{A}(3,1) = \{(3,1),(10,-1),(2,-1),(11,1)\},$$

are trivially left-homometric but $\pi_1(\mathcal{A}) = \{0,1,5,8\}$ and $\pi_1(\mathcal{B}) = \{2,3,10,11\}$ are not homometric in $\mathbb{Z}_{12}$.

This important proposition raises an interesting question: Is it possible, conversely, to find (left/right) homometric sets in D_n from homometric sets in $\mathbb{Z}_n$? In other words can we split two homometric sets $\mathcal{A}$ and $\mathcal{B}$ in $\mathbb{Z}_n$ both into two subsets A_+, A_-, B_+ and B_- so that the corresponding sets in D_n are homometric? We consider these questions in the following subsection with the definition of what we call a *lift*.

The Concept of Lift

We begin with a definition motivated by Prop. 2.4. $\mathcal{P}(E)$ corresponds to the sets containing all the subsets of a given set E.

Definition 2.8. *We call a* lift *an application* $l : \mathcal{P}(\mathbb{Z}_n) \longrightarrow \mathcal{P}(D_n)$ *so that* $\pi_1 \circ l = id$. *For a lift* l*, the set* $l(A)$ *is called* lift of a set $A \in \mathbb{Z}_n$.

We could say that l is a way to attribute to each number of a set either $+1$ or -1.

Example 2.10. Let us consider the set $\{0,1,4,6\}$ in $\mathbb{Z}_{12}$. A lift of $\{0,1,4,6\}$ corresponds to a set in D_{12} such that the base roots are exactly 0,1,4 and 6. The set

$$\{(0,-1),(1,1),(4,1),(6,-1)\}$$

is an example of lift.

We mention a result concerning triviality.

Proposition 2.5. *Let A and B be two non-trivial homometric sets in $\mathbb{Z}_n$. If $l(A)$ and $l(B)$ are two right-homometric lifts in D_n, they are also non-trivially homometric.*

Proof. It is the contraposition of the second part of Prop. 2.4. □

Then the former question is, in more formal terms: Given two homometric sets A and B in $\mathbb{Z}_n$, is there a lift l such that $l(A)$ and $l(B)$ are (left/right) homometric in D_n? And even further: Is there a lift l so that $l(A)$ and $l(B)$ are simultaneously left and right-homometric in D_n? Given the facts that the projection of right-homometric sets in D_n always gives homometric sets in $\mathbb{Z}_n$, and that we deduce all the homometric sets in D_n from the right-homometric sets, this question seems to be very relevant. In the rest of this section we will study this question in a particular case, using the discrete Fourier transform.

Using the Discrete Fourier Transform

Before we give the main results concerning lifting homometric sets in $\mathbb{Z}_n$ into homometric sets in D_n, we introduce again the discrete Fourier transform as defined in paragraph 1.2. We obtain from Thm. 2.2 the following characterization of homometry in the dihedral group.

Theorem 2.4. *Two sets $\mathcal{A}$ and $\mathcal{B}$ in D_n are homometric for the right action if and only if the two following equations hold:*

$$\begin{cases} |\mathcal{F}_{A_+}|^2 + |\mathcal{F}_{A_-}|^2 = |\mathcal{F}_{B_+}|^2 + |\mathcal{F}_{B_-}|^2 \\ \overline{\mathcal{F}_{A_+}}\mathcal{F}_{A_-} = \overline{\mathcal{F}_{B_+}}\mathcal{F}_{B_-}. \end{cases} \tag{34}$$

Two sets $\mathcal{A}$ and $\mathcal{B}$ in D_n are homometric for the left action if and only if the two following equations hold:

$$\begin{cases} |\mathcal{F}_{A_+}|^2 + |\mathcal{F}_{A_-}|^2 = |\mathcal{F}_{B_+}|^2 + |\mathcal{F}_{B_-}|^2 \\ \mathcal{F}_{A_+}\mathcal{F}_{A_-} = \mathcal{F}_{B_+}\mathcal{F}_{B_-}. \end{cases} \tag{35}$$

Proof. We use Thm. 2.2, Prop. 1.1 and Eq. 3. □

With these equations it is easier to see that left homometry is symmetric between A_+ and A_- and between B_+ and B_- whereas right homometry is not. We will give another proof of Prop. 2.4 to illustrate the convenience of Fourier transform, just after a remark.

Remark 2.4. Let $\mathcal{A}$ be a subset of D_n, A_1 and A_2 two subsets of $\mathbb{Z}_n$ such that $A = \pi_1(\mathcal{A}) = A_1 \cup A_2$. Then the Fourier transform of A is given by

$$\begin{aligned}|\mathcal{F}_A|^2 &= |\mathcal{F}_{A_1 \cup A_2}|^2\\ &= |\mathcal{F}_{A_1} + \mathcal{F}_{A_2}|^2.\end{aligned}$$

Let us recall that for $(z, z') \in \mathbb{C}^2$, we have

$$|z + z'|^2 = |z|^2 + |z'|^2 + 2\mathcal{R}e(\overline{z}z').$$

Then we obtain for the DFT of A:

$$|\mathcal{F}_A|^2 = |\mathcal{F}_{A_1}|^2 + |\mathcal{F}_{A_2}|^2 + 2\mathcal{R}e(\overline{\mathcal{F}_{A_1}}\mathcal{F}_{A_2}). \tag{36}$$

In particular if we consider the specific decomposition $A_1 = A_+$ and $A_2 = A_-$ we get

$$|\mathcal{F}_A|^2 = |\mathcal{F}_{A_+}|^2 + |\mathcal{F}_{A_-}|^2 + 2\mathcal{R}e(\overline{\mathcal{F}_{A_+}}\mathcal{F}_{A_-}). \tag{37}$$

Now we give another proof of Prop. 2.4. Let $\mathcal{A}$ and $\mathcal{B}$ be two right-homometric sets in D_n and $A = \pi_1(\mathcal{A}), B = \pi_1(\mathcal{B})$. Equation 37 gives

$$\begin{aligned}|\mathcal{F}_A|^2 &= |\mathcal{F}_{A_+}|^2 + |\mathcal{F}_{A_-}|^2 + 2\mathcal{R}e(\overline{\mathcal{F}_{A_+}}\mathcal{F}_{A_-})\\ &= |\mathcal{F}_{B_+}|^2 + |\mathcal{F}_{B_-}|^2 + 2\mathcal{R}e(\overline{\mathcal{F}_{B_+}}\mathcal{F}_{B_-}) \text{ (thanks to (34))}\\ &= |\mathcal{F}_B|^2\end{aligned}$$

and thus the projections A and B are homometric in $\mathbb{Z}_n$.

We will now analyse equations of Thm. 2.4 and give the main results of this part.

Proposition 2.6. *Let A and B be two sets in $\mathbb{Z}_n$ such that $A = A_1 \cup A_2$ and $B = B_1 \cup B_2$ for some subsets A_1, A_2, B_1 and B_2 in $\mathbb{Z}_n$. A and B are homometric if and only if*

$$|\mathcal{F}_{A_1}|^2 + |\mathcal{F}_{A_2}|^2 + 2\mathcal{R}e(\overline{\mathcal{F}_{A_1}}\mathcal{F}_{A_2}) = |\mathcal{F}_{B_1}|^2 + |\mathcal{F}_{B_2}|^2 + 2\mathcal{R}e(\overline{\mathcal{F}_{B_1}}\mathcal{F}_{B_2}). \tag{38}$$

Proof. We get the equivalence combining (4) and (36). □

We can now present one of the principal results.

Theorem 2.5. *Let A and B be two homometric subsets in $\mathbb{Z}_n$ such that $A = A_1 \cup A_2$ and $B = B_1 \cup B_2$ with $\boldsymbol{iv}(A_1) = \boldsymbol{iv}(B_1)$ and $\boldsymbol{iv}(A_2) = \boldsymbol{iv}(B_2)$. We can always lift A and B into (non-trivial) right-homometric sets in D_n.*

Proof. Let A and B be two homometric subsets verifying the conditions of the theorem. We know from Prop. 2.6 that

$$|\mathcal{F}_{A_1}|^2 + |\mathcal{F}_{A_2}|^2 + 2\mathcal{R}e(\overline{\mathcal{F}_{A_1}}\mathcal{F}_{A_2}) = |\mathcal{F}_{B_1}|^2 + |\mathcal{F}_{B_2}|^2 + 2\mathcal{R}e(\overline{\mathcal{F}_{B_1}}\mathcal{F}_{B_2}).$$

As $\mathbf{iv}(A_1) = \mathbf{iv}(B_1)$ and $\mathbf{iv}(A_2) = \mathbf{iv}(B_2)$ we deduce

$$|\mathcal{F}_{A_1}| = |\mathcal{F}_{B_1}| \text{ and } |\mathcal{F}_{A_2}| = |\mathcal{F}_{B_2}|. \tag{39}$$

It gives

$$\mathcal{R}e(\overline{\mathcal{F}_{A_1}}\mathcal{F}_{A_2}) = \mathcal{R}e(\overline{\mathcal{F}_{B_1}}\mathcal{F}_{B_2}).$$

Besides notice also that $|\overline{\mathcal{F}_{A_1}}\mathcal{F}_{A_2}| = |\overline{\mathcal{F}_{B_1}}\mathcal{F}_{B_2}|$, thanks to (39). We obtain finally the following two equations:

$$\begin{cases} \mathcal{R}e(\mathcal{F}_{A_1}\overline{\mathcal{F}_{A_2}}) = \mathcal{R}e(\mathcal{F}_{B_1}\overline{\mathcal{F}_{B_2}}) \\ |\mathcal{F}_{A_1}\overline{\mathcal{F}_{A_2}}| = |\mathcal{F}_{B_1}\overline{\mathcal{F}_{B_2}}|, \end{cases} \tag{40}$$

which are of the form: $\mathcal{R}e(z) = \mathcal{R}e(z')$ and $|z| = |z'|$. Then we must have $z = z'$ or $z = \overline{z'}$, i.e.

$$\mathcal{F}_{A_1}\overline{\mathcal{F}_{A_2}} = \mathcal{F}_{B_1}\overline{\mathcal{F}_{B_2}}$$
$$\text{or } \mathcal{F}_{A_1}\overline{\mathcal{F}_{A_2}} = \overline{\mathcal{F}_{B_1}}\mathcal{F}_{B_2}.$$

In the first case ($\mathcal{F}_{A_1}\overline{\mathcal{F}_{A_2}} = \mathcal{F}_{B_1}\overline{\mathcal{F}_{B_2}}$) if we choose $A_+ = A_2$, $A_- = A_1$, $B_+ = B_2$ and $B_- = B_1$ we get right-homometric sets in the dihedral group since (34) is verified. In the second case ($\mathcal{F}_{A_1}\overline{\mathcal{F}_{A_2}} = \overline{\mathcal{F}_{B_1}}\mathcal{F}_{B_2}$), with $A_+ = A_2$, $A_- = A_1$, $B_+ = B_1$ and $B_- = B_2$ we get also right-homometric sets in the dihedral group. Thanks to Prop. 2.5 we know that these lifts are not trivially homometric. □

This result proves not only the existence of right-homometric lifts but also gives a way to build these lifts, which is very practical. There is also an interesting corollary.

Corollary 2.2. *In $\mathbb{Z}_{4N}$, we can always lift homometric sets of cardinality equal to 4 into right-homometric sets in D_{4N}.*

Proof. Rosenblatt ([24]) proved that if A and B are homometric in $\mathbb{Z}_n$ with $\sharp(A) = \sharp(B) = 4$, they are of the following two types:

- In $\mathbb{Z}_{4N} : \exists a \in \{1, 2, ..., N-1\}, N \geq 2$,

$$A = \{0, a, a+N, 2N\} \text{ and } B = \{0, a, N, 2N+a\}; \tag{41}$$

- in $\mathbb{Z}_{13N}$: in that case we do not have any general formulation.

In the case of (41), if we choose $A_1 = \{0, 2N\}, A_2 = \{a, a+N\}, B_1 = \{a, 2N+a\}$ and $B_2 = \{0, N\}$, the conditions of Thm. 2.5 are verified since

$$B_1 = T_a A_1 \Longrightarrow \mathbf{iv}(A_1) = \mathbf{iv}(B_1)$$

and

$$B_2 = T_{-a} A_2 \Longrightarrow \mathbf{iv}(A_2) = \mathbf{iv}(B_2).$$

For the choice of lifts we will do, we need to see whether $\mathcal{F}_{A_1}\overline{\mathcal{F}_{A_2}} = \mathcal{F}_{B_1}\overline{\mathcal{F}_{B_2}}$ or $\mathcal{F}_{A_1}\overline{\mathcal{F}_{A_2}} = \overline{\mathcal{F}_{B_1}}\mathcal{F}_{B_2}$. We will show that $\mathcal{F}_{A_1}\overline{\mathcal{F}_{A_2}} = \overline{\mathcal{F}_{B_1}}\mathcal{F}_{B_2}$. We have

$$\mathcal{F}_{A_1}\overline{\mathcal{F}_{A_2}}(k) = e^{\frac{i\pi ka}{2N}}(1+e^{-i\pi k})(1+e^{\frac{i\pi k}{2}}),$$
$$\overline{\mathcal{F}_{B_1}}\mathcal{F}_{B_2}(k) = e^{\frac{i\pi ka}{2N}}(1+e^{i\pi k})(1+e^{\frac{-i\pi k}{2}}) = e^{\frac{i\pi ka}{2N}}\overline{(1+e^{-i\pi k})(1+e^{\frac{i\pi k}{2}})}$$

for $k \in \mathbb{Z}_n$. It is easy to see that $\overline{(1+e^{-i\pi k})} = (1+e^{-i\pi k})$. We then divide the cases

$$\begin{cases} k = 4p \\ k = 4p+1 \\ k = 4p+2 \\ k = 4p+3. \end{cases}$$

For $k = 4p$ and $k = 4p+2$ we have $\overline{(1+e^{\frac{i\pi k}{2}})} = (1+e^{\frac{i\pi k}{2}})$, and for $k = 4p+1$ and $k = 4p+3$ we have $(1+e^{-i\pi k}) = 0$. Thus $\mathcal{F}_{A_1}\overline{\mathcal{F}_{A_2}}(k) = \overline{\mathcal{F}_{B_1}}\mathcal{F}_{B_2}(k)$ for all k. Consequently the choice we have to do for the lifts is $A_+ = A_2$, $A_- = A_1$, $B_+ = B_1$ and $B_- = B_2$ or equivalently $A_+ = A_1$, $A_- = A_2$, $B_+ = B_2$ and $B_- = B_1$. □

In fact we see in the proof of this corollary that the result is even stronger, but since we do not know the form of homometric sets of cardinality equal to 4 in $\mathbb{Z}_{13N}$ we keep this formulation.

The following proposition will be useful when we will try to build simultaneous right and left-homometric sets, since it gives a link between these two homometries in a special case.

Proposition 2.7. *If two sets $\mathcal{A}$ and $\mathcal{B}$ in D_n are such that $I_0A_+ = A_+$ and $I_0B_+ = B_+$ (or $I_0A_- = A_-$ and $I_0B_- = B_-$), then the equations for right homometry and left homometry in D_n are the same. In other words*

$$\mathcal{A} \text{ and } \mathcal{B} \text{ are right-homometric } \Longleftrightarrow \mathcal{A} \text{ and } \mathcal{B} \text{ are left-homometric.}$$

Proof. If $I_0A_+ = A_+$ and $I_0B_+ = B_+$, we have with Prop. 1.1: $\overline{\mathcal{F}_{A_+}} = \mathcal{F}_{A_+}$ and $\overline{\mathcal{F}_{B_+}} = \mathcal{F}_{B_+}$. Thus the conditions of right and left homometry are the same (Eq. 34 and Eq. 35 are identical). If $I_0A_- = A_-$ and $I_0B_- = B_-$ it works exactly the same. □

In the next paragraph we present applications and examples for each one of these results.

Applications – The Cyclic Decomposition

Here we show the relevance (in the sense that they are useful in practice) of the previous results and we give illustrations for each one. We considered decompositions of homometric sets A and B in $\mathbb{Z}_n$ of the form $A = A_1 \cup A_2$ and $B = B_1 \cup B_2$, with $\mathbf{iv}(A_1)$=$\mathbf{iv}(B_1)$ and $\mathbf{iv}(A_2)$=$\mathbf{iv}(B_2)$. An important part of homometric sets in $\mathbb{Z}_n$ is of that form, as it is presented in [10]. In this work Goyette classifies the homometric sets in $\mathbb{Z}_n$ into four types, and two of them are actually of this form.

The idea of Goyette is to decompose two homometric sets in $\mathbb{Z}_n$ each into two smaller subsets, according to the existence of cyclic collections in $\mathbb{Z}_n$. We recall that a cyclic subset is generated by a single element. For instance in $\mathbb{Z}_8$, $\{0, 2, 4, 6\}$ is a cyclic subset generated by 2. Remark that if n is a prime number then $\mathbb{Z}_n$ contains only two trivial cyclic subsets, since n has no divisor: $\{0\}$ and $\mathbb{Z}_n$ itself.

Goyette Classification. Goyette says that two homometric sets in $\mathbb{Z}_n$ belong to one of the following four types.

- **Type 1**. Two homometric sets that share the same cyclic subsets (modulo translation), and that have residual subsets of the same set class modulo transposition and inversion.

More formally two homometric sets A and B in $\mathbb{Z}_n$ are of type 1 if there exists a cyclic subset $\phi \subset \mathbb{Z}_n$, a subset $\psi \subset \mathbb{Z}_n$, and $(p, q) \in \mathbb{Z}_n^2$ so that

$$A = \phi \cup \psi \tag{42}$$

$$B = T_p\phi \cup T_q\psi \text{ or } B = T_p\phi \cup I_q\psi. \tag{43}$$

In that case if we choose $A_1 = \phi$, $A_2 = \psi$, $B_1 = T_p\phi$ and $B_2 = T_q\psi$ (or $B_2 = I_q\psi$), we are in the situtation of Thm. 2.5 thus we can find right-homometric lifts.

Notation 1 *For two sets A and B in $\mathbb{Z}_n$ a decomposition like (42) and (43) will be called a* cyclic decomposition*. It will concern type 1, type 2 and type 3 of Goyette.*

- **Type 2**. Two homometric sets that share the same cyclic subsets (modulo translation), and that have residual subsets that are homometric.

More formally two homometric sets A and B in $\mathbb{Z}_n$ are of type 2 if there exists a cyclic subset $\phi \subset \mathbb{Z}_n$, two homometric sets ψ and ψ' in $\mathbb{Z}_n$, and $p \in \mathbb{Z}_n$ so that

$$A = \phi \cup \psi$$
$$B = T_p\phi \cup \psi'.$$

In that case if we choose $A_1 = \phi$, $A_2 = \psi$, $B_1 = T_p\phi$ and $B_2 = \psi'$ we are also in the situtation of Thm. 2.5 thus we can find right-homometric lifts. We give examples of sets of types 1 and 2 just below.

- **Type 3**. Two homometric sets that share the same cyclic subsets (modulo translation), and that have residual subsets that are neither of the same class nor homometric.

Goyette says that these sets are rare. They have the same formulation than type 2 but ψ and ψ' are not homometric hence those sets do not correspond to the situation of Thm. 2.5. Consequently we do not have any general process to find homometric lifts in D_n. Here is an example of sets of that type.

Example 2.11. In $\mathbb{Z}_{16}$ the sets U_1 and U_2 are homometric of type 3:

$$U_1 = \{0,1,4,5,7,8,10\} = \overbrace{\{0,8\}}^{cyclic} \cup [0,3,4,6,9]$$
$$U_2 = \{0,1,2,4,5,8,11\} = \{0,8\} \cup [0,1,3,4,10].$$

By computation we found non-trivial right-homometric lifts of U_1 and U_2 in D_{16}, but no left-homometric lifts. We give here the right-homometric lifts, which are not based on the cyclic decomposition:

$$\mathcal{U}_1 = \{(0,-1),(1,1),(4,1),(5,-1),(7,-1),(8,1),(10,-1)\}$$
$$\mathcal{U}_2 = \{(0,1),(1,-1),(2,1),(4,-1),(5,1),(8,-1),(11,1)\}.$$

We move now to the last type.

- **Type 4**. Two homometric sets that contain (and thus share) no cyclic collection subsets.

Those sets do not correspond to the situation of Thm. 2.5, they do not even give a decomposition of homometric sets into smaller subsets. Nevertheless we give an example of two homometric sets of that type, also found by Goyette.

Example 2.12. In $\mathbb{Z}_{24}$ the sets V_1 and V_2 are homometric of type 4:

$$V_1 = \{0, 1, 2, 6, 8, 11\}$$
$$V_2 = \{0, 1, 6, 7, 9, 11\}.$$

There is neither right-homometric lifts nor left-homometric lifts for these sets in D_{24}.

As we said, types 1 and 2 allow us to build right-homometric lifts. Interestingly, Goyette notices that type 1 corresponds to the majority of homometric sets, including all the homometric sets in $\mathbb{Z}_{12}$. We deduce then the following very interesting result in a musical perspective.

Theorem 2.6. *We can lift all the homometric sets in $\mathbb{Z}_{12}$ into right-homometric sets in D_{12}.*

It is particularly interesting for us since it concerns the musical case $n = 12$. In musical words, Thm. 2.6 means that given two homometric melodies in $\mathbb{Z}_{12}$, we can build two right-homometric chord sequences whose base roots correspond to notes of these melodies (meaning that they are lifts of these melodies). We will give an example, that will be also an illustration of Cor. 2.2. Just before we mention an important remark.

Remark 2.5. We talk essentially about lifts coming from decompositions of the form $A = A_1 \cup A_2$ and $B = B_1 \cup B_2$, with $\mathbf{iv}(A_1)$=$\mathbf{iv}(B_1)$ and $\mathbf{iv}(A_2)$=$\mathbf{iv}(B_2)$, or from cyclic decompositions, but there are obviously other kinds of lifts. For instance we said in Thm. 2.6 that we can always lift homometric sets in $\mathbb{Z}_{12}$, because they are of type 1 and we have a technique to lift sets of this type. But we can lift some sets differently. Consider for instance

$$W_1 = \{0, 1, 3, 4, 5, 7\}$$
$$W_2 = \{0, 2, 3, 4, 5, 8\}.$$

We can lift these sets in D_{12} into the sets

$$\mathcal{W}_1 = \{(0,1),(1,-1),(3,-1),(4,1),(5,-1),(7,-1)\}$$
$$\mathcal{W}_2 = \{(0,1),(2,1),(3,-1),(4,1),(5,-1),(8,1)\}.$$

$\mathcal{W}_1$ and $\mathcal{W}_2$ are interestingly both right-homometric and left-homometric (non-trivially). This is also the case of the lifts $\mathcal{U}_1$ and $\mathcal{U}_2$ that we saw for type 3.

We propose now some illustrations of the results we found.

Illustration of Cor. 2.2 – Musical Considerations. As we saw in the proof of Cor. 2.2, all the homometric sets of cardinality equal to 4 in $\mathbb{Z}_{4N}$ are of the form

$$A = \{0, a, a+N, 2N\} \text{ and } B = \{0, a, N, 2N+a\}.$$

These sets correspond to type 1 with $\phi = \{0,6\}$, $\psi = \{1,4\}$, $p = a$ and $q = -a$ (B is of the form $B = T_p\phi \cup T_q\psi$). We know from Cor. 2.2 that we can lift these sets on the right side. More precisely, if we look at the proof of the corollary we know that choosing $A_+ = \phi$, $A_- = \psi$, $B_+ = T_{-a}\psi$ and $B_- = T_a\phi$, we obtain two right-homometric lifts in D_{4N}. These lifts are

$$\mathcal{A} = \{(0,1),(a,-1),(a+N,-1),(2N,1)\}$$
$$\mathcal{B} = \{(0,1),(a,-1),(N,1),(2N+a,-1)\}.$$

With $N = 3$ – then it concerns sets in $\mathbb{Z}_{12}$ – and $a = 1$ we obtain the famous all interval tetrachords $\{0,1,4,6\}$ and $\{0,1,3,7\}$. They lift into the right-homometric sets

$$\mathcal{A} = \{(0,1),(1,-1),(4,-1),(6,1)\}$$
$$\mathcal{B} = \{(0,1),(1,-1),(3,1),(7,-1)\}.$$

By a musical point of view the homometric melodies $\{C, D^\flat, E, G^\flat\}$ and $\{C, D^\flat, E^\flat, G\}$ lift in the right-homometric chord sequences

$$\mathcal{A} = \{C, d^\flat, e, G^\flat\}$$
$$\mathcal{B} = \{C, d^\flat, E^\flat, g\}.$$

Is it possible to lift these sets into left-homometric sets? With this choice of decomposition we see, refering to Prop. 2.3, that we get two trivial left-homometric sets since $T_aA_+ = B_-$ and $T_{-a}A_- = B_+$. In fact with computation we see that there is not any non-trivial left-homometric lift of these sets.

Illustration of Thm. 2.5 with Type 2 of Goyette. We gave just above some illustrations of type 1 of Goyette. Let us move to the type 2. Goyette says that this type is more rare. He gives an example.

Example 2.13. In $\mathbb{Z}_{24}$ the sets

$$S_1 = \{0, 1, 5, 6, 8, 12, 14\}$$
$$S_2 = \{0, 3, 4, 5, 9, 11, 17\}$$

are of type 2. The cyclic subset associated is $\{0, 12\}$. We have

$$S_1 = C_1 \cup \{0, 12\}$$
$$S_2 = C_2 \cup \{5, 17\}$$
$$C_1 = \{1, 5, 6, 8, 14\} \text{ (set class [0,4,5,7,13])}$$
$$C_2 = \{0, 3, 4, 9, 11\} \text{ (set class [0,2,7,8,11])}$$
$$C_1 \text{ is homometric with } C_2.$$

We are in the case where $\phi = \{0, 12\}$, $\psi = \{1, 5, 6, 8, 14\}$, $\psi' = \{0, 3, 4, 9, 11\}$ and $p = 5$. The right-homometric lifts we obtain in D_{24} are

$$\mathcal{S}_1 = \{(0, 1), (1, -1), (5, -1), (6, -1), (8, -1), (12, 1), (14, -1)\}$$
$$\mathcal{S}_2 = \{(0, 1), (3, 1), (4, 1), (5, -1), (9, 1), (11, 1), (17, -1)\}.$$

There is not any left-homometric lifts for these sets.

Illustration of Prop. 2.7: About Simultaneous Right and left-homometric Sets. One interesting question is to find simultaneously right and left-homometric sets. We already gave an example when we presented the actions of the T/I-group and the PLR-group as left and right actions of the dihedral group on itself, and another example with the sets $\mathcal{W}_1$ and $\mathcal{W}_2$. As we said in Prop. 2.7, if $I_0 A_+ = A_+$ and $I_0 B_+ = B_+$ (or $I_0 A_- = A_-$ and $I_0 B_- = B_-$) the equations for right homometry and for left homometry are the same: it is then a very convenient situation to find simultaneously right and left-homometric sets. In fact it is the situation of the first example we gave. Recall that the sets were

$$\mathcal{D} = \{(0, -1), (1, 1), (3, 1), (4, -1), (8, -1)\} = \{c, D^\flat, E^\flat, e, a^\flat\}$$
$$\mathcal{E} = \{(0, -1), (3, 1), (4, -1), (5, 1), (8, -1)\} = \{c, E^\flat, e, F, a^\flat\}$$

in D_{12}. $\mathcal{D}$ and $\mathcal{E}$ are non trivially right and left-homometric in D_{12}, and correspond in fact to lifts of the sets $D = \{0,1,3,4,8\}$ and $E = \{0,3,4,5,8\}$ in $\mathbb{Z}_{12}$, with $D_+ = \{1,3\}$, $D_- = \{0,4,8\}$, $E_+ = \{3,5\}$ and $E_- = \{0,4,8\}$. Here $D_- = E_-$ and $I_0 D_- = D_-$ hence we are in the situation of Prop. 2.7.

In fact these sets are of type 1 with $\phi = D_-$, $\psi = D_+$, $p = 0$ and $q = 2$. Conversely every pair of type 1 with $p = 0$ is in the situation of Prop. 2.7, because every cyclic subset ϕ verifies the relation $I_0\phi = \phi$. In practice (is it a general result?) all the simultaneously right and left-homometric sets that we found by computation, and that correspond to the decomposition of Thm. 2.5, are of that form (type 1 with $p = 0$).

About Left Homometry

We did not talk a lot about left homometry because we have less convenient properties with them. For instance we can not always lift on the left two homometric sets in $\mathbb{Z}_n$ (whatever the type), the projection of left-homometric sets do not give homometric sets in $\mathbb{Z}_n$, etc.

With a calculation point of view the difficulty comes from the fact that the homometry condition in $\mathbb{Z}_n$ is of the form

$$\mathcal{F}_{A_1}\overline{\mathcal{F}_{A_2}} = \mathcal{F}_{B_1}\overline{\mathcal{F}_{B_2}} \tag{44}$$

$$\text{or } \mathcal{F}_{A_1}\overline{\mathcal{F}_{A_2}} = \overline{\mathcal{F}_{B_1}}\mathcal{F}_{B_2} \tag{45}$$

if we take the decomposition $A = A_1 \cup A_2$ and $B = B_1 \cup B_2$ with $\mathbf{iv}(A_1)$=$\mathbf{iv}(B_1)$ and $\mathbf{iv}(A_2)$=$\mathbf{iv}(B_2)$. Refering to the proof of Thm. 2.5: it is equivalent to right homometry but not at all to left homometry ($\mathcal{F}_{A_1}\mathcal{F}_{A_2} = \mathcal{F}_{B_1}\mathcal{F}_{B_2}$).

Since $\mathbf{iv}(A_1) = \mathbf{iv}(B_1)$ and $\mathbf{iv}(A_2) = \mathbf{iv}(B_2)$, we can write for instance

$$\begin{cases} \mathcal{F}_{A_1}(t) = e^{i\alpha(t)}\mathcal{F}_{B_1}(t) \\ \mathcal{F}_{A_2}(t) = e^{i\beta(t)}\mathcal{F}_{B_2}(t) \end{cases} \tag{46}$$

for some functions α, β. We deduce that

$$\mathcal{F}_{A_1}(t)\overline{\mathcal{F}_{A_2}(t)} = e^{i(\alpha(t)-\beta(t))}\mathcal{F}_{B_1}(t)\overline{\mathcal{F}_{B_2}(t)} \tag{47}$$

and with Eq. 44 we obtain

$$\mathcal{F}_{B_1}(t)\overline{\mathcal{F}_{B_2}(t)}\left(1 - e^{i(\alpha(t)-\beta(t))}\right) = 0. \tag{48}$$

Here it is complicated to go further (i.e. to do simplifications) since all the terms can be nil. For instance we know from [2] that $P \subset \mathbb{Z}_n$ is periodic (meaning

$P+m = P \iff T_m P = P$ for some m) if and only if $\mathcal{F}_P(t) = 0$ except if t belongs to some subgroup of $\mathbb{Z}_n$. A cyclic subset ϕ is periodic and we have $\mathcal{F}_\phi(t) = 0$ except if t belongs to some subgroup of $\mathbb{Z}_n$. Consequently if we consider sets of type 1, Eq. 48 is not easily solvable.

Table

To sum up we give below a table that recalls some properties we saw concerning right and left homometry in the dihedral group.

Table 3. Comparisons between right and left homometry in the dihedral group.

	Homometry for the right action	Homometry for the left action
Cardinality fixed and n fixed	Same number of homometric sets	
Invariance under... ... for $n = 12$	Left translation $\sim$ Action of the T/I-group	Right translation $\sim$ Action of the PLR-group
Projection π_1 on $\mathbb{Z}_n$	Z-rel. in $D_n \Longrightarrow$ Z-rel. in $\mathbb{Z}_n$ (trivial $\Longrightarrow$ trivial)	No
Decomposition $A = A_1 \cup A_2$ and $B = B_1 \cup B_2$ with $\mathbf{iv}(A_1)$=$\mathbf{iv}(B_1)$ and $\mathbf{iv}(A_2)$=$\mathbf{iv}(B_2)$		
Existence of lifts	Yes	Not always
Invariance with the choice A_+, A_-, B_+, B_-	No	Yes
Triviality conditions	$T_p A_+ = B_+$ and $T_p A_- = B_-$ $I_p A_+ = B_-$ and $I_p A_- = B_+$	$T_p A_+ = B_+$ and $T_{-p} A_- = B_-$ $T_p A_+ = B_-$ and $T_{-p} A_- = B_+$
Case $I_0 A_+ = A_+$ and $I_0 B_+ = B_+$	Same formulation for homometry with the first choice	

We move now to our computation results of enumeration of homometric sets in the dihedral group.

Table 4. Table with the number of homometric pairs, triples, t-uples in D_n for different values of n and p.

Cardinality	D_n	Homometric sets for the right/left action	Simulataneous right and left-homometric sets
$p = 4$	$n = 8$	2 pairs	0 pair
	$n = 12$	3 pairs	0 pair
	$n = 16$	4 pairs	0 pair
$p = 5$	$n = 8$	12 pairs	1 pair
	$n = 10$	20 pairs	2 pairs
	$n = 12$	8 pairs/2 triples	2 pairs
	$n = 14$	21 pairs	0 pair
	$n = 15$	15 pairs	3 pairs
	$n = 16$	40 pairs	2 pairs
	$n = 18$	30 pairs/3 triples	3 pairs
$p = 6$	$n = 8$	30 pairs/3 quadruples	9 pairs/1 quadruple
	$n = 9$	54 pairs/3 triples	0 pair/1 triple
	$n = 10$	70 pairs	4 pairs
	$n = 12$	358 pairs	53 pairs
	$n = 14$	252 pairs	14 pairs
	$n = 15$	225 pairs	18 pairs
	$n = 16$	500 pairs/6 quadruples	74 pairs/2 quadruples
	$n = 18$	906 pairs/6 triples	49 pairs/2 triples
$p = 7$	$n = 8$	36 pairs	12 pairs
	$n = 9$	63 pairs	5 pairs
	$n = 10$	102 pairs/5 quintuples	20 pairs/2 quintuples
	$n = 11$	55 pairs	0 pair
	$n = 12$	317 pairs/11 triples/10 quadruples/ 2 sextuples	63 pairs/8 triples/1 quadruple/ 1 sextuple
	$n = 13$	130 pairs	0 pair
	$n = 14$	539 pairs	140 pairs
	$n = 15$	405 pairs	36 pairs

2.4 Enumeration of Homometric Sets in D_n

Here we give the results of our computation in the quest to find all the homometric sets in the dihedral group. The problem of enumeration is open, we propose here a complete enumeration for small values of n and cardinality p, and we give all the homometric sets for $p = 4$ and $p = 5$ with $n = 12$ in musical form. In the Annex A at the end of this work we give some program codes and a more complete listing.

Enumeration of Tab. 4. In this table we give the number of homometric pairs, triples, t-uples in D_n for different values of n and p, where p corresponds to the cardinality of the sets. The column on the left gives the number of right-homometric t-uples (or left, the number is the same as we saw in Cor. 2.1), and the column on the right gives the number of simultaneously right and left-homometric t-uples.

The first homometric pair appears for $n = 8$ and $p = 4$. It could not be a surprise since the first homometric pair in $\mathbb{Z}_n$ appears also for $n = 8$ and $p = 4$, and we know from Prop. 2.4 that if we find a right-homometric pair in D_n we obtain with the projection π_1 a homometric pair in $\mathbb{Z}_n$. However we have to be careful because we can obtain a trivial homometric pair by projection of a non trivial right-homometric pair...

We notice the existence of homometric t-uples with $t > 2$: the first triple appears for $n = 12$, $p = 5$ (which is not the same in $\mathbb{Z}_n$), and the first simultaneously right and left pair appears for $n = 8$ and $p = 5$.

If the reader wants more details he can refer to Annex A where we give a complete listing of homometric sets for the right and for the left actions for small values of n and p. There is also a *Python* program code for finding homometric lifts in D_n from homometric sets in $\mathbb{Z}_n$.

Enumeration of Tab. 5. In Tab. 5 we give in a musical form the complete enumeration of homometric sets in D_{12} (i.e. for $n = 12$) for the right and the left actions for $p = 4$ and $p = 5$. We recall that they correspond to the action of the PLR and the T/I-groups. In boldface we see the simultaneously left and right-homometric pairs, and at the bottom of the table we see the triples.

We know from Prop. 2.4 that if we just keep the roots of the chords of homometric sets for the right action we obtain homometric sets in $\mathbb{Z}_{12}$: for instance the third pair with $p = 4$ is $\{C, d, f, G^\flat\}\&\{C, d, E^\flat, a^\flat\}$, hence we know that $\{C, D, F, G^\flat\}\&\{C, D, E^\flat, A^\flat\}$ is a homometric pair in $\mathbb{Z}_{12}$. Notice that if we do the projection of the first pair we get multisets since the chords contain both C

and c, and if we do the projection of the second pair we obtain the well-known "all interval tetrachords" $\{C, D^\flat, E, G^\flat\}\&\{C, D^\flat, E^\flat, G\}$ ($\{0,1,4,6\}\&\{0,1,3,7\}$).

Partial Conclusion

We gave all the results we found concerning homometry in the dihedral group. These results were in part found thanks to the efficiency of the discrete Fourier transform. The next subsection deals with the compositional aspects of homometry in the dihedral groups. By a musical point of view this subsection is fundamental because it gives a large generalization of the work we did to other kind of chords than major and minor triads.

Table 5. Left and right-homometric sets in D_{12} in musical form.

$N = 12$	Type	Homometric sets for the action of the T/I-group (left action)	Homometric sets for the action of the PLR-group (right action)
$p = 4$	Pairs	$\{C, d, e^\flat, G^\flat\}\&\{C, c, g^\flat, A\}$	$\{C, c, e^\flat, G^\flat\}\&\{C, c, E^\flat, g^\flat\}$
		$\{C, d^\flat, e, G^\flat\}\&\{C, d^\flat, g, A\}$	$\{C, d^\flat, e, G^\flat\}\&\{C, d^\flat, E^\flat, g\}$
		$\{C, d, f, G^\flat\}\&\{C, d, a^\flat, A\}$	$\{C, d, f, G^\flat\}\&\{C, d, E^\flat, a^\flat\}$
$p = 5$	Pairs	$\boldsymbol{\{C, c, d, E, A^\flat\}\&\{C, d, e, E, A^\flat\}}$	$\boldsymbol{\{C, c, d, E, A^\flat\}\&\{C, d, e, E, A^\flat\}}$
		$\boldsymbol{\{C, d^\flat, e^\flat, E, A^\flat\}\&\{C, e^\flat, E, f, A^\flat\}}$	$\boldsymbol{\{C, d^\flat, e^\flat, E, A^\flat\}\&\{C, e^\flat, E, f, A^\flat\}}$
		$\{C, c, d^\flat, f, G^\flat\}\&\{C, c, g^\flat, G, B\}$	$\{C, c, d^\flat, f, G^\flat\}\&\{C, c, D^\flat, F, g^\flat\}$
		$\{C, c, e^\flat, f, G^\flat\}\&\{C, c, E^\flat, g^\flat, B\}$	$\{C, c, e, f, G^\flat\}\&\{C, c, D^\flat, g^\flat, A^\flat\}$
		$\{C, c, D^\flat, g^\flat, A^\flat\}\&\{C, c, g^\flat, G, A^\flat\}$	$\{C, c, E, F, g^\flat\}\&\{C, c, E, g^\flat, B\}$
		$\{C, d^\flat, D^\flat, g, A^\flat\}\&\{C, d^\flat, g, G, A^\flat\}$	$\{C, d^\flat, E, F, g\}\&\{C, d^\flat, E, g, B\}$
		$\{C, D^\flat, d, a^\flat, A^\flat\}\&\{C, d, G, a^\flat, A^\flat\}$	$\{C, d, E, F, a^\flat\}\&\{C, d, E, a^\flat, B\}$
		$\{C, D^\flat, e^\flat, A^\flat, a\}\&\{C, e^\flat, G, A^\flat, a\}$	$\{C, e^\flat, E, F, a\}\&\{C, e^\flat, E, a, B\}$
	Triples	$\{C, c, d, e^\flat, G^\flat\}\&\{C, c, D, g^\flat, B^\flat\}$ $\&\{C, c, g^\flat, A^\flat, B^\flat\}$	$\{C, c, d, e, G^\flat\}\&\{C, c, D, E, g^\flat\}$ $\&\{C, c, D, g^\flat, B^\flat\}$
		$\{C, d^\flat, e^\flat, f, G^\flat\}\&\{C, d^\flat, D, g, B^\flat\}$ $\&\{C, d^\flat, g, A^\flat, B^\flat\}$	$\{C, d^\flat, e^\flat, f, G^\flat\}\&\{C, d^\flat, D, E, g\}$ $\&\{C, d^\flat, D, g, B^\flat\}$

2.5 Composing Homometric Music in D_n

We gave a musical interpretation to homometry in D_n only for the case $n = 12$, as homometry between sets of major and minor triads : we identified in D_{12} the pairs $(k, +1)$ with the major triads and the pairs $(k, -1)$ with the minor triads. What kind of musical interpretation can we give when $n \neq 12$? We present here an interesting generalization.

If we call $\langle X \rangle$ the set generated by the action of the T/I-group on a chosen chord X – i.e. $\langle X \rangle = \{X, T_1X, ..., T_{11}X, I_0X, ..., I_{11}X\}$ – we have a natural action of the T/I-group on $\langle X \rangle$

$$\begin{aligned} \lambda : T/I &\longrightarrow Sym(\langle X \rangle) \\ g &\longmapsto (x \mapsto gx) \end{aligned}$$

and a natural function

$$\begin{aligned} T/I &\longrightarrow \langle X \rangle \\ g &\longmapsto gX. \end{aligned}$$

We will suppose that this function is a bijection (the action of T/I on $\langle X \rangle$ is thus simply transitive). The action of T/I on $\langle X \rangle$ is essentially the same as left multiplication ($g(hX) = (gh)X$), and we know from [8] and [22] that we can define a second action

$$\begin{aligned} \rho : T/I &\longrightarrow Sym(\langle X \rangle) \\ g &\longmapsto (hX \mapsto hg^{-1}X), \end{aligned}$$

which is the same as right multiplication. The group $\rho(T/I)$ is the dual group to $\lambda(T/I)$, the functions $T/I \longrightarrow \langle X \rangle$ and ρ depend on X, but the group $\rho(T/I)$ does not. If we choose $X = C$-major, $\rho(T/I)$ is in fact the PLR-group which is, as we know, the dual group to the T/I-group. The operations P, L and R correspond to right multiplication by I_7, I_{11} and I_4. With this point of view we generalize the action of the PLR-group on the set S, seeing it as a right action of the T/I-group on $\langle X \rangle$. In this perspective, we obtain on Fig. 11 a new version of the diagrams (Fig. 6 and Fig. 9) with the sets $A = \{c, D^\flat, E^\flat, e, a^\flat\}$ and $B = \{c, E^\flat, e, F, a^\flat\}$.

With other choices of X such that the action of the T/I-group on X is simply transitive, we deduce from the same example that $\{X, I_8X, T_8X, T_4X, I_{10}X\}$ and $\{X, I_{10}X, T_8X, I_0X, T_4X\}$ are two left-homometric sets (top of Fig. 12). We can then generalize our interpretation of homometry in the dihedral groups

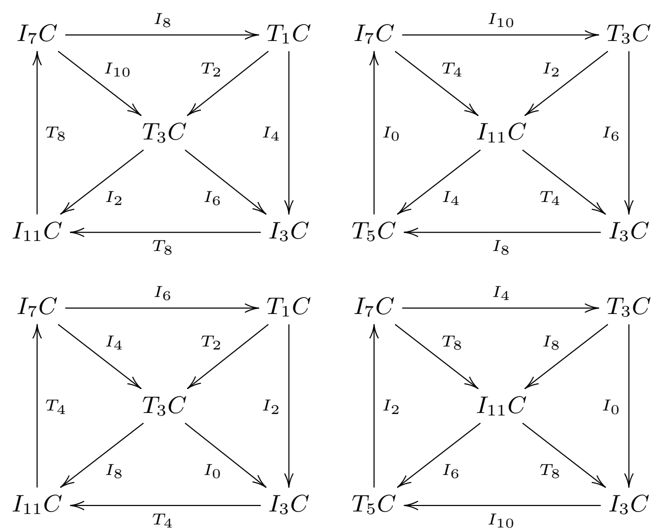


Fig. 11. Left-intervals (top) and right-intervals (bottom) in the T/I-group for the two sets $A = \{c, D^\flat, E^\flat, e, a^\flat\}$ and $B = \{c, E^\flat, e, F, a^\flat\}$ with $X = C$-major.

to chords with more than 3 notes, and also to unclassified chords. For instance with $X = C^7$, we obtain the two left-homometric chord sequences of Fig. 12 (bottom), which contain dominant and half-diminished chords. With $X = [0, 1, 5] = [C, D^\flat, F]$ we obtain the two left-homometric sets

$$\{[C, D^\flat, F], [A^\flat, A, E], [A^\flat, A, D^\flat], [E, F, A], [B^\flat, B, G^\flat]\}$$
$$\{[C, D^\flat, F], [B^\flat, B, G^\flat], [A^\flat, A, D^\flat], [C, D^\flat, A^\flat], [E, F, A]\}.$$

Besides, we can also obtain a bijection with the dihedral group D_{12}, choosing $(0, 1) = X$, $(k, 1) = T_kX$, $(0, -1) = I_0X$ and $(k, -1) = I_kX = T_kI_0X$. It allows us to give a musical interpretation to homometry in the dihedral groups for all n. Indeed the above identifications are valid for all n and give a bijection between the T/I-group in $\mathbb{Z}_n$ and the dihedral group D_n. Thus the left- and the right-homometry in D_n can be interpreted as the left- and the right-homometry coming from the left and the right actions of the T/I-group on some $\langle X \rangle$ for all n. Musically it allows us to use microtonality.

In the next paragraph we mention other possible musical interpretation for $n = 12$.

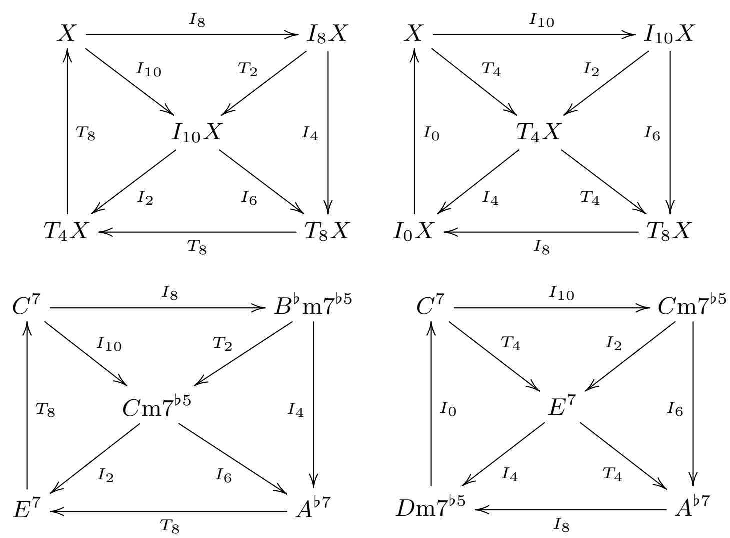


Fig. 12. (Top) Two left-homometric sets for X such that the T/I-group acts simply transitively on X. (Bottom) Application with $X = C^7$: we obtain two left-homometric chord progressions containing.

Some Other Musical Propositions with $n = 12$.

We can consider homometry in D_{12} as a musical concept without considering the neo-Riemannian groups and the set S. After a discussion with the composer Tom Johnson (who wrote pieces for homometry in $\mathbb{Z}_n$, see *Intervals* (2013)) and two composers working at Ircam (Karim Hadad and Mikhail Malt), it seems important for us to mention some other ways to interpret musically a set in D_{12}. An element in D_{12} is a pair $(s, \pm 1)$ with $s \in \mathbb{Z}_{12}$. It could designate:

- An ascending $(s, +1)$ or descending $(s, -1)$ interval of s pitch classes. Hence a set in D_{12} is a chord that we can build from an arbitrary note (C for instance) by adding notes corresponding to these intervals. For instance the set $\{(0,-1),(1,1),(4,1),(6,-1)\}$ will correspond to $\{-1,1,4,-6\}$, which is musically $\{B, D^\flat, E, G\flat\}$, and $\{(0,-1),(1,1),(3,-1),(7,1)\}$ will correspond to $\{-1,1,-3,7\}$ which is $\{B, D^\flat, A, G\}$: we obtain two homometric chords for the right action. If we transpose them to start with C we get

$$\begin{cases} \{C, D, F, G\} \\ \{C, D, B^\flat, A^\flat\}; \end{cases}$$

- A rhythmic impact played at time s with two velocities: $(s,-1)$ is a low impact and $(s,+1)$ is a loud impact. Hence a set in D_{12} is a rhythm: if we consider $\{(0,-1),(1,1),(4,1),(6,-1)\}$ and $\{(0,-1),(1,1),(3,-1),(7,1)\}$ we get the two rhythmic patterns given on Fig. 13, respectively of length 7 (first bar) and 8 (second bar), the unit 1 being the quarter note. We underlined the notes that have to be accentuated as they correspond to the component $+1$.

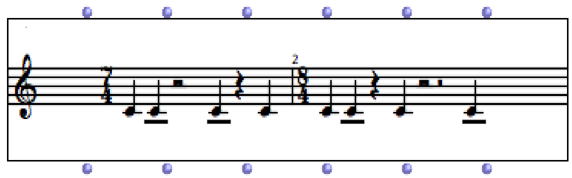


Fig. 13. Rhythmic representation of the right-homometric sets $\{(0,-1),(1,1),(4,1),(6,-1)\}$ and $\{(0,-1),(1,1),(3,-1),(7,1)\}$ in D_n.

- The note s played by a first instrument (symbolized by $+1$) or by a second instrument (symbolized by -1): hence a set in D_{12} corresponds to the union of two melodies played by two different instruments. With the same example than before and choosing for instance $+1$ to be a violin and -1 to be a cello, we get the two right-homometric sets: $\{\text{violin} : \{D^\flat, E\}, \text{cello} : \{C, G^\flat\}\}$ and $\{\text{violin} : \{D^\flat, G\}, \text{cello} : \{C, E^\flat\}\}$.

The two last propositions are very near the classical concept of homometry in $\mathbb{Z}_{12}$ (Z-relation) since the sets we keep finally are homometric in $\mathbb{Z}_{12}$. The fact that they lie in the dihedral group gives us a way to make distinctions and musical variations between elements inside these sets.

We propose in the next subsection a different way to study homometry in the dihedral group, which is based on polynomials. We do not give any new result, the main idea being more to show the practical aspect of polynomials for concrete calculations.

2.6 Using Polynomials for Homometry

We know from [19] or [3] that we can use polynomials to study homometry in $\mathbb{Z}_n$. For a set $A \subset \mathbb{Z}_n$ it consists in using the polynomial $A(x) = \sum_{k \in A} x^k \in \mathbb{Z}[x]$ with $\mathbb{Z}[x] := \mathbb{Z}[X]/(X^n - 1)$, where we note $x = X \mod X^n - 1$. The polynomials yield an isomorphism with the Fourier space and it is a convenient formulation for homometry in certain cases because it allows "operation on sets" via product and sum of polynomials.

First let us recall that $K[X]$ designates the ring of polynomials in one variable with coefficients in the ring K. $K[x] := K[X]/(X^n - 1)$ is the quotient ring for the equivalence relation $\mathcal{R}$

$$A \mathcal{R} B \Longleftrightarrow A \equiv B \bmod X^n - 1. \tag{49}$$

For more details about the general quotient $K[X]/(P)$ of $K[X]$ by an ideal generated by a polynomial $P \in K[X]$, please refer to [25]. Nevertheless we mention an important result for our study.

Proposition 2.8. *The ring $\mathbb{Z}[x] = \mathbb{Z}[X]/(X^n - 1)$ is not a field.*

Proof. We know from [25] that $K[X]/(P)$ is a field if and only if P is irreductible in $K[X]$. We have

$$X^n - 1 = (X - 1)(X^{n-1} + X^{n-2} + ...X + 1), \tag{50}$$

then $X^n - 1$ is not irreductible in $\mathbb{Z}[X]$. Consequently $\mathbb{Z}[x]$ is not a field. □

Example 2.14. If we look for instance at $\mathbb{Z}[X]/(X^{12} - 1)$, we have

$$(1 + X^6)(X - X^7) = 0$$

but $1 + X^6 \neq 0$ and $X - X^7 \neq 0$. This is not convenient and we will have to be careful in the rest of this section because we can not *a priori* symplify polynomial equations.

Definition 2.9. *Let A be a subset in $\mathbb{Z}_n$, we call* characteristic polynomial of *A the polynomial $A(x) = \sum_{k \in A} x^k \in \mathbb{Z}[x]$.*

We have an isomorphism between the Fourier space $\mathbb{C}^{\mathbb{Z}_n} \cong \mathbb{C}^n$ and the vector space $\mathbb{C}[X]/(X^n - 1)$. It is given by the relation

$$A(e^{-2ik\pi/n}) = \mathcal{F}_A(k)$$

for A a subset in $\mathbb{Z}_n$. For the proof refer to [3]. In particular the convolution product of two characteristic functions $\mathbb{1}_A$ and $\mathbb{1}_B$ is obtained by the product $A(X)B(X) \bmod X^n - 1$, transposition by p is simply multiplication by X^p and inversion consits in

$$I(A)(X) = X^n A(1/X).$$

This formulation is especially convenient when we consider homometry since the operations **ifunc** and **iv** are easily expressed.

Proposition 2.9. *For two sets A and B in $\mathbb{Z}_n$ we have*

$$\begin{cases} \boldsymbol{ifunc}(A,B) = A(x^{-1})B(x) \\ \boldsymbol{iv}(A) = A(x^{-1})A(x). \end{cases} \tag{51}$$

In particular two sets A and B are homometric if and only if

$$A(x^{-1})A(x) = B(x^{-1})B(x). \tag{52}$$

We can then give new formulations of the homometry conditions in the dihedral group, with the same notations.

Notation 2 *We associate to a set $\mathcal{A} \subset D_n$ two polynomials, namely the characteristic polynomial $A_+(x)$ of A_+ and the characteristic polynomial $A_-(x)$ of A_-, the sets A_+ and A_- being defined as in Def. 2.7.*

Example 2.15. With $\mathcal{A} = \{(0,-1),(1,1),(3,1),(4,-1),(8,-1)\}$ we saw that $A_+ = \{1,3\}$ and $A_- = \{0,4,8\}$, hence

$$\begin{cases} A_+(x) = x + x^3 \\ A_-(x) = 1 + x^4 + x^8. \end{cases} \tag{53}$$

Theorem 2.7. *Two sets $\mathcal{A}$ and $\mathcal{B}$ in D_n are homometric for the right action if and only if the two following equations hold:*

$$\begin{cases} A_+(x^{-1})A_+(x) + A_-(x^{-1})A_-(x) = B_+(x^{-1})B_+(x) + B_-(x^{-1})B_-(x) \\ A_+(x^{-1})A_-(x) = B_+(x^{-1})B_-(x). \end{cases} \tag{54}$$

Two sets $\mathcal{A}$ and $\mathcal{B}$ in D_n are homometric for the left action if and only if the two following equations hold:

$$\begin{cases} A_+(x^{-1})A_+(x) + A_-(x^{-1})A_-(x) = B_+(x^{-1})B_+(x) + B_-(x^{-1})B_-(x) \\ A_+(x)A_-(x) = B_+(x)B_-(x). \end{cases} \tag{55}$$

In order to practice the equivalence between the Fourier transform and the polynomial notations we give (one more time!) another proof of Prop. 2.4. If $\mathcal{A}$ and $\mathcal{B}$ are right-homometric in D_n, we have

$$\begin{aligned} A(x)A(x^{-1}) &= (A_+(x) + A_-(x))(A_+(x^{-1}) + A_-(x^{-1})) \\ &= A_+(x)A_+(x^{-1}) + A_+(x)A_-(x^{-1}) + A_-(x)A_+(x^{-1}) + A_-(x)A_-(x^{-1}) \\ &= B_+(x)B_+(x^{-1}) + B_+(x)B_-(x^{-1}) + B_-(x)B_+(x^{-1}) + B_-(x)B_-(x^{-1}) \\ &= B(x)B(x^{-1}). \end{aligned}$$

Thus $A = \pi_1(\mathcal{A})$ and $B = \pi_1(\mathcal{B})$ are homometric in $\mathbb{Z}_n$.

Unfortunately it is not always possible to use the same arguments with the polynomials than those used with the Fourier transform. For instance we did not manage to find an equivalent proof to Thm. 2.5. Instead of

$$\mathcal{R}e(\overline{\mathcal{F}_{A_1}}\mathcal{F}_{A_2}) = \mathcal{R}e(\overline{\mathcal{F}_{B_1}}\mathcal{F}_{B_2}),$$

we get

$$A_1(x)A_2(x^{-1}) + A_2(x)A_1(x^{-1}) = B_1(x)B_2(x^{-1}) + B_2(x)B_1(x^{-1})$$

with polynomials, which is much more complicated. Then there are advantages and drawbacks with the polynomial notations. It is hard to solve polynomial equations (it is even harder since $Z[x]$ is not a field), but on the contrary polynomials are efficient in practice for direct calculations.

If we consider for instance two sets A and B of type 1 of Goyette's classification, we get from Eq. (42) and Eq. (43)

$$\begin{aligned} &A(x) = \phi(x) + \psi(x) \\ &B(x) = x^p\phi(x) + x^q\psi(x) \text{ or } B(x) = x^p\phi(x) + x^q\psi(x^{-1}) \end{aligned}$$

and the equations of Thm. 2.7 for right homometry are (remark that ϕ is cyclic then $\phi(x^{-1}) = \phi(x)$)

$$\phi(x)\left(\psi(x) - x^{p-q}\psi(x)\right) = 0 \text{ or } \phi(x)\left(\psi(x) - x^{q-p}\psi(x^{-1})\right) = 0 \qquad (56)$$

according to the choice we do for A_+, A_-, B_+ and B_-. These last equations are equivalent to Eq. 48, and similarly we can not conclude that $\phi(x) = 0$ or $\psi(x) - x^{p-q}\psi(x) = 0$.

However it is possible to find the polynomials P such that $\phi(x)P(x) = 0$ for a given ϕ. We give the solutions in the following proposition.

Proposition 2.10. *If we consider the cyclic subset $\phi = \{0, n, ..., (p-1)n\}$ (with characteristic polynomial $\phi(x) = 1 + x^n + ... + x^{(p-1)n}$) in $\mathbb{Z}_{pn}$, we have*

$$\phi P = 0$$

$$\Longleftrightarrow$$

$$P(x) = P_1(x)(1 - x^n) + P_2(x)(1 - x^{2n}) + ... + P_{p-1}(x)(1 - x^{(p-1)n}),$$

where $P_i \in K[x]$ for all $i \in \{1, ..., p-1\}$.

Proof. We write

$$P(x) = a_0 + a_1 x + ... + a_{(p-1)n} x^{(p-1)n}.$$

If $\phi P = 0$ we have $P = -x^n P - ... - x^{(p-1)n} P$.

Besides for $k \in \{0, n, ..., (p-1)n\}$

$$-x^{kn} P(x) = -a_0 x^{kn} - a_1 x^{kn+1} - ... - a_{(p-k)n-1} x^{pn-1} - a_{(p-k)n} \\ - a_{(p-k)n+1} x - ... - a_{pn-1} x^{kn-1}.$$

It gives by identification $a_0 = -a_{(p-1)n} - a_{(p-2)n} - ... - a_n$ and

$$a_i = -\sum_{k=1}^{p-1} a_{(p-k)n+i}$$

for $i \in \{0, ..., pn-1\}$. This last expression can be written

$$a_{(p-1)n+i} = -a_i - a_{n+i} - ... - a_{(p-2)n+i}$$

and finally

$$P(x) = a_0 + a_1 x + ... + a_{(p-1)n-1} x^{(p-1)n-1} \\ - \sum_{i=0}^{n-1} (a_i - a_{n+i} - ... - a_{(p-2)n+i}) x^{(p-1)n+i}$$

$$\implies P(x) = P_1(x)(1 - x^n) + P_2(x)(1 - x^{2n}) + ... + P_{p-1}(x)(x^{1-(p-1)n}),$$

with $P_k(x) = \sum_{j=0}^{n-1} a_{(k-1)n+j} x^{(k-1)n+j}$.

Conversely if

$$P(x) = P_1(x)(1 - x^n) + P_2(x)(1 - x^{2n}) + ... + P_{p-1}(x)(x^{1-(p-1)n}),$$

it is clear that $\phi P = 0$. □

We will use this proposition in the following example.

Example 2.16. Consider again $\{0, 1, 4, 6\}$ and $\{0, 1, 3, 7\}$ in $\mathbb{Z}_{12}$. These sets are homometric of type 1 with $\phi = \{0, 6\}$, $\psi = \{1, 4\}$, $p = 1$ and $q = -1$.

We have

$$\begin{aligned} \psi(x) - x^{p-q} \psi(x^{-1}) &= x + x^4 - x^2 (x^{-1} + x^{-4}) \\ &= x + x^4 - x - x^{10} \\ &= x^4 (1 - x^6). \end{aligned}$$

Then we know thanks to Prop. 2.10 that $\phi(x) \left(\psi(x) - x^{p-q} \psi(x^{-1}) \right) = 0$ hence (56) is verified and we can lift these sets into right-homometric sets.

As we see in this example, the formulation with polynomials is useful for calculation. We can quickly check whether two sets are homometric or not. However this approach does not give any new result about homometry in the dihedral group.

Partial Conclusion

In this section we focused our attention on homometry in the dihedral group, for the right and the left actions. We first described the equivalence between these actions and the actions of the neo-Riemannian groups on the set S of major and minor triads. Then we gave the main results about what we called homometric *lifts*, the main one being that under some conditions that are not very constraining, we can always lift homometric sets in $\mathbb{Z}_n$ into right-homometric sets in D_n. We gave some corollaries and applications with concrete examples. We gave an enumeration of homometric sets in the dihedral group and we presented another way to work with homometry: the characteristic polynomials. We finished with musical considerations.

In the next secion we study non-commutative homometry in another semi-direct product: the time-spans group.

3 Homometry in the Time-spans Group

The time-spans group TS is a non-commutative group presented by Lewin in his book *Generalized Musical Intervals and Transformations* ([16]), and forms a GIS with the definition of the interval given in the book. Lewin considers only one definition of interval, that corresponds to a right action.

In fact the time-spans group is the semi-direct product $(\mathbb{R}, +) \rtimes (\mathbb{R}_+^*, .)$, and the interval defined by Lewin is ${}^r\mathbf{int}$ in the sense of Eq. 9. As far as we know, homometry in the time-spans group is only mentioned briefly in [20] where we find an example of homometric sets for the right action and a process to compute the interval vector of sets. This example is given in Ex. 3.1. In what we read, sets in the time-spans group are musically always considered as rhythms, that is why we will talk about rhythms as well. Our aim in this part is to study homometry for the left and for the right actions in this group, as we did for the dihedral group.

First we describe TS from both the mathematical and the musical points of view, and we introduce a discrete subgroup – isomorphic to the Baumslag-Solitar group $BS(1,2)$ – of TS on which we will focus our attention. We discuss some properties and graphical representations of homometric sets in this subgroup, and consider equations of homometry in the special case of rhythms with two durations. Finally we give some computational results.

3.1 Presentation of the Time-spans Group

In this section we define the time-spans group and explain precisely the kind of sets we work on, called *rhythms*. We then present a discrete subgoup TS^d of TS and discuss the notion of trivial homometry in this subgroup as we did in the dihedral group, showing again that we did not find any interval preserving operation except for the translations.

Mathematical Construction

We begin with the definition of the time-spans group, which was introduced by Lewin in [16], from the point of view of semi-direct products.

Definition 3.1. *The time-spans group (TS) is the semi-direct product* $(\mathbb{R}, +) \rtimes (\mathbb{R}_+^*, .)$. *Its elements are pairs* (t, Δ) *with* $t \in \mathbb{R}$, $\Delta \in \mathbb{R}_+^*$, *called* time spans.

The action of an element (u, δ) *on a time span* (t, Δ) *is given on the left by*

$$(u, \delta)(t, \Delta) = (u + \delta t, \delta\Delta) \tag{57}$$

and on the right by

$$(t, \Delta)(u, \delta) = (t + \Delta u, \delta\Delta). \tag{58}$$

The identity element is $(0, 1)$ *and the inverse of an element* (t, Δ) *is*

$$(t, \Delta)^{-1} = (-t/\Delta, 1/\Delta).$$

The right action is the contextual action described by Lewin in [16], whereas the left action is actually not considered by Lewin at all. We can calculate the right and left intervals between two time spans (t_1, Δ_1) and (t_2, Δ_2):

$$^{l}\mathbf{int}((t_1, \Delta_1), (t_2, \Delta_2)) = (t_2 - \Delta_2/\Delta_1 t_1, \Delta_2/\Delta_1),$$
$$^{r}\mathbf{int}((t_1, \Delta_1), (t_2, \Delta_2)) = ((t_2 - t_1)/\Delta_1, \Delta_2/\Delta_1).$$

The formulas are obviously the same as those for the dihedral group. Similarly we define two GISs, respectively for the right action and for the left action, and we use the traditional definitions for right and left homometries in TS.

Example 3.1. Here is an example of a pair of right-homometric sets in TS, drawn from [20]:

$$\mathcal{S}_1 = \{(0, 1), (1, 1), (2, \frac{1}{2}), (\frac{5}{2}, \frac{1}{2}), (\frac{7}{2}, \frac{1}{4})\}$$

and

$$\mathcal{S}_2 = \{(0, 1), (1, 1), (\frac{5}{2}, \frac{1}{2}), (3, \frac{1}{2}), (\frac{7}{2}, \frac{1}{4})\}.$$

Sets in the dihedral group could be seen for $n = 12$ as chord sequences, which describe the harmonic aspect of music. In this part we move to the rhythmic part. As we said, sets in TS can actually be seen as rhythms.

Musical Description: Sets in TS Considered as Rhythms

Musically a time span $(t, \Delta) \in TS$ can be interpreted as a note played at the onset $t \in \mathbb{R}$ with duration $\Delta \in \mathbb{R}_+^*$. By convention the time unit ($\Delta = 1$) is a quarter note: then $(0, 1)$ – the identity element – represents a quarter note played at the onset 0, while $(1, 1/2)$ represents an eighth note played one quarter note after the beginning.

A time span can also be represented as a half-open interval of $\mathbb{R}$ of the form $[t, t + \Delta[$. Two time spans (t_1, Δ_2) and (t_2, Δ_2) verify $(t_1, \Delta_1) \cap (t_2, \Delta_2) = \oslash$ if the two intervals have a null overlap: $[t_1, t_1 + \Delta_1[\cap [t_2, t_2 + \Delta_2[= \oslash$.

Following [22], we define a rhythm as follows.

Definition 3.2. *A* rhythm *is an almost countable collection* $\{(t_i, \Delta_i)\}_i$ *of time spans such that for all* i, j, $(t_i, \Delta_i) \cap (t_j, \Delta_j) = \emptyset$.

In Fig. 14 we picture such a rhythm with *OpenMusic*. OpenMusic is a visual programming language, based on Lisp, which was developed by the IRCAM Music Representation research group (1998 – 2013). Programs are created by assembling and connecting icons representing functions and data structures. Here we use for instance the function *ts-voice* which allows us to represent graphically a rhythm from the formal writting of the semi-direct product TS (this function was especially created for this purpose thus it is not an existing function in the current OpenMusic language).

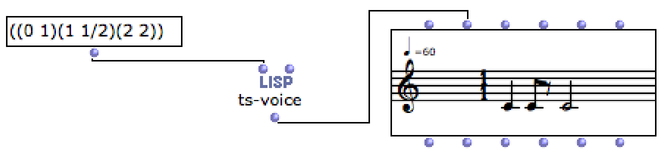


Fig. 14. Rhythmic representation of the set $\{(0,1)(1,1/2)(2,2)\} \in TS$ with OpenMusic.

In what follows we will focus exclusively on sets in TS that are rhythms, which is more interesting musically. Moreover we will work in a discrete subgroup of TS, since usual musical rhythms contain only discrete values of onsets and durations.

Trivial and non-Trival Homometry in a Discrete Subgroup in TS: the Baumslag-Solitar Group $BS(1,2)$

The group TS contains a discrete subgroup TS^d generated by the elements $a = (0, 1/2)$ and $b = (1, 1)$. This subgroup is isomorphic to the Baumslag-Solitar group $BS(1,2)$ which has the following presentation:

$$BS(1,2) = \langle a, b \mid a^{-1}ba = b^2 \rangle. \tag{59}$$

Note that the Baumslag-Solitar group $BS(1,2)$ is also isomorphic to the semidirect product $\mathbb{Z}[\frac{1}{2}] \rtimes \mathbb{Z}$, where $\mathbb{Z}[\frac{1}{2}]$ is the additive group of the dyadic rational (numbers of the form $\frac{p}{2^q}$, where p is an integer and q is a natural number), and where $\mathbb{Z}$ is isomorphic to the multiplicative group of powers of 2, i.e. the set $\{2^x \mid x \in \mathbb{Z}\}$, equipped with the multiplicative law. The action of $d \in \mathbb{Z}$ on $t \in \mathbb{Z}[\frac{1}{2}]$ is by multiplication.

Indeed we know from [4] that elements of $BS(1,2)$ are uniquely expressible as $a^m b^l a^{-n}$, where $m, n \geq 0$, and if $m > 0$ and $n > 0$ then 2 does not divide l. If we do the calculation with the multiplication law we obtain

$$a^m b^l a^{-n} = (\frac{l}{2^m}, 2^{n-m}), \tag{60}$$

then it is easy to see that $BS(1,2)$ and $\mathbb{Z}[\frac{1}{2}] \rtimes \mathbb{Z}$ are isomorphic.

The group TS^d is more appropriate in our context. It is simpler than TS and more efficient when describing concrete rhythms since it involves only discrete onsets and durations. Consequently we will consider only rhythms containing specific notes: whole notes, half notes, quarter notes, eight notes, sixteenth notes, etc. since we use divisions of the quarter note by 2^x for $x \in \mathbb{Z}$.

We have now to answer to a similar question than for the dihedral group: What does mean *trivial* homometry in TS^d? Are there interval preserving operation? As usual we know that left translation preserves homometry for the right action and that right translation preserves homometry for the left action. Inversion does not preserve neither homometry for the right action nor for the left action. As for the dihedral group we will study the automorphism group $\mathcal{A}ut(TS^d) \cong \mathcal{A}ut(BS(1,2))$ of TS^d in order to see if it contains some interval preserving operations.

Proposition 3.1. *The automorphism group $\mathcal{A}ut(BS(1,2))$ of $BS(1,2)$ is generated by A, B and T where A and B designate the inner automorphisms defined by a and b and T is the outer automorphism such that*

$$aT = a,\ bT = b^{-1}. \tag{61}$$

Proof. Collins describes in [4] the automorphism group of the groups of the form $G = \langle a, b \mid a^{-1}ba = b^s \rangle$ for $s \in \mathbb{Z}$. Taking into account this description and the fact that we are in the case $s = 2$ which is prime, we obtain the result of the proposition. □

Proposition 3.2. *There is no element of $\mathcal{A}ut(BS(1,2))$ that preserves intervals in $BS(1,2)$ except for the identity $(0,1)$.*

Proof. First we prove that neither T nor any inner automorphism is interval preserving, then we conclude that no general automorphism is interval preserving.

We look at the outer automorphism T which sends $a^m b^l a^{-n}$ to

$$(aT)^m(bT)^l(aT)^{-n} = a^m b^{-l} a^{-n}.$$

In other words $(\frac{p}{2^q}, 2^x)T = (\frac{-p}{2^q}, 2^x)$. It is easy to check that this operation is not interval preserving since it changes the sign of the first component.

Recall that the inner automorphisms correspond to conjugations by elements of the group, thus A and B correspond respectively to the conjugation by a and b. We show that there is no conjugation that preserves interval. Let us consider the conjugation by (λ, ω) on (t, Δ):

$$(\lambda, \omega)^{-1}(t, \Delta)(\lambda, \omega) = (\frac{\lambda(\Delta - 1)}{\omega} + \frac{t}{\omega}, \Delta).$$

If we look at the right interval between the two elements $(\lambda, \omega)^{-1}(t_1, \Delta_1)(\lambda, \omega)$ and $(\lambda, \omega)^{-1}(t_2, \Delta_2)(\lambda, \omega)$ we obtain

$$(\frac{\lambda(\Delta_2 - \Delta_1)}{\omega \Delta_1} + \frac{t_2 - t_1}{\omega \Delta_1}, \frac{\delta_2}{\Delta_1}).$$

Thus the right interval is preserved if and only if $\frac{\lambda(\Delta_2 - \Delta_1)}{\omega \Delta_1} + \frac{t_2 - t_1}{\omega \Delta_1} = \frac{t_2 - t_1}{\Delta_1}$ i.e.

$$\frac{\lambda}{\omega}(\Delta_2 - \Delta_1) + (\frac{1}{\omega} - 1)(t_2 - t_1) = 0.$$

Except for $\lambda = 0$ and $\omega = 1$ there is no global solution for every (t_1, Δ_1) and (t_2, Δ_2) belonging to TS^d.

Finally we know from [4] that any automorphism F in $\mathcal{A}ut(BS(1,2))$ can be expressed as

$$F = T^\lambda A^m B^l A^{-n},$$

with $m, m \geq 0$, $\lambda = 0$ or 1 (remark that $T^2 = id$), i.e. $F = T^\lambda C$ with C a conjugation. If $\lambda = 0$ we already proved that F is not interval preserving. If $\lambda = 1$ we deduce from the previous formulas that F is interval preserving if and only if

$$\frac{\lambda}{\omega}(\Delta_2 - \Delta_1) - (\frac{1}{\omega} - 1)(t_2 - t_1) = 0,$$

which has no global solution except for $\lambda = 0$ and $\omega = 1$.

If we look at the left intervals we obtain the same results. □

As we did not find any other interval preserving operation, we still keep the usual definition for non-trivial homometry in TS^d.

Definition 3.3. *We say that two sets in TS^d are* non-trivially *right (resp. left) homometric if there are right (resp. left) homometric and not linked by a left (resp. right) translation.*

As before, when we will say 'homometric' it will implicitly mean 'non-trivial homometric'. In the next part we give some (graphical) examples of homometric rhythms and some of their properties.

3.2 Some Properties and Examples of Homometric Sets in TS^d

We present here two different processes to build homometric sets for the right action in the time-spans group, starting from an example. Then with a graphical representation we disclose interesting parallelisms between homometric sets, leading to simple but isolated cases of simultaneously right and left-homometric sets.

A Way to Build right-homometric Rhythms

Let us look carefully at the two sets $\mathcal{S}_1$ and $\mathcal{S}_2$ of Ex. 3.1 that are homometric for the right action of TS^d. They contain a pair of subsets which are themselves homometric for the right group action, namely the sets

$$\{(0,1),(1,1),(2,\frac{1}{2}),(\frac{7}{2},\frac{1}{4})\}$$

and

$$\{(0,1),(1,1),(3,\frac{1}{2}),(\frac{7}{2},\frac{1}{4})\}.$$

Given $\alpha, \beta \in \mathbb{R}$, we now generalize this result and exhibit two sets which are always homometric for the right group action.

Theorem 3.1. *Let A be the set $A = \{(0,1),(\beta,1),(\frac{2\alpha-\beta}{3},\frac{1}{2}),(\alpha,\frac{1}{4})\}$ and B be the set $B = \{(0,1),(\beta,1),(\frac{2\alpha+2\beta}{3},\frac{1}{2}),(\alpha,\frac{1}{4})\}$. Then A and B are homometric for the right group action of TS^d.*

The proof is easily obtained by explicitly calculating the value of all the intervals between the elements of each set. Unfortunately, there is no pair of values (α, β) for which these two sets would also be homometric for the left action of G (excluding the trivial case $\alpha = \beta = 0$).

Other right-homometric pairs can be found.

Theorem 3.2. *Let A be the set $A = \{(0,\frac{1}{2}),(\alpha,\frac{1}{4}),(\beta,1),(2\beta+2\alpha,1)\}$ and B be the set $B = \{(0,\frac{1}{4}),(\beta-\alpha,\frac{1}{2}),(\beta+\alpha,\frac{1}{2}),(2\beta+\alpha,1)\}$. Then A and B are homometric for the right group action of TS^d.*

We move now to a graphical representation of rhythms that shows interesting parallelisms between homometric sets in TS^d.

Graphical Representation

In order to represent a rhythm we use the following graph: on the horizontal axis are the onsets of the time spans, and on the vertical axis are their durations.

By convention all the time spans sets we consider will have the time span $(0,1)$ for first element (same convention than with the dihedral group). We do not miss any homometric set since we can always, after right or left translation, obtain a set with first time span $(0,1)$ which has the same interval content. In Fig. 15 and Fig. 16 we show two examples of right-homometric rhythms. These examples reveal homometric sets containing respectively three different durations and two different durations.

There are obvious graphical parallelisms in each figure. Actually we have the following result.

Proposition 3.3. *Let $I = (t, \Delta)$, $J = (u, \delta)$, $I' = (t', \Delta')$ and $J' = (u', \delta')$ be four time spans, we have*

$$^r\boldsymbol{int}(I,J) = ^r\boldsymbol{int}(I',J') \iff \frac{1}{\Delta}\overrightarrow{IJ} = \frac{1}{\Delta'}\overrightarrow{I'J'}. \tag{62}$$

In particular $\overrightarrow{IJ}$ and $\overrightarrow{I'J'}$ are colinear, and even equal if I and I' have the same duration.

Proof. We have

$$\begin{aligned} ^r\mathbf{int}(I,J) = ^r\mathbf{int}(I',J') &\iff ((u-t)/\Delta, \delta/\Delta) = ((u'-t')/\Delta', \delta'/\Delta') \\ &\iff (u-t)/\Delta = (u'-t')/\Delta' \text{ and } \delta/\Delta = \delta'/\Delta' \\ &\iff \frac{1}{\Delta}\overrightarrow{IJ} = \frac{1}{\Delta'}\overrightarrow{I'J'}. \end{aligned}$$ □

If we go further and look at the extreme case of parallelism, we obtain *aligned* homometric rhythms, as shown in Fig. 17. These special cases give examples of simultaneously right and left-homometric sets.

In fact it is easy to show that for a fixed time span $I = (t, \Delta)$, the set of all the time spans that commute with I is exactly the line (OI), where O is the point $(0,1)$. Indeed

$$\begin{aligned} I = (t,\Delta) \text{ and } J = (u,\delta) \text{ commute } &\iff t + u\Delta = u + t\delta \\ &\iff \frac{\Delta - 1}{t} = \frac{\delta - 1}{u} \\ &\iff (OI) = (OJ). \end{aligned}$$

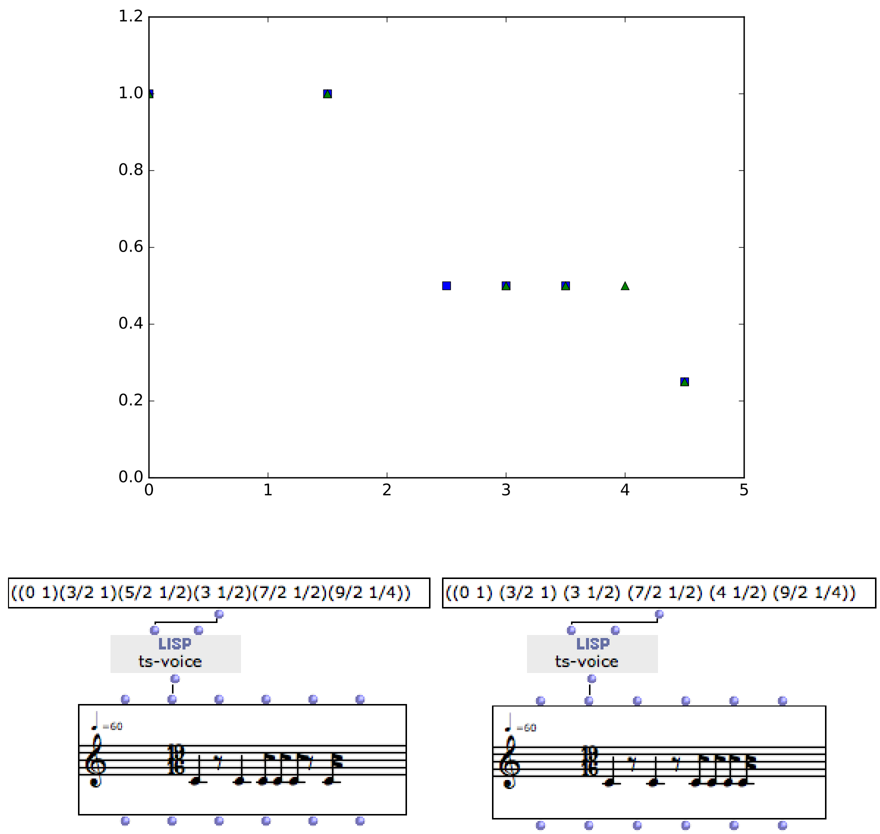


Fig. 15. Two right-homometric rhythms with six time spans and three different durations: $\{(0,1),(\frac{3}{2},1),(\frac{5}{2},\frac{1}{2}),(3,\frac{1}{2}),(\frac{7}{2},\frac{1}{2}),(\frac{9}{2},\frac{1}{4})\}$ and $\{(0,1),(\frac{3}{2},1),(3,\frac{1}{2}),(\frac{7}{2},\frac{1}{2}),(4,\frac{1}{2}),(\frac{9}{2},\frac{1}{4})\}$.

Then two aligned right (resp. left) homometric sets are sets where all the time spans commute, then they are also left (resp. right) homometric sets since the right and left interval are identical.

These cases are interesting but simple because all the time spans commute, and then all the durations of the time spans in a set are different. Anyway it gives a way to understand better homometry in TS^d. By a musical point of view such homometric sets correspond to rhythms with all the notes having a different duration.

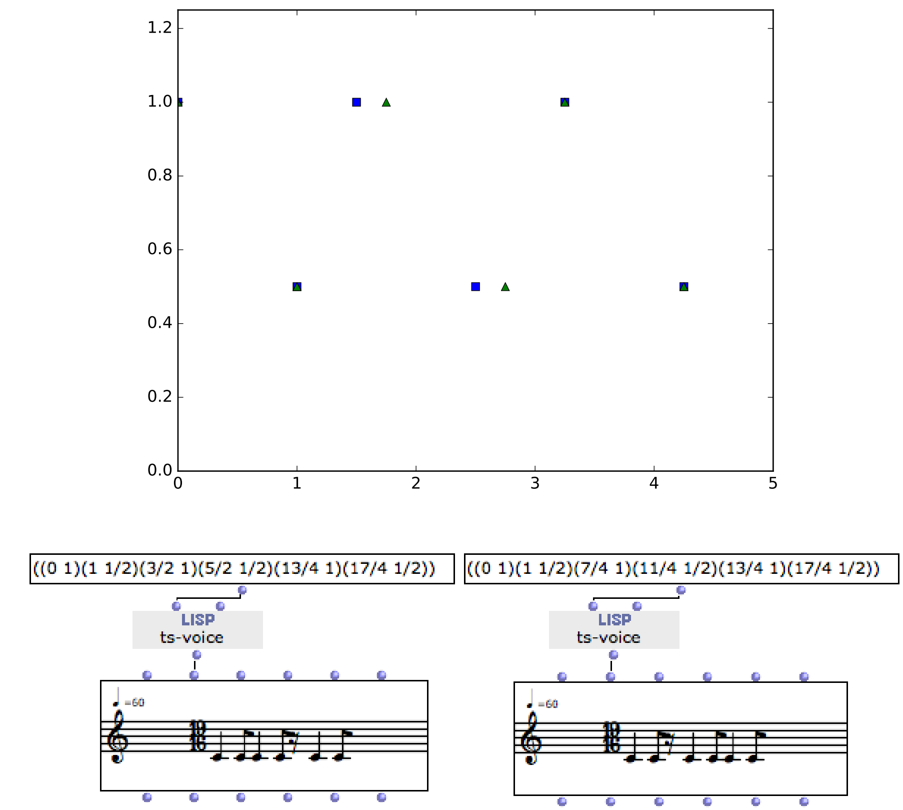


Fig. 16. Two right-homometric rhythms with six time spans and two different durations: $\{(0,1),(1,\frac{1}{2}),(\frac{3}{2},1),(\frac{5}{2},\frac{1}{2}),(\frac{13}{4},1),(\frac{17}{4},\frac{1}{2})\}$ and $\{(0,1),(1,\frac{1}{2}),(\frac{7}{4},1),(\frac{11}{4},\frac{1}{2}),(\frac{13}{4},1),(\frac{17}{4},\frac{1}{2})\}$.

This brief graphical study disclosed an important property: there exists simultaneously right and left-homometric sets in the time spans group. However these sets were all *aligned*, which is a simple case where all the time spans commute. Then an interesting question (not solved!) is in order: Are two simultaneously right and left-homometric sets necessarily aligned? Our computation research did not bring any counterexample, but we did not manage to prove this conjecture.

Besides this study revealed that it may be interesting to use a graphical approach to solve the question of homometry in the time spans group. However we will let here this approach (we did not find any other consistent result) and use the same framework than the one we used with the dihedral group.

3.3 Right and Left Homometry Conditions in TS^d in the Special Case with Two Durations

Our aim is to find equivalent equations to those of Thm. 2.2 for the time-spans group. We will work exclusively on the subgroup TS^d, and on a simple case: when rhythms have only two durations. Why? Because in that case we are in a situation quite similar to the previous situation with the dihedral group, where we had two possible natures for chords: +1 and -1. The two durations we choose

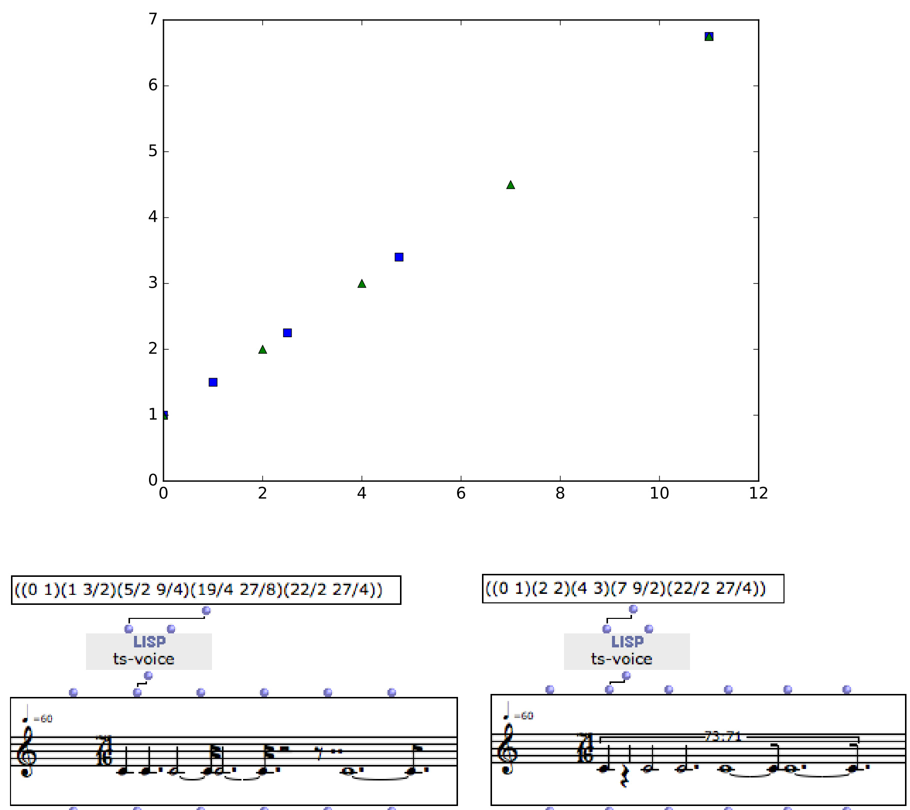


Fig. 17. Two left-and-right-homometric sets with five time spans: $\{(0,1),(1,\frac{3}{2}),(\frac{5}{2},\frac{9}{4}),(\frac{19}{4},\frac{27}{8}),(\frac{22}{2},\frac{27}{4})\}$ and $\{(0,1),(2,2),(4,3),(7,\frac{9}{2}),(\frac{22}{2},\frac{27}{4})\}$.

are 1 and 1/2. This choice is not really important since the results are quite similar for other choices.

Notations. The set of rhythms with durations 1 and 1/2 in TS^d will be written TS^d_2. A set in TS^d will be written with the typography '$\mathcal{A}$' while a set in $\mathbb{Z}[\frac{1}{2}]$ will be written 'A' (same conventions than those used in the dihedral group).

A set $\mathcal{A}$ in TS^d_2 is the union of two subsets corresponding respectively to the part with rhythms having duration 1 and rhythms having durations 1/2. We will call A_1 the set in $\mathbb{Z}[\frac{1}{2}]$ of onsets having duration 1 and $A_{1/2}$ the set in $\mathbb{Z}[\frac{1}{2}]$ of onsets having duration 1/2.

More formally we have the following definitions.

Definition 3.4. *Let $p_1 : TS^d_2 \longrightarrow \mathbb{Z}[\frac{1}{2}]$ be the first projection, $p_2 : TS^d_2 \longrightarrow \mathbb{Z}$ be the second projection, and $\mathcal{A} \subset TS^d_2$. We call $\mathcal{A}_1$ (resp. $\mathcal{A}_{1/2}$) the subset of $\mathcal{A}$ defined by $\mathcal{A}_1 = \{a \in \mathcal{A} \mid p_2(a) = 1\}$ (resp. $\mathcal{A}_{1/2} = \{a \in \mathcal{A} \mid p_2(a) = 1/2\}$). Hence $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_{1/2}$.*

Definition 3.5. *Let $\mathcal{A}$ be a subset in TS^d_2. We call*

$$A_1 := p_1(\mathcal{A}_1) \text{ and } A_{1/2} := p_1(\mathcal{A}_{1/2}).$$

We have then $p_1(\mathcal{A}) = A_1 \cup A_{1/2}$. If there is no confusion we will write A for the set $p_1(\mathcal{A})$.

Example 3.2. Let

$$\mathcal{A} = \{(0,1),(1,\frac{1}{2}),(\frac{3}{2},1),(\frac{5}{2},\frac{1}{2})\} \in TS^d_2.$$

We have $\mathcal{A}_1 = \{(0,1),(\frac{3}{2},1)\}$ and $\mathcal{A}_{1/2} = \{(1,\frac{1}{2}),(\frac{5}{2},\frac{1}{2})\}$, hence $A_1 = \{0,\frac{3}{2}\}$, $A_{1/2} = \{1,\frac{5}{2}\}$ and $A = \{0,1,\frac{3}{2},\frac{5}{2}\}$.

Characterization of Homometry in TS^d_2. Our purpose here is to give the equations of homometry with the different points of view we already used, namely with the functions **iv** and **ifunc**, with the Fourier transform and with polynomials.

Theorem 3.3. *Two rhythms $\mathcal{A}$ and $\mathcal{B}$ in TS^d_2 are homometric for the right action if and only if the two following equations hold:*

$$\begin{cases} \boldsymbol{iv}(A_1) + \boldsymbol{iv}(2A_{1/2}) = \boldsymbol{iv}(B_1) + \boldsymbol{iv}(2B_{1/2}) \\ \boldsymbol{ifunc}(A_1, A_{1/2}) = \boldsymbol{ifunc}(B_1, B_{1/2}). \end{cases} \tag{63}$$

Two rhythms $\mathcal{A}$ and $\mathcal{B}$ in TS_2^d are homometric for the left action if and only if the two following equations hold:

$$\begin{cases} \boldsymbol{iv}(A_1) + \boldsymbol{iv}(A_{1/2}) = \boldsymbol{iv}(B_1) + \boldsymbol{iv}(B_{1/2}) \\ \boldsymbol{ifunc}(A_1, 2A_{1/2}) = \boldsymbol{ifunc}(B_1, 2B_{1/2}). \end{cases} \tag{64}$$

Proof. Let $\mathcal{A}, \mathcal{B}$ be two right-homometric sets in TS_2^d. Let us recall that

$$^r\mathbf{int}((t_1, \Delta_1), (t_2, \Delta_2)) = ((t_2 - t_1)/\Delta_1, \Delta_2/\Delta_1)$$

for $(t_1, \Delta_1), (t_2, \Delta_2)$ in $\mathcal{A}$. Then we have three cases, which give the two equations Eq. 63 and 64: $\Delta_2/\Delta_1 = 1$, $\Delta_2/\Delta_1 = 1/2$ or $\Delta_2/\Delta_1 = 2$.

If $\Delta_2/\Delta_1 = 1$ **i.e.** $\Delta_2 = \Delta_1$. Either $\Delta_1 = 1 = \Delta_2$ then

$$^r\mathbf{int}((t_1, \Delta_1), (t_2, \Delta_2)) = ((t_2 - t_1), 1)$$

for t_1, t_2 in A_1. To obtain all the intervals of that type we have to calculate $\mathbf{iv}(A_1)$. Or $\Delta_1 = 1/2 = \Delta_2$ then

$$^r\mathbf{int}((t_1, \Delta_1), (t_2, \Delta_2)) = (2(t_2 - t_1), 1)$$

for t_1, t_2 in $A_{1/2}$. To obtain all the intervals of that type we have to calculate $\mathbf{iv}(2A_{1/2})$. Then we must have

$$\mathbf{iv}(A_1) + \mathbf{iv}(2A_{1/2}) = \mathbf{iv}(B_1) + \mathbf{iv}(2B_{1/2}).$$

If $\Delta_2/\Delta_1 = 1/2$ **i.e.** $\Delta_1 = 1$ **and** $\Delta_2 = 1/2$. Then

$$^r\mathbf{int}((t_1, \Delta_1), (t_2, \Delta_2)) = ((t_2 - t_1), 1/2)$$

for $t_1 \in A_1, t_2 \in A_{1/2}$. To obtain all the intervals of that type we have to calculate $\mathbf{ifunc}(A_1, A_{1/2})$. Then we must have

$$\mathbf{ifunc}(A_1, A_{1/2}) = \mathbf{ifunc}(B_1, B_{1/2})$$

If $\Delta_2/\Delta_1 = 2$ **i.e.** $\Delta_1 = 1/2$ **and** $\Delta_2 = 1$. Then

$$^r\mathbf{int}((t_1, \Delta_1), (t_2, \Delta_2)) = (2(t_2 - t_1), 2)$$

for $t_1 \in A_{1/2}, t_2 \in A_1$ Hence it gives the same result than the latter case.

This gives the two equations of Thm. 3.3. Reciprocally if two sets verify Eq. 63 and Eq. 64, then they have the same right interval content. It works exactly the same with the left intervals. □

These equations have obvious common points with the equations in the dihedral group but they work differently. For instance right homometry in TS_2^d does not imply homometry in $\mathbb{Z}[\frac{1}{2}]$, left homometry neither. However if two sets $\mathcal{A}$ and $\mathcal{B}$ are both right and left-homometric, then we have in particular

$$\begin{cases} \mathbf{iv}(A_1) + \mathbf{iv}(A_{1/2}) = \mathbf{iv}(B_1) + \mathbf{iv}(B_{1/2}) \\ \mathbf{ifunc}(A_1, A_{1/2}) = \mathbf{ifunc}(B_1, B_{1/2}), \end{cases} \tag{65}$$

hence the sets A and B are homometric in $\mathbb{Z}[\frac{1}{2}]$ (same proof than for Prop. 2.4).

We will now use the discrete Fourier transform. We can define the DFT $\mathcal{F}(A)$ of a finite subset A in $\mathbb{Z}$ with cardinality equal to N as

$$\mathcal{F}(A)(t) = \sum_{k \in A} e^{\frac{-2i\pi kt}{N}} \tag{66}$$

for $t \in \mathbb{Z}[\frac{1}{2}]$. We have the same results than those mentioned in Prop. 1.1. We obtain the following equations.

Theorem 3.4. *Two rhythms $\mathcal{A}$ and $\mathcal{B}$ in TS_2^d are homometric for the right action if and only if the two following equations hold:*

$$\begin{cases} |\mathcal{F}_{A_1}(t)|^2 + |\mathcal{F}_{A_{1/2}}(2t)|^2 = |\mathcal{F}_{B_1}(t)|^2 + |\mathcal{F}_{B_{1/2}}(2t)|^2 \\ \overline{\mathcal{F}_{A_1}(t)}\mathcal{F}_{A_{1/2}}(t) = \overline{\mathcal{F}_{B_1}(t)}\mathcal{F}_{B_{1/2}}(t). \end{cases} \tag{67}$$

Two rhythms $\mathcal{A}$ and $\mathcal{B}$ in TS_2^d are homometric for the left action if and only if the two following equations hold:

$$\begin{cases} |\mathcal{F}_{A_1}(t)|^2 + |\mathcal{F}_{A_{1/2}}(t)|^2 = |\mathcal{F}_{B_1}(t)|^2 + |\mathcal{F}_{B_{1/2}}(t)|^2 \\ \overline{\mathcal{F}_{A_{1/2}}(2t)}\mathcal{F}_{A_1}(t) = \overline{\mathcal{F}_{B_{1/2}}(2t)}\mathcal{F}_{B_1}(t). \end{cases} \tag{68}$$

We can thus deduce the equivalent of Thm. 2.5.

Theorem 3.5. *Let A and B be homometric sets in $\mathbb{Z}[\frac{1}{2}]$ such that $A = A_u \cup A_v$ and $B = B_u \cup B_v$ for some subsets A_u, A_v, B_u and B_v, with $\boldsymbol{iv}(A_u) = \boldsymbol{iv}(B_u)$ and $\boldsymbol{iv}(A_v) = \boldsymbol{iv}(B_v)$. We can always lift A and B into right-homometric sets in TS_2.*

Proof. The proof is more or less exactly the same than the proof of Thm. 2.5. □

Example 3.3. Consider the two homometric sets

$$A = \{0, 1, \frac{3}{2}, \frac{5}{2}, \frac{13}{4}, \frac{17}{4}\} = \underbrace{\{0, \frac{3}{2}, \frac{13}{4}\}}_{A_u} \cup \underbrace{\{1, \frac{5}{2}, \frac{17}{4}\}}_{A_v}$$

and

$$B = \{0, 1, \frac{7}{4}, \frac{11}{4}, \frac{13}{4}, \frac{17}{4}\} = \underbrace{\{0, \frac{7}{4}, \frac{13}{4}\}}_{B_u} \cup \underbrace{\{1, \frac{11}{4}, \frac{17}{4}\}}_{B_v}.$$

We see that $\mathbf{iv}(A_u) = \mathbf{iv}(B_u)$ and $\mathbf{iv}(A_v) = \mathbf{iv}(B_v)$. Consequently we are in the situation of Thm. 3.5. A and B lift in TS_2^d into the right-homometric sets

$$\mathcal{A} = \{(0,1), (1, \frac{1}{2}), (\frac{3}{2}, 1), (\frac{5}{2}, \frac{1}{2}), (\frac{13}{4}, 1), (\frac{17}{4}, \frac{1}{2})\}$$

and

$$\mathcal{B} = \{(0,1), (1, \frac{1}{2}), (\frac{7}{4}, 1), (\frac{11}{4}, \frac{1}{2}), (\frac{13}{4}, 1), (\frac{17}{4}, \frac{1}{2})\},$$

which are actually the sets of Fig. 16.

The sets A and B have another very specific particularity: $A_v = T_1(A_u)$ and $B_v = T_1(B_u)$. It implies

$$\mathbf{iv}(A_v) = \mathbf{iv}(A_u) = \mathbf{iv}(B_u) = \mathbf{iv}(B_v).$$

Besides, A_u and B_u have a remarkable property. They correspond to the sets

$$\{0, \frac{\frac{p}{4} - \frac{1}{4}}{2}, \frac{p}{4}\} \text{ and } \{0, \frac{\frac{p}{4} + \frac{1}{4}}{2}, \frac{p}{4}\},$$

where $p = 13$, or modulo multiplication by 4 to the also homometric sets

$$\{0, \frac{p-1}{2}, p\} \text{ and } \{0, \frac{p+1}{2}, p\}. \tag{69}$$

In fact two sets of the form of Eq. 69 are homometric for all $p \in \mathbb{R}$.

Example 3.4. There are several examples of homometric sets with the same properties. We consider such an example in the case where the durations are 1 and 2, which does not change the theoretical aspects. Let

$$C = \{0, 1, 3, 4, 7, 8\} = \underbrace{\{0, 3, 7\}}_{C_u} \cup \underbrace{\{1, 4, 8\}}_{C_v}$$

and

$$D = \{0, 1, 4, 5, 7, 8\} = \underbrace{\{0, 4, 7\}}_{D_u} \cup \underbrace{\{1, 5, 8\}}_{D_v}.$$

C and D are homometric, $C_v = T_1(C_u)$ and $D_v = T_1(D_u)$. Besides we have $\mathbf{iv}(C_u) = \mathbf{iv}(D_u)$ and $C_u = \{0, \frac{p-1}{2}, p\}$, $D_u = \{0, \frac{p+1}{2}, p\}$ with $p = 7$. C and D lift into the right-homometric rhythms

$$\mathcal{C} = \{(0,1), (1,2), (3,1), (4,2), (7,1), (8,2)\}$$

and

$$\mathcal{D} = \{(0,1), (1,2), (4,1), (5,2), (7,1), (8,2)\}.$$

Example 3.5. We give a last example of that type with the sets

$$\{(0,1),(\frac{3}{2},\frac{1}{2}),(2,1),(\frac{7}{2},\frac{1}{2}),(\frac{17}{4},1),(\frac{23}{4},\frac{1}{2})\}$$

and

$$\{(0,1),(\frac{3}{2},\frac{1}{2}),(\frac{9}{4},1),(\frac{13}{4},\frac{1}{2}),(\frac{17}{4},1),(\frac{23}{4},\frac{1}{2})\}.$$

There are represented on Fig. 18.

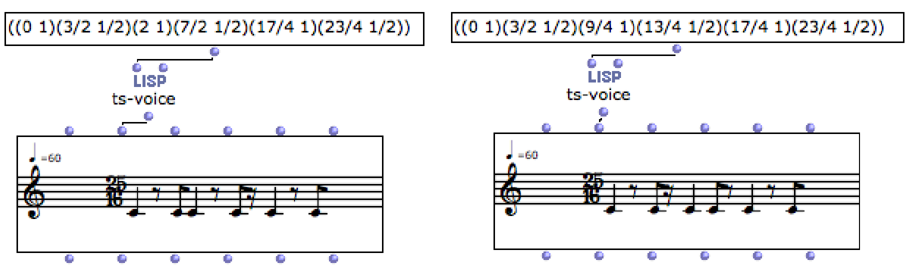


Fig. 18. The two right-homometric rhythms $\{(0,1),(\frac{3}{2},\frac{1}{2}),(2,1),(\frac{7}{2},\frac{1}{2}),(\frac{17}{4},1),(\frac{23}{4},\frac{1}{2})\}$ and $\{(0,1),(\frac{3}{2},\frac{1}{2}),(\frac{9}{4},1),(\frac{13}{4},\frac{1}{2}),(\frac{17}{4},1),(\frac{23}{4},\frac{1}{2})\}$.

Remark 3.1. Note that the conclusion of the theorem is that we can lift in TS_2 and not in TS_2^d *a priori*, because we are not sure that the homometric sets we obtain are rhythms since there can be overlaps between the time spans.

Polynomial notations. Concerning the polynomial notations it is easier to define them in this context. For a set $A \in \mathbb{Z}[\frac{1}{2}]$, we define the characteristic polynomial $A(X)$ of A as the element

$$A(X) = \sum_{k \in A} X^k \tag{70}$$

with the usual multiplication law. We know from [15] that two sets A and B are homometric if

$$A(X)A(X^{-1}) = B(X)B(X^{-1}).$$

We obtain then the following equations.

Theorem 3.6. *Two rhythms $\mathcal{A}$ and $\mathcal{B}$ in TS_2^d are homometric for the right action if and only if the two following equations hold:*

$$\begin{cases} A_1(X)A_1(X^{-1}) + A_{1/2}(X^2)A_{1/2}(X^{-2}) \\ \qquad\qquad = B_1(X)B_1(X^{-1}) + B_{1/2}(X^2)B_{1/2}(X^{-2}) \\ A_{1/2}(X)A_1(X^{-1}) = B_{1/2}(X)B_1(X^{-1}). \end{cases} \quad (71)$$

Two rhythms $\mathcal{A}$ and $\mathcal{B}$ in TS_2^d are homometric for the left action if and only if the two following equations hold:

$$\begin{cases} A_1(X)A_1(X^{-1}) + A_{1/2}(X)A_{1/2}(X^{-1}) \\ \qquad\qquad = B_1(X)B_1(X^{-1}) + B_{1/2}(X)B_{1/2}(X^{-1}) \\ A_{1/2}(X^{-2})A_1(X) = B_{1/2}(X^{-2})B_1(X). \end{cases} \quad (72)$$

Again the polynomial notations do not bring any new result, but it can be useful in practice for concrete calculations.

The Case with more than Two Durations. The previous results concern TS_2^d, in fact the generalization of these results to TS^d (i.e. two rhythms with more than two durations) is complicated. Consider for instance rhythms with the three following durations: 1, 1/2 and 2. When we look at the right intervals in TS^d we obtain several types of intervals that have 2 as second component: between time spans of the form $(t, 1)$ and $(t', 2)$, but also between time spans of the form $(t, 1/2)$ and $(t', 1)$. As a consequence the equations of homometry involve more terms and then are more difficult to solve. In this work we will not talk about these more general cases.

Partial Conclusion

We studied the time spans group which is a non-commutative group whose sets can be interpreted as musical rhythms. We saw interesting graphical properties that lead to simultaneously right and left-homometric sets. In the case of only two durations, the equations of right and left homometries are like those in the dihedral group, thus the results have similarities, for instance concerning the existence of right-homometric lifts.

In the last section of this part we will use the same procedure we used with both the dihedral group and the time spans group, in order to do a generalization to semi-direct products.

4 Generalization to Semi-direct Products

We saw that we can consider in a general semi-direct product $G = Z \rtimes H$, the two GISs associated to the right and to the left actions of G on itself, and as a consequence that we can study right and left homometry in G. In this part we want to generalize the process we used with the time spans-group and with the dihedral group, to give a characterization of homometry in the more general context of a semi-direct product. The idea is again to decompose a set belonging to G into subsets according to elements of H, and then to study the relationships between homometry in G and homometry with the projections via π_1 in Z. We do not solve the given equations since they involve many terms and may hardly be simplified (they were already complicated to solve in D_n and in $TS_2^d!$). The idea is on the one hand to generalize in a more abstract context some properties we already saw, and on the other hand to give a systematic technique to have a better comprehension of homometry in a semi-direct product.

We first define some notations in the continuity of the previous ones, then we present the homometric equations for the right and for the left actions in a semi-direct product. Finally we give two results that are generalizations of former results.

4.1 Notations

Let $G = Z \rtimes H$ be the semi-direct product of the two commutative groups $(Z, +)$ and $(H, .)$ as defined in subsection 1.3. A set A in G will be refered to as '$\mathcal{A}$' while a set A in Z will be refered to as 'A' (same notations than previously). We want to define a decomposition of a set $\mathcal{A}$ in G similar to the decomposition with A_+ and A_- that we used in the dihedral group, or with A_1 and $A_{1/2}$ in the time-spans group. It consists in decomposing the set $\mathcal{A}$ into subsets according to elements of H.

Definition 4.1. *We write $\pi_1 : G \longrightarrow Z$ (resp. $\pi_2 : G \longrightarrow H$) the projection of $G = Z \rtimes H$ on Z (resp. on H).*

Definition 4.2. *Let $h \in H$ and $\mathcal{A} \subset G$. We call $\mathcal{A}_h := \{a \in \mathcal{A} \mid \pi_2(a) = h\}$. It implies that $\mathcal{A} = \bigsqcup_{h\in H} \mathcal{A}_h$.*

Definition 4.3. *Let $\mathcal{A}$ be a subset of G. We define $A_h = \pi_1(\mathcal{A}_h)$. In particular we have $\pi_1(A) = \bigsqcup_{h\in H} A_h$.*

We will now present the equations of homometry in G, that are generalizations of Thm. 2.2 and Thm. 3.3.

4.2 Characterization of Homometry in G

The following theorem presents the homometry conditions for the right and for the left actions of G on itself.

Theorem 4.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be two finite and right-homometric subsets of G. Then for each $h \in H$:*

$$\sum_{\substack{(h_1,h_2)\in\pi_2^2(A)\\ h_1^{-1}h_2=h}} \boldsymbol{ifunc}(h_1^{-1}A_{h_1}, h_1^{-1}A_{h_2}) = \sum_{\substack{(h_1,h_2)\in\pi_2^2(B)\\ h_1^{-1}h_2=h}} \boldsymbol{ifunc}(h_1^{-1}B_{h_1}, h_1^{-1}B_{h_2}). \tag{73}$$

Let $\mathcal{A}$ and $\mathcal{B}$ be two finite and left-homometric subsets of $G = Z \rtimes H$. Then for each $h \in H$:

$$\sum_{\substack{(h_1,h_2)\in\pi_2^2(A)\\ h_2h_1^{-1}=h}} \boldsymbol{ifunc}(hA_{h_1}, A_{h_2}) = \sum_{\substack{(h_1,h_2)\in\pi_2^2(B)\\ h_2h_1^{-1}=h}} \boldsymbol{ifunc}(hB_{h_1}, B_{h_2}) \tag{74}$$

Proof. Let $(z_1, h_1), (z_2, h_2) \in \mathcal{A}$, we saw that

$${}^r\mathbf{int}((z_1, h_1), (z_2, h_2)) = (h_1^{-1}(z_2 - z_1), h_1^{-1}h_2).$$

Then for a fixed $h \in H$:

$$\{h_1^{-1}(z_2 - z_1) \mid ((z_1, h_1), (z_2, h_2)) \in A^2 \text{ and } h_2h_1^{-1} = h\}$$

must be equal to

$$\{t_1^{-1}(y_2 - y_1) \mid ((y_1, t_1), (y_2, t_2)) \in \mathcal{B}^2 \text{ and } t_2t_1^{-1} = h\},$$

which gives the result.

Similarly we saw that

$${}^l\mathbf{int}((z_1, h_1), (z_2, h_2)) = (z_2 - h_2h_1^{-1}z_1, h_2h_1^{-1}).$$

Then for a fixed $h \in H$:

$$\{z_2 - h_2h_1^{-1}z_1 \mid ((z_1, h_1), (z_2, h_2)) \in A^2 \text{ and } h_2h_1^{-1} = h\}$$

must be equal to

$$\{y_2 - h_2h_1^{-1}y_1 \mid ((y_1, t_1), (y_2, t_2)) \in \mathcal{B}^2 \text{ and } t_2t_1^{-1} = h\}.$$ □

Theorem 4.1 is at first glance a complex way to describe homometry in semi-direct products. However it gives a characterization in the general case and a systematic way to study homometry in this context. As we saw with the dihedral group and with the time-spans group, Eq. 73 and Eq. 74 consist each in two equations when H has two elements. When we consider groups with more elements those equations contain more terms hence the resolution is harder.

We can not conclude from Eq. 73 and Eq. 74 – contrary to the dihedral group or to TS_2^d – that the projections of two homometric sets, for the right action and for the left action, on Z are homometric, since we have disturbing coefficients h^{-1} in Eq. 73. However the property is true with some conditions when we restrict the study to a subset $H_2 \subset H$ with $\sharp(H_2) = 2$.

Proposition 4.1. *Let H_2 be a subset of H such that:*

- $\sharp(H_2) = 2$, *we note* $H_2 = \{h_1, h_2\}$;
- $1_H \in H_2$ *(for instance* $h_1 = 1_H$*)*;
- $h_2^{-1} \neq h_2$.

If $\mathcal{A}$ and $\mathcal{B}$ are two homometric sets in G for both the right and the left actions, then their projection $A = \pi_1(\mathcal{A})$ and $B = \pi_1(\mathcal{B})$ are homometric in Z.

Proof. Eq. 74 gives for $h = 1_H$:

$$\mathbf{iv}(A_{h_1}) + \mathbf{iv}(A_{h_2}) = \mathbf{iv}(B_{h_1}) + \mathbf{iv}(B_{h_2}).$$

Eq. 73 gives for $h = h_2$ (remark that $h_2^{-1} \neq h_2$ hence if $t_1^{-1}t_2 = h_2$ we must have $t_1 = 1_H, t_2 = h_2$):

$$\mathbf{ifunc}(A_{h_1}, A_{h_2}) = \mathbf{ifunc}(B_{h_1}, B_{h_2}).$$

We obtain equivalent equations to Eq. 65 then we conlude that A and B are homometric in Z. □

For example TS_2^d verifies the conditions of Prop. 4.1. The dihedral group does not since $h_2 = -1 = h_2^{-1}$. In fact in this latter group $H = \mathbb{Z}_2$ and we have a stronger result that we already proved.

Proposition 4.2. *Let $H_2 = \mathbb{Z}_2$. If $\mathcal{A}$ and $\mathcal{B}$ are two homometric sets in G for the right action then their projections $A = \pi_1(\mathcal{A})$ and $B = \pi_1(\mathcal{B})$ are homometric in Z.*

Proof. For $h = 1$, Eq. 73 gives

$$\mathbf{iv}(A_1) + \mathbf{iv}(IA_{-1}) = \mathbf{iv}(B_1) + \mathbf{iv}(IB_{-1}),$$

where IA designates the inversion of the set A. As $\mathbf{iv}(IA_{-1}) = \mathbf{iv}(A_{-1})$ we get

$$\mathbf{iv}(A_1) + \mathbf{iv}(A_{-1}) = \mathbf{iv}(B_1) + \mathbf{iv}(B_{-1}).$$

For $h = -1$, Eq. 73 gives

$$.\mathbf{ifunc}(A_1, A_{-1}) + \mathbf{ifunc}(IA_{-1}, IA_1) = \mathbf{ifunc}(B_1, B_{-1}) + \mathbf{ifunc}(IB_{-1}, IB_1)$$

But $\mathbf{ifunc}(IA_{-1}, IA_1) = \mathbf{ifunc}(A_1, A_{-1})$ hence

$$\mathbf{ifunc}(A_1, A_{-1}) = \mathbf{ifunc}(B_1, B_{-1})$$

and A and B are homometric in Z. □

Remark 4.1. Notice that the results of this last section give partial answers to the questions raised in Rmk. 1.1 concerning the relationships between homometries in G, in Z and in H.

Part II

Distances in Chord Spaces

Presentation

In this part we have in mind a double objective. The first one is to study some topological and other mathematical properties of the chord space of Tymocko (cf. [30], [31], [32]). This space, whose elements can be seen as musical chords, is mathematically an orbifold that has interesting specificities. It is for instance a complete metric space. This particularity leads us to define a distance between musical chords, which is our second – and main – objective. More precisely, we want to define a measure of distances between chords that do not have the same number of notes. Tymoczko, who defines a distance in its musical chord space, does not mention the possibility to build such a distance. Beyond this mathematical and theoretical interesting problem, we also want to present some computational and concrete applications of our distance.

In the first section we present already existing definitions of distances between chords in other musical spaces. In the second section we describe and study some mathematical properties of the chord space of Tymoczko. Then we define in the third section, a measure of distances in the general space containing chords of all cardinalities. In the fourth section we present some musical applications of this distance.

5 Some Distances between Chords

Even if our approach is exclusively inspired by the work of Tymoczko and his chord space based on voice-leading, we present in this section other definitions of distances between chords. We will not use them after, but on the one hand it seems important for us to mention that there are various ways to approach the problem of building distances between chords, and on the other hand the distances we present have in fact some links.

The two types of distances we describe are the distance in the *Tonnetz*, and distances that rely on the interval content, which make a bridge with the first part concerning homometry.

5.1 The Tonnetz

This model comes from Euler in 1739. It has been used – and is always used of course – with many different musical approaches. A tonnetz is a discrete structure that uses lattices, with adjacent points on a particular axis being separated by the same interval. On Fig. 19 we have a familiar Tonnetz, where the two diagonal axis represent acoustically major and minor thirds, while the horizontal axis represents pure perfect fifths.

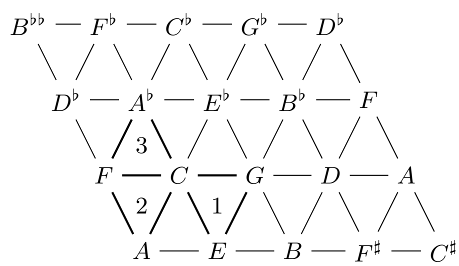


Fig. 19. A tonnetz. The two diagonal axis represent acoustically major and minor thirds. The horizontal axis represents pure perfect fifths.

On that kind of Tonnetz a chord is represented by a polygon, for instance triads are represented by triangles. We measure distance in accordance with neo-Riemannian theory, which considers triangles sharing an edge to be one unit apart – the link with neo-Riemannian theory consists in the fact that such triangles are in fact linked by a simple operation of the neo-Riemannian groups (cf. subsection 2.2) – and which decomposes larger distances into sequences of one-unit moves. Globally, the more common tones chords have, the closer they will be. For instance on Fig. 19 we see that F-major (triangle 2) is closer to C-major (triangle 1) than F-minor (triangle 3). More precisely, C-major is two units away from F-major but three units away from F-minor.

We can also build a Tonnetz in dimension 3, as shown on Fig. 20. On this tonnetz there is one added axis, the z axis, representing the seventh. Four-notes chords are represented as tetrahedron.

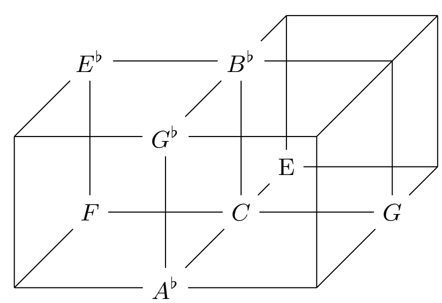


Fig. 20. A tonnetz in dimension 3. 4-notes chords are represented as tetrahedron. For instance C^7 is the tetrahedron whose vertices are C, E, G and $B^\flat$.

These tonnetz are in general based on consonant intervals as thirds, fourth and fifths. Thus it encodes a very different information than the chord space of Tymozcko that we will see later, based on voice-leading. In this latter space, two notes separated by one semitone are very near: for instance the distance between the two-notes chords $[C, G]$ and $[C^\sharp, G^\sharp]$ is small, while it is the contrary in the tonnetz. However, as explained in [32], when we consider chords that are nearly even [9] or related by inversion, which is the case of major and minor triads for example, the distance in the tonnetz and the distance based on voice leading are close.

Remark 5.1. As we said, in the tonnetz chords are represented as polygons. We can also use a different representation in which chords are points – the vertices of the lattice – and edges are for instance, for triads, the operations in one of the two neo-Riemannian groups that link adjacent triads. This is the case of Douthett and Steinbach's graph (Fig. 8) we saw in the first part. Distance can be

[9] A chord is *even* if its notes are distributed as evenly as possible, cf. Def. 6.10

measured as the infimum number of edges that separates two chords. That kind of graph contains more or less the same information than the above-mentioned tonnetz.

We will describe now another way to measure a distance between chords, that relies on the interval content.

5.2 Measures Based on the Interval Content

We presented in the first part of this work the concept of homometry which relies on the interval vectors of sets. We did not mention the possiblity of using it as a distance between pitch classes sets (*p.c. sets*) in $\mathbb{Z}_{12}$. There are various ways to build a measure of similarity, dissimilarity, between two p.c. sets from their intervals, as described in [26], [21] or [32]. Our goal is not to give a complete description of these measures, we will focus briefly on one of them to express their common specificities.

We will describe the mesure **Angle** presented by Scott in [26], and that was first introduced by David Rogers in 1992. As Scott says, this measure "looks and feels very much like a measure taken from physics", similar to the measure used in astronomy between stars. Consider two p.c. sets A and B in $\mathbb{Z}_{12}$. The interval content of each can be seen as a vector in $\mathbb{R}^6$ with origin O. The function **Angle** is exactly the angle in $\mathbb{R}^6$ between these two vectors:

$$\mathbf{Angle}(A, B) := \langle \mathbf{ic}(A), \mathbf{ic}(B) \rangle,$$

where $\langle u, v \rangle$ designates the angle between the vectors u and v, expressed in radians or degrees. It is pictured on Fig. 21.

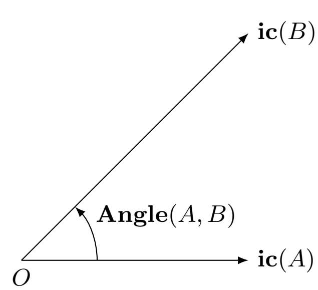


Fig. 21. The measure $\mathbf{Angle}(A, B)$.

The measure **Angle** has several properties. It returns a value between 0° and 90°. In particular $\mathbf{Angle}(A, B) = 0$ when $A = B$, when A and B are translated or inverted one from the other, when A and B are *homometric*, or when $\mathbf{ic}(A)$ is a scalar multiple of $\mathbf{ic}(B)$. Hence it has links with homometry. The largest value is 90°, which happens when A and B have no interval in common.

Interestingly, this measure is able to tell "how far apart" are two p.c. sets with *different cardinalities*, which is a central point in our own future quest.

Scott also recalls that this measure does not take into account the length of the vectors (which is the usual parameter for distances): as angle between stars tell us how far apart stars appear from our observations on Earth, the measure **Angle** tells us how far apart the "aparent sounds" of p.c. sets are.

As we said there are other measures based on the interval content – for instance the measure **Weight**, cf. [21] – but we will not give any more details here. We just want to point out that Tymoczko, in [32], presents a measure based on the length ("magnitude") of the Fourier transform of p.c. sets. As we saw in subsection 1.2, there are profound links between the magnitude of the DFT of a p.c. set and its interval vector. Hence this measure has some similarities with the measure **Angle**. For instance, the distance between two homometric sets equals zero. At this point it seems interesting to mention that the Fourier coefficients of the DFT of a p.c. set contain specific information about the set's harmonic character. For example the third Fourier coefficient of a chord which is close to an augmented triad will be large. For more details refer to Tymoczko and the work of Jason Yust (for instance [35]).

Partial Conclusion

We presented different sorts of measures of distances between chords, that rely on the tonnetz, on the interval content of p.c. sets or on their Fourier transform. Tymoczko does some comparisons in [32]. One of the main result he wants to prove is that the distances he describes are close in some specific cases.

In the present work the objective is not to do comparisons. Our idea is to propose a quick review of existing distances, and to focus our attention on a particular one, which is a distance in the continuous chord space of Tymoczko. This is the aim of the following section.

6 The chord space $\mathcal{A}_n$ of Tymoczko

Tymoczko defines a mathematical chord space in his book *A Geometry Of Music* ([30]) based on voice leading. As he says in [31]:" Voice leading is the technique of connecting the individual notes in a series of chords so as to form simultaneous melodies. Chords are usually connected so that these lines (or voices) move independently (not all in the same direction by the same amount), efficiently (by short distances), and without voice crossings (along non-intersecting paths). These features facilitate musical performance, engage explicit aesthetic norms, and enable listeners to distinguish multiple simultaneous melodies." Elements in this space are musical chords considered modulo octave identification and musical inversion [10]. Topologically it is an orbifold that has specific musical properties, and its mathematical construction allows us to define a mathematical distance on it, as we will show it in the next section.

Here we recall some mathematical aspects concerning properly discontinuous actions and the topology of an orbifold, which will be useful later to understand precisely the mathematical structure of the chord space $\mathcal{A}_n$ of Tymoczko. In [28], Slavich already proposed a topological analysis of the space $\mathcal{A}_n$. Here we present similar results with our own approach, and we give also new proofs to other results of Tymoczko.

6.1 Recall about Properly Discontinuous Action and Orbifolds - Topological Considerations

We will give some classical definitions such as *orbit* or *stabilizer* but we assume that the reader is familiar with the concept of topological group action. For further explanation about mathematical aspects please refer to the literature.

Properly Discontinuous Actions

Let X be a topological space and G a topological group acting on X. The left action of an element $g \in G$ on $x \in X$ will be written gx, and we will note $\mathcal{U}(x)$ the set of open neighborhoods of x in X.

[10] We make a distinction between *inversion* as employed in the previous part and *musical inversion* employed in this part. *Musical inversion* means that we change the order of notes in a chord, for instance the chord C-major is $[C, E, G]$, the first musical inversion is $[E, G, C]$ and the second inversion is $[G, C, E]$. Considering a chord modulo musical inversion means that we identify these three chords.

Definition 6.1. *The* orbit *of $x \in X$ under the action of G is*

$$Orb(x) = \{gx \mid g \in G\} \subset X.$$

Definition 6.2. *The* stabilizer *of $x \in X$ under the action of G is*

$$Stab(x) = \{g \in G \mid gx = x\} \subset G.$$

Definition 6.3. *We say that the action of G on X is* continuous *if the following application is continuous:*

$$\begin{aligned} \phi : G \times X &\longrightarrow X \\ (g, x) &\longmapsto gx. \end{aligned}$$

If G is discrete, it is equivalent to say that for all $g \in G$, the map $x \longmapsto gx$ is an homeomorphism. If X is a differentiable manifold and if this application is a diffeomorphism, we will say that G acts by diffeomorphism.

Definition 6.4. *Let G be a discrete group acting continuously on X. We say that the action is* properly discontinuous *if for all $(x, y) \in X^2$, there exists $U_x \in \mathcal{U}(x)$ and $U_y \in \mathcal{U}(y)$ such that the set*

$$\{g \in G \mid g(U_x) \cap U_y \neq \emptyset\}$$

is finite.

From now, we will assume that X is Hausdorff and locally compact. The following remark will be useful to understand better the local topology of an orbifold.

Remark 6.1. If G acts properly discontinuously on X, then for all $x \in X$ there exists $U_x \in \mathcal{U}(x)$ such that

$$Stab(x) = \{g \in G \mid g(U_x) \cap U_x \neq \emptyset\}.$$

Proof. This result is not our own contribution. However we did not find a clear proof in the literature, that is why we give here our own proof. By definition, there exists $U \in \mathcal{U}(x)$ such that $\{g \in G \mid g(U) \cap U \neq \emptyset\}$ is finite, let's say

$$\{g \in G \mid g(U) \cap U \neq \emptyset\} = \{id, g_1, ..., g_n\}.$$

If for some $i \in \{1, ..., n\}$ we have $g_i x \neq x$, there exists – since X is Hausdorff – $U_i \in \mathcal{U}(x)$ with $U_i \subset U$ and $W_i \in \mathcal{U}(g_i x)$ with $W_i \subset U$ such that $U_i \cap W_i = \emptyset$.

As the action is continuous we can find $V_i \in \mathcal{U}(x)$ with $V_i \subset U_i$ such that $g_i(V_i) \subset W_i$. Then $g_i(V_i) \cap V_i = \oslash$.

Since there are a finite number of g_i, we can do this procedure for each g_k such that $g_k x \neq x$, and finally choose $U_x = \cap_{i=1}^n V_i$ – where we write $V_i = U$ if $g_i x = x$. □

We can now move to the construction of an orbifold.

Orbifolds

This is a brief introduction to orbifolds. For a general study about orbifold theory see [29]. We begin with some definitions.

Definition 6.5. *An n-dimensional* orbifold chart *on the topological space X is a 3-uple $(\tilde{U}, G, \pi)$ where*

- *$\tilde{U}$ is open in $\mathbb{R}^n$;*
- *G is a finite group of homeomorphisms of $\tilde{U}$;*
- *$\pi : \tilde{U} \longrightarrow X$ is a map defined by $\pi = \overline{\pi} \circ p$, where $p : \tilde{U} \longrightarrow \tilde{U}/G$ is the orbit map and $\overline{\pi} : \tilde{U}/G \longrightarrow X$ is a map that induces an homeomorphism of $\tilde{U}/G$ onto an open subset $U \subset X$.*

An *embedding* $\lambda : (\tilde{U}_1, G_1, \pi_1) \longrightarrow (\tilde{U}_2, G_2, \pi_2)$ between two charts is a smooth embedding $\lambda : \tilde{U}_1 \longrightarrow \tilde{U}_2$ such that $\pi_2 \circ \lambda = \pi_1$.

For $i = 1, 2$, let $(\tilde{U}_i, G_i, \pi_i)$ be two orbifold charts on X such that $U_i = \pi(\tilde{U}_i)$, and x be in $U_1 \cap U_2$. We say that these charts are *compatible* if there exists an open neighborhood $V \in U_1 \cap U_2$ of x and a chart $(\tilde{V}, H, \phi)$ with $\phi(\tilde{V}) = V$ such that there are two embeddings $\lambda_i : (\tilde{V}, H, \phi) \longrightarrow (\tilde{U}_i, G_i, \pi_i)$.

Definition 6.6. *An n-dimensional* orbifold atlas *on X is a collection*

$$\mathcal{U} = \{(\tilde{U}_\alpha, G_\alpha, \pi_\alpha)\}_{\alpha \in I}$$

of compatible n-dimensional orbifold charts which cover X.

Definition 6.7. *An* orbifold $\mathcal{O}$ *of dimension n consists of a paracompact Hausdorff space $X_{\mathcal{O}}$ together with an n-dimensional orbifold atlas of charts $\mathcal{U}_{\mathcal{O}}$.*

Remark 6.2. A manifold is an orbifold where each G_α is the trivial group, so that $\tilde{U_\alpha}$ is homeomorphic to U_α.

Proposition 6.1. *If M is a manifold and G is a group acting properly discontinuously on M, then the quotient M/G is an orbifold.*

For the proof the reader can refer to [**?**]. This proposition is very useful and could have been taken as a simpler definition of an orbifold since the orbifold we will study is of that form.

Example 6.1. Let us give an example of orbifold. Take the quotient of the unit disc D^2 in $\mathbb{R}^2$ by the action of a finite order rotation r_k. The unit disc is open in $\mathbb{R}^2$, then it can be seen as an orbifold. The group $\langle r_k \rangle$ acts properly discontinuously on D^2 (the continuity is obvious and $\langle r_k \rangle$ is finite then the other conditions are easily filled). $D^2/\langle r_k \rangle$ is then an orbifold of dimension 2 thanks to the proposition. If we look at the stabilizers, we remark that $Stab(x)$ is trivial for all $x \in D^2$ except for the origin O where $Stab(O) = \langle r_k \rangle$. It turns this element into a special element in a topological point of view. We will explain it right now.

We want to describe the local topology of an orbifold $\mathcal{O}$, more precisely we want to describe the orbifold in the neighborhood of a given element. Let $x \in \mathcal{O}$. We have to consider two cases:

- **If** $Stab(x)$ **is trivial** (i.e. $Stab(x) = \{id_G\}$). If we consider an orbifold chart in the neighborhood of x, there exists an homeomorphism between $\tilde{U}/G$ and U_x, where $\tilde{U}$ is open in $\mathbb{R}^n$ and $V_x \in \mathcal{U}(x)$. From the above-mentioned remark, we also know that there exists $V_x \supset U_x \in \mathcal{U}(x)$ such that

$$\{g \in G \mid g(U_x) \cap U_x \neq \oslash\} = Stab(x) = \{id_G\}.$$

  Then the local topology of $\mathcal{O}$ in the neighborhood of x is the same than the topology of $\tilde{U}/\{id_G\}$ i.e. the same than the topology of $\tilde{U}$.

- **If** $Stab(x)$ **is not trivial**. Again we use an orbifold chart in the neighborhood of x thus we get an homeomorphism between $\tilde{U}/G$ and U_x, where $\tilde{U}$ is open in $\mathbb{R}^n$ and $V_x \in \mathcal{U}(x)$. There exists $V_x \supset U_x \in \mathcal{U}(x)$ such that

$$\{g \in G \mid g(U_x) \cap U_x \neq \oslash\} = Stab(x).$$

  Then the local topology of $\mathcal{O}$ in the neighborhood of x is the same than the topology of $\tilde{U}/Stab(x)$, which is a "non-trivial" quotient space. Such elements are called *singular*.

More intuitively, the topology in a neighborhood of $x \in \mathcal{O}$ is the topology of $\tilde{U}/Stab(x)$ where $\tilde{U}$ is open in $\mathbb{R}^n$.

Example 6.2. We consider again $D^2/\langle r_k \rangle$. For $x \neq O$, $Stab(x)$ is trivial then the local topology of x is the same as in D^2. For O the local topology is given by $\tilde{U}/Stab(O) \cong \tilde{U}/\langle r_k \rangle$. Finally $D^2/\langle r_k \rangle$ looks like a cone, and the only singular point is the origin O, as shown on Fig. 22.

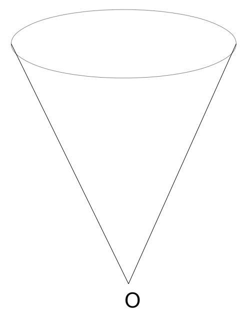


Fig. 22. The orbifold $D^2/\langle r_k \rangle$ is topologically a cone.

Now we will use this analysis to study the chord space $\mathcal{A}_n$ of Tymoczko.

6.2 The Chord Space $\mathcal{A}_n$

We are interested in studying the n-notes chord space of Tymoczko $\mathcal{A}_n$ ([30], [31], [32]). The human pitch perception is logarithmic and periodic, for instance two notes separated by an octave (frequencies f and $2f$) are heard to possess the same quality. To model this logarithmic aspect we associate a real number p to a frequency f according to the equation

$$p = 69 + 12 \log_2(f/440).$$

We obtain a linear space in which a semitone has size 1 and an octave has size 12. Distances in this space reflect physical distances on keyboard instruments. This formula is made to assign 69 to the middle A, which is often used as reference with the frequency 440 Hz (which explains the formula), and middle C is assigned 60. In this space the notes are encoded by integers, thus a n-notes chord can be represented as a vector with n components (each component corresponding to a note of the chord), i.e. as a point in an n-dimensional linear space.

As we are interested in the topological properties of our chord space, we will however consider chords as points in the continuous space $\mathbb{R}^n$. Besides we will take into account two equivalences. The first one is the equivalence modulo the group of translations by integers T^n (which corresponds musically to the equivalence modulo octaves). The space we obtain is $\mathbb{R}^n/T^n$ which is the n-dimensional torus $\mathbb{T}^n$. The second one is the equivalence modulo the group of permutations of n coordinates, written S_n (which means that we consider unordered chords). That gives the final space $\mathbb{T}^n/S_n$.

Recall that $\mathbb{T}^n$ is a differentiable manifold of dimension n. The action of S_n on $\mathbb{T}^n$ is by diffeomorphism and properly discontinuous (S_n is finite). Consequently we know from Prop. 6.1 that $\mathbb{T}^n/S_n$ is an orbifold.

Definition 6.8. *A n-*notes chord *is an element of the orbifold*

$$\mathcal{A}_n := \mathbb{T}^n/S_n.$$

As a differentiable manifold, $\mathbb{T}^n$ has locally the same topology than a point in $\mathbb{R}^n$. Then if we want to study the local topology $\mathcal{A}_n$, we know from the previous subsection that we have to study $Stab(x)$ – for the action of S_n – for each $x \in \mathcal{A}_n$. Let x be a point of $\mathbb{T}^n$, we have two cases:

- **If all the coordinates of x are different.** Then $Stab(x) = \{id\}$ and the local topology in the neighborhood x is the same than the topology in $\mathbb{T}^n$ (i.e. the same than an open set in $\mathbb{R}^n$).

- **If x has identical coordinates.** We write $x = ([x_1], ..., [x_n])$, where $[y]$ is the equivalence class of $y \in \mathbb{R}^n$ in $\mathbb{T}^n$, and we consider the partition P_x of $\{1, ..., n\}$ such that

  $$\{i_1, ..., i_m\} \in P_x \Longleftrightarrow [x_{i_j}] = [x_{i_k}] \text{ for } j, k = 1, ..., m.$$

  Then if P_x consists in p sets $R_1, ..., R_p$ such that $\sharp(R_j) = c_j$, we have

  $$Stab(x) \simeq S_{c_1} \times .... \times S_{c_p},$$

  where S_{c_k} is the group of permutations of c_k elements. Finally the local topology of x is given by the topology of $U_x/S_{c_1} \times .... \times S_{c_p}$, where U_x is homeomorphic to an open set in $\mathbb{R}^n$.

A space like $U_x/S_{c_1} \times .... \times S_{c_p}$ is not easily representable in the general case. In the following examples we study two concrete cases with $\mathcal{A}_2$ and $\mathcal{A}_3$, i.e. in dimension 2 and 3.

Example 6.3. Dimension 2. We start with the torus $\mathbb{T}^2$ whose points are ordered chords of two notes. To form the space of unordered pairs $\mathbb{T}^2/S_2$ we have to identify (x, y) and (y, x). For instance the two points B and D on the 2-torus on Fig. 23–left have to be identified as in Fig. 23–right. The space we get is a triangle with two edges identified, which is a Mobius strip.

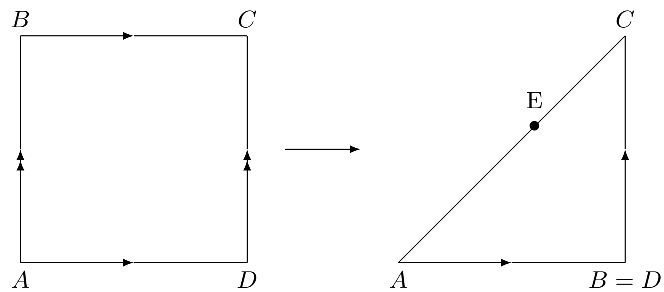


Fig. 23. Identification $B \equiv C$ in the 2-torus. We obtain a Mobius strip.

On Fig. 23, E is a point of the form $([x], [x])$, hence $Stab(E) = S^2$. As a consequence this point is singular, as every point on the line (AC). In the neighborhood of E the topology is the same than the topology of a point belonging to the boundary of the half-space $\{x \geq y\}$ in $\mathbb{R}^2$.

Example 6.4. Dimension 3. In $\mathbb{R}^3$ the topology in the neighborhood of an element of the form $([x], [x], [y])$ will be the same than for an element belonging to the boundary of the half space $\{x \geq y\} \subset \mathbb{R}^3$ (point B on Fig. 24).

Near an element of the form $([x], [x], [x])$ the topology will be the same than near an element in $\{x \geq y, x \geq z\}$ belonging to the intersection $x = y = z$ (point C on Fig. 24).

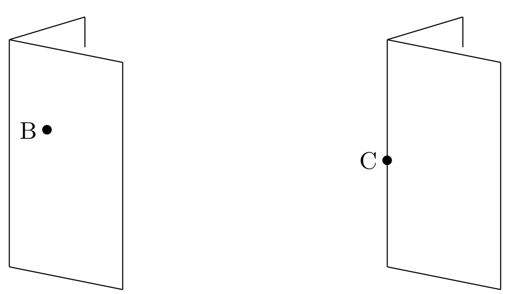


Fig. 24. Local topology in $\mathcal{A}_3$.

We will now explain more precisely the local structure of $\mathcal{A}_n$, describing a fundamental domain of the space. We give first the definition of a fundamental domain.

Definition 6.9. *Given an action of a group G on a topological space X by homeomorphism, a* fundamental domain *for this action is a set of representatives for the orbits.*

Example 6.5. If we take $X = \mathbb{R}^n$ the Euclidean space, and $G = \mathbb{Z}^n$ the lattice acting on it by translations, the quotient X/G is a n-torus. A fundamental domain here can be taken to be $[0,1)^n$.

Theorem 6.1. *$\mathcal{A}_n$ is a metric space whose fundamental domain is a n-dimensional prism whose base is a $(n-1)$-dimensional simplex. The vertices are identified according to a circular permutation.*

Proof. The theorem is Tymoczko's one, but he does not give the proof. We give it here.

We have to do the quotient space of $\mathbb{R}^n$ by the group $G = \langle T^n, S_n \rangle$. Let's denote by τ_i the translation

$$(x_1, ..., x_i, ..., x_n) \mapsto (x_1, ..., x_i + 1, ..., x_n)$$

and by σ_{ij} the permutation between coordinates x_i and x_j.

We call

$$C_t = \{(x_1, ..., x_n) \mid \sum_{i=1}^{n} x_i = t\}$$

and

$$C = \cup_{t \in [0,1]} C_t$$
$$D = \{(x_1, ..., x_n) \mid x_i \geq x_{i+1}, \forall i = 1, ..., n-1, x_1 \leq x_n + 1\}.$$

We want to show that a fundamental domain is given by $P = D \cap C$. We will describe the transformation sending $x \in \mathbb{R}^n$ in a unique point in P. Before this explanation we see that (here $m \in \mathbb{Z}$)

$$\begin{aligned}
\tau_j^m \sigma_{ij} \tau_j^{-m}(x_1, ..., x_i, ..., x_j, ..., x_n) &= \tau_j^m \sigma_{ij}(x_1, ..., x_i, ..., x_j - m, ..., x_n) \\
&= \tau_j^m (x_1, ..., x_j - m, ..., x_i, ..., x_n) \\
&= (x_1, ..., x_j - m, ..., x_i + m, ..., x_n),
\end{aligned}$$

which is a reflection – belonging to G – regarding to the hyperplane $x_j = x_i + m$. Hence all the transformations $\tau_j^m \sigma_{ij} \tau_j^{-m}$ fix the hyperplanes C_t, for all t (the planes $x_j = x_i + m$ are perpendicular to C_t).

Let $x \in \mathbb{R}^n$, and

$$t_x := \sum_{i=1}^{n} x_i - \lfloor \sum_{i=1}^{n} x_i \rfloor \in [0,1].$$

We can send x in C_{t_x} with a translation along x_p for instance, more precisely there exists a unique $k \in \mathbb{Z}$ such that $\tau_p^k(x) \in C_{t_x}$. Then we want to move in C_{t_x}, to get to $D \cap C_{t_x}$. We will use for this purpose the reflections regarding to the hyperplanes $x_j = x_i + m$, $m \in \mathbb{Z}$. We do two remarks. First if we fix (i,j), the planes in $H_{ij} := \{x_j = x_i + m \mid m \in \mathbb{Z}\}$ are parallel. Besides, the boundary of D is given by the set of planes $x_1 = x_2, x_2 = x_3, ...x_{n-1} = x_n$ and $x_1 = x_n + 1$ which belong to $H := \cup_{(i,j)\in[1,n]^2} H_{ij}$.

The planes in H decompose C_{t_x} in $(n-1)$-dimensional simplexes. We know from the previous remark that $D \cap C_{t_x}$ is exactly one of them. We call v_k the vertices of $D \cap C_0$. They are of the following forms.

- for v_0: $$\left| \begin{matrix} x_{n-1} + 1 \neq x_0 \\ x_0 = x_1 \\ x_1 = x_2 \\ . \\ . \\ x_{n-2} = x_{n-1}; \end{matrix} \right| \bigcap \{\textstyle\sum_i x_i = 0\} \Longrightarrow v_0 = (0, ..., 0)$$

- for $v_k, k = 1, ..., n-1$: $$\left| \begin{matrix} x_{n-1} + 1 = x_0 \\ x_0 = x_1 \\ . \\ . \\ x_{n-k-2} = x_{n-k-1} \\ x_{n-k-1} = x_{n-k} - 1 \\ x_{n-k} = x_{n-k+1} \\ . \\ . \\ x_{n-2} = x_{n-1} \end{matrix} \right| \bigcap \{\textstyle\sum_i x_i = 0\}$$

$$\Longrightarrow v_k = (\underbrace{\frac{k}{n}, ..., \frac{k}{n}}_{(n-k)-times}, \underbrace{\frac{k-n}{n}, ..., \frac{k-n}{n}}_{k-times}),\ k = 1, ..., n-1.$$

Since we obtain $D \cap C_t$ just with the translation by $(\frac{t}{n}, ..., \frac{t}{n})$ of $D \cap C_0$, we deduce that the vertices of $D \cap C_t$ are $\{v_k + (\frac{t}{n}, ..., \frac{t}{n})\}_k$. Then C_{t_x} is a union of simplexes of the geometrical form $(v_0, ..., v_{n-1})$, and using the reflections (regarding to the sides of these simplexes) we have one and only one point in $D \cap C_{t_x}$ (so in $D \cap C$) corresponding to x. Since we only use translations by integers, the final point in $D \cap C_{t_x}$ does not depend on the first translation sending x in C_{t_x}. Finally, the vertices of $D \cap C_1$ are $(v'_0, ..., v'_{n-1}) = \{v_k + (\frac{1}{n}, ..., \frac{1}{n})\}_k$

$$\begin{aligned}
v'_0 &= (\frac{1}{n}, ..., \frac{1}{n}) \equiv v_1 \\
v'_1 &= (\frac{2}{n}, ..., \frac{2}{n}, \frac{2}{n} - 1) \equiv v_2 \\
&. \\
&. \\
&. \\
v'_{n-2} &= (\frac{n-1}{n}, \frac{n-1}{n}, \frac{-1}{n} ..., , \frac{-1}{n}) \equiv v_{n-1} \\
v'_{n-1} &= (1, 0, ..., 0) \equiv v_0,
\end{aligned}$$

so we have an identification between $D \cap C_0$ and $D \cap C_1$ according to a circular permutation. □

Before giving a corollary, we recall a musical definition, that comes from [2].

Definition 6.10. *A n-notes chord is said to be maximally even if – in comparison to other chords of the same cardinality – its notes are distributed as evenly as possible.*

Example 6.6. In $\mathcal{A}_3$ the augmented triad $[C, E, G\sharp]$ is maximally even. In $\mathcal{A}_4$ the diminished chord $[C, E^\flat, G^\flat, A]$ is maximally even.

We deduce from Thm. 6.5 the following corollary.

Corollary 6.1. *The chords lying in the central line of the fundamental domain are the maximally even chords.*

Proof. The theorem is Tymoczko's one, but he does not give the proof. We give it here.

The vertices of $D \cap C_0$ are

$$v_k = (v_k^0, ..., v_k^{n-1}) = (\underbrace{\frac{k}{n}, ..., \frac{k}{n}}_{(n-k)-times}, \underbrace{\frac{k-n}{n}, ..., \frac{k-n}{n}}_{k-times}) \text{ for } k = 0, ..., n-1.$$

We calculate the coordinates of the center of gravity $G = (x_0^G, ..., x_{n-1}^G)$ of the simplex $D \cap C_0$:

$$x_p^G = \frac{1}{n}(\sum_{k=0}^{n-1} v_k^p) = \frac{1}{n}(\sum_{k=0}^{n-p-1} \frac{k}{n} + \sum_{k=n-p}^{n-1} \frac{k-n}{n}) = \frac{1}{2n}(n - 2p - 1).$$

Then we see that the difference between two consecutive coordinates x_p^G and x_{p+1}^G is exaclty $\frac{1}{n}$. More precisely

$$\begin{aligned}
x_1^G &= x_0^G - \frac{1}{n} \\
x_2^G &= x_1^G - \frac{1}{n} \\
&. \\
&. \\
x_{i+1}^G &= x_i^G - \frac{1}{n} \\
&. \\
&. \\
x_{n-1}^G &= x_0^G - 1 + \frac{1}{n};
\end{aligned}$$

which is exactly the definition of a maximaly even chord. This is for the point just in one end of the central line. Since we obtain $D \cap C_t$ just with the translation by $(\frac{t}{n}, ..., \frac{t}{n})$ of $D \cap C_0$, we obtain all the elements of the central line with the set $\{G + (\frac{t}{n}, ..., \frac{t}{n}), t \in [0, 1]\}$, and each member of the set is then maximaly even.

Conversely let $A = (x_0, ..., x_{n-1})$ be a maximaly even chord in $D \cap C$, we have $A = G + (x_0 - x_0^G, ..., x_0 - x_0^G)$ which belongs to the central line. □

Remark 6.3. The vertices v_k correspond to points of the form $([x], ..., [x])$, which are singular and such that $v_{k+1}^i - v_k^i = \frac{1}{n}$. It means that all the vertices are n different chords with only one note (repeated n times) and such that these different notes form a maximaly even chord.

Besides the nearer chords are to the boundary of the fundamental domain, the more they "look like" $([x], ..., [x])$. The nearer chords are to the central line, the more they are maximaly even. For instance the major and minor chords are near the central line in $\mathcal{A}_3$.

Musical point of view. These results are illustrated on Fig. 25, 26 and 27, drawn from the work of Timoczko. On these figures there are the fundamental domains or slices of fundamental domains, with points corresponding to "concrete" musical notes.

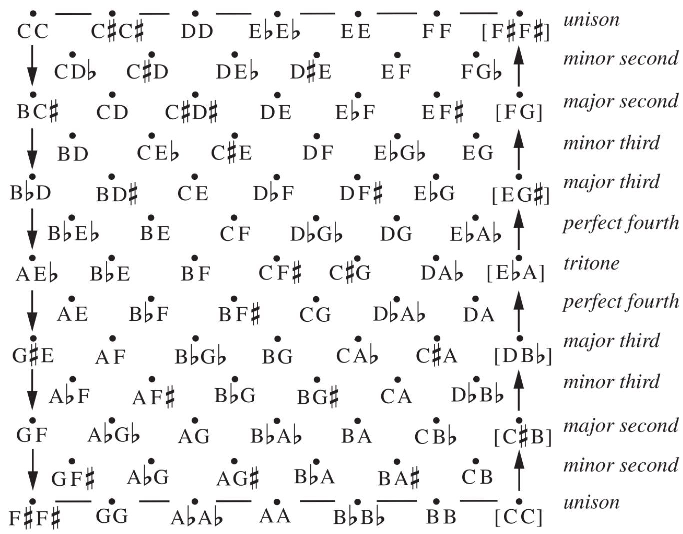


Fig. 25. Representation of $\mathcal{A}_2$. $D \cap C_0$ is the vertical line on the left and $D \cap C_1$ is the vertical line on the right. Transposition corresponds to a motion on the horizontal line. The central horizontal line corresponds to tritones.

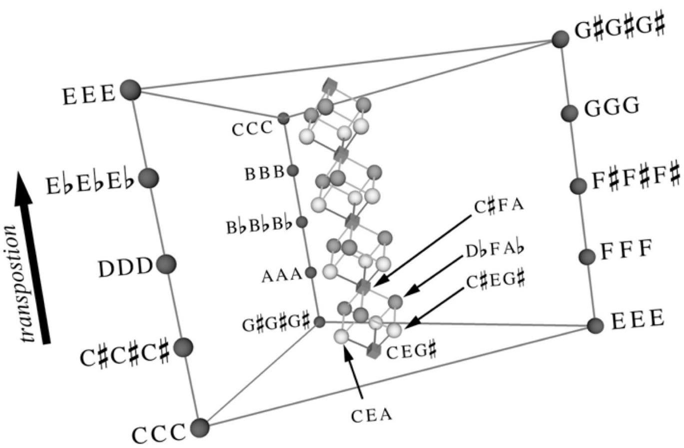


Fig. 26. Representation of $\mathcal{A}_3$. $D \cap C_0$ is the lower horizontal face and $D \cap C_1$ is the upper horizontal face. Transposition corresponds to a vertical motion. The central line corresponds to maximaly even chords i.e. to augmented triads. Major and minor triads are near the central line.

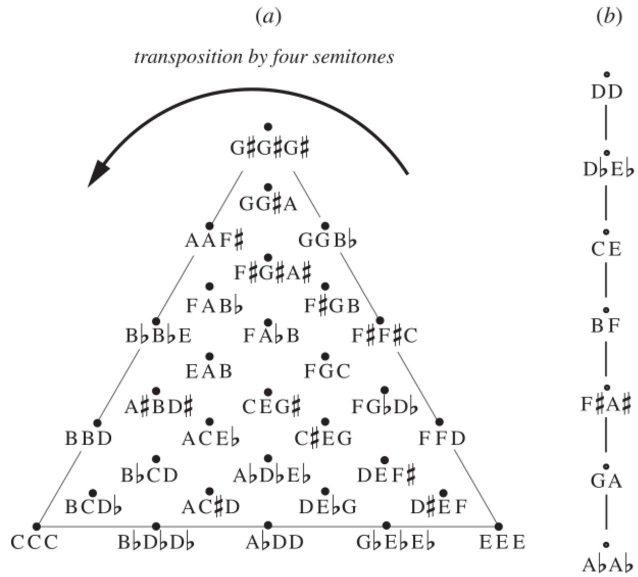


Fig. 27. (a) Representation of the slice $D \cap C_0$ of $\mathcal{A}_3$. We only see the point $[CEG^\sharp]$ of the central line, all the points sum to 0. (b) Representation of the slice $D \cap C_0$ of $\mathcal{A}_2$.

We described the local structure and topology of the orbifold chords space of Tymoczko. We will now build a distance on it.

7 Building Distances on Chord Spaces

We saw the chord space $\mathcal{A}_n$ as an orbifold. As we will see in the beginning of this section, it allows us to build a distance between chords, hence a way to measure how far two chords are the ones from the others. As the space is based on voice-leading, the distance between two chords is small if we can go from the first to the second with a small voice-leading. We will see that with this distance, $\mathcal{A}_n$ is a complete metric space. However, the distance can be used only with chords lying in the same chord space, i.e. chords having the same number of notes. If we want to analyse classical music or pop music, it can be sufficient in some cases, but if we consider contemporary music or jazz music – which often involve complex chords with various numbers of notes – we will need to generalize our first distance. That is the aim of the second subsection, where we propose a definition of mathematical distance in the whole space containing all the chords.

7.1 A Distance in $\mathcal{A}_n$

In this subsection we define a distance on $\mathcal{A}_n$ from the euclidean distance in $\mathbb{R}^n$, we show that with this distance $\mathcal{A}_n$ is a complete metric space, and finally we give a table with all the distances between major and minor triads in $\mathcal{A}_3$.

For a metric space X with metric d_X, $x \in X$ and $\epsilon > 0$, $B(x, \epsilon)$ will designate the open ball of radius ϵ centered at x. We begin with the following proposition.

Proposition 7.1. *Let X be a metric space and d_X the metric of X, and G a group acting properly discontinuously and by isometry on X. Then X/G is a metric space and we can define a distance d on it by*

$$d([x], [y]) = \inf_{g \in G} d_X(x, gy). \tag{75}$$

Besides, the canonical application $q : (X, d_X) \longrightarrow (X/G, d)$ is continuous and such that

$$\forall x \in X, \forall \epsilon > 0, q(B(x, \epsilon)) = B(q(x), \epsilon).$$

As a consequence the topology associated to d is the quotient topology on X/G.

Proof. This proposition is inspired from already existing results. We adapted them and joined them together.

- The distance is well defined because the action is isometric. Indeed if we choose another member of $[x]$, for instance $g'x$, we have

$$\begin{aligned}\inf_{g\in G} d_X(g'x, gy) &= \inf_{g\in G} d_X(g'x, g'(g'^{-1}gy)) \\ &= \inf_{g\in G} d_X(x, g'^{-1}gy) \\ &= \inf_{g\in G} d_X(x, gy)\end{aligned}$$

since the application $G \longrightarrow G, g \mapsto g'^{-1}g$ is an isomorphism.
We have now to check that this definition is a distance. First we clearly see that $d([x],[y]) > 0$. Then for all $g, g' \in G$, we have (we use here the fact that the action is by isometry)

$$\begin{aligned}d([x],[z]) &\le d_X(gx, g'z) \\ &\le d_X(gx, y) + d_X(y, g'z).\end{aligned}$$

If we take the infimum on $g' \in G$ we obtain, because G acts properly discontinuously,

$$d([x],[z]) \le d_X(gx, y) + d([y],[z])$$

and then the infimum on $g \in G$

$$d([x],[z]) \le d([x],[y]) + d([y],[z]),$$

meaning that the triangle inequality is verified.
- Now we will show that

$$d([x],[y]) = 0 \Longrightarrow [x] = [y].$$

We suppose that $d([x],[y]) = 0$ for some x and y. The action is properly discontinuous then there exists $U_x \in \mathcal{V}(x)$ and $U_y \in \mathcal{V}(y)$ such that

$$\{g \in G, g(U_x) \cap U_y \neq \oslash\}$$

is finite. As

$$\inf_{g\in G} d_X(x, gy) = 0,$$

there exists $g \in G$ such that $gy = x$, then $[x] = [y]$.
- The continuity of q is clear since q is 1-lipschitz

$$d([x],[y]) \le d_X(x, y).$$

- We have $q(B(x,\epsilon)) \subseteq B(q(x),\epsilon)$ because $d([x],[y]) \leq d_X(x,y)$. Conversely

$$q(y) \in B(q(x),\epsilon) \Longleftrightarrow d(q(x),q(y)) \leq \epsilon \Longleftrightarrow d([x],[y]) \leq \epsilon,$$

  then there exists $g \in G$ such that $d_X(x,gy) \leq \epsilon$, i.e. $gy \in B(x,\epsilon)$, and we conclude $q(y) = q(gy) \in q(B(x,\epsilon))$. We proved that

$$\forall x \in X, \forall \epsilon > 0, q(B(x,\epsilon)) = B(q(x),\epsilon).$$

- Concerning the topology. We note $\mathcal{T}_q$ the quotient topology and $\mathcal{T}_d$ the topology associated to d on X/G. As q is continuous we have $\mathcal{T}_d \subset \mathcal{T}_q$. But with $q(B(x,\epsilon)) = B(q(x),\epsilon)$ we conclude that q is an open application, and then $V = q(q^{-1}(V))$ and $\mathcal{T}_q \subset \mathcal{T}_d$. Finally $\mathcal{T}_d = \mathcal{T}_q$. □

In our case, we first look at the torus $\mathbb{T}^n = \mathbb{R}^n/T^n$. The action of T^n is isometric and properly discontinuous, consequently we have a metric on $\mathbb{T}^n$. Then we look at $\mathbb{T}^n/S_n$. Again the action of S_n is isometric and properly discontinuous (S_n is finite). Thank to Prop. 7.1, we can then define a distance $d_{\mathcal{A}_n}$ on $\mathcal{A}_n$ and the topology associated to this distance is the quotient topology of $\mathcal{A}_n$. The distance is given by

$$d_{\mathcal{A}_n}([x],[y]) = \inf_{\sigma \in S_n} d_{\mathbb{T}^n}(\overline{x}, \overline{\sigma(y)}),$$

where $\overline{x}$ designates the class of x in $\mathbb{T}^n$. Again with the proposition we have

$$d_{\mathbb{T}^n}(\overline{x}, \overline{\sigma y}) = \inf_{k \in T^n} d_{\mathbb{R}^n}(x, k + \sigma(y)).$$

Finally we obtain

$$d_{\mathcal{A}_n}([x],[y]) = \inf_{\sigma \in S_n} \left(\inf_{k \in T^n} |x - \sigma(y) + k|_{\mathbb{R}^n} \right),$$

where $|x|_{\mathbb{R}^n}$ corresponds to $\sqrt{\sum_i x_i^2}$.

We will see that $(\mathcal{A}_n, d_{\mathcal{A}_n})$ is complete for all $n \in \mathbb{N}$, with the following proposition.

Proposition 7.2. *Let X be a complete metric space, and G be a finite group acting properly discontinuously and by isometry on X. With the distance d defined as in Prop. 7.1, X/G is a complete space.*

Proof. Let $([x_n])_{n \in \mathbb{N}}$ be a Cauchy sequence in X/G. It implies that for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$d([x_p],[x_N]) < \epsilon$$

for all $p > N$. It means

$$\inf_{g \in G} d_X(gx_p, x_N) < \epsilon.$$

Consequently there exists $g_p \in G$ such that $d_X(g_p x_p, x_N) < \epsilon$, since the action is properly discontinuous. This way we can construct the sequence $(g_n x_n)_n$ in X. This is a Cauchy sequence

$$p, q > N \Longrightarrow d_X(g_p x_p, g_q x_q) < d_X(g_p x_p, x_N) + d_X(x_N, g_q x_q) < 2\epsilon$$

in the complete space X, so it has a limit $x \in X$. More precisely there exists $N' \in \mathbb{N}$ such that

$$n > N' \Longrightarrow d_X(g_n x_n, x) < \epsilon.$$

For $p > \max(N, N')$, we have

$$d([x_p], [x]) = \inf_{g \in G} d_X(gx_p, x) \leq d_X(g_p x_n, x) < \epsilon,$$

i.e. $[x_p] \longrightarrow_{n\to\infty} [x]$. Hence X/G is a complete space. □

Corollary 7.1. *For each $n \in \mathbb{N}$, the metric space $(\mathcal{A}_n, d_{\mathcal{A}_n})$ with*

$$d_{\mathcal{A}_n}([x], [y]) = \inf_{\sigma \in S_n} \left(\inf_{k \in T^n} |x - \sigma(y) + k|_{\mathbb{R}^n} \right) \tag{76}$$

is a complete space.

We stop here the topological study of $\mathcal{A}_n$. For more concrete considerations we give all the distances between major and minor triads – elements of $\mathcal{A}_3$ – in Tab. 6. We do not interpret this table now, we will propose some short musical analyses in the next section.

We will now define a distance on the global chord space $\mathcal{A}$.

7.2 A Distance on the Chord Space $\mathcal{A}$

In this subsection we define a distance between chords that have not the same number of notes, i.e. chords that lie in different $\mathcal{A}_n$. For this purpose we first define the space $\mathcal{A}$ of all chords.

Table 6. All the distances between major and minor triads.

	C	$D^\flat$	D	$E^\flat$	E	F	$G^\flat$	G	$A^\flat$	A	$B^\flat$	B	c	$d^\flat$	d	$e^\flat$	e	f	$g^\flat$	g	$a^\flat$	a	$b^\flat$	b
C	0																							
$D^\flat$	1.7	0																						
D	3.5	1.7	0																					
$E^\flat$	2.2	3.5	1.7	0																				
E	1.4	2.2	3.5	1.7	0																			
F	2.2	1.4	2.2	3.5	1.7	0																		
$G^\flat$	3.7	2.2	1.4	2.2	3.5	1.7	0																	
G	2.2	3.7	2.2	1.4	2.2	3.5	1.7	0																
$A^\flat$	1.4	2.2	3.7	2.2	1.4	2.2	3.5	1.7	0															
A	2.2	1.4	2.2	3.7	2.2	1.4	2.2	3.5	1.7	0														
$B^\flat$	3.5	2.2	1.4	2.2	3.7	2.2	1.4	2.2	3.5	1.7	0													
B	1.7	3.5	2.2	1.4	2.2	3.7	2.2	1.4	2.2	3.5	1.7	0												
c	1	2.5	3.3	2	1.7	2.8	3	1.4	1	2.5	3	1.4	0											
$d^\flat$	1.4	1	2.5	3.3	2	1.7	2.8	3	1.4	1	2.5	3	1.7	0										
d	3	1.4	1	2.5	3.3	2	1.7	2.8	3	1.4	1	2.5	3.5	1.7	0									
$e^\flat$	2.5	3	1.4	1	2.5	3.3	2	1.7	2.8	3	1.4	1	2.2	3.5	1.7	0								
e	1	2.5	3	1.4	1	2.5	3.3	2	1.7	2.8	3	1.4	1.4	2.2	3.5	1.7	0							
f	1.4	1	2.5	3	1.4	1	2.5	3.3	2	1.7	2.8	3	2.2	1.4	2.2	3.5	1.7	0						
$g^\flat$	3	1.4	1	2.5	3	1.4	1	2.5	3.3	2	1.7	2.8	3.7	2.2	1.4	2.2	3.5	1.7	0					
g	2.8	3	1.4	1	2.5	3	1.4	1	2.5	3.3	2	1.7	2.2	3.7	2.2	1.4	2.2	3.5	1.7	0				
$a^\flat$	1.7	2.8	3	1.4	1	2.5	3	1.4	1	2.5	3.3	2	1.4	2.2	3.7	2.2	1.4	2.2	3.5	1.7	0			
a	2	1.7	2.8	3	1.4	1	2.5	3	1.4	1	2.5	3.3	2.2	1.4	2.2	3.7	2.2	1.4	2.2	3.5	1.7	0		
$b^\flat$	3.3	2	1.7	2.8	3	1.4	1	2.5	3	1.4	1	2.5	3.5	2.2	1.4	2.2	3.7	2.2	1.4	2.2	3.5	1.7	0	
b	2.5	3.3	2	1.7	2.8	3	1.4	1	2.5	3	1.4	1	1.7	3.5	2.2	1.4	2.2	3.7	2.2	1.4	2.2	3.5	1.7	0

The Space $\mathcal{A}$ of all Chords

We begin with a definition.

Definition 7.1. *The space* $\mathcal{A}$ *of all chords is defined by* $\mathcal{A} := \bigsqcup_{n\in\mathbb{N}} \mathcal{A}_n$.

Then the question is: Is it possible to define a distance on $\mathcal{A}$? *A priori* there is just one canonical way to define a mathematical distance in such a space. For $A \in \mathcal{A}_m$ and $B \in \mathcal{A}_n$, it is given by

$$\delta_\infty(A,B) = \begin{cases} d_{\mathcal{A}_n}(A,B), & \text{if } m = n, \\ +\infty, & \text{if } m \neq n. \end{cases}$$

It has a good property which is $\delta_\infty|_{\mathcal{A}_n} = d_{\mathcal{A}_n}$, but it is not interesting since it does not provide a reasonable way to calculate distances when $n \neq m$. A reasonable distance d on $\mathcal{A}$ should verify:

(i) $d_{|\mathcal{A}_n} = d_{\mathcal{A}_n}$ for all $n \in \mathbb{N}$;

(ii) $0 < d(A,B) < +\infty$ for all $(A,B) \in \mathcal{A}^2$;

(iii) d verifies the triangle inequality;

(iv) the topology associated to d is the topology of the union $\bigsqcup_{n\in\mathbb{N}} \mathcal{A}_n$ i.e. an open ball in $(\mathcal{A}, d)$ is the union of open balls in the $\mathcal{A}_n$ for $n \in \mathbb{N}$.

For this purpose we will use the Hausdorff distance between sets.

A Metric with the Hausdorff Distance

The Hausdorff distance is a measure of distances between sets.

Definition 7.2. *Let (E,d) be a metric space. The Hausdorff distance between two subsets* X *and* Y *of* E *is defined by*

$$d_H(X,Y) = \max\{\sup_{x\in X}\inf_{y\in Y} d(x,y), \sup_{y\in Y}\inf_{x\in X} d(x,y)\}.$$

To have an idea of what measures the Hausdorff distance we picture an example on Fig. 28 with two sets X (dark grey) Y (light grey). We see that the two values $\sup_{x\in X}\inf_{y\in Y} d(x,y)$ (top) and $\sup_{y\in Y}\inf_{x\in X} d(x,y)$ (bottom) are different.

In general, $d_H(X,Y)$ may be infinite. If both X and Y are bounded, then $d_H(X,Y)$ is guaranteed to be finite and (ii) is verified. Moreover, $d_H(X,Y) = 0$

if and only if X and Y have the same closure. In practice we will only consider bounded and finite subsets, then $d_H(X,Y)$ will be finite and $d_H(X,Y) = 0$ if and only if $X \cap Y \neq \oslash$.

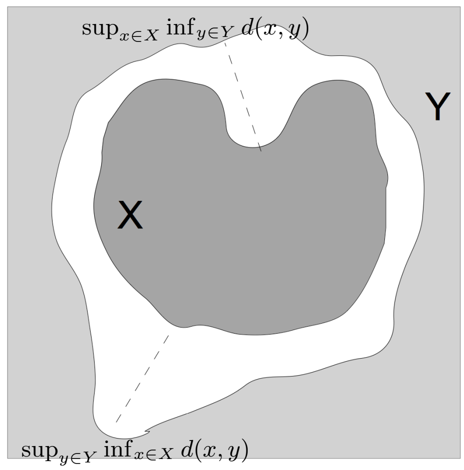


Fig. 28. The values $\sup_{x\in X} \inf_{y\in Y} d(x,y)$ (top) and $\sup_{y\in Y} \inf_{x\in X} d(x,y)$ (bottom) are different.

Suppose we have to calculate the distance between a chord $A \in \mathcal{A}_m$ and a chord $B \in \mathcal{A}_n$, with $m < n$. We would like to use the Hausdorff distance. Our idea is to "project" B and A into a smaller space $\mathcal{A}_t$ $(t \leq m < n)$ so that we can evaluate the distance between these projections with the distance $d_{\mathcal{A}_t}$.

Definition 7.3. *For B in $\mathcal{A}$ and $t \leq \sharp(B)$ we define*

$$P_t(B) := \{Y \subset B \mid \sharp(Y) = t\}.$$

Example 7.1. If we choose $G^7 = [G, B, D, F] = [7, 11, 2, 5]$ we obtain

$$P_3(G^7) = \{[11, 2, 5], [7, 2, 5], [7, 11, 5], [7, 11, 2]\}$$
$$P_2(G^7) = \{[7, 11], [7, 2], [7, 5], [11, 2], [11, 5], [2, 5]\}$$

Now if we use the Hausdorff distance with $X = P_t(A)$ and $Y = P_t(B)$ we obtain a distance d between A and B:

$$d(A,B) = \max\{\sup_{R\in P_t(A)} \inf_{S\in P_t(B)} d_{\mathcal{A}_t}(R,S), \sup_{S\in P_t(B)} \inf_{R\in P_t(A)} d_{\mathcal{A}_t}(R,S)\}.$$

However with this definition we do not verify condition (i) because $t < \min(n,m)$. That is why we choose $t = \min(n,m)$ (here $t = m$). Hence $P_m(A) = A$ and we have

$$d(A,B) = \max\{\inf_{S\in P_m(B)} d_{\mathcal{A}_m}(A,S), \sup_{S\in P_m(B)} d_{\mathcal{A}_m}(A,S)\}.$$

Finally

$$d(A,B) = \sup_{S\in P_m(B)} d_{\mathcal{A}_m}(A,S) \tag{77}$$

We will keep this distance but before moving to Prop. 7.3 we just do a remark.

Remark 7.1. The distance d of (77) can not be a mathematical distance because:

$$d(A,B) = 0 \nRightarrow A = B \text{ in } \mathcal{A}.$$

Indeed A and B do not belong to the same chord space if $m \neq n$. A distance that verifies all the mathematical axioms except the above property is called a *pseudometric*. However if we identify $A \sim B$ when $d(A,B) = 0$, the quotient space $\mathcal{A}/\sim$ may be a metric space. This is called a *metric identification*. In our case, if $d(A,B) = 0$ then $\forall S \in P_m(B)$ we have $A = S$. It implies that there exists $x \in \mathbb{R}$ such that $A = ([x],....,[x]) \in \mathcal{A}_m$ and $B = ([x],...,[x]) \in \mathcal{A}_n$. Then

$$A \in \mathcal{A}_m \sim B \in \mathcal{A}_n \Longleftrightarrow \exists x \in \mathbb{R},\ A = ([x],....,[x]) \in \mathcal{A}_m \text{ and } B = ([x],...,[x]) \in \mathcal{A}_n.$$

With the metric identification, $\mathcal{A}/\sim$ is a metric space (as we will see in the proof of the next proposition, we proved the triangle inequality in a specific case and not in general, thus we are not *sure* that it is verified, even if we strongly think that it is).

Proposition 7.3. *With the distance*

$$d(A,B) := \sup_{S\in P_m(B)} d_{\mathcal{A}_m}(A,S) \tag{78}$$

for $A \in \mathcal{A}_m$ and $B \in \mathcal{A}_n$, with $m \leq n$, the space ($\mathcal{A}$, d) verifies the conditions (i), (ii), (iii) and (iv).

Proof. Condition (ii) is clearly verified. We look at condition (i). If $m = n$ then $P_m(B) = B$ and

$$d(A,B) = \sup_{S\in P_m(B)} d_{\mathcal{A}_m}(A,S) = d_{\mathcal{A}_m}(A,B)$$

hence (i) is verified.

Concerning condition (iii): we did not manage to prove that the triangle inequality is verified in the general case. We are pretty sure that it is and we did not find any counterexample by computation. We mention a situtation where we are sure that the inequality is verified. For A a chord in $\mathcal{A}_m$, C a chord in $\mathcal{A}_n$ with $m < n$ and B be a chord in some $\mathcal{A}_p$, we actually know that if $p = m$ we have

$$d(A,C) \leq d(A,B) + d(B,C).$$

Indeed

$$d(A,C) = \sup_{S \in P_m(C)} d_{\mathcal{A}_m}(A,S)$$

hence for $S \in P_m(C)$ we can apply the triangle inequality in $\mathcal{A}_m$, and we obtain for all $T \in P_m(B)$:

$$\begin{aligned} & d_{\mathcal{A}_m}(A,S) \leq d_{\mathcal{A}_m}(A,T) + d_{\mathcal{A}_m}(T,S) \\ \Longrightarrow & d_{\mathcal{A}_m}(A,S) - d_{\mathcal{A}_m}(T,S) \leq d_{\mathcal{A}_m}(A,T) \\ \Longrightarrow & d_{\mathcal{A}_m}(A,S) - d_{\mathcal{A}_m}(T,S) \leq \sup_{T \in P_m(B)} d_{\mathcal{A}_m}(A,T) = d(A,B). \end{aligned}$$

Thus

$$d_{\mathcal{A}_m}(A,S) \leq d(A,B) + d_{\mathcal{A}_m}(T,S). \tag{79}$$

If $p = m$ we have $B = T$ and consequently

$$d_{\mathcal{A}_m}(B,S) = d_{\mathcal{A}_m}(T,S) \leq \sup_{S \in P_m(C)} d_{\mathcal{A}_m}(T,S) = d(B,C).$$

We obtain

$$d_{\mathcal{A}_m}(A,S) \leq d(A,B) + d(B,C).$$

Taking the supremum for $S \in P_m(C)$ we obtain the triangle inequality. For the other situations ($p \neq m$) we did not find the proof, however Eq. 79 might be a good starting point to find the result.

To prove condition (iv), we want to see if we can write an open ball in $\mathcal{A}$ as a union of open sets in $\mathcal{A}_n$ for $n \in \mathbb{N}$ with the topology given by the distance d.

Let A be a chord in $\mathcal{A}_m$ and $r > 0$, we have

$$\begin{aligned}
B(A,r) &= \{X \in \mathcal{A} \mid d(A,X) < r\} \\
&= \bigcup_{n\in\mathbb{N}} \{X \in \mathcal{A}_n \mid d(A,X) < r\} \\
&= \bigcup_{p<m} \{X \in \mathcal{A}_p \mid d(A,X) < r\} \cup \bigcup_{p\geq m} \{X \in \mathcal{A}_p \mid d(A,X) < r\} \\
&= \bigcup_{p<m} \{X \in \mathcal{A}_p \mid \sup_{S\in P_p(A)} d_{\mathcal{A}_p}(S,X) < r\} \\
&\qquad \cup \bigcup_{p\geq m} \{X \in \mathcal{A}_p \mid \sup_{R\in P_m(X)} d_{\mathcal{A}_m}(A,R) < r\} \\
&:= \bigcup_{p<m} W_p^1(A) \cup \bigcup_{p\geq m} W_p^2(A).
\end{aligned}$$

Let us look at the sets $W_p^1(A)$.

$$\begin{aligned}
W_p^1(A) &= \{X \in \mathcal{A}_p \mid \forall S \in P_p(A), d_{\mathcal{A}_p}(S,X) < r\} \\
&= \bigcap_{S\in P_p(A)} B_{\mathcal{A}_p}(S,r)
\end{aligned}$$

but $\{S \in P_p(A)\}$ is finite so $\bigcap_{S\in P_p(A)} B_{\mathcal{A}_p}(S,r)$ is open in $\mathcal{A}_p$. Then for all $p < m$, $W_p^1(A)$ is open.

We look now at the sets $W_p^2(A)$ for $p > m$. We will show that they are open using the definition of an open set. Let $p > m$ be a fixed integer, $Y \in W_p^2(A)$ and $Y' \in \mathcal{A}_p$ such that $d_{\mathcal{A}_p}(Y,Y') < \eta$ for some $\eta > 0$. We want to see if for η small enough, $Y' \in W_p^2(A)$ i.e. we have to prove

$$\sup_{R'\in P_m(Y')} d_{\mathcal{A}_m}(A,R') < r. \tag{80}$$

Let $R' \in P_m(Y')$. We can choose $R \in P_m(Y)$ such that

$$d_{\mathcal{A}_m}(R,R') \leq d_{\mathcal{A}_p}(Y,Y') < \eta.$$

We use the triangle inequality:

$$d_{\mathcal{A}_m}(A,R') \leq d_{\mathcal{A}_m}(A,R) + d_{\mathcal{A}_m}(R,R').$$

Let's write $d_{\mathcal{A}_m}(A,R) = \rho$. We obtain

$$d_{\mathcal{A}_m}(A,R') \leq \rho + \eta.$$

As $Y \in W_p^2(A)$ we have $\rho < r$, so if we take $\eta < r - \rho$, we have

$$d_{\mathcal{A}_m}(A,R') < r.$$

The set $P_m(Y')$ is finite so we can do it for each $R' \in P_m(Y')$, and then Eq. 80 is verified.

We conclude that $W_p^2(A)$ is open in $\mathcal{A}_p$ for all $p > m$.

Finally $B(A, r)$ can be written as a union of open sets belonging to the $\mathcal{A}_n$ for $n \in \mathbb{N}$. □

We give here an example with a explicit calculation of the distance d.

Example 7.2. We consider the C-major chord $[C, E, G] = [0, 4, 7]$ in $\mathcal{A}_3$ and $G^7 = [G, B, D, F] = [7, 11, 2, 5]$ in $\mathcal{A}_4$. We obtain

$$d(C\text{-major}, G^7) = 3$$

because

$$\begin{aligned} &d([0,4,7],[11,2,5]) = 3 \longrightarrow \text{ sup} \\ &d([0,4,7],[7,11,2]) = 2.2 \\ &d([0,4,7],[7,2,5]) = 2.2 \\ &d([0,4,7],[7,11,5]) = 1.4. \end{aligned}$$

From a mathematical point of view, d is thus a good distance. However for musical reasons we will use a modified version of it. It is explained in the following subsection. The modifications we will do are personal and subjective choices, that is why we want to warn the reader that from the only mathematical point of view one should keep the distance d. The modified version of d will be written d^*, and will be refered to as a *measure* and not a *distance*. Before moving to this subsection we mention that we also studied other definitions of measures of distances in $\mathcal{A}$:. We present in Annex B two alternative measures.

7.3 Modifying *d* for Musical Reasons

In order to understand better the distances we built, we tested it on special cases. We found some undesired properties in a musical point of view, that is why we chose to keep a modified version of d. The modification of d consists in removing the duplicated notes from the chords before calculating the distance. We explain it in the following paragraph with examples. Let us recall that the order of the notes in the chords is not important since we look at those chords modulo the action of the group of permutations.

An Example between a 3-notes Chord and a 4-notes Chord.

Let us consider $[G, G, B, D] = [7, 7, 11, 2]$ and $[G, B, D] = [7, 11, 2]$. These chords do not have the same number of notes and the first chord has the duplicated notes G. We have

$$d([G, G, B, D], [G, B, D]) = 5,$$

which is very large because it corresponds to the choice

$$d([G, B, G], [G, B, D]).$$

It is not natural since the notes between the two chords are the same: G, B and D. If we remove the duplicated notes we obtain

$$d([G, B, D], [G, B, D]) = 0,$$

which seems better.

An Example of a Cadence $V - I$ between two 4-notes Chords.

We study the two chords $[D, D, C, F^\sharp] = [2, 2, 0, 6]$ and $[G, D, B, G] = [7, 2, 11, 7]$. They have the same number of notes and have both a duplicated note: D in the fisrt chord, G in the second. We have

$$d([D, D, C, F^\sharp], [G, D, B, G]) = 5.2$$

It is a large number: the smallest voice leading correspond to

$$\begin{aligned}
&D \longrightarrow G \text{ (distance: 7-2=5)}\\
&D \longrightarrow D \text{ (distance: 0)}\\
&C \longrightarrow B \text{ (distance: 1)}\\
&F^\sharp \longrightarrow G \text{ (distance: 1).}
\end{aligned}$$

Then $d([D, D, C, F^\sharp], [G, D, B, G]) = \sqrt{5^2 + 1^2 + 1^2} \approx 5.2$.

These chords correspond musically to a cadence $V - I$: $D^7 - G$-major, then it seems not natural that the distance is large. If we remove the duplicated notes we obtain

$$d([D, C, F^\sharp], [D, B, G]) = \delta([D, C, F^\sharp], [D, B, G]) = 1.4,$$

which seems more natural.

These examples lead us to redefine d, by removing duplicated notes before computing the distances. The *measure* (not *distance* since it is a modified version of the distance d) we obtain will be called d^*. In the next section we propose some musical applications of d^*.

Partial Conclusion

In this section we first defined a distance on the space $\mathcal{A}_n$, and we proved that with this distance $(\mathcal{A}_n, d_{\mathcal{A}_n})$ is a complete metric space. We gave a table with an explicit calculation of distances between the major and minor triads. Then we built a measure of distances on the global space $\mathcal{A}$ in order to evaluate "how far apart" are chords that do not have the same number of notes, and we finally modified this distance to get a measure more appropriated to musical analysis. In the next section we study some concrete musical applications of this measure.

8 Musical Applications

Here we discuss some musical applications of the measures d^* on $\mathcal{A}$. We create what we call *graphs of distances* in order to propose two short analyses of Bach's choral 7 and 8. We keep sometimes the denomination *distance* in order to be clearer, even if we consider d^* which is a modification of the distance d.

8.1 Graphs of Distances

Music is always a delicate mix between melodic, harmonic and rhythmic aspects. As our distance is exclusively based on the harmonic side, we will however turn musical pieces into simple chord sequences, without taking into account the temporal aspect (rhythm, duration of the notes, etc.). We give an example.

Example 8.1. We consider the seventh Bach's choral entitled: "Nun lob',mein Seel',den Herren". The tonality is A major. The sheetmusic on Fig. 29 shows the first part of this choral.

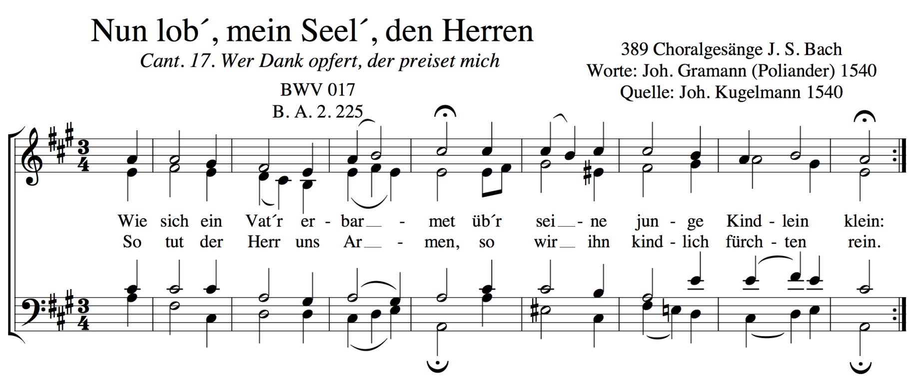


Fig. 29. First part of Choral 7.

It contains, obviously, notes with different durations and also counterpoint. If we just look at the beginning until the first chord of the fourth bar (i.e. the ten first chords of the choral), what we keep for our analysis is the chord sequence given in Fig. 30: if there are sustained notes in the original piece, we repeat them in the next chord for our analysis – as for instance with the fourth and fifth chords, where in the sheetmusic the upper D belonging to the chord $[D, A, D, F^\sharp]$ moves to the $C^\sharp$ (counterpoint) and then to the B of the next chord $[D, G^\sharp, B, E]$. On Fig. 30 we repeat all the sustained notes to get the chord sequence: $[D, A, D, F^\sharp] - [D, A, C^\sharp, F^\sharp] - [D, G^\sharp, B, E]$.

Besides, we do not take into account the rhythmic aspect: all the chords have the same duration, for instance a quarter note. We obtain finally

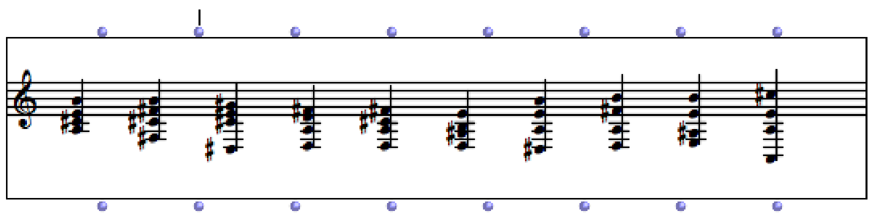


Fig. 30. Chord sequence corresponding to the first bars of Choral 7.

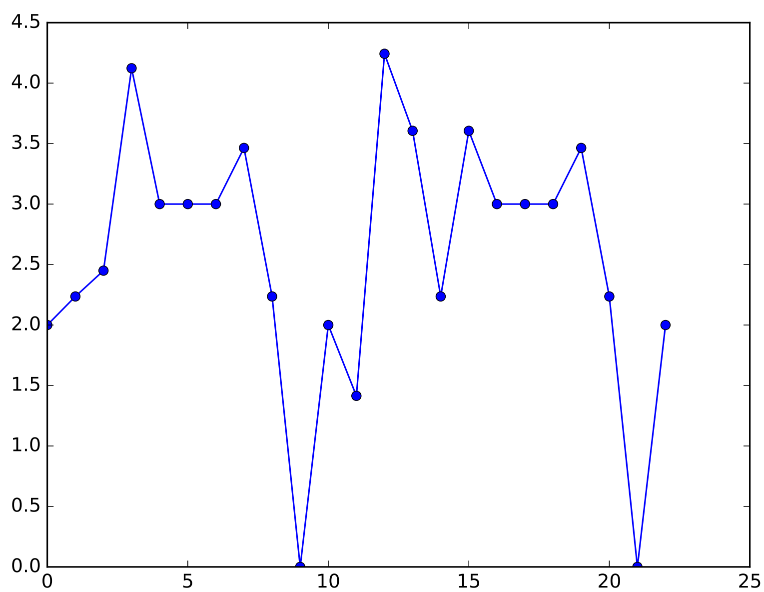


Fig. 31. Graph of distances of the first part of Choral 7 with d^*.

From a chord sequence such as Fig. 30, we can calculate the distances between all the consecutive chords and represent them on a graph called *graph of distances*: on the vertical axis we picture the values of the distances and on the horizontal axis is the time.

For example on Fig. 31 we picture the graph of distances of the first part of the Choral 7 – corresponding to Fig. 29 – with d^*. The first point (with the value 2 at time 0) means that the distance between the first chord and the second chord is 2. The second point means that the distance between the second chord and the third chord is 2.3, etc. Hence the first ten points until the value 0 correspond to the representation of the chord sequence given in Fig. 30.

These graphs can be a starting point for harmonic analysis of musical pieces. Indeed we can wonder if there are links between graphical patterns and musical elements. For instance we see on Fig. 31 some repetitive patterns: is there any interpretation to it? Does a graphical shape correspond to a precise musical property? In order to give some answers to these questions, we propose in the next subsections two short analyses of Bach's choral 7 and 8.

8.2 Analysis of Choral 7: "Nun lob',mein Seel',den Herren"

The whole sheetmusic of choral 7 is given on Fig. 34, and the graph of distances is given on Fig. 35, for d^*. We split our analysis into two parts, corresponding respectively to the first eight bars and to the rest of the choral.

The First Eight Bars.

If we look at the beginning of Fig. 35 we clearly see the repetition of the whole pattern of Fig. 31, which is normal since there is a repetition of this part in the choral (repetition bar line). But we clearly see inside this pattern the repetition of a smaller pattern: the first apparition is in a red box while the second is in a green box (Fig. 33 and in Fig. 32).

The fact that there is a common pattern on a graph of distances does not imply *a priori* that the chords are the same: in our situation for instance the first chord is not the same (it begins with a D^{M7} for the first pattern and with a $F^\sharp m/E$ in the second pattern), however the other chords are actually identical. The chord sequence is:

$$\left\{ \begin{array}{l} D^{M7} \\ F^\sharp m^7 \end{array} \right\} - E^7 - A - Bm - E \text{ (with 2 notes: } E \text{ and } G^\sharp) - A - A - F^\sharp m.$$

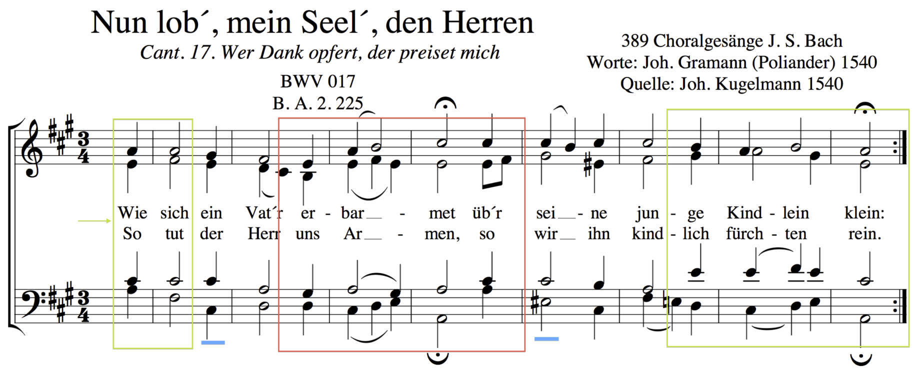


Fig. 32. Repetition of a smaller pattern inside the first part.

The graph is interesting because it clearly shows an internal structure of this part into two sections, that share an (almost) common harmonic sequence. The

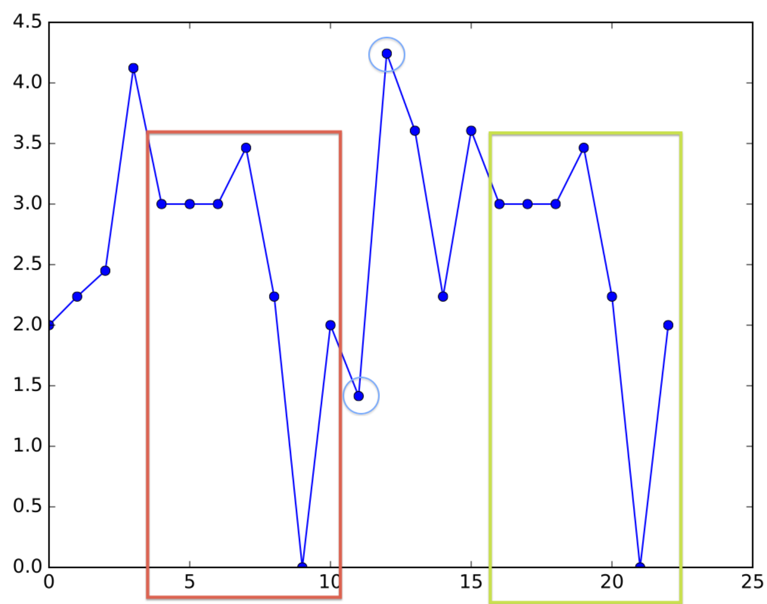


Fig. 33. Repetition of a smaller pattern inside the first part.

fact that the chord sequences are identical is not obvious at first glance when we look at Fig. 32 while it appears clearly on the graph of distances.

Another interesting remark is that the extrema in Fig. 33 (if we do not count when the distance is equal to 0 i.e. when a chord is repeated in that case) correspond to the two points just after the first pattern in red (they are encircled on Fig. 33). In the choral it corresponds exactly at the moment where appears the first musical alteration (the $E^\sharp$), which is not in the main tonality of A major. The first chord where $E^\sharp$ appears is a C-major, which is the "majorization" of the C-minor that plays the same role after the other pattern (and that belongs to the main tonality, both of them are underlined in Fig. 32).

The Rest of the Choral.

We look now at the rest of the choral. On Fig. 35 we see the graph of the whole choral. Let us look at the extrema. We draw red boxes around the maxima and blue boxes around the minima both on Fig 35 and 34. The analysis of these extrema is not relevant: the maxima correspond respectively to the moves

$$[E,A] - [D, B, E, G^\sharp]$$
$$E \longrightarrow E \text{ (distance: 0)}$$
$$A \longrightarrow D \text{ (dist :5)}$$

and

$$[D,A^\sharp, E, C^\sharp] - [B, D^\sharp]$$
$$D \longrightarrow D^\sharp \text{ (dist: 1)}$$
$$E \longrightarrow B \text{ (dist: 5).}$$

The distances are large because it is between 4-notes chords and 2-notes chords. The first minimum is 0 (repetition of a chord) while the others correspond respectively to the moves

$$[A,D, B] - [A, C^\sharp, B]$$
$$A \longrightarrow A \text{ (dist: 0)}$$
$$D \longrightarrow C^\sharp \text{ (dist: 1)}$$
$$B \longrightarrow B \text{ (dist: 0)}$$

and

$$
\begin{aligned}
&[D,B,E,C^\sharp] - [D,A^\sharp,E,C^\sharp]\\
&\quad D \longrightarrow D \text{ (dist: 0)}\\
&\quad B \longrightarrow A^\sharp \text{ (dist: 1)}\\
&\quad E \longrightarrow E \text{ (dist: 0)}\\
&\quad C^\sharp \longrightarrow C^\sharp \text{ (dist: 0).}
\end{aligned}
$$

Thus the study of extrema does not reveal any relevant point. If we focus on more general behaviours – more general graphical pattern – it seems more interesting. On Fig. 36 we show the different parts of the choral according to the fermatas, which is a natural musical decomposition. On this graph this decomposition seems to be quite relevant: indeed it isolates more or less different types of graphical behaviours. Then our distance might be used more for general considerations such as dividing a piece into parts. We confirm this property in the following analysis of choral 8.

Fig. 34. Sheetmusic of Choral 7.

Fig. 35. Graph of distances of Choral 7 with d^*.

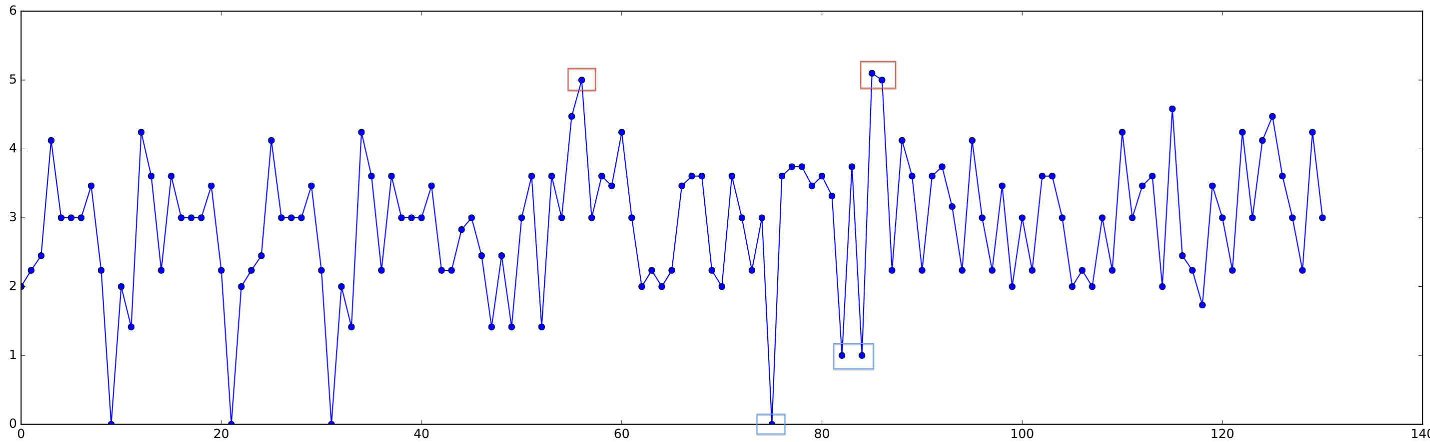
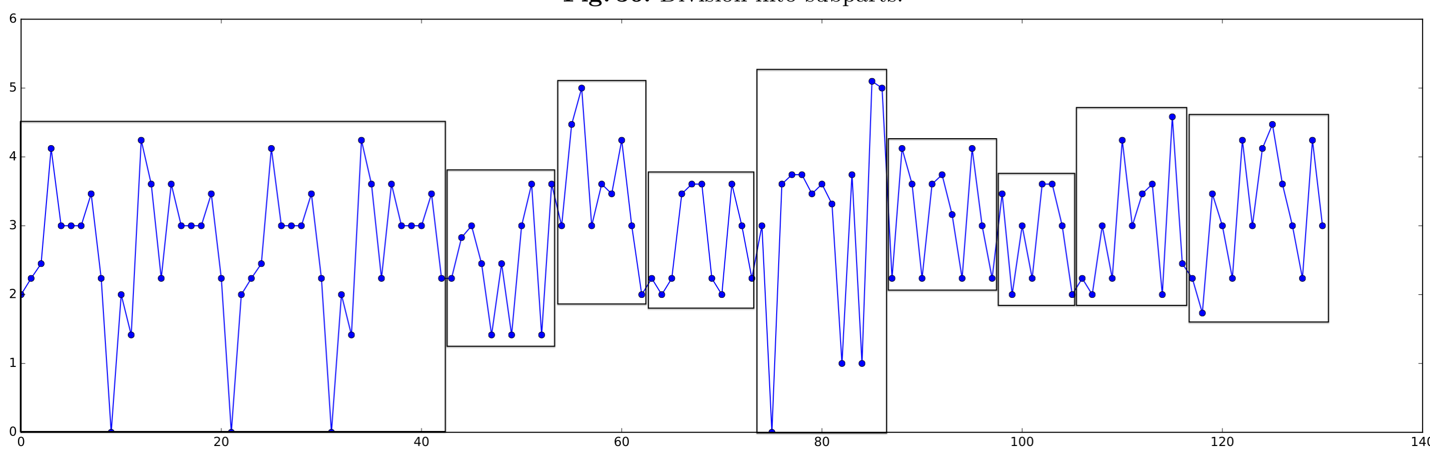


Fig. 36. Division into subparts.



8.3 Analysis of Choral 8: "Freuet euch, ihr Christen"

We study now the eighth Choral: "Freuet euch, ihr Christen". The sheetmusic is given on Fig. 37, while the graph of distances is on Fig. 38.

We see several things on Fig. 38. First there are small patterns that are frequent and have approximatively the same shape (they look like crenels): they are inside green boxes, both on the graph and on the sheetmusic. Musically they correspond to parts with a fixed chord and a diatonic motion of the bass line, hence it is not surprising to find identical shapes on the graph of distances. There is also a similar case with the patterns inside blue boxes: they have a similar shape on the graph of distances. Similarly for the two patterns inside the red boxes which look like "suspended" little variations. They do not correspond to the same chord sequence, but muscically they have obvious common points: a first bar with ascending or descending melodies on the left hand and repeated chords on the right hand, followed by a second bar with similar counterpoints patterns. Thus graphical similarities correspond to similar musical motions.

On Fig. 39 we show the graphical decomposition of the choral according to the fermatas of the sheetmusic (which is again a natural decomposition of the choral, as on Fig. 36). Clearly we see a correspondence between this musical decomposition and the shapes on the graph: each musical part is associated to a recognizable graphical pattern. Moreover, some of these patterns have common points. For instance the parts number 2 and 7 are very similar – they are in fact exactly the parts inside the red boxes on the previous graph –, the parts 4 and 5 as well. Parts 1, 3 and 6 have also some common points.

Consequently the graph of distances seems to be a good tool to identify musical similarities from graphical patterns, i.e. to analyze a global behaviour. It may be interesing to study more precisely which type of graphical pattern corresponds to which musical particularity.

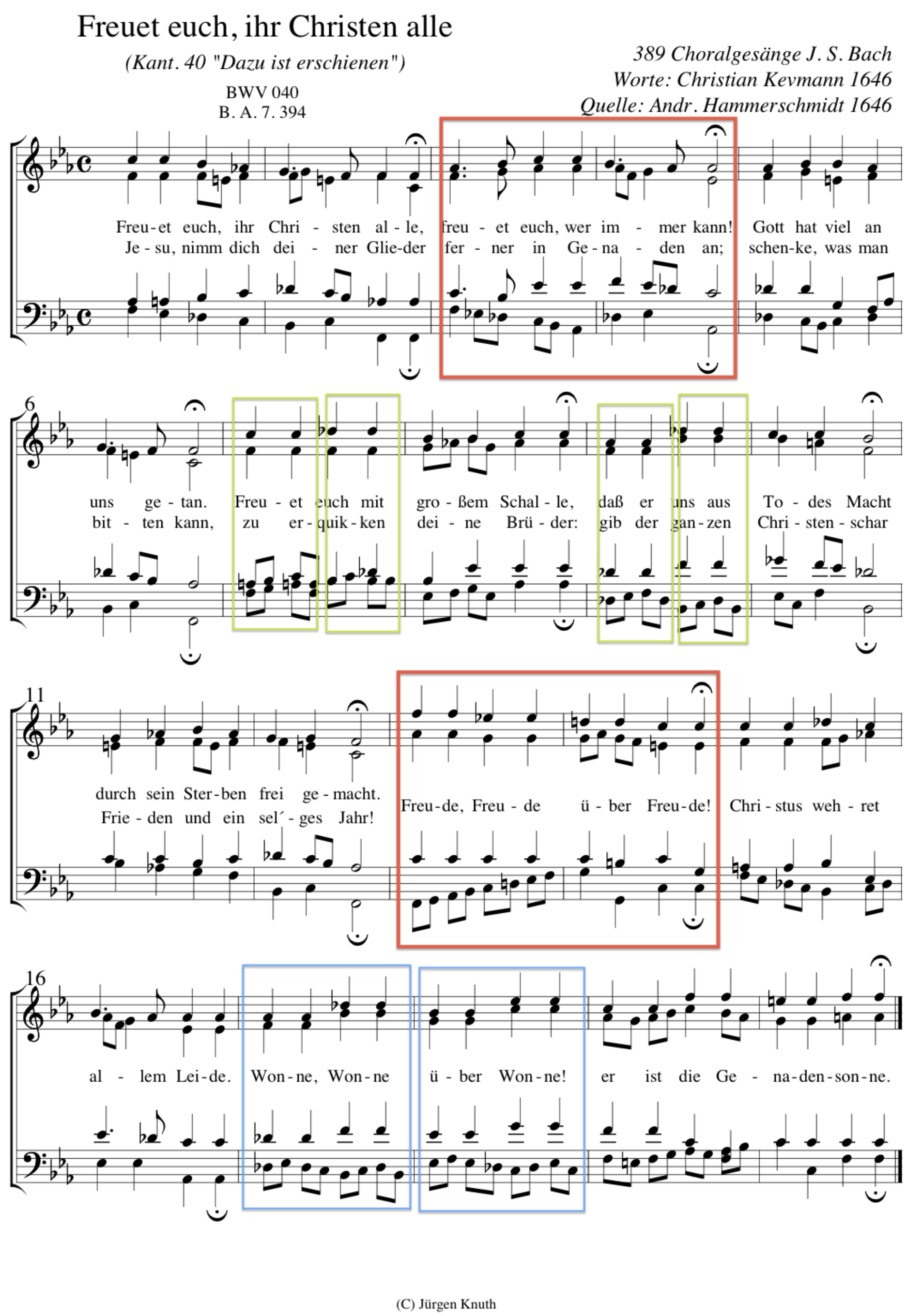


Fig. 37. Sheetmusic of Choral 8.

Fig. 38. Graph of distances of Choral 8 with d^*.

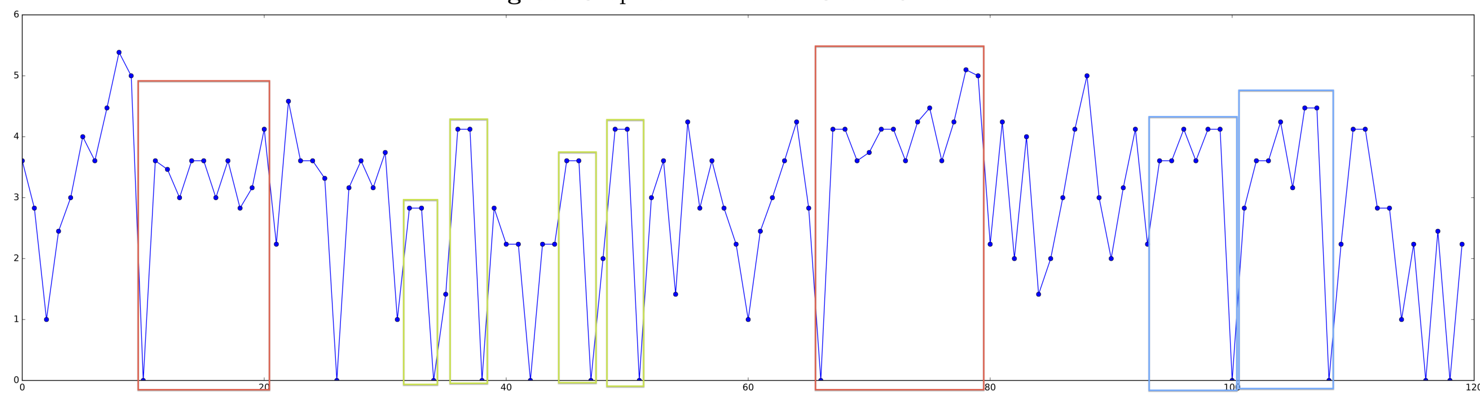
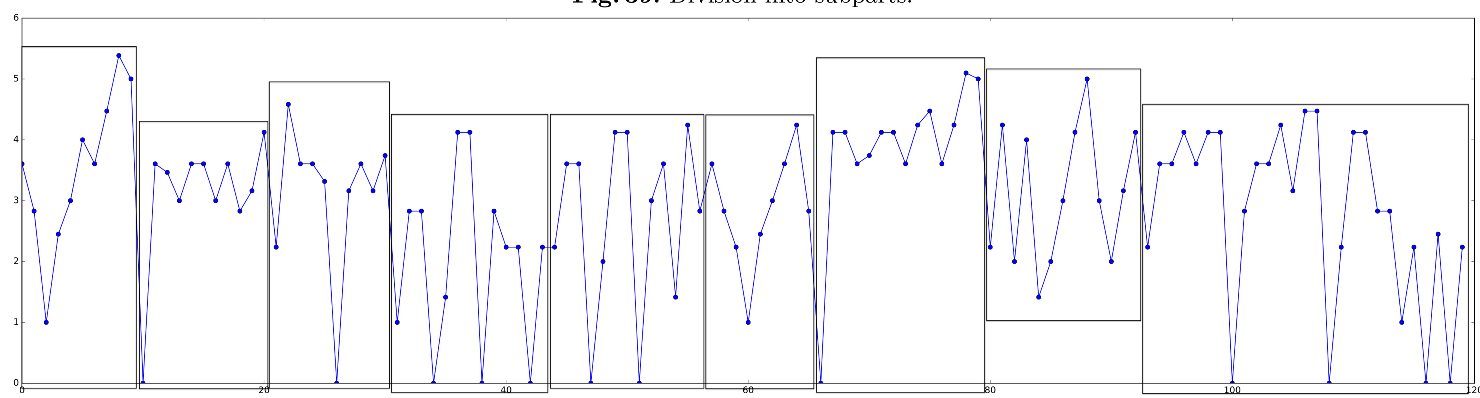


Fig. 39. Division into subparts.



Partial Conclusion

In this section we presented some musical applications of d^*. What we saw from two brief analyses of Bach's choral is that the measure d^* may be used for global analysis (for instance to identify graphical patterns and then decompose a piece into different parts) more than for precise analysis on isolated values. It should be interesting to go further and:

- identify which kind of graphical pattern correspond to which musical property;
- modify again the distance to get more accurate and musical results;
- analyze other Bach's pieces to do a comparison with the chorals;
- analyze other kind of music, such as jazz or contemporary music.

We tried actually to look at jazz music, but it is in fact quite difficult. As our distance is based on a choice of notes between two chords, it may be relevant for written music, which is not the case of jazz pieces. For such pieces we may, before analyzing, choose the notes for each chord, which would become very subjective, and a very different kind of work than the one we proposed. Indeed our objective was at first to build a mathematical distance on the chord space of Tymozcko and propose a topological study.

Conclusion

As a global conclusion we would like to mention the open questions that would be interesting to work on for future works.

Concerning the use of groupoids, it would be interesting to go further in the theoretical aspect and see if we can transpose the homometric results with semi-direct products (cf. section 4) to the categorical generalization of semi-direct products. Besides it would be stimulating to find concrete musical applications with groupoids. Concerning homometry in the dihedral group, a future work should explore more the general behaviour of left-homometry and see if we can weaken the hypothesis of Thm. 2.5. We have also an intuition that might be interesting to prove (or to invalidate): the right-and-left-homometric sets of the form of Thm. 2.5 are always of type 1 of Goyette classification with $p = 0$. For the time-spans group we think that a future work should study homometric rhythms with more than two durations. Concerning the distances between chords, we already mentioned in the last *partial conclusion* some ideas for future studies. We want to add that it would be enriching to build, for instance, an OpenMusic function that evaluates the distance between two chords. This could be an interesting tool for composers.

This work falls within a recent approach in the field of Music Information Research that merges systematic computational analysis and advanced mathematical disciplines such as algebra, topology and category theory. It raises promising and challenging questions such as the automatic classification of musical styles (our distance may be for instance a computational tool that distinguishes jazz music from contemporary or classical music), and stresses the necessity to develop computer-aided environments to compare theoretical constructions and computational models. The mathematical concepts constitute a framework to modelize musical properties and conversely musical questions are starting points for new mathematical researches. This interplay is asking for the development of ad-hoc computational tools.

Finally we would like to thank all the readers for their attention, hoping that they found some interests in our researches!

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Annex A

We give in Tab. 7, Tab. 9 and Tab. 8 a complete listing of homometric sets in D_n for the right and for the left actions for small values of p and n. They are written with another convention than in the rest of this work: a pair $(s, +1)$ is designated by $s+$ and a pair $(s, -1)$ is designated by $s-$. For instance 0+ means C-major while $0-$ means C-minor if $n = 12$.

We give below the program code we used to find homometric lifts for the right and for the left actions in D_n from homometric sets in $\mathbb{Z}_n$. It is written with the langage *Python*. We put the symbol '?' at the places where the reader has to enter his own datas.

Table 7. Listing of all homometric sets in D_n for $p = 4$ until $n = 18$.

D_n	Homometric sets for the left action	Homometric sets for the right action
n=8	[0+,4+,0-,2-] & [0+,6+,0-,4-]	[0+,4+,0-,2-] & [0+,2+,0-,4-]
	[0+,4+,1-,3-] & [0+,6+,1-,5-]	[0+,4+,1-,3-] & [0+,2+,1-,5-]
n=12	[0+,6+,0-,3-] & [0+,9+,0-,6-]	[0+,6+,0-,3-] & [0+,3+,0-,6-]
	[0+,6+,1-,4-] & [0+,9+,1-,7-]	[0+,6+,1-,4-] & [0+,3+,1-,7-]
	[0+,6+,2-,5-] & [0+,9+,2-,8-]	[0+,6+,2-,5-] & [0+,3+,2-,8-]
n=16	[0+,8+,0-,4-] & [0+,12+,0-,8-]	[0+,8+,0-,4-] & [0+,4+,0-,8-]
	[0+,8+,1-,5-] & [0+,12+,1-,9-]	[0+,8+,1-,5-] & [0+,4+,1-,9-]
	[0+,8+,2-,6-] & [0+,12+,2-,10-]	[0+,8+,2-,6-] & [0+,4+,2-,10-]
	[0+,8+,3-,7-] & [0+,12+,3-,11-]	[0+,8+,3-,7-] & [0+,4+,3-,11-]

```
###### ALGORITHM TO FIND HOMOMETRIC LIFTS ######

import numpy as np
import matplotlib.pyplot as plt
from random import randint

### Calculation of the left and of the right intervals between \\
                              \\ two pairs z1 and z2 in D2n ###
```

```
def left_interval_D2n (z1,z2,n):
    n1,k1 = z1
    n2,k2 = z2
    return ((n2-k2*k1*n1) % n,k2*k1 % n)

def right_interval_D2n (z1,z2,n):
    n1,k1 = z1
    n2,k2 = z2
    return (((n2-n1)*k1) % n,k1*k2 % n)

### Calculation of the left and of the right interval vectors \\
                                    \\ of a set theVect in D2n ###

def leftIV_D2n (theVect,n):
    return [left_interval_D2n(z1,z2,n) for i,z1 \\
           \\ in enumerate(theVect) for j,z2 in enumerate(theVect)]

def rightIV_D2n (theVect,n):
    return [right_interval_D2n(z1,z2,n) for i,z1 \\
            \\ in enumerate(theVect) for j,z2 in enumerate(theVect)]

### Test if two interval vectors theVect1 and theVect2 \\
                                        \\ in D2n are equal ###

def is_homometric(theVect1,theVect2):
    try:
        for x in theVect1:
            theVect2.remove(x)
        flag=1
    except Exception:
        flag=0
    return flag

### Give a nice print of a vector under the form s+ or s- ###

def niceprint(theVect1):
    rep = ""
```

```
    for n,h in theVect1:
        if h!=1:
            rep+=str(n)+"- "
        else:
            rep+=str(n)+"+ "
    return rep

###### MAIN: ALGORITHM THAT FINDS THE HOMOMETRIC LIFTS IN D2N \\
                                \\ OF HOMOMETRIC SETS IN ZN ######

N= ?     ## N correspond to the value in D2N ##
card = ? ## card correspond to the value p i.e. to the \\
                                    \\ cardinality of the sets ##

binaryList=np.zeros((2**card,card))

for x in range(2**card):
    for y in range(card):
        if (x & 2**y)>0:
            binaryList[x][y]=1.0
        else:
            binaryList[x][y]=(N-1)

base_A = [?]   ## base_A and base_B correspond to the two \\
base_B = [?]           \\ homometric sets in ZN that we want to lift

for u in range(2**card):
    for v in range(2**card):
        sign1=binaryList[u]
        sign2=binaryList[v]

        vect_A=zip(base_A,binaryList[u])
        vect_B=zip(base_B,binaryList[v])
        homom_gauche = is_homometric(leftIV_D2n(vect_A,N),leftIV_D2n(vect_B,N))
        homom_droite = is_homometric(rightIV_D2n(vect_A,N),rightIV_D2n(vect_B,N))
        if homom_gauche:
            print "Gauche: ",niceprint(vect_A),"/ ",niceprint(vect_B)
        if homom_droite:
```

```
        print "Droite: ",niceprint(vect_A),"/ ",niceprint(vect_B)
    if homom_gauche or homom_droite:
        print "------------------"
```

Table 8. Listing of all homometric sets in D_n for $p = 5$ and $n = 12$.

D_n	Homometric sets for the left action	Homometric sets for the right action
n=12	[0+,4+,8+,0-,2-] & [0+,4+,8+,2-,4-]	[0+,4+,8+,0-,2-] & [0+,4+,8+,2-,4-]
	[0+,4+,8+,1-,3-] & [0+,4+,8+,3-,5-]	[0+,4+,8+,1-,3-] & [0+,4+,8+,3-,5-]
	[0+,6+,0-,2-,4-] & [0+,2+,10+,0-,6-]	[0+,6+,0-,2-,4-] & [0+,2+,4+,0-,6-]
	[0+,6+,0-,2-,4-] & [0+,8+,10+,0-,6-]	[0+,6+,0-,2-,4-] & [0+,2+,10+,0-,6-]
	[0+,6+,0-,1-,5-] & [0+,7+,11+,0-,6-]	[0+,6+,0-,1-,5-] & [0+,1+,5+,0-,6-]
	[0+,6+,1-,3-,5-] & [0+,2+,10+,1-,7-]	[0+,6+,1-,3-,5-] & [0+,2+,4+,1-,7-]
	[0+,6+,1-,3-,5-] & [0+,8+,10+,1-,7-]	[0+,6+,1-,3-,5-] & [0+,2+,10+,1-,7-]
	[0+,6+,0-,4-,5-] & [0+,4+,11+,0-,6-]	[0+,6+,0-,4-,5-] & [0+,1+,8+,0-,6-]
	[0+,1+,8+,0-,6-] & [0+,7+,8+,0-,6-]	[0+,2+,4+,0-,6-] & [0+,2+,10+,0-,6-]
	[0+,2+,10+,0-,6-] & [0+,8+,10+,0-,6-]	[0+,4+,5+,0-,6-] & [0+,4+,11+,0-,6-]
	[0+,1+,8+,1-,7-] & [0+,7+,8+,1-,7-]	[0+,2+,4+,1-,7-] & [0+,2+,10+,1-,7-]
	[0+,2+,10+,1-,7-] & [0+,8+,10+,1-,7-]	[0+,4+,5+,1-,7-] & [0+,4+,11+,1-,7-]
	[0+,1+,8+,2-,8-] & [0+,7+,8+,2-,8-]	[0+,4+,5+,2-,8-] & [0+,4+,11+,2-,8-]
	[0+,1+,8+,3-,9-] & [0+,7+,8+,3-,9-]	[0+,4+,5+,3-,9-] & [0+,4+,11+,3-,9-]

Table 9. Listing of all homometric sets in D_n for $p = 5$ and $n = 8$, $n = 10$.

D_n	Homometric sets for the left action	Homometric sets for the right action
n=8	[0+,1+,4+,0-,2-] & [0+,1+,5+,1-,3-]	[0+,1+,4+,0-,2-] & [0+,2+,0-,1-,4-]
	[0+,3+,4+,0-,2-] & [0+,1+,4+,2-,4-]	[0+,3+,4+,0-,2-] & [0+,4+,5+,2-,4-]
	[0+,1+,5+,0-,2-] & [0+,3+,7+,0-,2-]	[0+,1+,5+,0-,2-] & [0+,3+,7+,0-,2-]
	[0+,4+,5+,0-,2-] & [0+,3+,4+,2-,4-]	[0+,4+,5+,0-,2-] & [0+,4+,7+,2-,4-]
	[0+,4+,7+,0-,2-] & [0+,6+,0-,1-,4-]	[0+,4+,7+,0-,2-] & [0+,3+,7+,1-,3-]
	[0+,1+,4+,1-,3-] & [0+,1+,5+,2-,4-]	[0+,3+,4+,1-,3-] & [0+,2+,0-,1-,5-]
	[0+,3+,4+,1-,3-] & [0+,1+,4+,3-,5-]	[0+,4+,5+,1-,3-] & [0+,4+,7+,3-,5-]
	[0+,4+,5+,1-,3-] & [0+,6+,0-,1-,5-]	[0+,4+,7+,1-,3-] & [0+,3+,7+,2-,4-]
	[0+,4+,0-,1-,3-] & [0+,5+,7+,0-,4-]	[0+,4+,0-,1-,3-] & [0+,1+,3+,0-,4-]
	[0+,4+,0-,2-,3-] & [0+,2+,7+,0-,4-]	[0+,4+,0-,2-,3-] & [0+,1+,6+,0-,4-]
	[0+,1+,6+,0-,4-] & [0+,5+,6+,0-,4-]	[0+,2+,3+,0-,4-] & [0+,2+,7+,0-,4-]
	[0+,1+,6+,1-,5-] & [0+,5+,6+,1-,5-]	[0+,2+,3+,1-,5-] & [0+,2+,7+,1-,5-]
n=10	[0+,5+,1-,3-,4-] & [0+,7+,8+,1-,6-]	[0+,2+,6+,0-,2-] & [0+,2+,4+,0-,4-]
	[0+,1+,7+,0-,5-] & [0+,6+,7+,0-,5-]	[0+,4+,6+,0-,2-] & [0+,2+,4+,0-,6-]
	[0+,2+,9+,0-,5-] & [0+,7+,9+,0-,5-]	[0+,4+,8+,0-,2-] & [0+,2+,8+,0-,4-]
	[0+,2+,4+,1-,5-] & [0+,2+,6+,3-,5-]	[0+,2+,6+,1-,3-] & [0+,2+,4+,1-,5-]
	[0+,4+,6+,3-,5-] & [0+,2+,8+,1-,7-]	[0+,4+,6+,1-,3-] & [0+,2+,4+,1-,7-]
	[0+,1+,7+,1-,6-] & [0+,6+,7+,1-,6-]	[0+,4+,8+,1-,3-] & [0+,2+,8+,1-,5-]
	[0+,1+,7+,2-,7-] & [0+,6+,7+,2-,7-]	[0+,5+,0-,1-,3-] & [0+,2+,9+,0-,5-]
	[0+,2+,6+,0-,2-] & [0+,2+,8+,0-,4-]	[0+,5+,0-,2-,3-] & [0+,2+,3+,0-,5-]
	[0+,4+,6+,0-,2-] & [0+,6+,8+,0-,6-]	[0+,6+,8+,0-,4-] & [0+,4+,8+,2-,4-]
	[0+,4+,8+,0-,2-] & [0+,6+,8+,0-,4-]	[0+,5+,0-,1-,4-] & [0+,1+,4+,0-,5-]
	[0+,2+,6+,1-,3-] & [0+,2+,8+,1-,5-]	[0+,4+,6+,2-,4-] & [0+,2+,8+,0-,6-]
	[0+,4+,6+,1-,3-] & [0+,6+,8+,1-,7-]	[0+,5+,1-,2-,4-] & [0+,2+,9+,1-,6-]
	[0+,4+,8+,1-,3-] & [0+,6+,8+,1-,5-]	[0+,5+,0-,3-,4-] & [0+,1+,7+,0-,5-]
	[0+,5+,0-,1-,3-] & [0+,1+,8+,0-,5-]	[0+,5+,1-,3-,4-] & [0+,2+,3+,1-,6-]
	[0+,5+,0-,2-,3-] & [0+,7+,8+,0-,5-]	[0+,1+,3+,0-,5-] & [0+,1+,8+,0-,5-]
	[0+,2+,4+,0-,4-] & [0+,2+,6+,2-,4-]	[0+,3+,4+,0-,5-] & [0+,3+,9+,0-,5-]
	[0+,5+,0-,1-,4-] & [0+,6+,9+,0-,5-]	[0+,6+,8+,1-,5-] & [0+,4+,8+,3-,5-]
	[0+,4+,6+,2-,4-] & [0+,2+,8+,0-,6-]	[0+,4+,6+,3-,5-] & [0+,2+,8+,1-,7-]
	[0+,5+,1-,2-,4-] & [0+,1+,8+,1-,6-]	[0+,3+,4+,1-,6-] & [0+,3+,9+,1-,6-]
	[0+,5+,0-,3-,4-] & [0+,3+,9+,0-,5-]	[0+,3+,4+,2-,7-] & [0+,3+,9+,2-,7-]

Annex B

In this annex we present two alternative measures to d on $\mathcal{A}$. The first one is near the construction of d, while the second one uses the lowest common multiple.

The measure $\boldsymbol{\delta}$. This measure (we do not say *distance* since it is not a distance) is, in a certain way, dual to d. For $A \in \mathcal{A}_m$ and $B \in \mathcal{A}_n$ with $m \leq n$, we "immerse" the chord A into the bigger space $\mathcal{A}_n$ in which lies B, so that we can use the distance $d_{\mathcal{A}_n}$. For this purpose we define for $t \geq \sharp(A)$,

$$Q_t(A) := \{Z \supset A \mid \sharp(Z) = t \text{ and } Z \text{ contains only notes belonging to } A\}.$$

δ is then defined by

$$\delta(A,B) := \inf_{Z \in Q_n(A)} d_{\mathcal{A}_n}(Z,B).$$

As explained below, δ verifies (i), (ii) and (iv). However it does not verify (iii) (triangle inequality). Consider the chords $A = [0,6]$, $B = [8,10]$ and $C = [9,10,10]$ for a counterexample. We have

$$\begin{aligned}
\delta(A,C) &= \inf\{d_{\mathcal{A}_3}([0,0,6],[9,10,10]), d_{\mathcal{A}_3}([0,6,6],[9,10,10])\} \\
&= \inf\{\sqrt{17}, \sqrt{29}\} \\
&= \sqrt{17},
\end{aligned}$$

$\delta(A,B) = \sqrt{8}$ and $\delta(B,C) = 1$. We deduce $\delta(A,C) > \delta(A,B) + \delta(B,C)$, thus the triangle inequality is not verified.

Conditions (i) and (ii) are verified for equivalent reasons than those used with the distance d. The proof of (iv) is also similar to the one used for d. We fix $r > 0$, $A \in \mathcal{A}_m$, we have

$$\begin{aligned}
B(A,r) &= \bigcup_{p<m} \{X \in \mathcal{A}_p \mid \delta(A,X) < r\} \cup \bigcup_{p \geq m} \{X \in \mathcal{A}_p \mid \delta(A,X) < r\} \\
&= \bigcup_{p<m} \{X \in \mathcal{A}_p \mid \inf_{T \in Q_m(X)} d_{\mathcal{A}_m}(A,T) < r\} \\
&\qquad\qquad \cup \bigcup_{p \geq m} \{X \in \mathcal{A}_p \mid \inf_{Z \in Q_p(A)} d_{\mathcal{A}_p}(Z,X) < r\} \\
&:= \bigcup_{p<m} \Omega_p^1(A) \cup \bigcup_{p \geq m} \Omega_p^2(A).
\end{aligned}$$

First we look at the sets $\Omega_p^2(A)$. We have

$$\begin{aligned}
\Omega_p^2(A) &= \{X \in \mathcal{A}_p \mid \exists Z_p \in Q_p(A), X \in B_p(Z_p,r)\} \\
&= \bigcup_{Z_p \in Q_p(A)} B_{\mathcal{A}_p}(Z_p,r)
\end{aligned}$$

hence for all $p < m$, $\Omega_p^2(A)$ is open.

We look now at the sets $\Omega_p^1(A)$ for $p \geq m$. Let $Y \in \Omega_p^1(A)$ and $Y' \in \mathcal{A}_p$ such that $d_{\mathcal{A}_p}(Y, Y') < \eta$ for some $\eta > 0$. We will prove that for η small enough, $Y' \in \Omega_p^1(A)$ i.e. we have to prove

$$\inf_{T' \in Q_m(Y')} d_{\mathcal{A}_m}(A, T') < r. \tag{81}$$

As $Y \in \Omega_p^1(A)$, there exists $T_Y \in Q_m(Y)$ such that $d_{\mathcal{A}_m}(A, T_Y) = \rho < r$. Let $\eta' > 0$, as $d_{\mathcal{A}_p}(Y, Y') < \eta$ there exists $T_{Y'} \in Q_m(Y')$ such that

$$d_{\mathcal{A}_m}(T_{Y'}, T_Y) < \eta + \eta'.$$

Then we have with the triangle inequality

$$\begin{aligned} d_{\mathcal{A}_m}(A, T_{Y'}) &\leq d_{\mathcal{A}_m}(A, T_y) + d_{\mathcal{A}_m}(T_Y, T_{Y'}) \\ &\leq \rho + \eta + \eta'. \end{aligned}$$

If we choose $\eta + \eta' < r - \rho$ we obtain $d_{\mathcal{A}_m}(A, T_{Y'}) < r$ hence Eq. 81 is verified. We conclude that $\Omega_p^1(A)$ is open.

Finally $B(A, r)$ can be written as a union of open sets belonging to the $\mathcal{A}_n$ for $n \in \mathbb{N}$.

Measure with the lowest common multiple. The second measure we propose uses the lowest common multiple of m and n for two chords $A \in \mathcal{A}_m$ and $B \in \mathcal{A}_n$ with $m \leq n$. We write *lcm* for *lowest common multiple*.

We call $d = \text{lcm}(m, n)$. We can immerse A and B into the chord space $\mathcal{A}_d$, with the following process. We explain it for instance with A. There exists $\alpha \in \mathbb{N}$ such that $d = m\alpha$. We add α times each note of A to obtain A' that lies in $\mathcal{A}_d$. For instance consider the chord $A = C$-major $= [0, 4, 7]$ in $\mathcal{A}_3$ and the chord $B = G^7 = [7, 11, 2, 5]$ in $\mathcal{A}_4$, we have $\text{lcm}(3, 4) = 12$. In order to obtain the chord A' we will then add 4 times each note of A, to obtain B' we will add 3 times each note of B to get finally two chords with 12 notes. We obtain

$$A' = [C, C, C, C, E, E, E, E, G, G, G, G] = [0, 0, 0, 0, 4, 4, 4, 4, 7, 7, 7, 7]$$
$$B' = [G, G, G, B, B, B, D, D, D, F, F, F] = [7, 7, 7, 11, 11, 11, 2, 2, 2, 5, 5, 5]$$

We define the distance ∂ between A and B by

$$\partial(A, B) := d_{\mathcal{A}_{12}}(A', B'). \tag{82}$$

This distance seems mathematically interesting since we do not "loose" or "change" information about the chords A and B. However in practice it is computationally long to evaluate and from the future musical analysis we will do we know that it is not very relevant to keep chords with repetition of notes.