The Mystery of Anatol Vieru's Periodic Sequences Unveiled

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Abstract. In [10], Anatol Vieru proposed a compositional technique based on an algorithmic manipulation of periodic sequences in $\mathbb{Z}_{12}$. This technique was translated in mathematical terms in ([3,4,8]). Two mathematical problems arose starting from the so called Vieru's sequence V: period of primitives and proliferation of values. In this paper we announce, providing only the sketch of the proofs, the solution of these questions in a purely algebraic way.

Introduction

In the *Book of Modes* [10], the romanian composer Anatol Vieru collects periodic sequences by iteratively applying a finite sum operator starting from the constant sequence (6) on $\mathbb{Z}_{12}$, corresponding to the triton interval. Then he decodes from each sequence a musical aspect, giving rise to a composition: *Zone d'Oubli*.

This was the starting point for a prolific math-music research area ([1–4,8]). Vieru highlighted two remarkable phenomena about the particular sequence (so called Vieru's sequence):

$$V = (2, 1, 2, 4, 8, 1, 8, 4) \in \mathbb{Z}_{12}$$

originated from the initial sequence $(2, 1)$ corresponding to Messiaen's second mode of limited transpositions. Vieru repeatedly applied to V the operator Σ_8 (see Eq. 1). He noticed that in the obtained sequences the period tends to increase and it is always a power of 2. Moreover the values 4 and 8 tend to proliferate among the coefficients of the sequences (recovering in some cases more than 99% of the coefficients [4]). In [4] the authors faced the problem using the Fitting Lemma and explicit computations, providing the main reference for this work, but leaving open the two problems. In a recent article submitted to a mathematical journal, we completely solved these questions. The main new idea consists in linking periodic sequences to binomial coefficients, which have been studied using Kummer's Theorem [7] and the generalisation of Lucas' Theorem ([5,6]). In this paper, we announce these results providing only a sketch of the proofs.

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1 Anatol Vieru's Periodic Sequence: A New Formalization

Let us recall some definitions.

A sequence $f \in \mathbb{Z}_m^{\mathbb{N}} := (\mathbb{Z}/m\mathbb{Z})^{\mathbb{N}}$ is called *periodic* if there exists $j \geq 1$ such that $\theta^j(f) = f$ where θ is the shift operator defined by

$$\theta(f)(n) := f(n+1) \quad \forall n \in \mathbb{N}.$$

The minimal $j \geq 1$ satisfying this condition is called the *period* of f and it is denoted it by $\tau(f)$ (we use the notation $(a_0, \ldots, a_{n-1})$ for a sequence of period n). The set $P_m := \bigcup_{j \geq 1} \ker(\theta^j - \mathrm{id})$ of all periodic sequences over $\mathbb{Z}_m$ is a $\mathbb{Z}_m$-module with point-wise sum and multiplication.

Let us consider on P_m the operators $\Delta := \theta - \mathrm{id}$ (discrete derivation) and Σ_c for $c \in \mathbb{Z}_m$ (discrete integration) defined as

$$\Sigma_c f(n) := \begin{cases} c & \text{if } n = 0 \\ f(n-1) + \Sigma_c f(n-1) & \text{if } n > 0. \end{cases} \tag{1}$$

We will write Σ instead of Σ_0 to keep the notation clean. We denote by (c) the constant sequence (i.e., periodic sequence of period 1) having all entries equal to $c \in \mathbb{Z}_m$. Hence $\Sigma_c f = \Sigma f + (c)$ and $\Delta(\Sigma_c f) = f$ for every $f \in P_m$ and $c \in \mathbb{Z}_m$. Observe that in particular $\Sigma_c(f_1 + f_2) = \Sigma f_1 + \Sigma_c f_2$ for any $f_1, f_2 \in P_m$ and so $\Sigma_c^s(f_1 + f_2) = \Sigma^s f_1 + \Sigma_c^s f_2$ for any $s \geq 1$.

Given the constant sequence $f = (c)$ with $c \neq 0$, we have: $\Sigma f(0) = 0, \Sigma f(1) = c$ and iterating $\Sigma f(n) = nc$ so $\Sigma f = (0, c, 2c, \ldots, (m-1)c)$ has period m. More generally one has:

Lemma 1. *If (c) is a constant sequence in P_m, then $\Sigma^s(c)(n) \equiv_m c\binom{n}{s}$.*

The period never decreases when applying Σ. Indeed the following holds:

Lemma 2. *Given $f \in P_m$ of period τ, let us denote by $\mathrm{tr}(f) := \sum_{i=0}^{\tau-1} f(i)$. For each $c \in \mathbb{Z}_m$ the period of $\Sigma_c f$ is $h\tau$ where h is the minimum positive integer such that $h \cdot \mathrm{tr}(f) \equiv 0 \mod m$.*

We say that a periodic sequence $f \in P_m$ is *nilpotent* (resp. *idempotent*) if there exists $n \geq 1$ such that $\Delta^n f = 0$ (resp. $\Delta^n f = f$). These two kinds of sequences are called resp. *reducible* and *reproducible* sequences in [2]. The *nilpotency* (resp. *idempotency*) *index* of f is the minimal n satisfying the previous condition. We denote by N_m^{Δ} and I_m^{Δ} the $\mathbb{Z}_m$-submodules of nilpotent resp. idempotent sequences.

Example 1. 1. Consider the sequence $f = (0, 1, 2, 3) \in P_4$. We have:

$$\begin{aligned} \Delta f = &\ \theta f - f = (1, 2, 3, 0) - (0, 1, 2, 3) = (1) \\ \Delta(1) = &\ \theta(1) - (1) = (1) - (1) = (0). \end{aligned}$$

Hence f is nilpotent with nilpotency index 2.

2. The sequence $g = (2, 1) \in P_3$ is idempotent of idempotency index 1, since:

$$\Delta g = \theta g - g = (1, 2) - (2, 1) = (2, 1) = g.$$

2 Decomposing P_m

We use three decompositions: the decomposition into primes, the decomposition in nilpotent and idempotent parts and lastly the decomposition of nilpotent sequences using constants.

2.1 Decomposition with Primes

Given $m \in \mathbb{N}$, $m \geq 2$ with prime factorization $m = \prod_{i=1}^{t} p_i^{\ell_i}$, the group isomorphism $\mathbb{Z}/m\mathbb{Z} \to \bigoplus_{i=1}^{t} \mathbb{Z}/p_i^{\ell_i}\mathbb{Z}$ gives rise to an isomorphism of abelian groups (see [2, Th. 5])

$$\begin{aligned} P_m &\longrightarrow \bigoplus_{i=1}^{t} P_{p_i^{\ell_i}} \\ f &\longmapsto (f_{p_i})_{1\leq i\leq t} \end{aligned}$$

where f_{p_i} is the projection of f in $P_{p_i^{\ell_i}}$ and we will call it the p_i-part of f. Its inverse is given by the Chinese remainder theorem.

Lemma 3. *([2, Prop. 6, Prop. 13]) Following the previous notation, f is nilpotent (resp. idempotent) if and only if its p_i-part f_{p_i} is nilpotent (resp. idempotent) for any $1 \leq i \leq t$. The nilpotency (resp. idempotency) index coincides with the maximum of the nilpotency (resp. idempotency) indices of f_{p_i} and the period $\tau(f) = \operatorname{lcm}\{\tau(f_{p_i})\}_{1\leq i\leq t}$.*

Example 2. Since $\mathbb{Z}/12\mathbb{Z} \simeq \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$, we obtain $P_{12} \simeq P_3 \oplus P_4$ and Vieru's sequence $V = (2, 1, 2, 4, 8, 1, 8, 4)$ decomposes as:

$$V_3 = (2, 1) \in P_3, \qquad V_2 = (2, 1, 2, 0, 0, 1, 0, 0) \in P_4.$$

2.2 Decomposition in Nilpotent and Idempotent Part

Let us recall [4, Prop. 1] that, by the Fitting Lemma, $P_m = I_m^{\Delta} \oplus N_m^{\Delta}$. The primes decomposition and Lemma 3 imply the following isomorphisms:

$$P_m = \bigoplus_{i=1}^{t} I_{p_i^{\ell_i}}^{\Delta} \oplus N_{p_i^{\ell_i}}^{\Delta} \qquad I_m^{\Delta} = \bigoplus_{i=1}^{t} I_{p_i^{\ell_i}}^{\Delta} \qquad N_m^{\Delta} = \bigoplus_{i=1}^{t} N_{p_i^{\ell_i}}^{\Delta}.$$

Thus we can always reduce to study sequences on $\mathbb{Z}_{p^\ell}$.

Lemma 4. *If $f \in P_{p^\ell}$, then:*

1. *[4, Th. 3] $f \in N_{p^\ell}^{\Delta}$ if and only if $\tau(f) = p^t$ for $t \in \mathbb{N}$;*
2. *if $f \in N_{p^\ell}^{\Delta}$ with period p^t and nilpotency index η, then $\eta \leq \ell p^t$.*

2.3 Decomposition of Nilpotent Sequences Using Constants

The nilpotent sequences decompose in sums of primitives of constant sequences:

Lemma 5. *A nilpotent sequence $f \in N_m^{\Delta}$ of nilpotency index η can be written in a unique way as*

$$f = c_0 + \Sigma^1 c_1 + \cdots + \Sigma^{\eta-1} c_{\eta-1}$$

for suitable constants $c_0, \ldots, c_{\eta-1} \in \mathbb{Z}_m$.

Applying the previous decompositions to Vieru's sequence

$$V = (2, 1, 2, 4, 8, 1, 8, 4) \in P_{12}$$

we find the 2-part $V_2 = (2, 1, 2, 0, 0, 1, 0, 0) \in P_4$ and the 3-part $V_3 = (2, 1) \in P_3$. In this case V_2 is nilpotent of index 5 while V_3 is idempotent with index 1.

Therefore the nilpotent and idempotent components are:

$$\widetilde{V_2} = (6, 9, 6, 0, 0, 9, 0, 0) \equiv (-3) \cdot V_2 \mod 12, \qquad \widetilde{V_3} = (8, 4) \equiv 4 \cdot V_3 \mod 12.$$

Vieru repeatedly used the operator Σ_8 applied to $V = \widetilde{V_2} + \widetilde{V_3}$ in order to generate new periodic sequences. Since $\Sigma_8 \widetilde{V_3} = \widetilde{V_3}$, we get

$$\Sigma_8^s V = \Sigma^s \widetilde{V_2} + \Sigma_8^s \widetilde{V_3} = \Sigma^s \widetilde{V_2} + \widetilde{V_3}. \tag{2}$$

Thus we are reduced to study the operator Σ applied to the nilpotent sequence V_2 in P_4. In particular the period of $\Sigma_8^s V$ in P_{12} coincides with the period of $\Sigma^s V_2$ in P_4. Since $\Sigma_8 \widetilde{V_3} = \widetilde{V_3} = (8, 4)$, the proliferation of the values $8, 4$ in $\Sigma_8^s V$ in P_{12} is equivalent to the proliferation of the value 0 in $\Sigma^s V_2$ in P_4.

Finally, the last decomposition provided by Lemma 5 gives $V_2 = (2) + \Sigma(3) + \Sigma^2(2) + \Sigma^3(3) + \Sigma^4(2)$.

3 Unveiling the Period and the Proliferation of Values

3.1 Period of the Primitives of Vieru's Sequence

The study of the period is based on the following lemma:

Lemma 6. *For every $s \in \mathbb{N}$, the sequence $\Sigma^s(2) \in P_4$ has period 2^{k_s} while $\Sigma^s(3) \in P_4$ has period 2^{k_s+1} where $k_s := \lfloor \log_2(s) \rfloor + 1$ is the number of figures in the representation of s in base* 2.

Proof. We prove the statement for the primitives of (3) proceeding by induction on the primitive index s. As observed before Lemma 1, the period of $\Sigma(3)$ is $4 = 2^{k_1+1}$. Suppose the statement true for s, let us prove it for $s + 1$.

– If $k_{s+1} = k_s$, it is possible to show that $\mathrm{tr}(\Sigma^s(3)) = 0$ using Kummer's Theorem [7]. By Lemma 2 one obtains

$$\tau(\Sigma^{s+1}(3)) = \tau(\Sigma^s(3)) = 2^{k_s} = 2^{k_{s+1}}.$$

– If $k_{s+1} = k_s + 1$, again by Kummer's Theorem one gets $\operatorname{tr}(\Sigma^s(3)) = 2$. By Lemma 2 one obtains

$$\tau(\Sigma^{s+1}(3)) = 2 \cdot \tau(\Sigma^s(3)) = 2 \cdot 2^{k_s} = 2^{k_s+1} = 2^{k_{s+1}}.$$

Reducing $(2) \in P_4$ to $(1) \in P_2$ via the isomorphism $2\mathbb{Z}/4\mathbb{Z} \xrightarrow{\sim} \mathbb{Z}/2\mathbb{Z}$ the previous argument proves the statement.

The following result solves completely the problem of the period of Vieru's sequence $V = (2, 1, 2, 4, 8, 1, 8, 4)$.

Theorem 1. *The period of $\Sigma^s V$ is $2^{k_{s+3}+1}$ (with k_{s+3} is the number of figures of $s+3$ in base 2). Notice that the period changes whenever $s = 2^r - 3$ for $r \geq 2$.*

Proof. As previously observed, the period of $\Sigma^s V$ coincides with the period of $\Sigma^s V_2$. The period of the sequences

$$\Sigma^s V_2 = \Sigma^s(2) + \Sigma^{s+1}(3) + \Sigma^{s+2}(2) + \Sigma^{s+3}(3) + \Sigma^{s+4}(2)$$

clearly divides the least common multiple of the periods of its summands, which coincides with the period of $\Sigma^{s+3}(3)$. Using the previous theorem and Lemma 2, an accurate analysis permits to prove that they are in fact equal.

3.2 Proliferation of Values

Let us study the proliferation of 0 in $\Sigma^s V_2$ with $s \geq 1$. This will allow us to evaluate the number of 4, 8 in the primitives of Vieru's sequence V. Rather than the absolute value of occurrences inside the period, it is much more interesting to study the ratio with respect to the period. In [4], the authors explicitly computed the first 61 primitives of V and they remarked that *"at level 61 of period 128, more than* 90% *of the elements belong to the set* $\{4, 8\}$*. This percentage dramatically decreases in the following level which is the last one having period equal to 128"*. The level 61 above corresponds to $s = 59$ and it has period 128 by Theorem 1, as confirmed by the computation in [4].

Lemma 7. *Let us denote by $z(s)$ the number of zeros in the sequence $\Sigma^{s-3}V_2$ inside its period. Then for every $r \geq 3$ the following inequalities hold:*

$$z(2^r - 1) < z(2^r - 2) = 2^{r+1} - 8.$$

More precisely, one has:

$$\Sigma^{2^r-5}V_2 = (\underbrace{0, \ldots, 0}_{2^r-5}, 2, 3, 1, 0, 0, \underbrace{0, \ldots, 0}_{2^{r-1}-4}, 2, 2, 0, 0, \underbrace{0, \ldots, 0}_{2^{r-1}-5}, 2, 1, 3, 0, 0).$$

Proof. The proof is based on Kummer's Theorem [7] and on the generalization of Lucas' Theorem proved by Davis and Webb in [5, Th. 1]. The second statement is proved by a component-wise analysis. In the under-braced positions all summands of V_2 are equal to zero by Kummer's Theorem. The remaining 14 coefficients can be explicitly computed using Davis and Webb result.

The following result solves completely the problem of the proliferation of $\{4,8\}$ in Vieru's sequence $V = (2,1,2,4,8,1,8,4)$.

Theorem 2. *The ratio between the number of* $\{4,8\}$ *in the primitives of* $\Sigma_8^{2^r-5}V$ *(with* $r \geq 3$*) and their period is* $\frac{2^{r+1}-8}{2^{r+1}}$*, which tends to* 1 *for* $r \to \infty$*.*

More precisely, $\Sigma_8^{2^r-5}V$ *is equal to:*

$$(\underbrace{8,4,8,4,\ldots,8}_{2^r-5},10,11,1,8,4,\underbrace{8,4,8,4,\ldots,4}_{2^{r-1}-4},2,10,8,4,\underbrace{8,4,8,4,\ldots,8}_{2^{r-1}-5},10,5,7,8,4).$$

Proof. Since the number of $\{4,8\}$ in $\Sigma_8^{2^r-5}V$ coincides with the number of zeros in $\Sigma^{2^r-5}V_2$, the first part of the statement follows from the previous lemma and Theorem 1. The explicit form for $\Sigma_8^{2^r-5}V$ follows from the previous lemma and Eq. (2), where

$$\Sigma^{2^r-5}\widetilde{V_2} \equiv (-3)\cdot\Sigma^{2^r-5}V_2 \mod 12.$$

Remark 1. It is nice to compare the formula of Theorem 2 for $r = 3,4,6$ with the explicit computation of the corresponding levels $5, 13, 61$ provided in [4, App. A].

4 Recursive Formulas for the Number of $\{4,8\}$ in $\Sigma_8^{s-3}V$

In the last days we proved a recursive formula for the number $z(s)$ of zeroes in the $(s-3)$-primitive of V_2 in P_4. The number $z(s)$ coincides also with the number of $\{4,8\}$ in $\Sigma_8^{s-3}V$ in P_{12}. We chose this shift by -3 in order to have all sequences of the same period 2^{r+2} when $2^r \leq s < 2^{r+1}$. In this interval of primitives, one can compute the percentage of $\{4,8\}$ as $z(s)/2^{r+2}$.

We need first to introduce a tuple $\mathfrak{d}_r$ of integers. Denote by $\mathrm{wt}(\ell)$ the Hamming weight of ℓ, i.e. the number of 1's in the binary expansion of ℓ. Then we set

$$\mathfrak{d}_r(m) = 2^{\mathrm{wt}(2^r+2^{r-1}-1-m)+1}.$$

We will be mainly interested in the values of $\mathfrak{d}_r(m)$ when $2^r + 2^{r-2} + 3 \leq m < 2^r + 2^{r-1} - 1$. For brevity we write

$$\mathfrak{d}_r := (\mathfrak{d}_r(m))_{2^r+2^{r-2}+3\leq m<2^r+2^{r-1}-1}.$$

For $\mathfrak{d}_r$ the following equalities hold:

$$\mathfrak{d}_5 = (4,8,4,4) \quad \text{and} \quad \mathfrak{d}_{r+1} = (2\times\mathfrak{d}_r, 4, 2^{r-1}, 2^{r-2}, 2^{r-2}, \mathfrak{d}_r)\ \forall r \geq 5.$$

We can now enunciate the recursive formula. Define:

$$\begin{aligned} u_i &:= 2^{r-i} \qquad i = 1,2,3 \\ t &:= s - u_1 \qquad 2^r \leq s < 2^{r+1} \\ (c_1,c_2,c_3,c_4) &:= 2^{r-3}(12,8,10,11) \\ (c_1',c_2',c_3',c_4') &:= 2^{r-3}(12,10,11,12). \end{aligned}$$

The initial condition for the recursive formula is the 2^5-tuple $(z(s))_s$ for $2^5 \leq s < 2^6$:

$$(32, 48, 64, 88, 64, 80, 88, 92, 64, 80, 88, 104, 92, 104, 108, 94,$$
$$78, 88, 96, 108, 96, 104, 108, 110, 102, 108, 112, 118, 114, 118, 120, 64).$$

In this interval, the period of the sequences $\Sigma_8^{s-3}V$ is constantly equal to 128.

For $2^r \leq s < 2^{r+1}$ with $r \geq 6$, the 2^r-tuple $(z(s))_s$ concides with:

$$(\underbrace{2z(t), \ldots, 2z(t)}_{2^{r-2}-1}, \underbrace{z(t-u_3)+c_1, \ldots, z(t-u_3)+c_4}_{4}, \underbrace{2z(t)-\mathfrak{d}_r(s), \ldots, 2z(t)-\mathfrak{d}_r(s)}_{2^{r-2}-4},$$
$$\underbrace{z(t-u_2)+c_1', \ldots, z(t-u_2)+c_4'}_{4}, \underbrace{z(t-u_1)+2^{r+1}, \ldots, z(t-u_1)+2^{r+1}}_{2^{r-1}-4}, \underbrace{2z(t-u_1)}_{1}).$$

Let us recall that $t = s - 2^{r-1}$ and so in the tuple above the first coefficient is computed using $s = 2^r$, the second one using $s = 2^r + 1$, the last one using $s = 2^{r+1} - 1$.

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