Master I.C.A.

Master I.C.A.

Spécialité A.S.T.

Art
Sciences
Technologies

Traitement interactif de l'image et du son

Méthodes mathématiques pour la création musicale : aspects théoriques, informatiques et cognitifs

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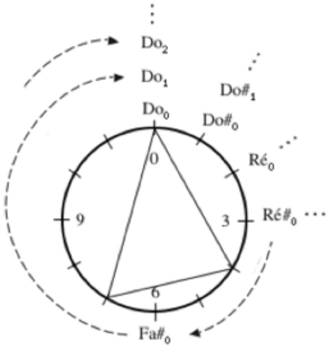
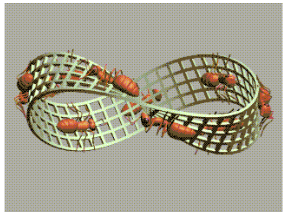
Equipe Représentations Musicales
IRCAM/CNRS/UPMC UMR 9912
`Moreno.Andreatta@ircam.fr`

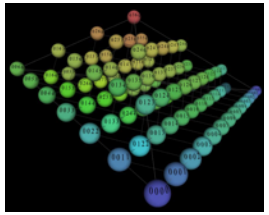
Plan du cours

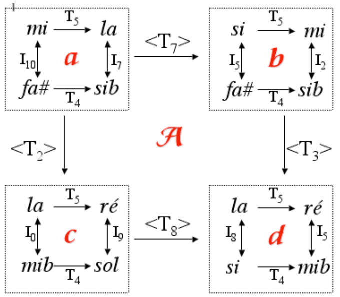
1.) Rappel historique et premiers éléments de musicologie computationnelle
2.) Représentation et énumération des structures musicales : *Set* Theory et théories transformationnelles
3.) Pavages en composition : la construction des canons rythmiques mosaïques
4.) Ramifications philosophiques et cognitives de l'approche algébrique en musique



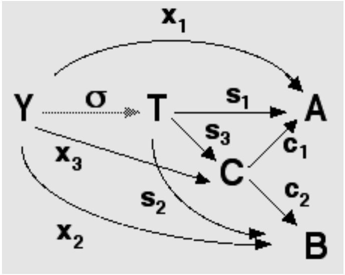
Do₂
Do₁
Do#₁
Do₀
Do#₀
Ré₀
Ré#₀ …
0
9
3
6
Fa#₀



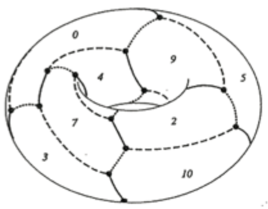


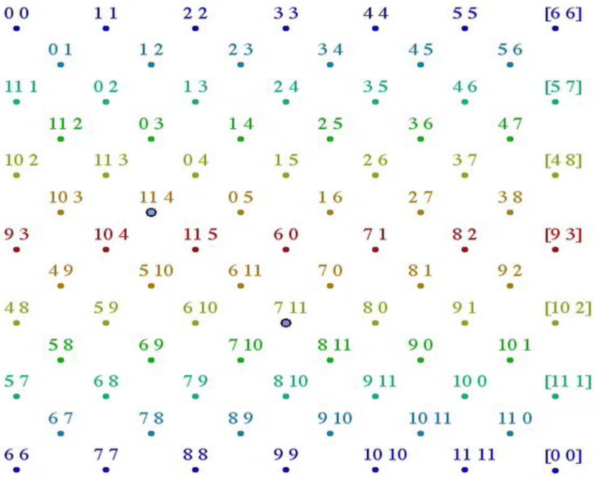


mi, T5, la, I10, a, I7, fa#, T4, sib
<T7>
si, T5, mi, I5, b, I2, fa#, T4, sib
<T2>
𝒜
<T3>
la, T5, ré, I0, c, I9, mib, T4, sol
<T8>
la, T5, ré, I8, d, I5, si, T4, mib



x1
Y
σ
T
s1
A
s3
x3
c1
C
s2
c2
x2
B





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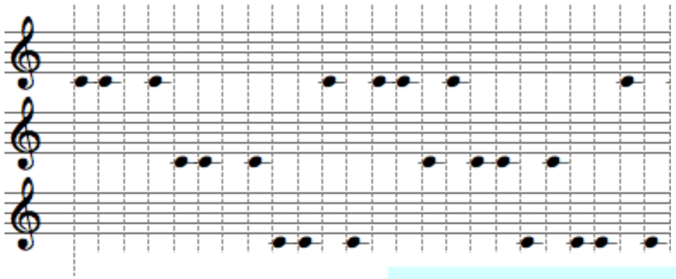
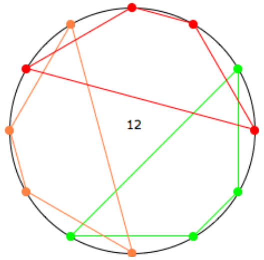
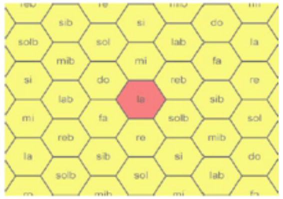
0 1 1 2 2 3 3 4 4 5 5 6
11 1 0 2 1 3 2 4 3 5 4 6 [5 7]
11 2 0 3 1 4 2 5 3 6 4 7
10 2 11 3 0 4 1 5 2 6 3 7 [4 8]
10 3 11 4 0 5 1 6 2 7 3 8
9 3 10 4 11 5 6 0 7 1 8 2 [9 3]
4 9 5 10 6 11 7 0 8 1 9 2
4 8 5 9 6 10 7 11 8 0 9 1 [10 2]
5 8 6 9 7 10 8 11 9 0 10 1
5 7 6 8 7 9 8 10 9 11 10 0 [11 1]
6 7 7 8 8 9 9 10 10 11 11 0
6 6 7 7 8 8 9 9 10 10 11 11 [0 0]







$Df(x)=f(x)-f(x-1).$

f = 7 11 10 11 7 2 7 11 10 11 7 2 7 11…

Df = 4 11 1 8 7 5 4 11 1 8 7 5 4 11…

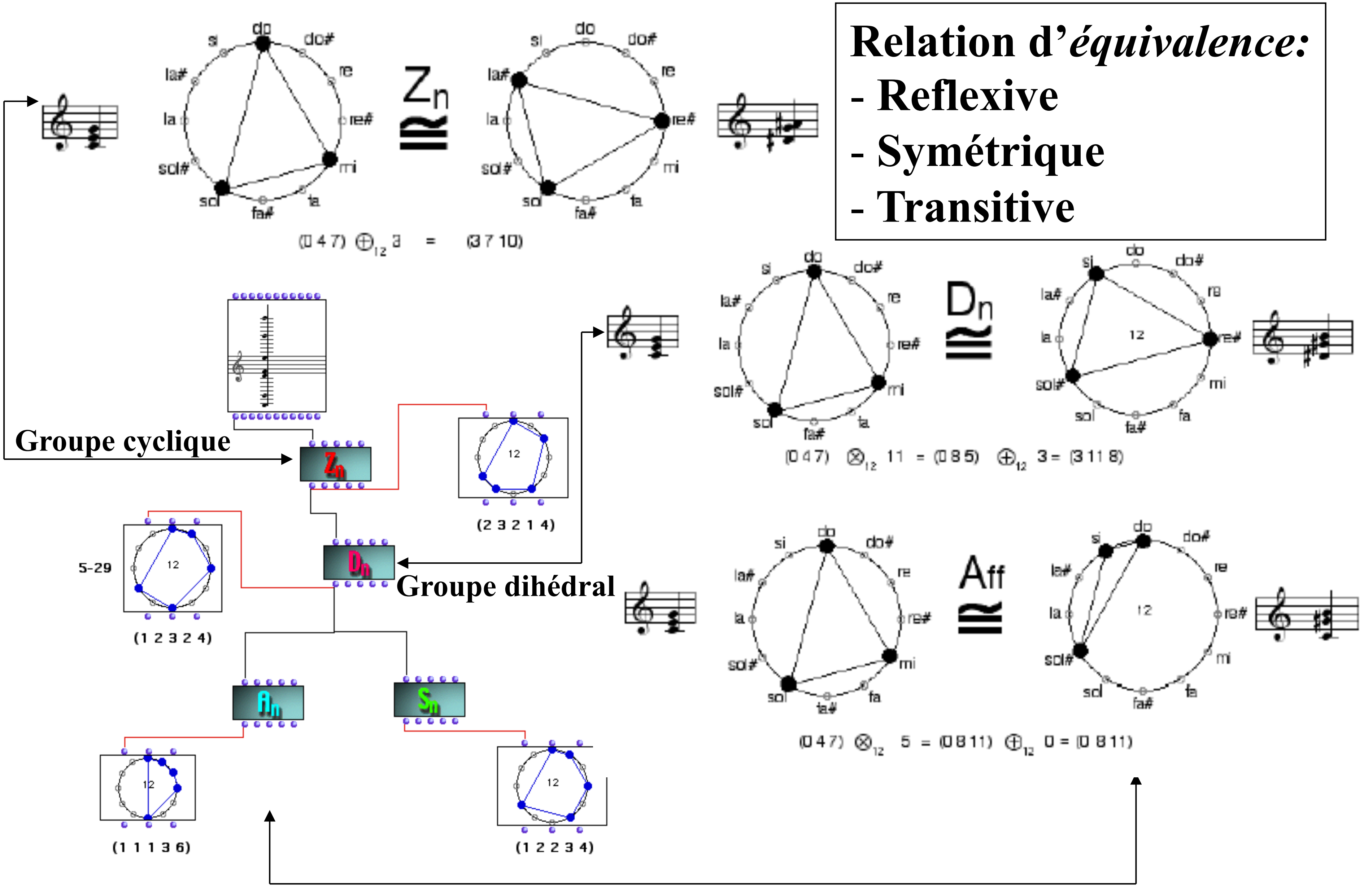
$D^2 f$ = 7 2 7 11 10 11 7 2 7 11 10 11…

$D^3 f$ = 7 5 4 11 1 8 7 5 4 11 1 8…

$D^k f$ = ……

Transformations géométriques et relation d’équivalence

Relation d’*équivalence:*
- **Reflexive**
- **Symétrique**
- **Transitive**


do
do#
re
re#
mi
fa
fa#
sol
sol#
la
la#
si
Zn
≅
$(0\ 4\ 7) \oplus_{12} 3 = (3\ 7\ 10)$
Dn
≅
12
$(0\ 4\ 7) \otimes_{12} 11 = (0\ 8\ 5) \oplus_{12} 3 = (3\ 11\ 8)$
Aff
≅
$(0\ 4\ 7) \otimes_{12} 5 = (0\ 8\ 11) \oplus_{12} 0 = (0\ 8\ 11)$
Zn
Dn
An
Sn
(2 3 2 1 4)
5-29
(1 2 3 2 4)
(1 1 1 3 6)
(1 2 2 3 4)

Groupe cyclique

Groupe dihédral

Groupe Affine

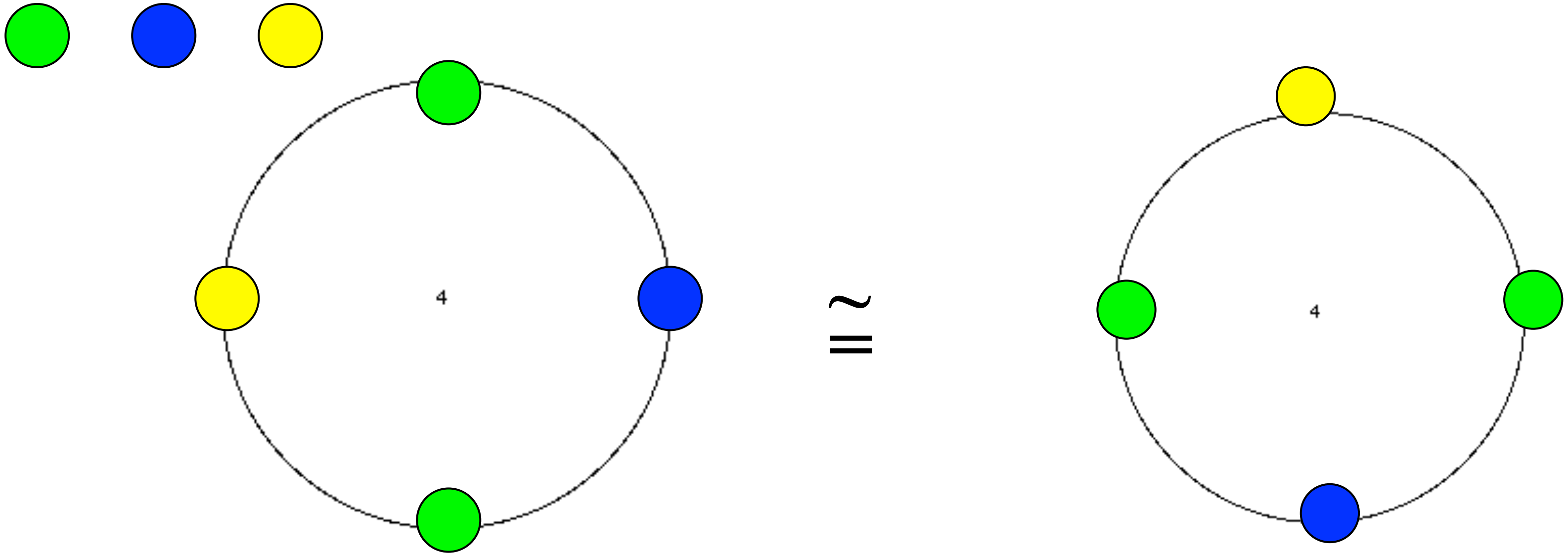
Architecture paradigmatique

Enumeration des orbites par rapport à l'action d'un groupe

Lemme de Burnside

$$n = \frac{1}{|G|}\sum_{g\in G} |X^g|.$$

$$X^g = \{x\in X : gx=x\}$$



?

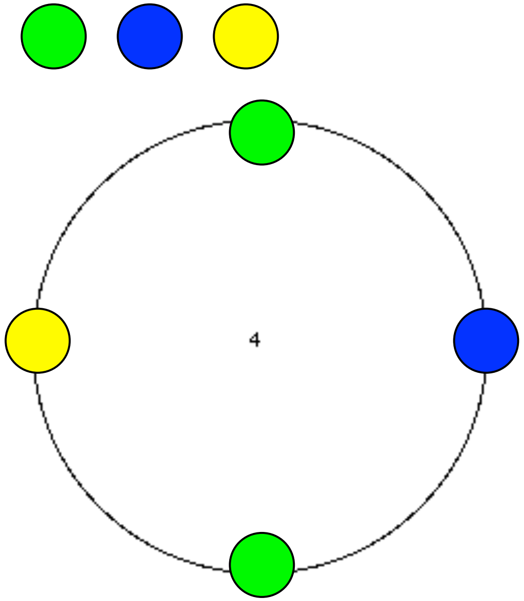
Trouver le nombre de configurations possibles

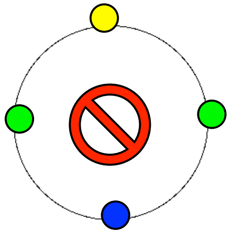
Enumération d'orbites par rapport à l'action d'un groupe

Lemme de Burnside

$$n = \frac{1}{|G|}\sum_{g\in G}|X^g|.$$

$X^g = \{x\in X : gx=x\}$





Action de **Z/4Z**

T_0 = identité

T_1 = rotation de 90°

T_2 = rotation de 180°

T_3 = rotation de 270°

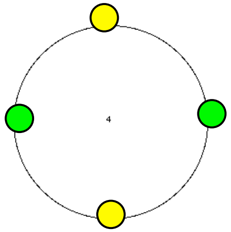
Configurations possibles = $3^4 = 81$

T_0 fixe toute configuration => $|X^{T_0}|=81$

T_1 fixe toute configuration monochromes => $|X^{T_1}|=3$

T_3 idem

T_2 fixe toute configuration «double-diamètre » => $|X^{T_1}| = 3^2 = 9$



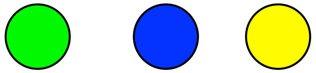
⇨ $n = 1/4\,(81+3+3+9) = 24$

Enumération d'orbites par rapport à l'action d'un groupe

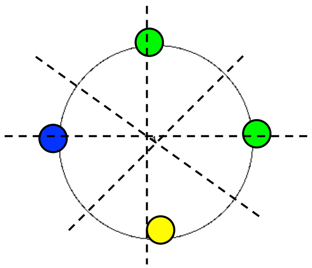
Lemme de Burnside

$$n = \frac{1}{|G|}\sum_{g\in G} |X^g|.$$

$$X^g = \{x \in X : gx = x\}$$



Action de $\mathbf{Z}_4$



Transformation	*Action*	*Cycle representation*	*No. of cycles*	*Fixed configs.*	*Cycle type*	*Cycle index*
T_0	0→0, 1→1, 2→2, 3→3	(0)(1)(2)(3)	4	$3^4 = 81$	1^4	$t_1^{\,4}$
T_1	0→1→2→3→0	(0 1 2 3)	1	$3^1 = 3$	4^1	$t_4^{\,1}$
T_2	0→2→0, 1→3→1	(0 2)(1 3)	2	$3^2 = 9$	2^2	$t_2^{\,2}$
T_3	0→3→2→1→0	(0 3 2 1)	1	$3^1 = 3$	4^1	$t_4^{\,1}$

Julian Hook, « Why are there 29 Tetrachords? A Tutorial on Combinatorics and Enumeration in Music Theory », MTO, 13(4), 2007

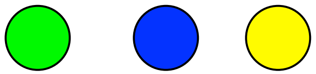
$$n = 1/4\ (81+3+3+9) = 24$$

Enumération d’orbites par rapport à l’action d’un groupe

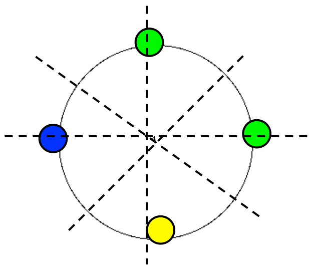
Lemme de Burnside

$$n = \frac{1}{|G|}\sum_{g\in G} |X^g|.$$

$$X^g = \{x\in X : gx=x\}$$



Action de $\mathbf{D_4}$



Transformation	*Action*	*Cycle representation*	*No. of cycles*	*Fixed configs.*	*Cycle type*	*Cycle index*
T_0	$0\to0,\ 1\to1,\ 2\to2,\ 3\to3$	(0)(1)(2)(3)	4	$3^4 = 81$	1^4	$t_1{}^4$
T_1	$0\to1\to2\to3\to0$	(0 1 2 3)	1	$3^1 = 3$	4^1	$t_4{}^1$
T_2	$0\to2\to0,\ 1\to3\to1$	(0 2)(1 3)	2	$3^2 = 9$	2^2	$t_2{}^2$
T_3	$0\to3\to2\to1\to0$	(0 3 2 1)	1	$3^1 = 3$	4^1	$t_4{}^1$
I	$0\to0,\ 1\to3\to1,\ 2\to2$	(0)(1 3)(2)	3	$3^3 = 27$	$1^2 2^1$	$t_1{}^2 t_2{}^1$
T_1I	$0\to1\to0,\ 2\to3\to2$	(0 1)(2 3)	2	$3^2 = 9$	2^2	$t_2{}^2$
T_2I	$0\to2\to0,\ 1\to1,\ 3\to3$	(0 2)(1)(3)	3	$3^3 = 27$	$1^2 2^1$	$t_1{}^2 t_2{}^1$
T_3I	$0\to3\to0,\ 1\to2\to1$	(0 3)(1 2)	2	$3^2 = 9$	2^2	$t_2{}^2$

Julian Hook, « Why are there 29 Tetrachords? A Tutorial on Combinatorics and Enumeration in Music Theory », MTO, 13(4), 2007

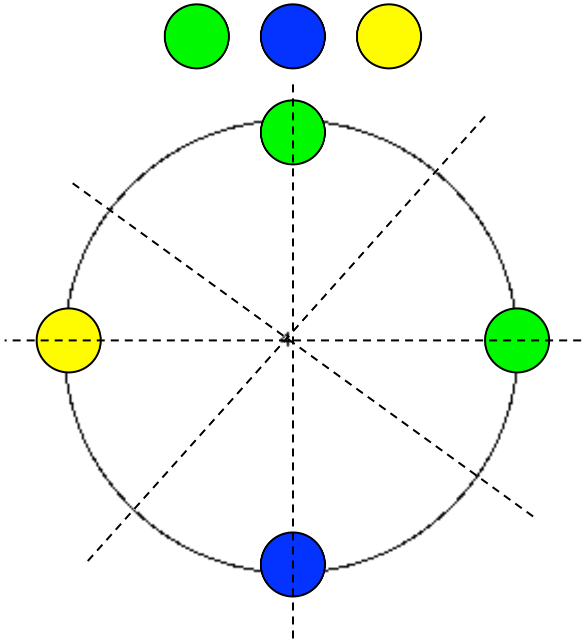
$$n = 1/8\ (81+3+3+9+27+9+27+9) = 168/8=21$$

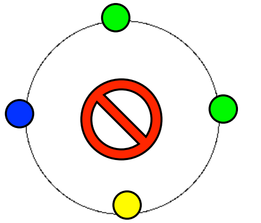
Enumération d'orbites par rapport à l'action d'un groupe

Lemme de Burnside

$$n = \frac{1}{|G|}\sum_{g\in G} |X^g|.$$

$$X^g = \{x\in X : gx=x\}$$





Action de $\mathbf{D_4}$

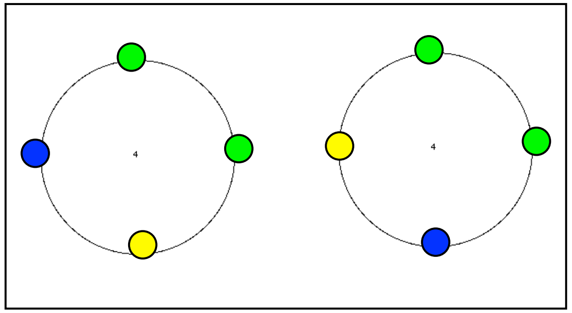
T_0 = id $T_0 I$ = inversion

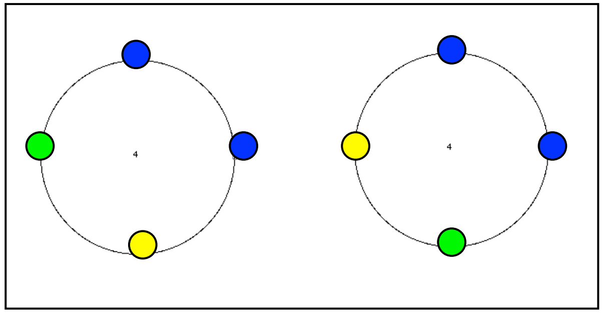
T_1 = rot 90° $T_1 I$ = inv.

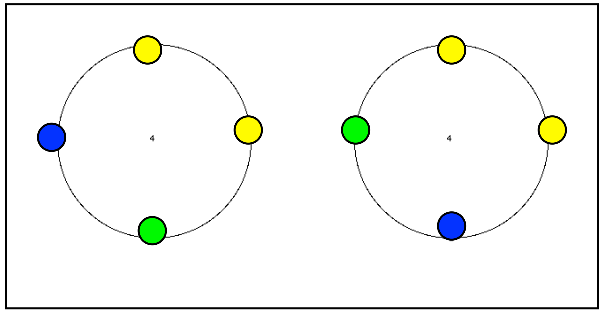
T_2 = rot 180° $T_2 I$ = inv.

T_3 = rot 270° $T_3 I$ = inv.

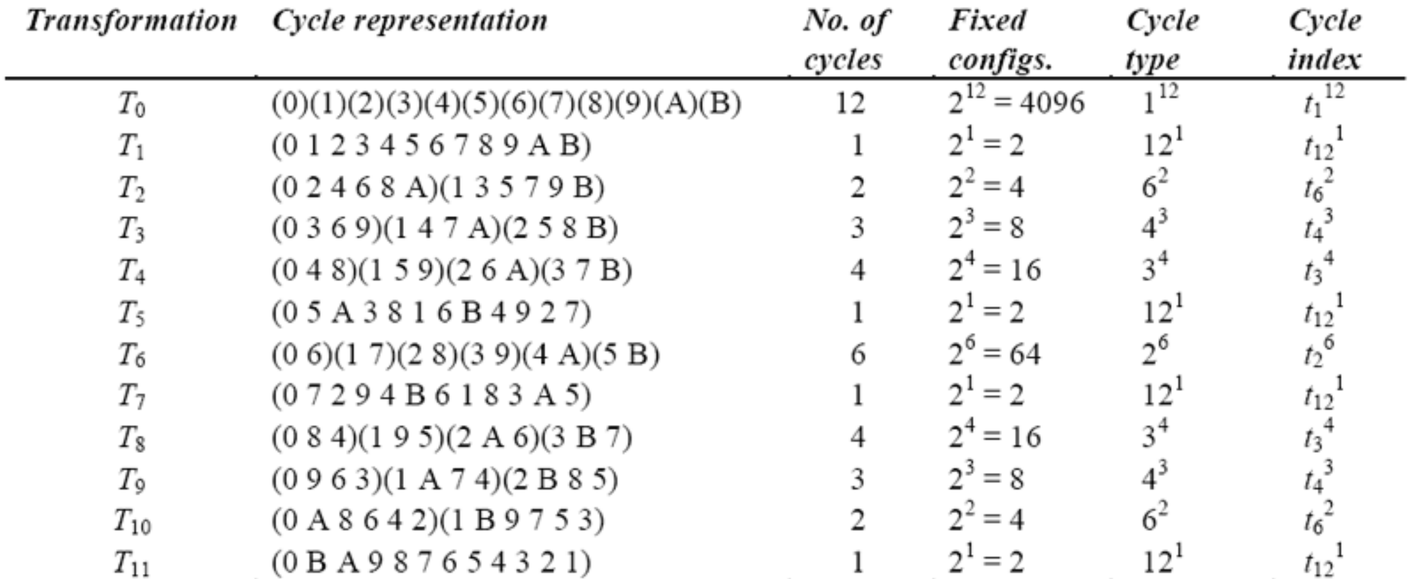
21=24-3







Enumération d'accords par rapport à l'action du groupe dihédrale



Transformation	*Cycle representation*	*No. of cycles*	*Fixed configs.*	*Cycle type*	*Cycle index*
T_0	(0)(1)(2)(3)(4)(5)(6)(7)(8)(9)(A)(B)	12	$2^{12} = 4096$	1^{12}	$t_1{}^{12}$
T_1	(0 1 2 3 4 5 6 7 8 9 A B)	1	$2^1 = 2$	12^1	$t_{12}{}^1$
T_2	(0 2 4 6 8 A)(1 3 5 7 9 B)	2	$2^2 = 4$	6^2	$t_6{}^2$
T_3	(0 3 6 9)(1 4 7 A)(2 5 8 B)	3	$2^3 = 8$	4^3	$t_4{}^3$
T_4	(0 4 8)(1 5 9)(2 6 A)(3 7 B)	4	$2^4 = 16$	3^4	$t_3{}^4$
T_5	(0 5 A 3 8 1 6 B 4 9 2 7)	1	$2^1 = 2$	12^1	$t_{12}{}^1$
T_6	(0 6)(1 7)(2 8)(3 9)(4 A)(5 B)	6	$2^6 = 64$	2^6	$t_2{}^6$
T_7	(0 7 2 9 4 B 6 1 8 3 A 5)	1	$2^1 = 2$	12^1	$t_{12}{}^1$
T_8	(0 8 4)(1 9 5)(2 A 6)(3 B 7)	4	$2^4 = 16$	3^4	$t_3{}^4$
T_9	(0 9 6 3)(1 A 7 4)(2 B 8 5)	3	$2^3 = 8$	4^3	$t_4{}^3$
T_{10}	(0 A 8 6 4 2)(1 B 9 7 5 3)	2	$2^2 = 4$	6^2	$t_6{}^2$
T_{11}	(0 B A 9 8 7 6 5 4 3 2 1)	1	$2^1 = 2$	12^1	$t_{12}{}^1$

Lemme de Burnside

$$n = \frac{1}{|G|}\sum_{g\in G} |X^g|.$$

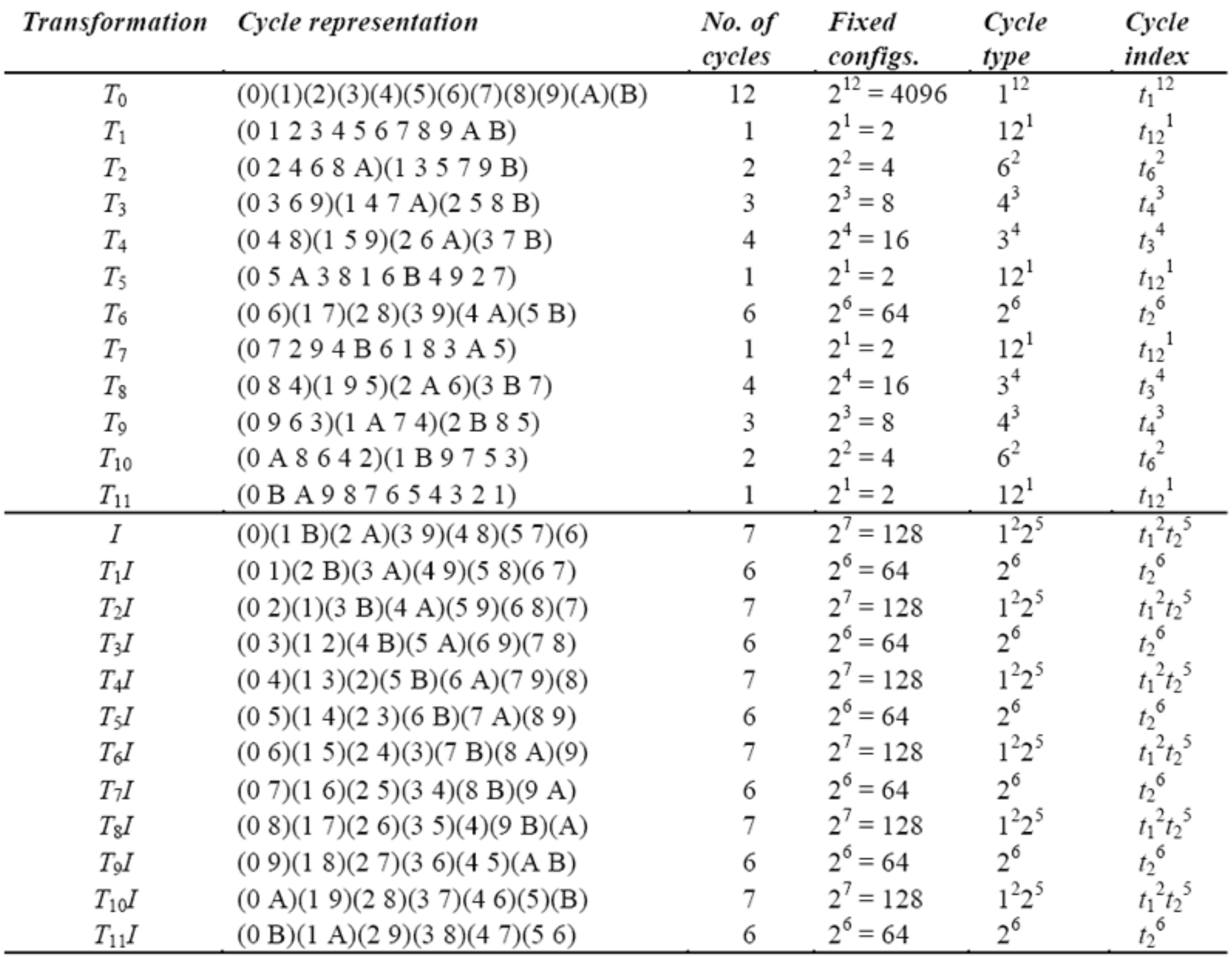
$X^g = \{x\in X : gx=x\}$

Action de $\mathbf{Z}_{12}$

(Hook, MTO)

# d'accords = 1/12[4096+2+4+8+16+2+64+2+16+8+4+2]=4224/12=352

Enumération d'accords par rapport à l'action du groupe dihédrale



Transformation	Cycle representation	No. of cycles	Fixed configs.	Cycle type	Cycle index
T_0	(0)(1)(2)(3)(4)(5)(6)(7)(8)(9)(A)(B)	12	$2^{12} = 4096$	1^{12}	t_1^{12}
T_1	(0 1 2 3 4 5 6 7 8 9 A B)	1	$2^1 = 2$	12^1	t_{12}^1
T_2	(0 2 4 6 8 A)(1 3 5 7 9 B)	2	$2^2 = 4$	6^2	t_6^2
T_3	(0 3 6 9)(1 4 7 A)(2 5 8 B)	3	$2^3 = 8$	4^3	t_4^3
T_4	(0 4 8)(1 5 9)(2 6 A)(3 7 B)	4	$2^4 = 16$	3^4	t_3^4
T_5	(0 5 A 3 8 1 6 B 4 9 2 7)	1	$2^1 = 2$	12^1	t_{12}^1
T_6	(0 6)(1 7)(2 8)(3 9)(4 A)(5 B)	6	$2^6 = 64$	2^6	t_2^6
T_7	(0 7 2 9 4 B 6 1 8 3 A 5)	1	$2^1 = 2$	12^1	t_{12}^1
T_8	(0 8 4)(1 9 5)(2 A 6)(3 B 7)	4	$2^4 = 16$	3^4	t_3^4
T_9	(0 9 6 3)(1 A 7 4)(2 B 8 5)	3	$2^3 = 8$	4^3	t_4^3
T_{10}	(0 A 8 6 4 2)(1 B 9 7 5 3)	2	$2^2 = 4$	6^2	t_6^2
T_{11}	(0 B A 9 8 7 6 5 4 3 2 1)	1	$2^1 = 2$	12^1	t_{12}^1
I	(0)(1 B)(2 A)(3 9)(4 8)(5 7)(6)	7	$2^7 = 128$	$1^2 2^5$	$t_1^2 t_2^5$
T_1I	(0 1)(2 B)(3 A)(4 9)(5 8)(6 7)	6	$2^6 = 64$	2^6	t_2^6
T_2I	(0 2)(1)(3 B)(4 A)(5 9)(6 8)(7)	7	$2^7 = 128$	$1^2 2^5$	$t_1^2 t_2^5$
T_3I	(0 3)(1 2)(4 B)(5 A)(6 9)(7 8)	6	$2^6 = 64$	2^6	t_2^6
T_4I	(0 4)(1 3)(2)(5 B)(6 A)(7 9)(8)	7	$2^7 = 128$	$1^2 2^5$	$t_1^2 t_2^5$
T_5I	(0 5)(1 4)(2 3)(6 B)(7 A)(8 9)	6	$2^6 = 64$	2^6	t_2^6
T_6I	(0 6)(1 5)(2 4)(3)(7 B)(8 A)(9)	7	$2^7 = 128$	$1^2 2^5$	$t_1^2 t_2^5$
T_7I	(0 7)(1 6)(2 5)(3 4)(8 B)(9 A)	6	$2^6 = 64$	2^6	t_2^6
T_8I	(0 8)(1 7)(2 6)(3 5)(4)(9 B)(A)	7	$2^7 = 128$	$1^2 2^5$	$t_1^2 t_2^5$
T_9I	(0 9)(1 8)(2 7)(3 6)(4 5)(A B)	6	$2^6 = 64$	2^6	t_2^6
$T_{10}I$	(0 A)(1 9)(2 8)(3 7)(4 6)(5)(B)	7	$2^7 = 128$	$1^2 2^5$	$t_1^2 t_2^5$
$T_{11}I$	(0 B)(1 A)(2 9)(3 8)(4 7)(5 6)	6	$2^6 = 64$	2^6	t_2^6

Lemme de Burnside

$$n = \frac{1}{|G|}\sum_{g\in G} |X^g|.$$

$X^g = \{x \in X : gx = x\}$

Action de $\mathbf{D_{12}}$

(Hook, MTO)

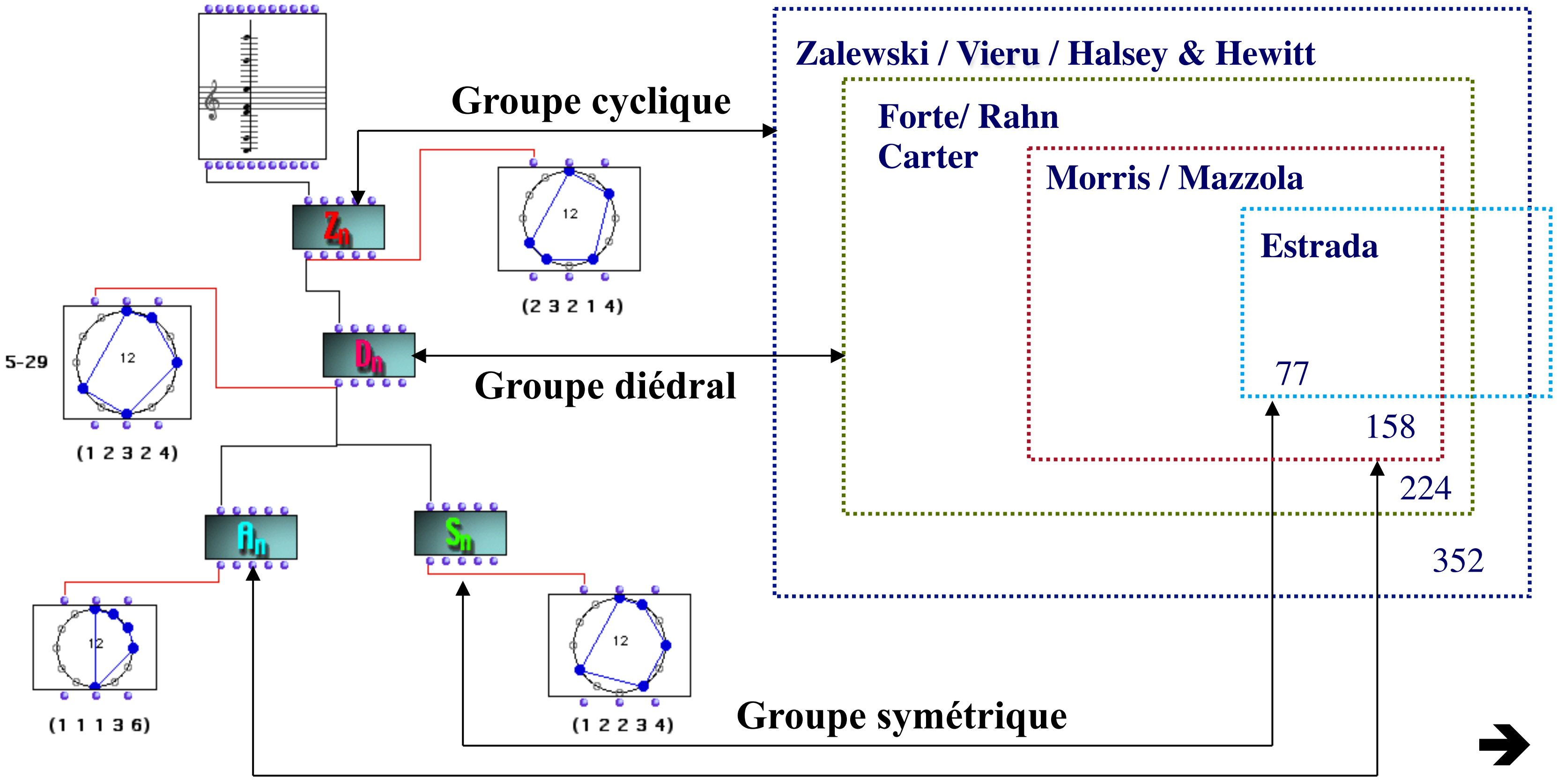
Z_n # d'accords = 1/12[4096+2+4+8+16+2+64+2+16+8+4+2]=4224/12=352

D_n # d'accords = 1/24[4224+1152] = 224

Classification paradigmatique des structures musicales

$G\backslash k$	1	2	3	4	5	6	7	8	9	10	11	12
C_{12}	1	6	19	43	66	80	66	43	19	6	1	1
D_{12}	1	6	12	29	38	50	38	29	12	6	1	1
$\mathrm{Aff}_1(Z_{12})$	1	5	9	21	25	34	25	21	9	5	1	1

Set Theory


Zn
12
(2 3 2 1 4)
5-29
12
(1 2 3 2 4)
Dn
An
Sn
12
(1 1 1 3 6)
12
(1 2 2 3 4)

Groupe cyclique

Zalewski / Vieru / Halsey & Hewitt

Forte/ Rahn
Carter

Morris / Mazzola

Estrada

77

158

224

352

Groupe diédral

Groupe symétrique

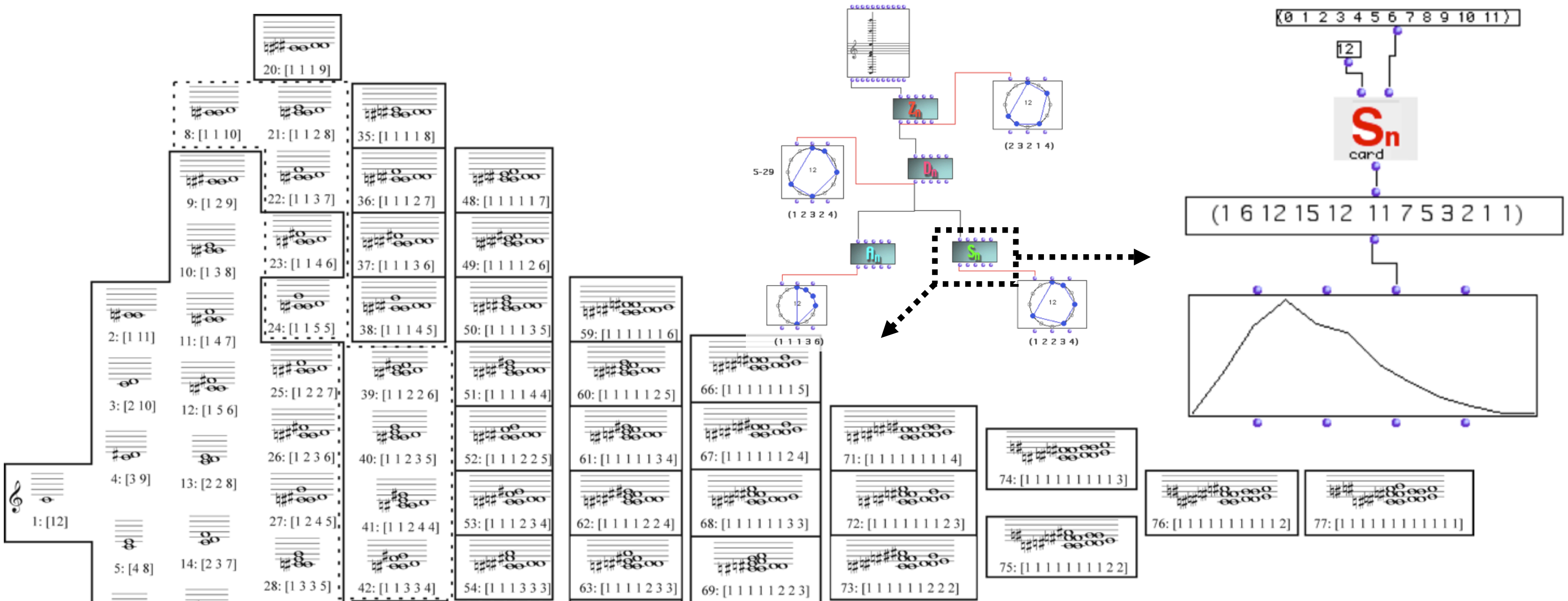
Groupe affine

Architecture paradigmatique

➔

02-paradigms

Permutoèdre comme catalogue d'accords


1: [12]
2: [1 11]
3: [2 10]
4: [3 9]
5: [4 8]
8: [1 1 10]
9: [1 2 9]
10: [1 3 8]
11: [1 4 7]
12: [1 5 6]
13: [2 2 8]
14: [2 3 7]
20: [1 1 1 9]
21: [1 1 2 8]
22: [1 1 3 7]
23: [1 1 4 6]
24: [1 1 5 5]
25: [1 2 2 7]
26: [1 2 3 6]
27: [1 2 4 5]
28: [1 3 3 5]
35: [1 1 1 1 8]
36: [1 1 1 2 7]
37: [1 1 1 3 6]
38: [1 1 1 4 5]
39: [1 1 2 2 6]
40: [1 1 2 3 5]
41: [1 1 2 4 4]
42: [1 1 3 3 4]
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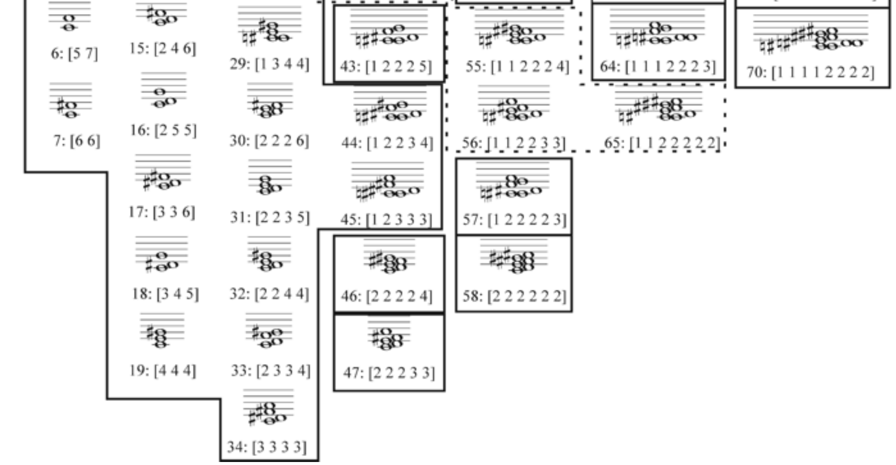
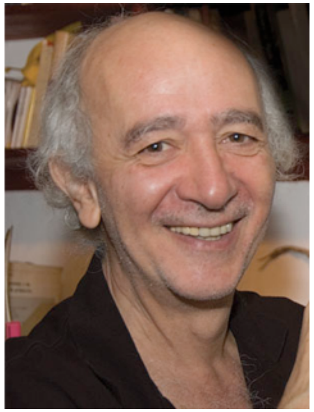
49: [1 1 1 1 2 6]
50: [1 1 1 1 3 5]
51: [1 1 1 1 4 4]
52: [1 1 1 2 2 5]
53: [1 1 1 2 3 4]
54: [1 1 1 3 3 3]
59: [1 1 1 1 1 1 6]
60: [1 1 1 1 1 2 5]
61: [1 1 1 1 1 3 4]
62: [1 1 1 1 2 2 4]
63: [1 1 1 1 2 3 3]
66: [1 1 1 1 1 1 1 5]
67: [1 1 1 1 1 1 2 4]
68: [1 1 1 1 1 1 3 3]
69: [1 1 1 1 1 2 2 3]
71: [1 1 1 1 1 1 1 1 4]
72: [1 1 1 1 1 1 1 2 3]
73: [1 1 1 1 1 1 2 2 2]
74: [1 1 1 1 1 1 1 1 1 3]
75: [1 1 1 1 1 1 1 1 2 2]
76: [1 1 1 1 1 1 1 1 1 1 2]
77: [1 1 1 1 1 1 1 1 1 1 1 1]
(0 1 2 3 4 5 6 7 8 9 10 11)
12
Sn
card
(1 6 12 15 12 11 7 5 3 2 1 1)
Zn
Dn
An
Sn
12
5-29
(1 2 3 2 4)
(2 3 2 1 4)
(1 1 3 6)
(1 2 2 3 4)


6: [5 7]
7: [6 6]
15: [2 4 6]
16: [2 5 5]
17: [3 3 6]
18: [3 4 5]
19: [4 4 4]
29: [1 3 4 4]
30: [2 2 2 6]
31: [2 2 3 5]
32: [2 2 4 4]
33: [2 3 3 4]
34: [3 3 3 3]
43: [1 2 2 2 5]
44: [1 2 2 3 4]
45: [1 2 3 3 3]
46: [2 2 2 2 4]
47: [2 2 2 3 3]
55: [1 1 2 2 2 4]
56: [1 1 2 2 3 3]
57: [1 2 2 2 2 3]
58: [2 2 2 2 2 2]
64: [1 1 1 2 2 2 3]
65: [1 1 2 2 2 2 2]
70: [1 1 1 1 2 2 2 2]

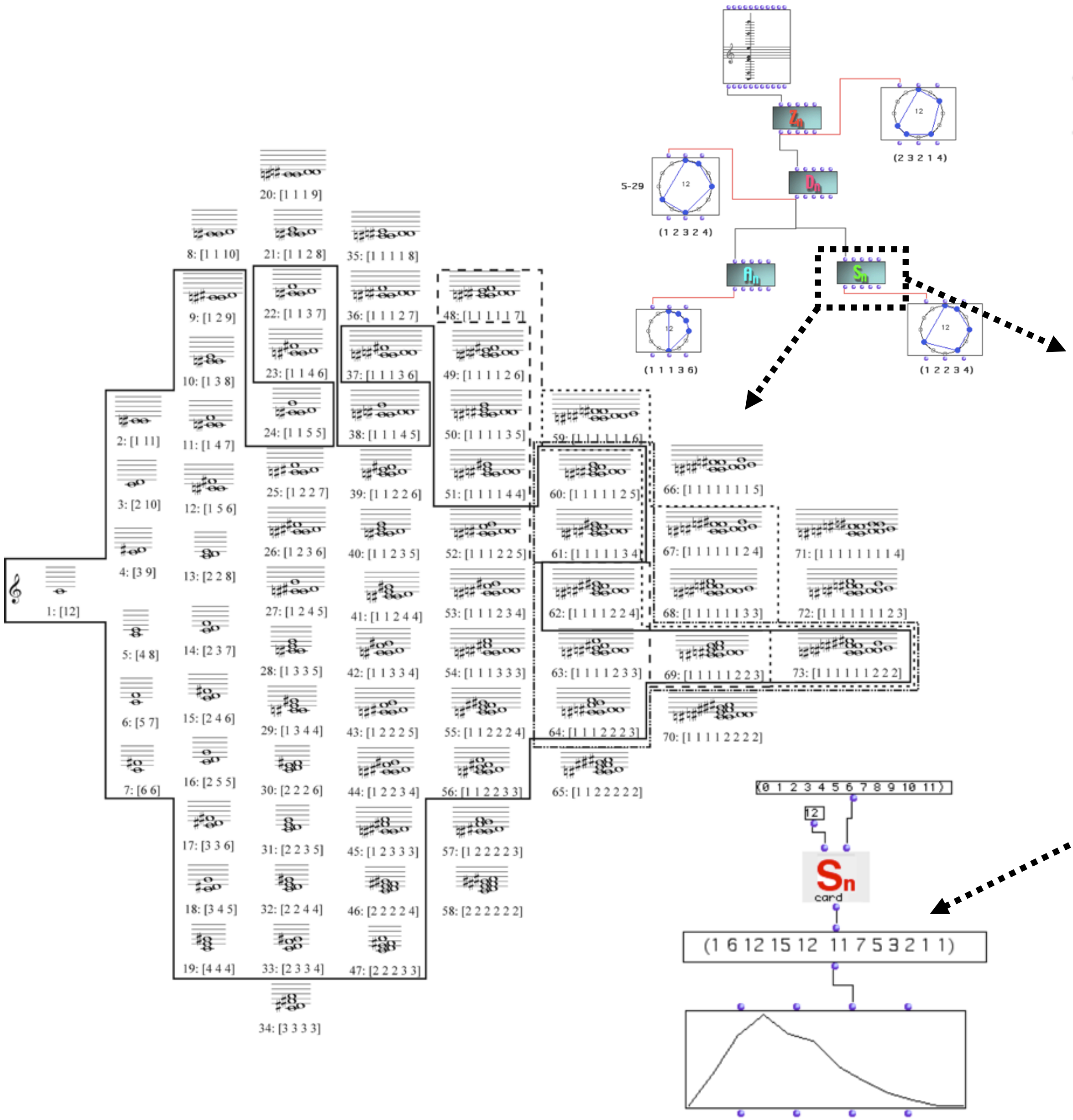
L. Van Beethoven,
Quatuor n° 17



Julio Estrada

Julio Estrada, Modèles mathématiques en composition et analyse musicales, intervention au Master ATIAM / Cursus de Composition, Ircam, 21 Novembre 2012

Permutoèdre comme catalogue d’accords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8: [1 1 10]
21: [1 1 2 8]
35: [1 1 1 1 8]
9: [1 2 9]
22: [1 1 3 7]
36: [1 1 1 2 7]
48: [1 1 1 1 1 7]
10: [1 3 8]
23: [1 1 4 6]
37: [1 1 1 3 6]
49: [1 1 1 1 2 6]
2: [1 11]
11: [1 4 7]
24: [1 1 5 5]
38: [1 1 1 4 5]
50: [1 1 1 1 3 5]
59: [1 1 1 1 1 1 6]
3: [2 10]
12: [1 5 6]
25: [1 2 2 7]
39: [1 1 2 2 6]
51: [1 1 1 1 4 4]
60: [1 1 1 1 1 2 5]
66: [1 1 1 1 1 1 1 5]
4: [3 9]
13: [2 2 8]
26: [1 2 3 6]
40: [1 1 2 3 5]
52: [1 1 1 2 2 5]
61: [1 1 1 1 1 3 4]
67: [1 1 1 1 1 1 2 4]
71: [1 1 1 1 1 1 1 1 4]
1: [12]
27: [1 2 4 5]
41: [1 1 2 4 4]
53: [1 1 1 2 3 4]
62: [1 1 1 1 2 2 4]
68: [1 1 1 1 1 1 3 3]
72: [1 1 1 1 1 1 1 2 3]
5: [4 8]
14: [2 3 7]
28: [1 3 3 5]
42: [1 1 3 3 4]
54: [1 1 1 3 3 3]
63: [1 1 1 1 2 3 3]
69: [1 1 1 1 1 2 2 3]
73: [1 1 1 1 1 1 2 2 2]
6: [5 7]
15: [2 4 6]
29: [1 3 4 4]
43: [1 2 2 2 5]
55: [1 1 2 2 2 4]
64: [1 1 1 2 2 2 3]
70: [1 1 1 1 2 2 2 2]
7: [6 6]
16: [2 5 5]
30: [2 2 2 6]
44: [1 2 2 3 4]
56: [1 1 2 2 3 3]
65: [1 1 2 2 2 2 2]
17: [3 3 6]
31: [2 2 3 5]
45: [1 2 3 3 3]
57: [1 2 2 2 2 3]
18: [3 4 5]
32: [2 2 4 4]
46: [2 2 2 2 4]
58: [2 2 2 2 2 2]
19: [4 4 4]
33: [2 3 3 4]
47: [2 2 2 3 3]
34: [3 3 3 3]
Zn
Dn
An
Sn
12
5-29
(1 2 3 2 4)
(2 3 2 1 4)
(1 1 1 3 6)
(1 2 2 3 4)
(0 1 2 3 4 5 6 7 8 9 10 11)
12
Sn
card
(1 6 12 15 12 11 7 5 3 2 1 1)

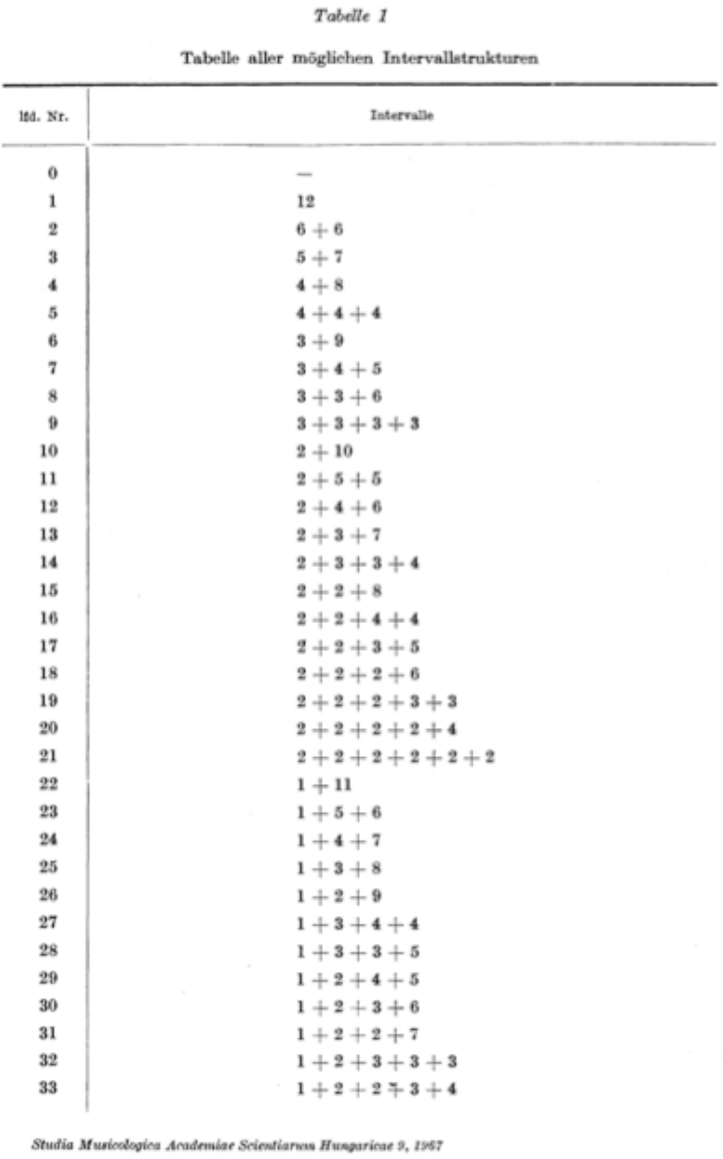


Tabelle 1

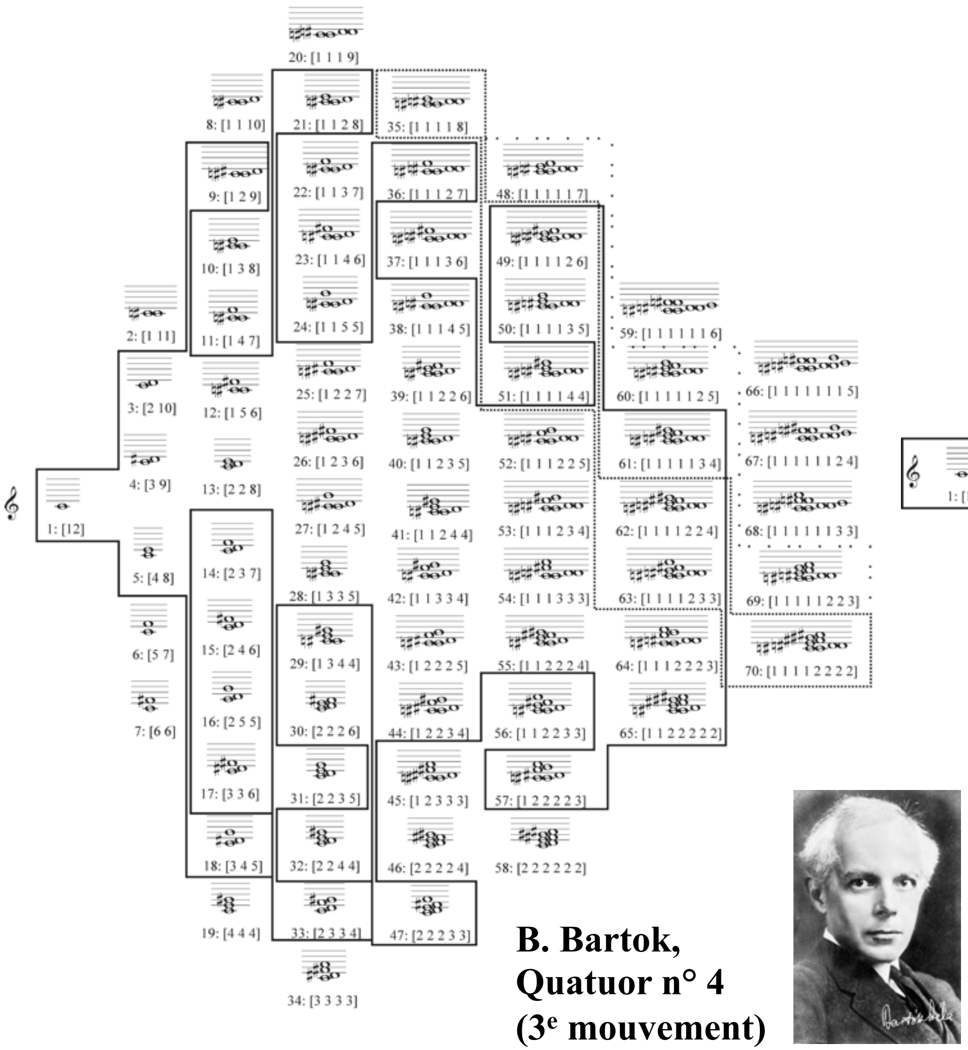
Tabelle aller möglichen Intervallstrukturen

lfd. Nr.	Intervalle
0	—
1	12
2	6 + 6
3	5 + 7
4	4 + 8
5	4 + 4 + 4
6	3 + 9
7	3 + 4 + 5
8	3 + 3 + 6
9	3 + 3 + 3 + 3
10	2 + 10
11	2 + 5 + 5
12	2 + 4 + 6
13	2 + 3 + 7
14	2 + 3 + 3 + 4
15	2 + 2 + 8
16	2 + 2 + 4 + 4
17	2 + 2 + 3 + 5
18	2 + 2 + 2 + 6
19	2 + 2 + 2 + 3 + 3
20	2 + 2 + 2 + 2 + 4
21	2 + 2 + 2 + 2 + 2 + 2
22	1 + 11
23	1 + 5 + 6
24	1 + 4 + 7
25	1 + 3 + 8
26	1 + 2 + 9
27	1 + 3 + 4 + 4
28	1 + 3 + 3 + 5
29	1 + 2 + 4 + 5
30	1 + 2 + 3 + 6
31	1 + 2 + 2 + 7
32	1 + 2 + 3 + 3 + 3
33	1 + 2 + 2 + 3 + 4

Studia Musicologica Academiae Scientiarum Hungaricae 9, 1967

W. Reckziegel, “Musikanalyse und Wissenschaft.” Studia Musicologica 9(1-2), 1967, p. 163-186

…et émancipation de la dissonance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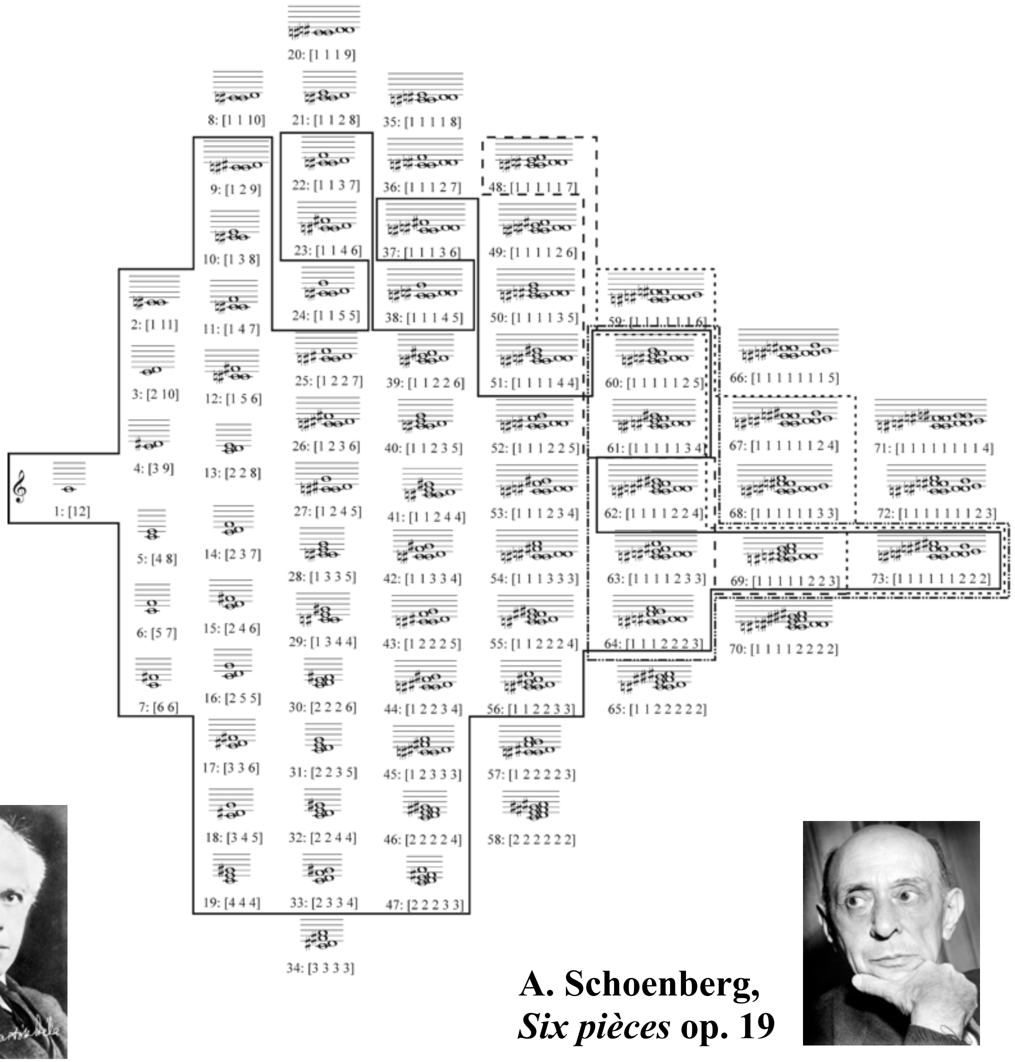
3: [2 10]
4: [3 9]
5: [4 8]
6: [5 7]
7: [6 6]
8: [1 1 10]
9: [1 2 9]
10: [1 3 8]
11: [1 4 7]
12: [1 5 6]
13: [2 2 8]
14: [2 3 7]
15: [2 4 6]
16: [2 5 5]
17: [3 3 6]
18: [3 4 5]
19: [4 4 4]
20: [1 1 1 9]
21: [1 1 2 8]
22: [1 1 3 7]
23: [1 1 4 6]
24: [1 1 5 5]
25: [1 2 2 7]
26: [1 2 3 6]
27: [1 2 4 5]
28: [1 3 3 5]
29: [1 3 4 4]
30: [2 2 2 6]
31: [2 2 3 5]
32: [2 2 4 4]
33: [2 3 3 4]
34: [3 3 3 3]
35: [1 1 1 1 8]
36: [1 1 1 2 7]
37: [1 1 1 3 6]
38: [1 1 1 4 5]
39: [1 1 2 2 6]
40: [1 1 2 3 5]
41: [1 1 2 4 4]
42: [1 1 3 3 4]
43: [1 2 2 2 5]
44: [1 2 2 3 4]
45: [1 2 3 3 3]
46: [2 2 2 2 4]
47: [2 2 2 3 3]
48: [1 1 1 1 1 7]
49: [1 1 1 1 2 6]
50: [1 1 1 1 3 5]
51: [1 1 1 1 4 4]
52: [1 1 1 2 2 5]
53: [1 1 1 2 3 4]
54: [1 1 1 3 3 3]
55: [1 1 2 2 2 4]
56: [1 1 2 2 3 3]
57: [1 2 2 2 2 3]
58: [2 2 2 2 2 2]
59: [1 1 1 1 1 1 6]
60: [1 1 1 1 1 2 5]
61: [1 1 1 1 1 3 4]
62: [1 1 1 1 2 2 4]
63: [1 1 1 1 2 3 3]
64: [1 1 1 2 2 2 3]
65: [1 1 2 2 2 2 2]
66: [1 1 1 1 1 1 1 5]
67: [1 1 1 1 1 1 2 4]
68: [1 1 1 1 1 1 3 3]
69: [1 1 1 1 1 2 2 3]
70: [1 1 1 1 2 2 2 2]

B. Bartok,
Quatuor n° 4
(3e mouvement)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7: [6 6]
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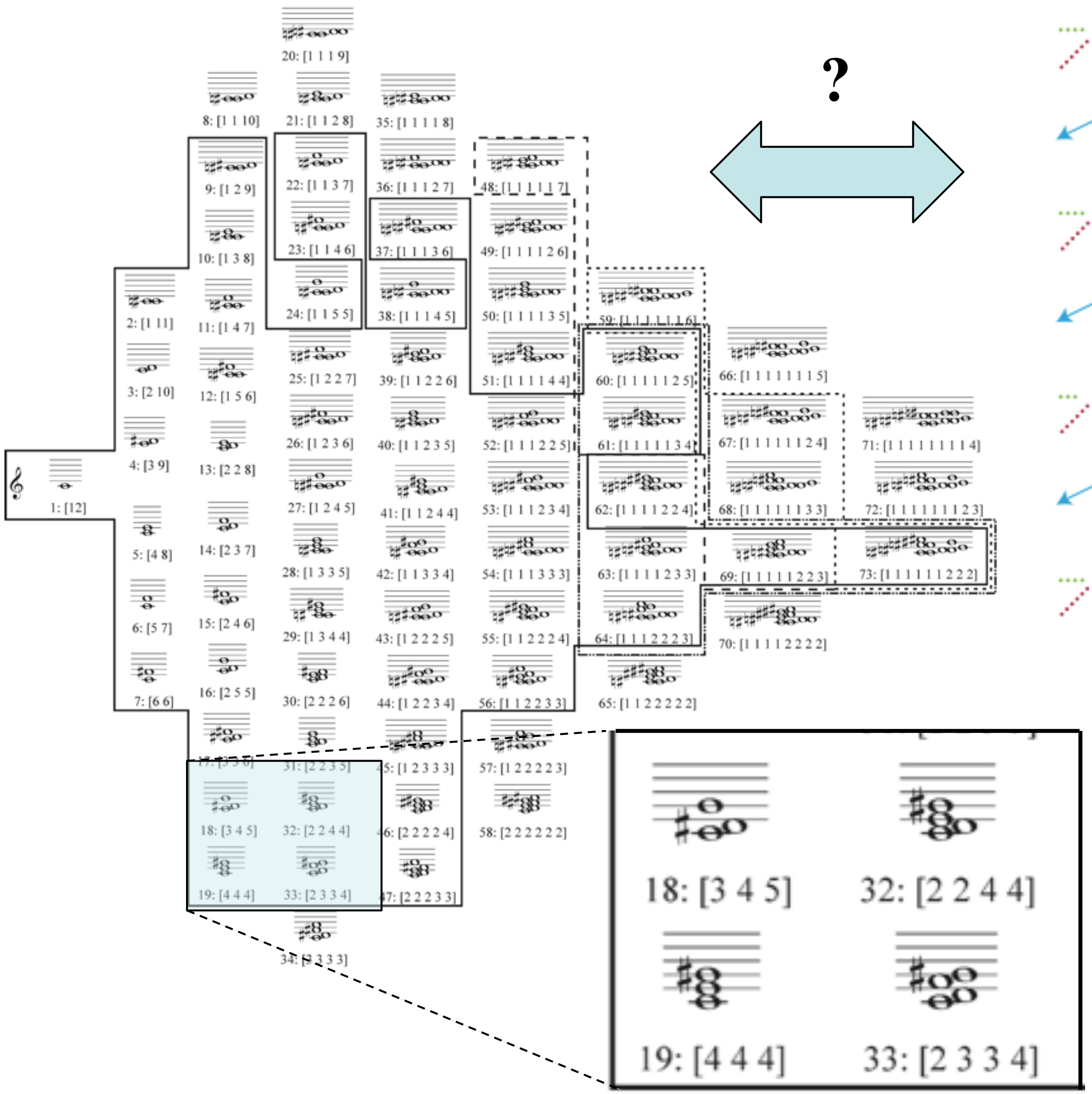
9: [1 2 9]
10: [1 3 8]
11: [1 4 7]
12: [1 5 6]
13: [2 2 8]
14: [2 3 7]
15: [2 4 6]
16: [2 5 5]
17: [3 3 6]
18: [3 4 5]
19: [4 4 4]
20: [1 1 1 9]
21: [1 1 2 8]
22: [1 1 3 7]
23: [1 1 4 6]
24: [1 1 5 5]
25: [1 2 2 7]
26: [1 2 3 6]
27: [1 2 4 5]
28: [1 3 3 5]
29: [1 3 4 4]
30: [2 2 2 6]
31: [2 2 3 5]
32: [2 2 4 4]
33: [2 3 3 4]
34: [3 3 3 3]
35: [1 1 1 1 8]
36: [1 1 1 2 7]
37: [1 1 1 3 6]
38: [1 1 1 4 5]
39: [1 1 2 2 6]
40: [1 1 2 3 5]
41: [1 1 2 4 4]
42: [1 1 3 3 4]
43: [1 2 2 2 5]
44: [1 2 2 3 4]
45: [1 2 3 3 3]
46: [2 2 2 2 4]
47: [2 2 2 3 3]
48: [1 1 1 1 1 7]
49: [1 1 1 1 2 6]
50: [1 1 1 1 3 5]
51: [1 1 1 1 4 4]
52: [1 1 1 2 2 5]
53: [1 1 1 2 3 4]
54: [1 1 1 3 3 3]
55: [1 1 2 2 2 4]
56: [1 1 2 2 3 3]
57: [1 2 2 2 2 3]
58: [2 2 2 2 2 2]
59: [1 1 1 1 1 1 6]
60: [1 1 1 1 1 2 5]
61: [1 1 1 1 1 3 4]
62: [1 1 1 1 2 2 4]
63: [1 1 1 1 2 3 3]
64: [1 1 1 2 2 2 3]
65: [1 1 2 2 2 2 2]
66: [1 1 1 1 1 1 1 5]
67: [1 1 1 1 1 1 2 4]
68: [1 1 1 1 1 1 3 3]
69: [1 1 1 1 1 2 2 3]
70: [1 1 1 1 2 2 2 2]
71: [1 1 1 1 1 1 1 1 4]
72: [1 1 1 1 1 1 1 2 3]
73: [1 1 1 1 1 1 2 2 2]

A. Schoenberg,
***Six pièces* op. 19**

Julio Estrada, Modèles mathématiques en composition et analyse musicales, intervention au Master ATIAM / Cursus de Composition, Ircam, 21 Novembre 2012

Permutoèdre et *Tonnetz*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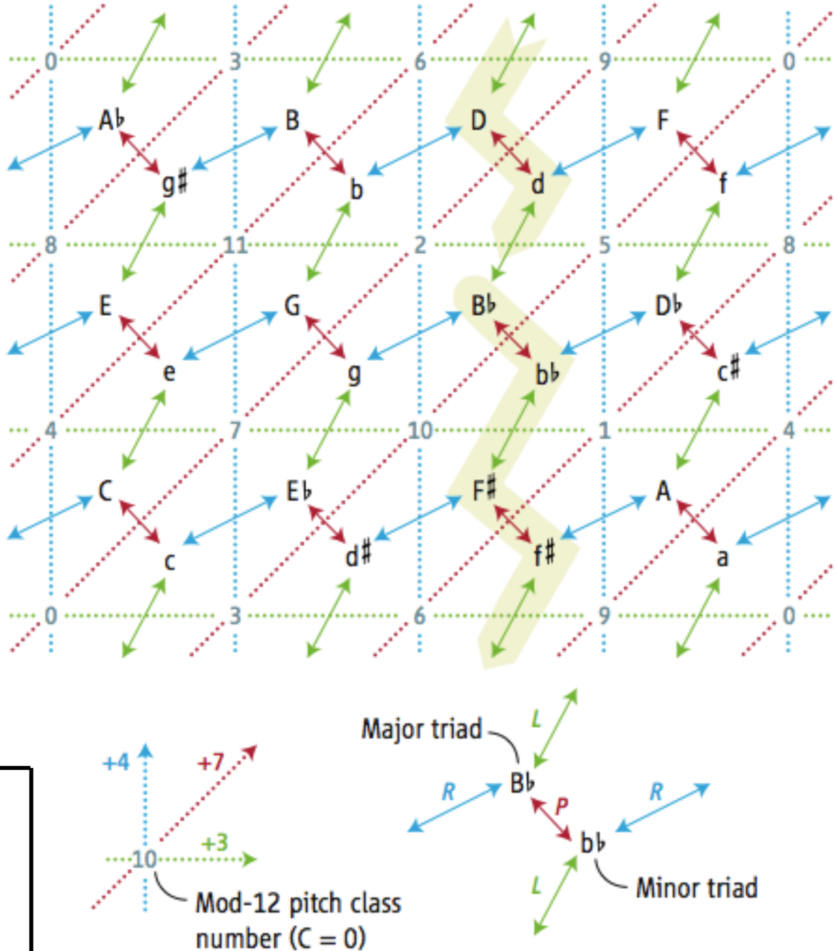
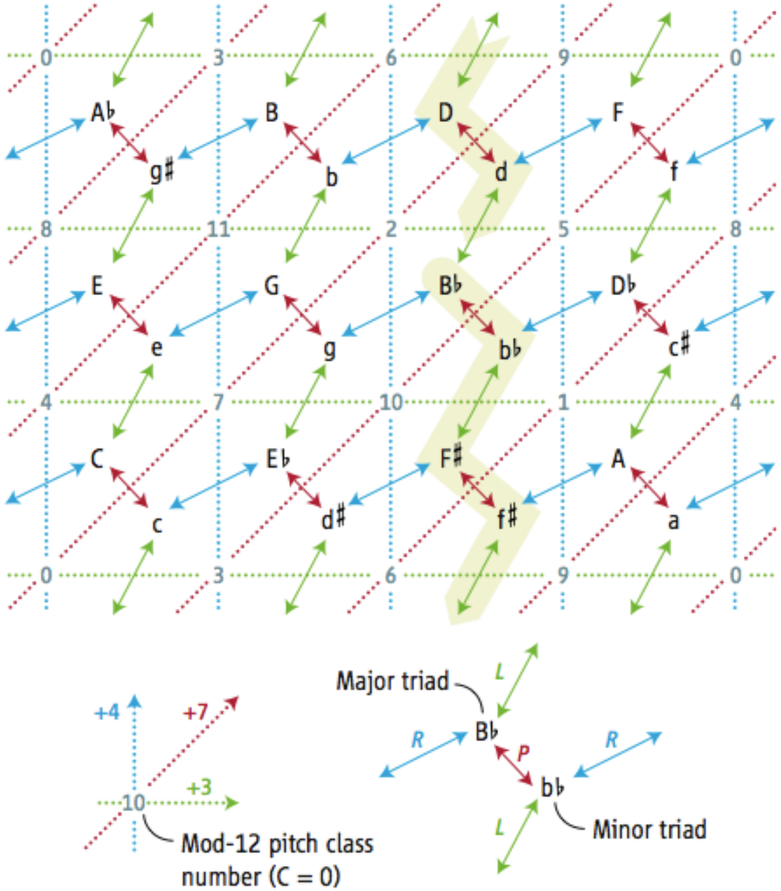
3: [2 10]
4: [3 9]
5: [4 8]
6: [5 7]
7: [6 6]
8: [1 1 10]
9: [1 2 9]
10: [1 3 8]
11: [1 4 7]
12: [1 5 6]
13: [2 2 8]
14: [2 3 7]
15: [2 4 6]
16: [2 5 5]
17: [3 3 6]
18: [3 4 5]
19: [4 4 4]
20: [1 1 1 9]
21: [1 1 2 8]
22: [1 1 3 7]
23: [1 1 4 6]
24: [1 1 5 5]
25: [1 2 2 7]
26: [1 2 3 6]
27: [1 2 4 5]
28: [1 3 3 5]
29: [1 3 4 4]
30: [2 2 2 6]
31: [2 2 3 5]
32: [2 2 4 4]
33: [2 3 3 4]
34: [3 3 3 3]
35: [1 1 1 1 8]
36: [1 1 1 2 7]
37: [1 1 1 3 6]
38: [1 1 1 4 5]
39: [1 1 2 2 6]
40: [1 1 2 3 5]
41: [1 1 2 4 4]
42: [1 1 3 3 4]
43: [1 2 2 2 5]
44: [1 2 2 3 4]
45: [1 2 3 3 3]
46: [2 2 2 2 4]
47: [2 2 2 3 3]
48: [1 1 1 1 1 7]
49: [1 1 1 1 2 6]
50: [1 1 1 1 3 5]
51: [1 1 1 1 4 4]
52: [1 1 1 2 2 5]
53: [1 1 1 2 3 4]
54: [1 1 1 3 3 3]
55: [1 1 2 2 2 4]
56: [1 1 2 2 3 3]
57: [1 2 2 2 2 3]
58: [2 2 2 2 2 2]
59: [1 1 1 1 1 1 6]
60: [1 1 1 1 1 2 5]
61: [1 1 1 1 1 3 4]
62: [1 1 1 1 2 2 4]
63: [1 1 1 1 2 3 3]
64: [1 1 1 2 2 2 3]
65: [1 1 2 2 2 2 2]
66: [1 1 1 1 1 1 1 5]
67: [1 1 1 1 1 1 2 4]
68: [1 1 1 1 1 1 3 3]
69: [1 1 1 1 1 2 2 3]
70: [1 1 1 1 2 2 2 2]
71: [1 1 1 1 1 1 1 1 4]
72: [1 1 1 1 1 1 1 2 3]
73: [1 1 1 1 1 1 2 2 2]
18: [3 4 5]
32: [2 2 4 4]
19: [4 4 4]
33: [2 3 3 4]

?


0
3
6
9
0
A♭
g♯
B
b
D
d
F
f
8
11
2
5
8
E
e
G
g
B♭
b♭
D♭
c♯
4
7
10
1
4
C
c
E♭
d♯
F♯
f♯
A
a
0
3
6
9
0
+4
+7
+3
10
Mod-12 pitch class number (C = 0)
Major triad
L
R
B♭
P
R
b♭
L
Minor triad

Le *Tonnetz* en tant que GIS


0
3
6
9
0
A♭
B
D
F
g♯
b
d
f
8
11
2
5
8
E
G
B♭
D♭
e
g
b♭
c♯
4
7
10
1
4
C
E♭
F♯
A
c
d♯
f♯
a
0
3
6
9
0
+4
+7
+3
10
Mod-12 pitch class number (C = 0)
Major triad
L
R
B♭
P
R
b♭
L
Minor triad

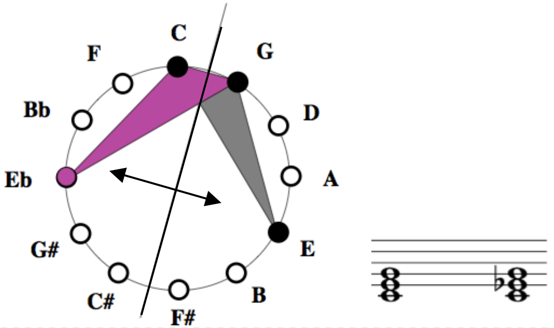
$$\rho = \langle L, R \mid L^2 = (LR)^{12} = 1 \; ; \; LRL = L(LR)^{-1} \rangle$$

- ρ opère de façon simplement transitive sur l'ensemble *S* des 24 triades consonantes

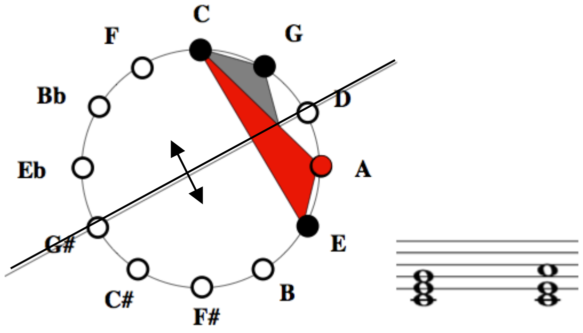
⇒ (*S*, ρ, int) est un GIS

[Noll04]

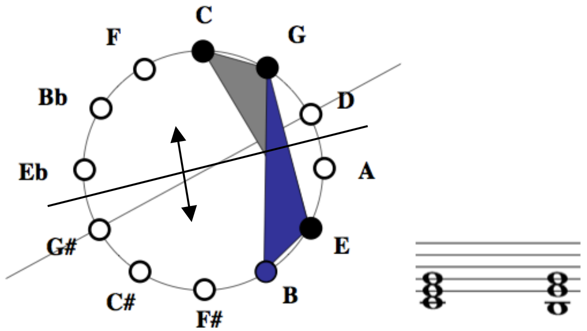
(Neo-)Riemannian Operation **P** = **„Parallel“**

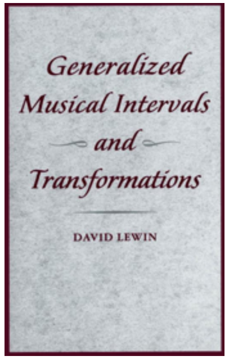

C
G
D
A
E
B
F#
C#
G#
Eb
Bb
F

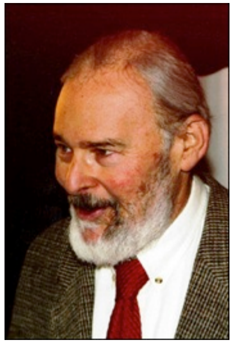
(Neo-)Riemannian Operation **R** = **„Relative“**


C
G
D
A
E
B
F#
C#
G#
Eb
Bb
F

(Neo-)Riemannian Operation **L** = **„Leading-Tone“**


C
G
D
A
E
B
F#
C#
G#
Eb
Bb
F


Generalized
Musical Intervals
and
Transformations
DAVID LEWIN



Système d'Intervalles Généralisés - Système Généralisé d'Intervalles
David Lewin's *Generalized Interval System* [GMIT, 1987]

GIS = (S, G, int)

S = ensemble

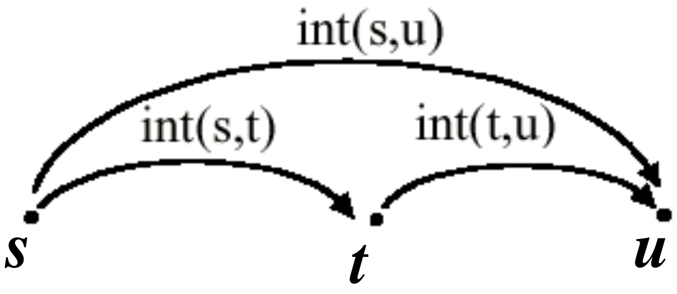
(G,•) = groupe d'intervalles

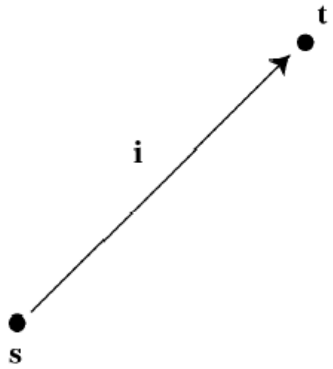
int = fonction intervallique

$$S \times S \xrightarrow{\text{int}} G$$

1. Pour tout objets s, t, u dans S :
int(s,t)•int(t,u) = int(s,u)

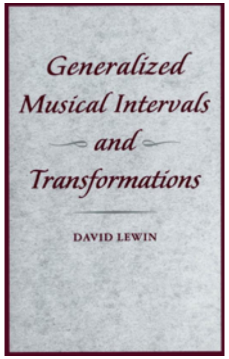
2. Pour tout objet s dans S et tout intervalle i dans G il y a un seul objet t dans S tel que int(s,t) = i


int(s,u)
int(s,t)
int(t,u)
s
t
u

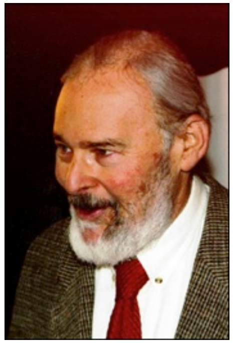


- S={…, *do*, *do*$_{\#}$=*ré*$_{b}$, *ré*, …, *si*, *do'*,…}, G=**Z**, int(*do*,*ré*)=2, int(*fa*,*do*)=-5 etc.
- S={…, *do*, *ré*, *mi*, *fa*, *sol*, *la*, *si*, *do'*, …}, G=**Z**, int(*do*,*ré*)=1, int(*fa*,*do*)=-3 etc.
- S=G=$\mathbf{Z}_{12}$={*do*, *do*$_{\#}$=*ré*$_{b}$, *ré*, …, *si*}, int(*do*,*ré*)=2, int(*fa*,*do*)=7 etc.

Premières généralisations : transposition


Generalized Musical Intervals and Transformations

DAVID LEWIN



GIS = (*S*, *G*, int)

S = ensemble

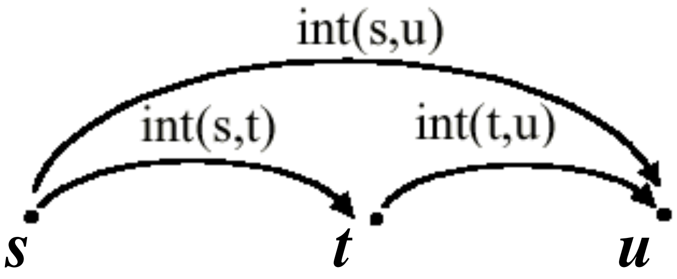
(G,•) = groupe d'intervalles

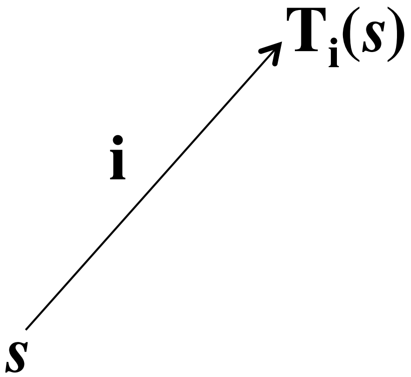
int = fonction intervallique

$$S \times S \xrightarrow{\text{int}} G$$

1. Pour tout objets s, t, u dans S :
$\text{int}(s,t) \bullet \text{int}(t,u) = \text{int}(s,u)$

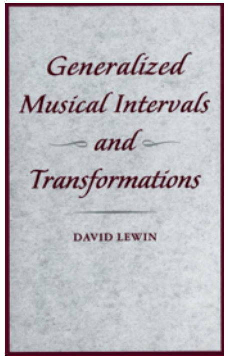
2. Pour tout objet s dans S et tout intervalle i dans G il y a un seul objet t dans S tel que $\text{int}(s,t) = i$





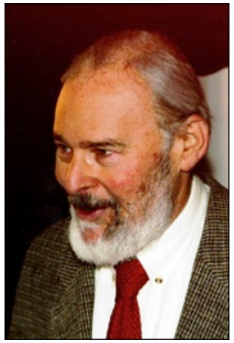
Généralisation de la notion de transposition (musicale)

Pour tout élément i dans G, la transposition T_i est une application
$T_i : S \dashrightarrow S$ telle que $\text{int}(s, T_i(s)) = i$ pour tout élément s dans S



Generalized Musical Intervals and Transformations

DAVID LEWIN



Premières propriétés intervalliques

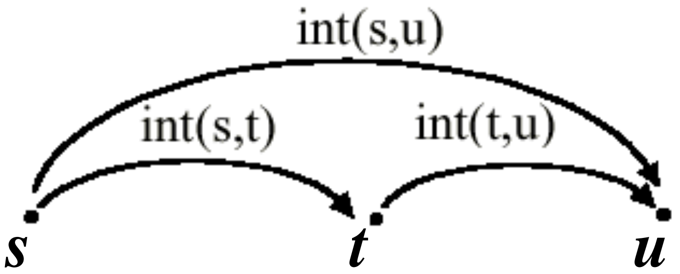
GIS = (S, G, int)

S = ensemble

(G,•) = groupe d'intervalles

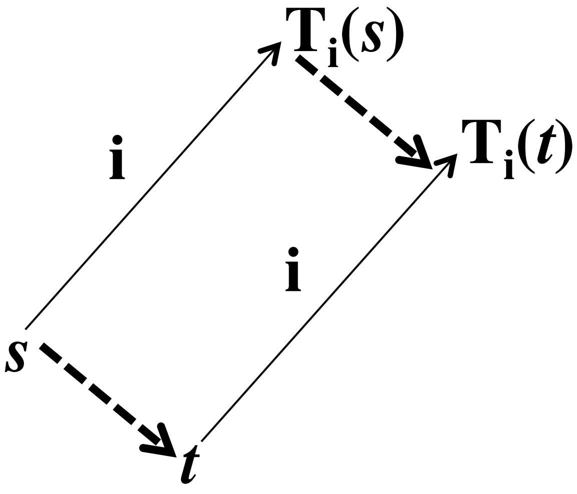
int = fonction intervallique

$$S \times S \overset{\text{int}}{\dashrightarrow} G$$



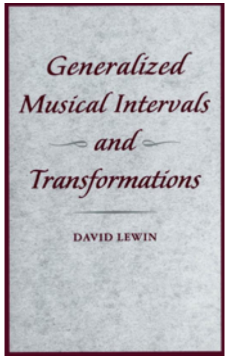
1. Pour tout objets s, t, u dans S :
int(s,t)•int(t,u) = int(s,u)

2. Pour tout objet s dans S et tout intervalle i dans G il y a un seul objet t dans S tel que int(s,t) = i



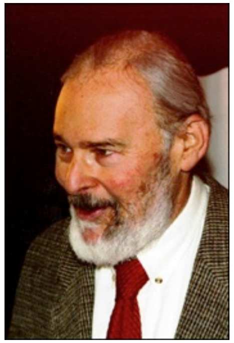
Dans un GIS commutatif, la transposition préserve les rapports intervallaires

int (s, t) = int (T$_i$ (s), T$_i$ (t))



Generalized
Musical Intervals
and
Transformations

DAVID LEWIN



Premières propriétés d'un GIS non commutatif

GIS = (S, G, int)

S = ensemble

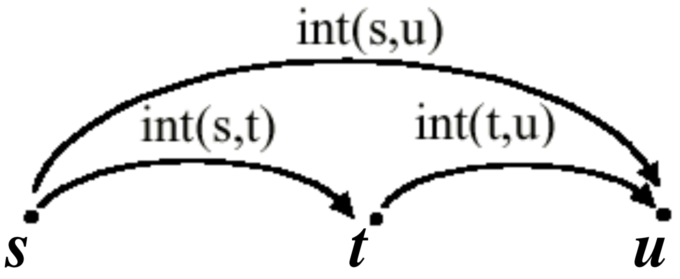
(G,•) = groupe non commutatif

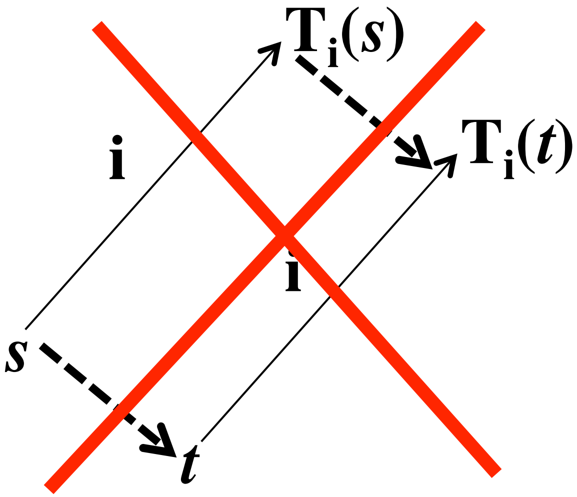
int = fonction intervallique

$$S \times S \xrightarrow{\text{int}} G$$

1. Pour tout objets s, t, u dans S :
$\text{int}(s,t)\bullet\text{int}(t,u) = \text{int}(s,u)$

2. Pour tout objet s dans S et tout intervalle i dans G il y a un seul objet t dans S tel que $\text{int}(s,t) = i$





Il a des transpositions qui ne préservent pas les intervalles et il y a des transformations qui préservent les intervalles et qui ne sont pas des transpositions [GMIT, p. 50]

Equivalence entre GIS et action de groupe

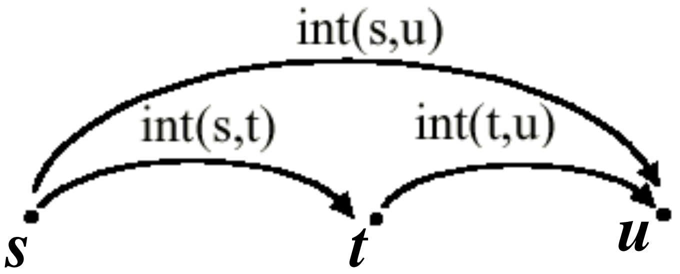
GIS = (S, G, int)

S = ensemble

(G,•) = groupe d'intervalles

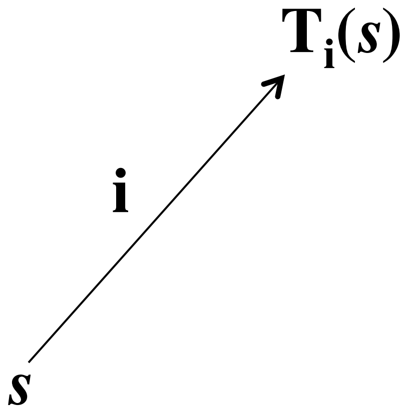
int = fonction intervallique

$$S \times S \xrightarrow{\text{int}} G$$



Action simplément transitive

1. Pour tous objets s, t, u dans S : int(s,t)•int(t,u) = int(s,u)
2. Pour tout objet s dans S et tout intervalle i dans G il y a un seul objet t dans S tel que int(s,t) = i



Soit $\tau = \{T_i \,;\, i \in G\}$ le groupe des transpositions

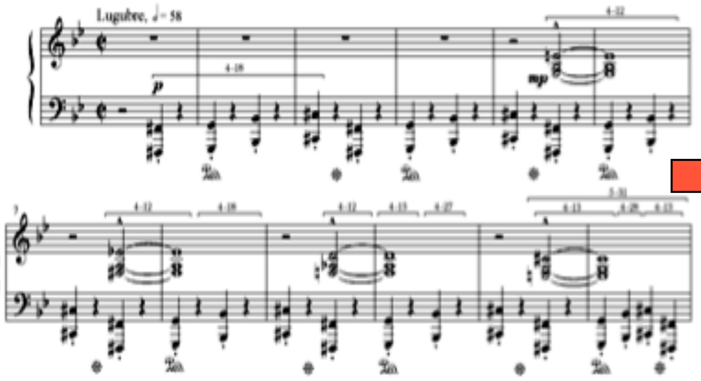
GIS = (S, G, int) ⇔ $\tau \times S \rightarrow S$ telle que $(T_i\,,\, s) \rightarrow T_i(s)$

Terminologies équivalentes :

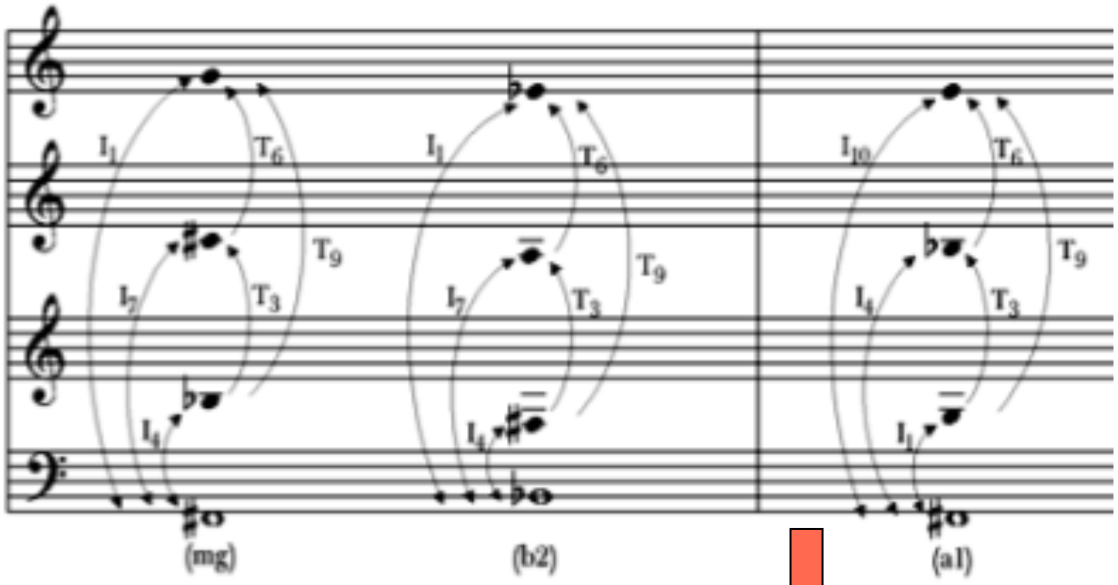
- Un GIS est un G-torseur à gauche
- S est un ensemble principal homogène [Bourbaki]

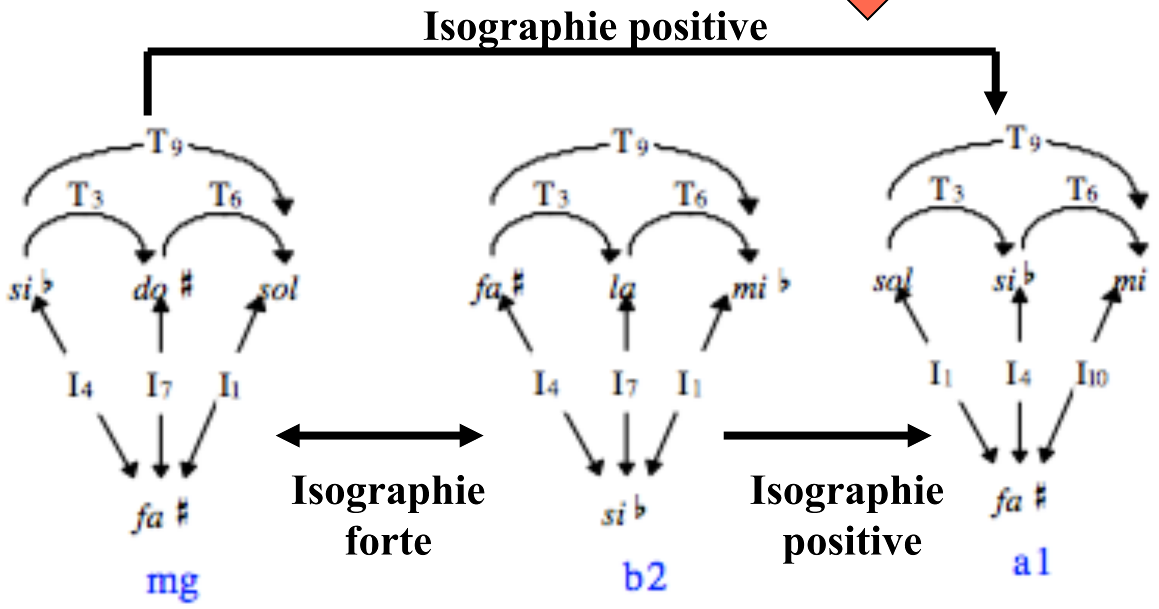
Klumpenhouwer Networks (K-réseaux)

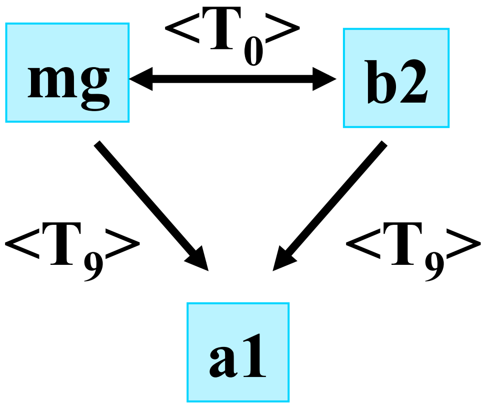
Xavier Hascher: « Liszt et les sources de la notion d'agrégat », Analyse Musicale, 43, 2002


Lugubre, ♩ = 58
p
mp

Ex. 1 - « Ladislaus Teleki » (*Historische ungarische Bildnisse* n° 4), mes. 1-7
Les agrégats dans la classification de Forte

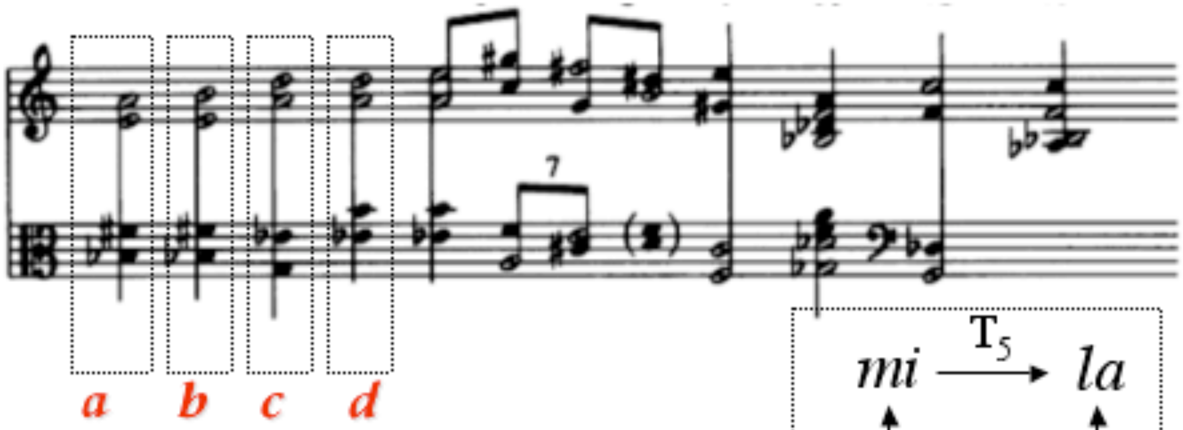

I_1
T_6
T_9
I_7
T_3
I_4
(mg)
I_1
T_6
T_9
I_7
T_3
I_4
(b2)
I_{10}
T_6
T_9
I_4
T_3
I_1
(a1)


Isographie positive
T_9
T_3
T_6
si ♭
do ♯
sol
I_4
I_7
I_1
fa ♯
mg
Isographie forte
T_9
T_3
T_6
fa ♯
la
mi ♭
I_4
I_7
I_1
si ♭
b2
Isographie positive
T_9
T_3
T_6
sol
si ♭
mi
I_1
I_4
I_{10}
fa ♯
a1



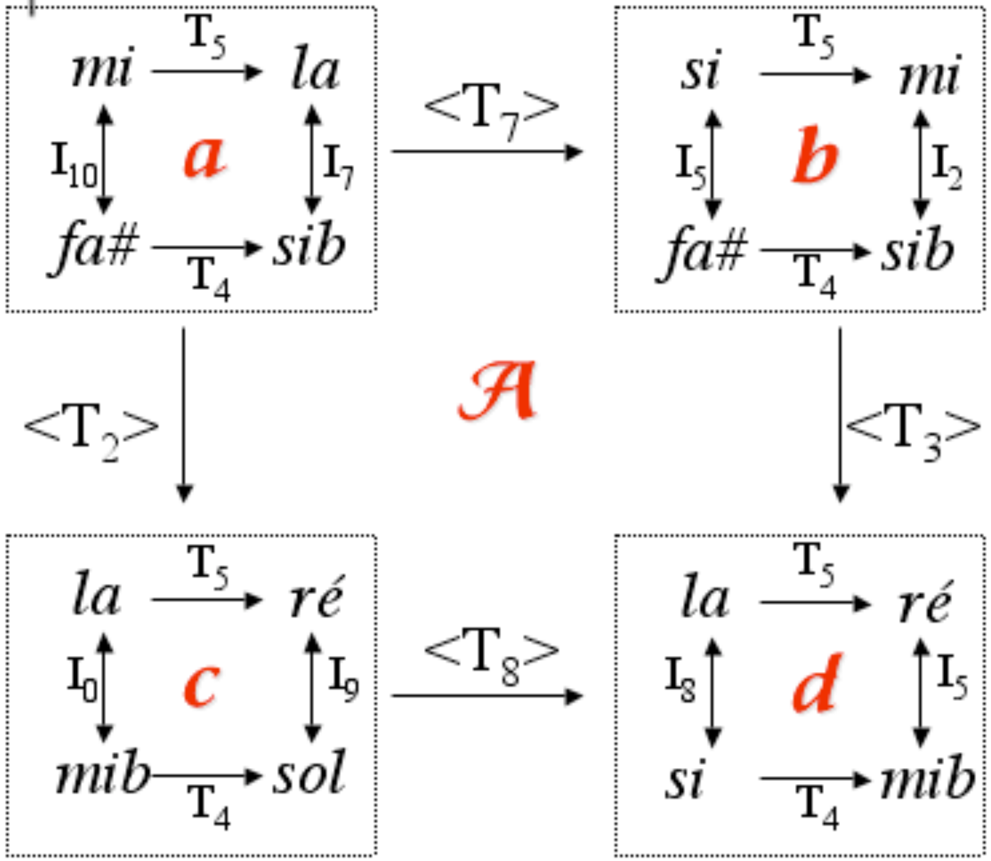
Klumpenhouwer Networks (K-réseaux) : isographies positives et récursivité

David Lewin: «A Tutorial on K-nets using the Chorale in Schoenberg's Op.11, N°2 », JMT, 1994



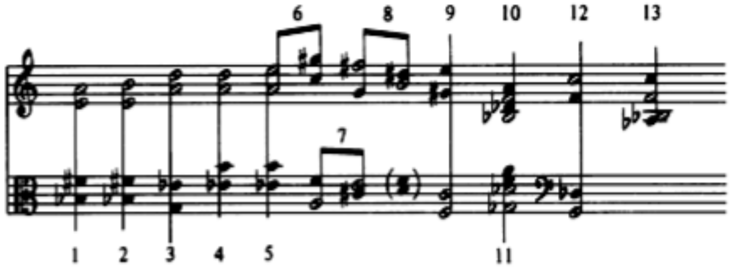
$$\langle T_k \rangle : T_m \to T_m$$
$$I_m \to I_{k+m}$$

$$\begin{tikzcd} \mathcal{A} \arrow[r, "?"] \arrow[d, "?"'] & \mathcal{B} \arrow[d, "?"] \\ \mathcal{C} \arrow[r, "?"'] & \mathcal{D} \end{tikzcd}$$

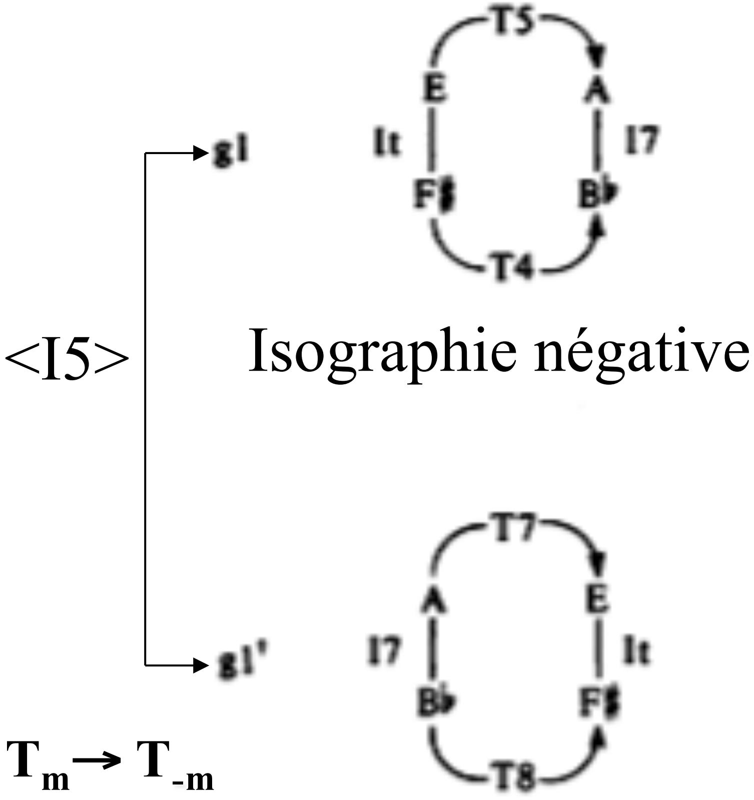

mi
T5
la
I10
a
I7
fa#
T4
sib
<T7>
si
T5
mi
I5
b
I2
fa#
T4
sib
<T2>
A
<T3>
la
T5
ré
I0
c
I9
mib
T4
sol
<T8>
la
T5
ré
I8
d
I5
si
T4
mib

Klumpenhouwer Networks (K-nets) : isographies négatives

David Lewin: «A Tutorial on K-nets using the Chorale in Schoenberg's Op.11, N°2 », JMT, 1994

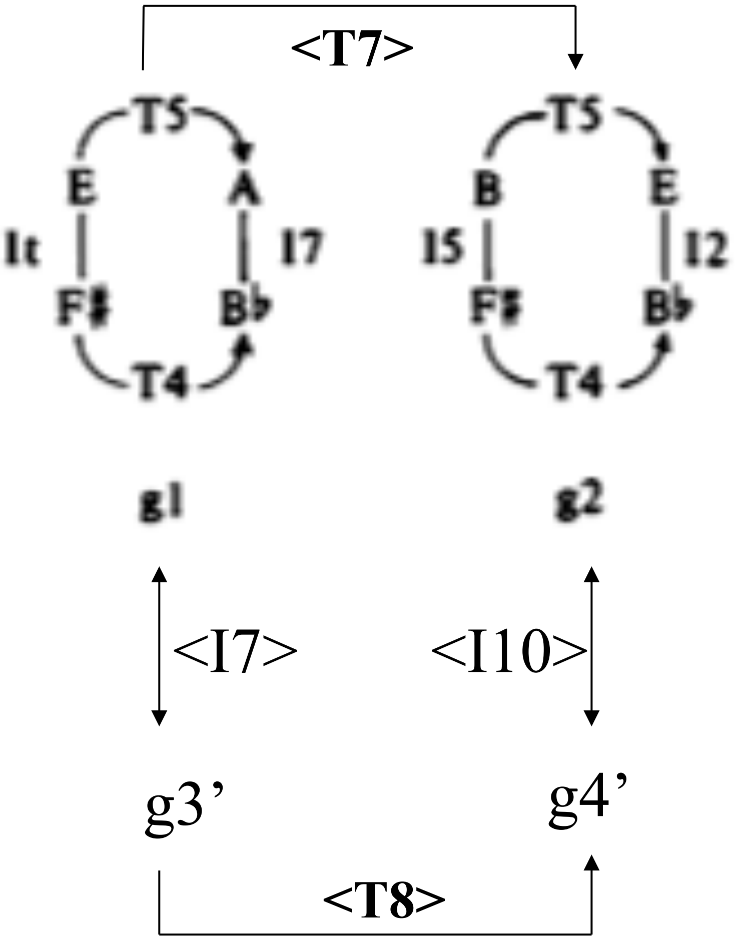

6
8
9
10
12
13
7
1
2
3
4
5
11

Example 9


T5
E
A
It
I7
F♯
B♭
T4
g1
T7
A
E
I7
It
B♭
F♯
T8
g1'

<I5>

Isographie négative


T5
E
A
It
I7
F♯
B♭
T4
T5
B
E
I5
I2
F♯
B♭
T4
g1
g2

<T7>

<I7>

<I10>

g3'

g4'

<T8>

$$\langle I_k\rangle : T_m \rightarrow T_{-m}$$
$$I_m \rightarrow I_{k-m}$$

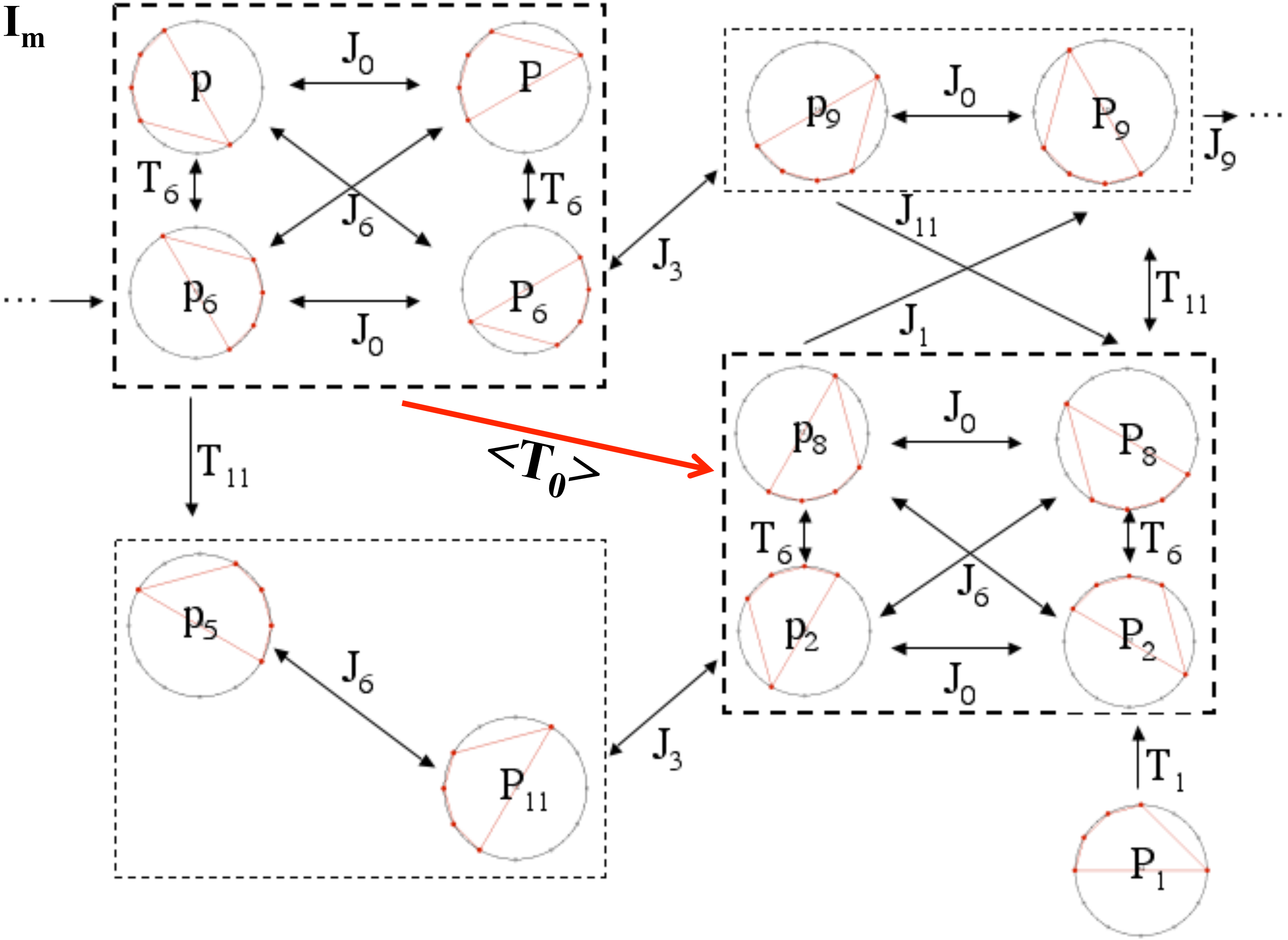
$$\langle T_k\rangle : T_m \rightarrow T_m$$
$$I_m \rightarrow I_{k+m}$$

Isographies « fortes » dans le réseau transformationnel

Stockhausen: *Klavierstück III* (Analyse de D. Lewin)

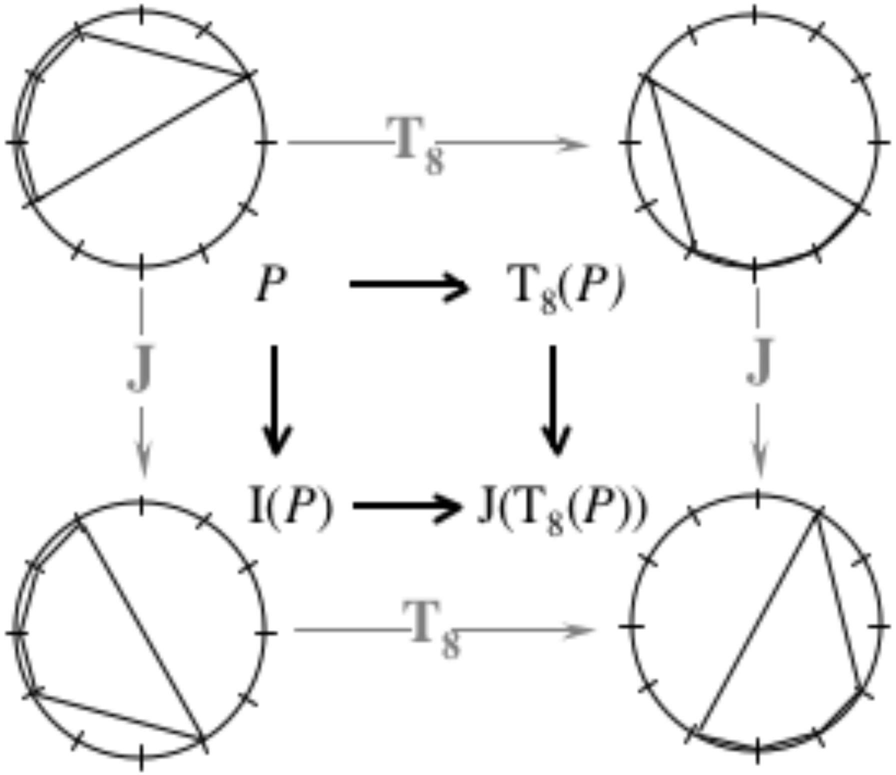
$\langle T_0 \rangle : T_m \rightarrow T_m$

$I_m \rightarrow I_m$



p
J_0
P
T_6
J_6
T_6
p_6
J_0
P_6
J_3
p_9
J_0
P_9
J_9
J_{11}
T_{11}
J_1
T_{11}
$\langle T_0 \rangle$
p_8
J_0
P_8
T_6
J_6
T_6
p_2
J_0
P_2
T_1
P_1
p_5
J_6
P_{11}
J_3

Action de groupe et commutativité des diagrammes



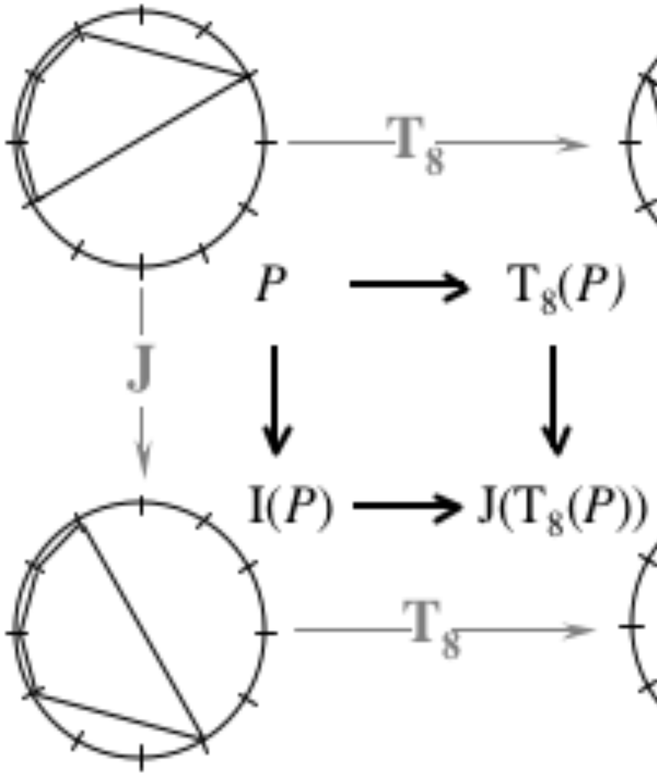
$$\begin{tikzcd}
\bigcirc \arrow[r, "f"] \arrow[d, "g"] & \bigcirc \arrow[d, "g"] \\
\bigcirc \arrow[r, "f"] & \bigcirc
\end{tikzcd}$$

Tout diagramme commute

$$\forall f, g \in <\mathrm{T}, \mathrm{J}>$$

Le groupe des 24 transformations $\sigma = \{T_0, T_1, \ldots, T_{11}, T_0J, T_1J, \ldots, T_{11}J\}$ est commutatif et opère de manière simplement transitive sur l'espace *S* des 24 formes du pentacorde de base (i.e. l'ensemble de ses 12 transpositions et de ses 12 inversions)

Inversions « contextuelles » et commutativité des diagrammes



$$\begin{tikzcd}
\bigcirc \arrow[r, "f"] \arrow[d, "g"] & \bigcirc \arrow[d, "g"] \\
\bigcirc \arrow[r, "f"] & \bigcirc
\end{tikzcd}$$

Tout diagramme commute

$\forall f\,, g \in \, < \mathrm{T}, \mathrm{J} >$

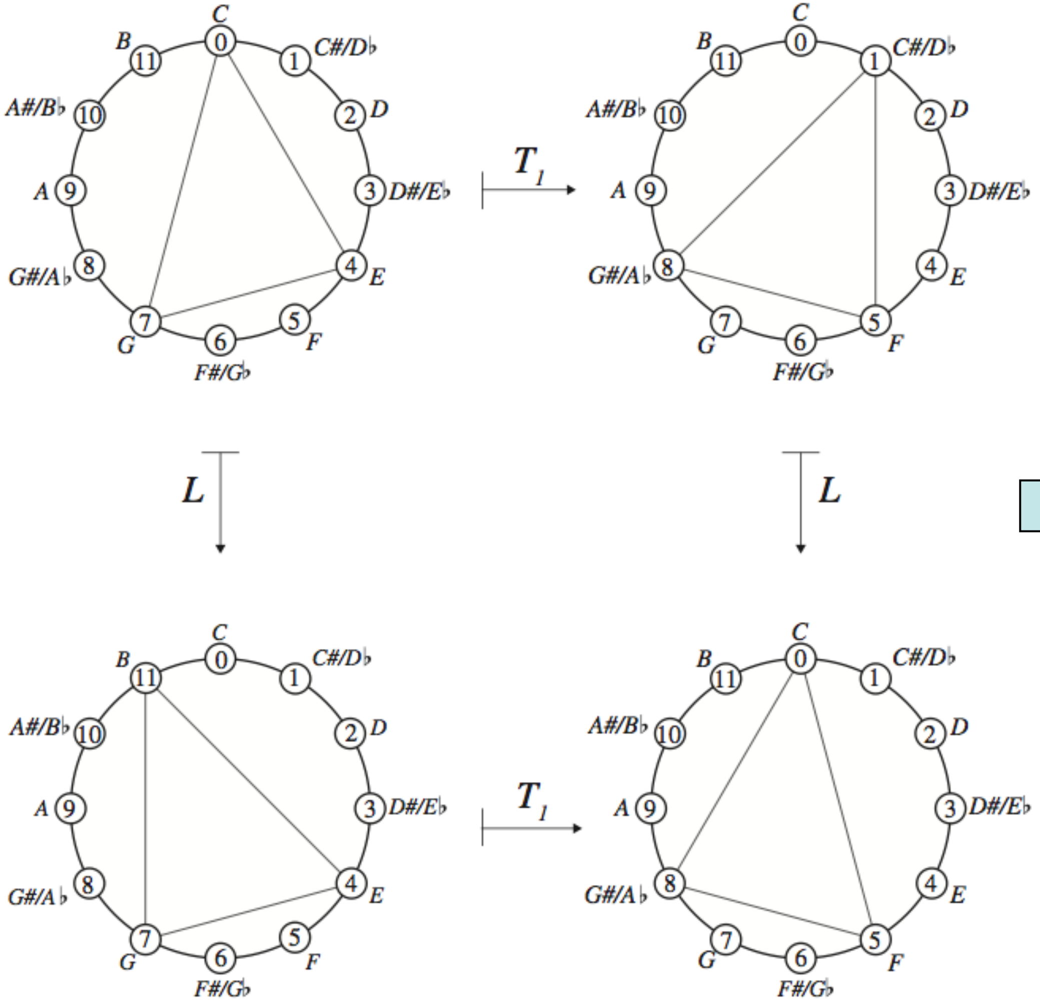
Le groupe des 24 transformations $\sigma = \{T_0, T_1, \ldots, T_{11}, T_0J, T_1J, \ldots, T_{11}J\}$ est commutatif et opère de manière simplement transitive sur l'espace *S* des 24 formes du pentacorde de base (i.e. l'ensemble de ses 12 transpositions et de ses 12 inversions)

$\Rightarrow$ (S, σ, int) est un GIS

Dualité entre (S, ρ, int) et (S, D_{12}, int)

$\rho = \langle L, R \mid L^2 = (LR)^{12} = 1 \,;\, LRL = L(LR)^{-1} \rangle$ ↔ **$D_{12} = \langle I, T \mid I^2 = T^{12} = 1 \,;\, ITI = I(IT)^{-1} \rangle$**

⇒ ρ et D_{12} sont l'un le *centralisateur* de l'autre (dans le groupe symétrique *Sym*(S))



T_1
L
L
T_1
C 0
C#/D♭ 1
D 2
D#/E♭ 3
E 4
F 5
F#/G♭ 6
G 7
G#/A♭ 8
A 9
A#/B♭ 10
B 11

[Crans, Fiore & Satyendra, 2008]

(S, ρ, int) ≠ (S, D_{12}, int)
[cf. équivalence entre GIS]

$$\begin{tikzcd}
\bigcirc \arrow[r, "f"] \arrow[d, "g"] & \bigcirc \arrow[d, "g"] \\
\bigcirc \arrow[r, "f"] & \bigcirc
\end{tikzcd}$$

Tout diagramme commute

$\forall f \in D_{12}$
$\forall g \in \rho$

Fonction Intervallique IFUNC dans un GIS

GIS = (S, G, int)

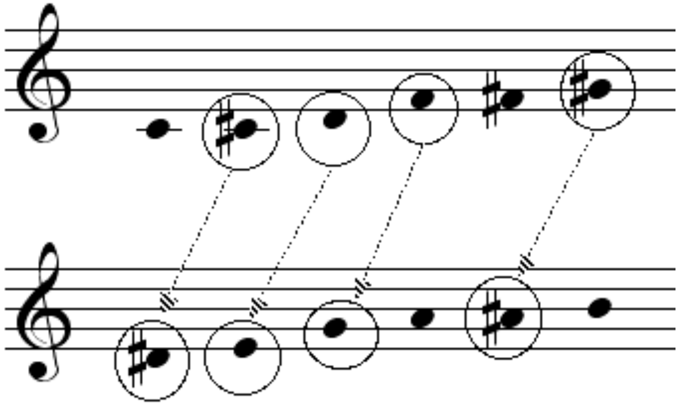
S ensemble

H et H’ dans S

IFUNC(H, H’)(i) =
$= \#\{(a,b) \in H \times H' \mid \mathrm{int}(a,b)=i\}$

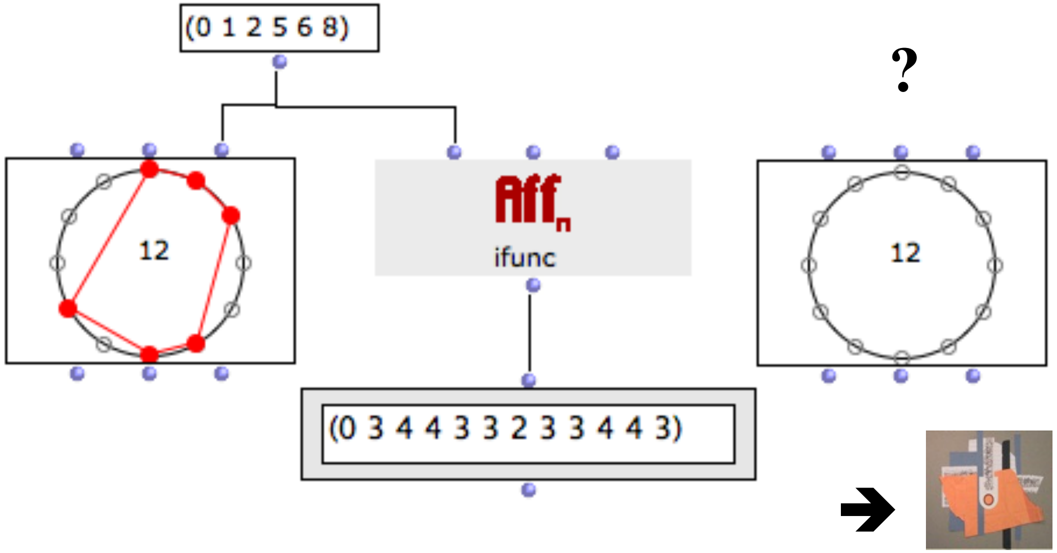
IFUNC(H, H ’)(2) = 4

H = {0, 1, 2, 5, 6, 8}



H ’={3, 4, 7, 9, 10, 11}

Problème de Lewin (1959)
Est-on capable, si l’on connaît l’ensemble H et, pour tout i, les valeurs IFUNC(H,H’)(i) de reconstruire H’ ?



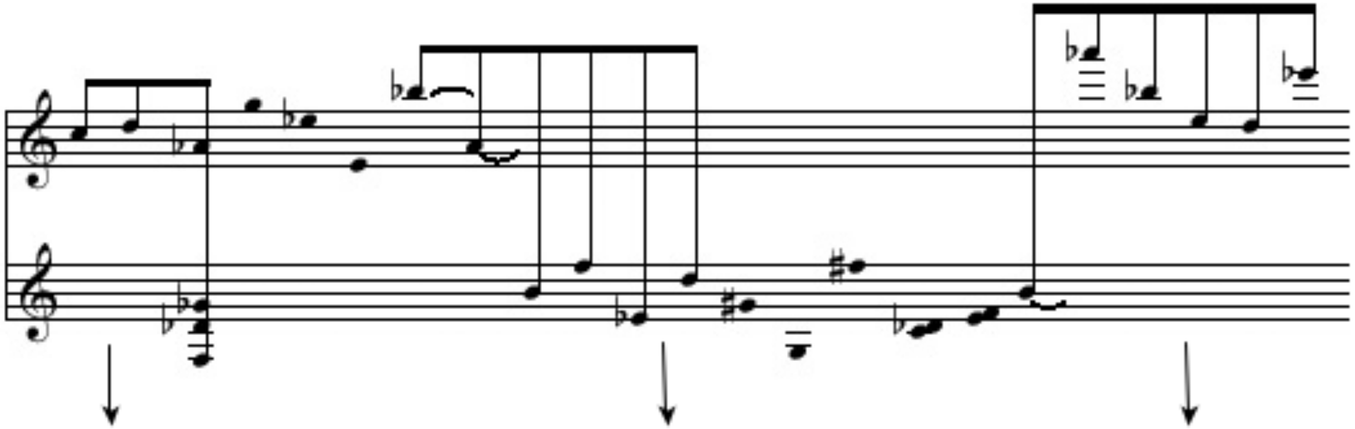
(0 1 2 5 6 8)

(0 3 4 4 3 3 2 3 3 4 4 3)

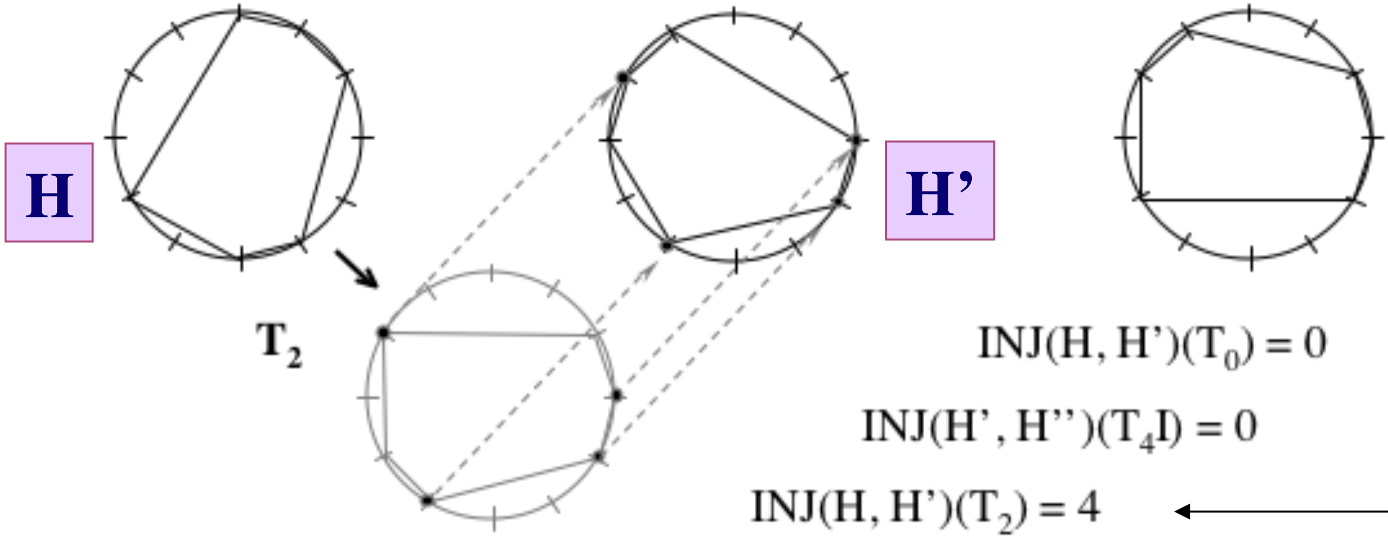
➔

09-ifunc

Fonction d'Injection comme généralisation de relation d'inclusion



$H = \{0, 1, 2, 5, 6, 8\}$ $H' = \{3, 4, 7, 9, 10, 11\}$ $H'' = T_4I(H) = \{2, 3, 4, 8, 10, 11\}$



$\mathrm{INJ}(H, H')(T_0) = 0$

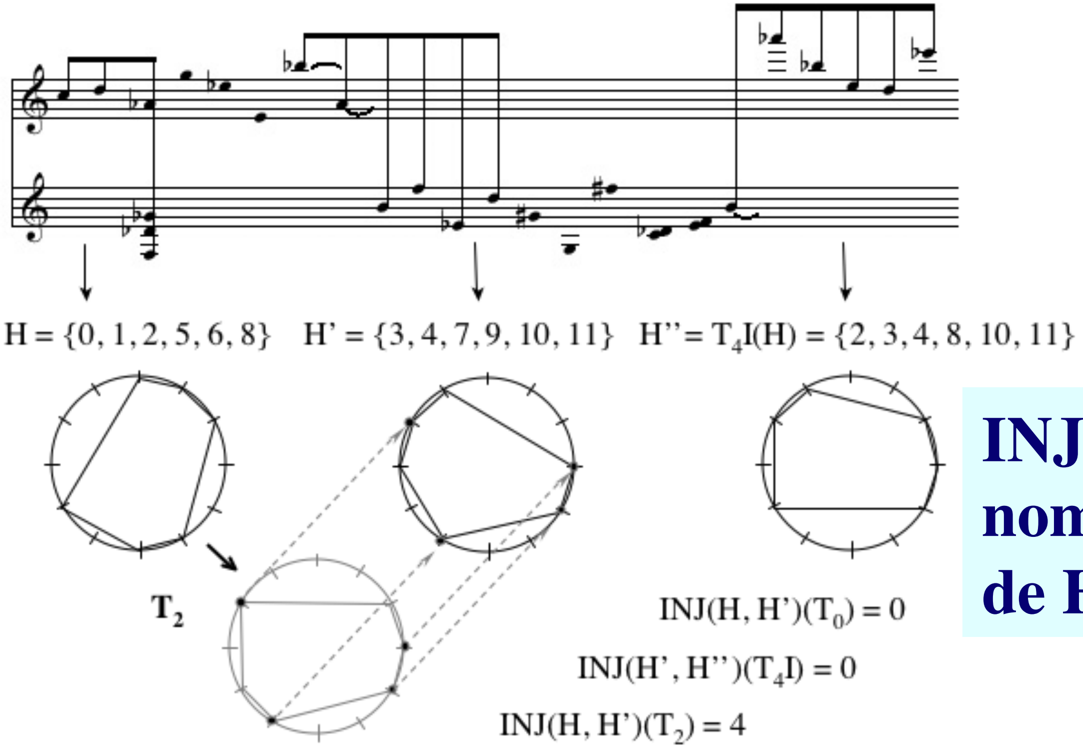
$\mathrm{INJ}(H', H'')(T_4I) = 0$

$\mathrm{INJ}(H, H')(T_2) = 4$

$i=2$

$$\mathrm{INJ}(H,H')(T_i) = \#\{a \in H \mid T_i(a) \in H'\}$$

Fonction d’Injection (INJ) et fonction intervallaire (IFUNC)

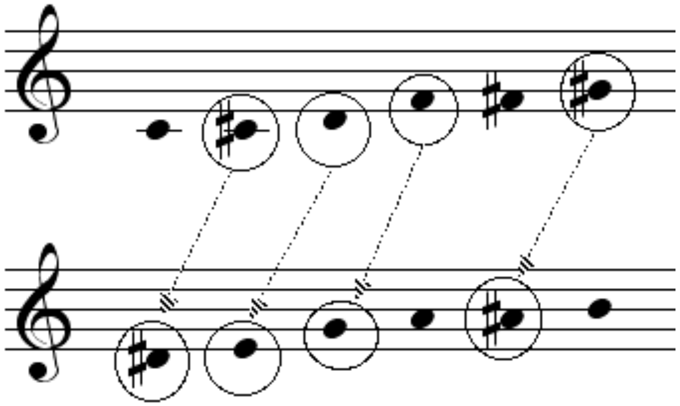

H = {0, 1, 2, 5, 6, 8}
H’ = {3, 4, 7, 9, 10, 11}
H’’= T₄I(H) = {2, 3, 4, 8, 10, 11}
T₂
INJ(H, H’)(T₀) = 0
INJ(H’, H’’)(T₄I) = 0
INJ(H, H’)(T₂) = 4

$\mathrm{INJ(H,H')}(\mathrm{T}_i)$ =
nombre d’éléments *a*
de H tels que $\mathrm{T}_i(a) \in \mathrm{H'}$

||

H

H’



IFUNC(H, H’)(*i*) =
nombre d'éléments
(*a*,*b*) dans H × H ’ tels
que int(*a*,*b*)=*i*

« Here the basic hierarchical scope of the (twelve-tone) system is contained essentially in the simple theorem that:

Given a collection of pitches (pitch classes), the multiplicity of occurrence of any interval (…) determines the number of common pitches between the original collection and the transposition by the interval »

(Milton Babbitt, *Past and Present Concepts*, 1961)

« ...on peut remplacer entièrement le concept d'intervalle dans un GIS avec le concept de transposition dans un espace »

« ...on peut remplacer le concept même de GIS avec l'idée d'un espace S sur lequel on a un groupe d'opérations qui opère »

(David Lewin, *Generalized Musical Intervals and Transformations*, 1987)

Babbitt

$\updownarrow$ $GIS = \mathrm{Z}_{12}$

$\updownarrow$ GIS

Lewin

$$\mathbf{INJ}(A, B)(\mathrm{T_i}) \quad = \quad \mathbf{IFUNC}(A, B)(\mathrm{i})$$

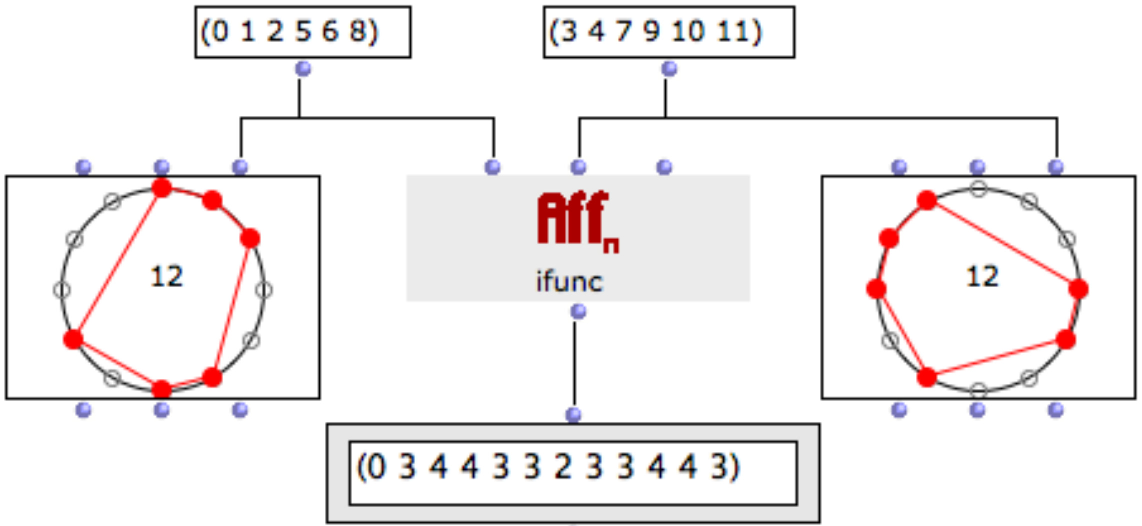
Approche transformationnelle et théorie de l'homométrie

$$\mathbf{INJ}(A,B)(\mathrm{T}_k) \quad = \quad \mathbf{IFUNC}(A,B)(k)$$

- IFUNC peut s'écrire comme un produit de convolution de fonctions caractéristiques (Lewin, *JMT*, 1958)

$$\mathbf{IFUNC}(A,B)(k) = \#\{(a,b) \in A\times B \mid a=b+k\}$$

$$\mathbb{1}_A * \mathbb{1}_{-B}(k) = \sum_{x\in\mathbb{Z}_c} \mathbb{1}_A(x).\mathbb{1}_{-B}(k-x) = \sum_{x\in\mathbb{Z}_c} \mathbb{1}_A(x).\mathbb{1}_B(x-k) = \sum_{\substack{(x,y)\in A\times B\\ y=x-k}} 1$$



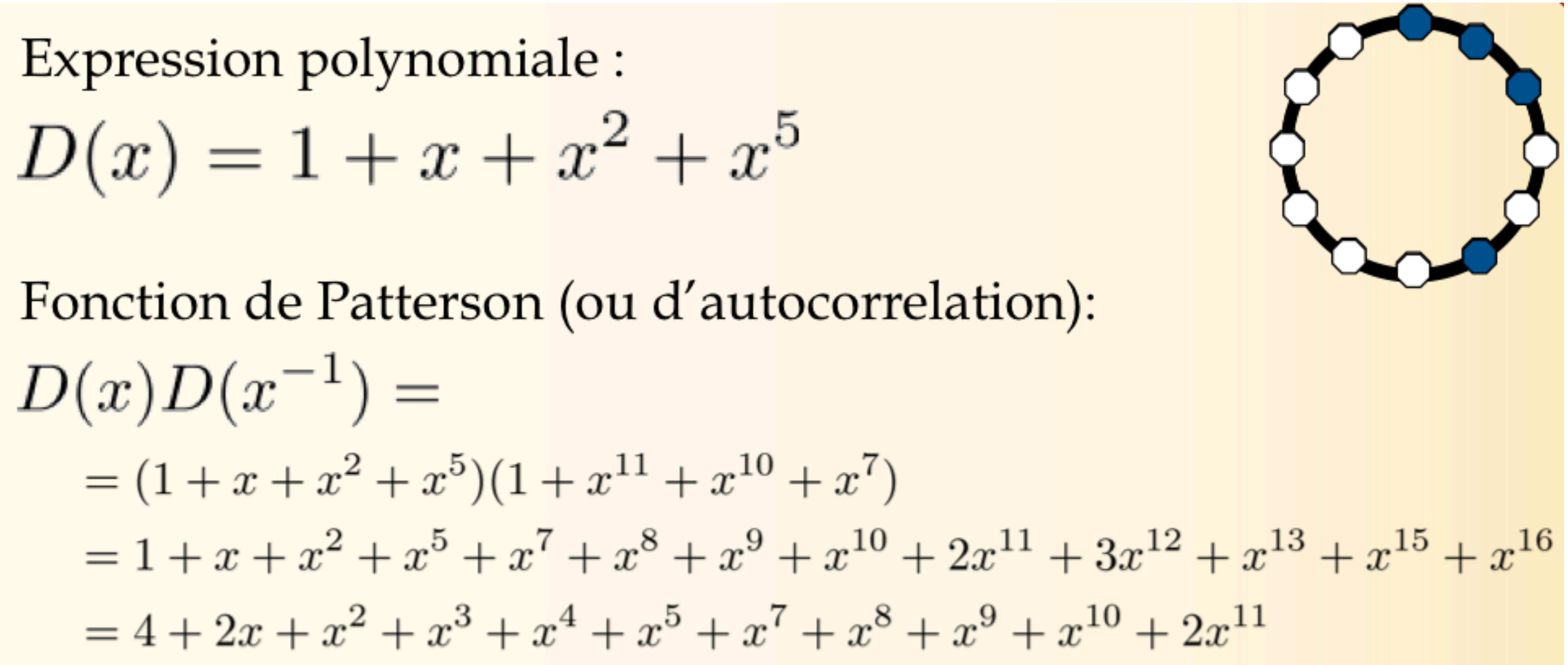
```
(0 1 2 5 6 8)
(3 4 7 9 10 11)
(0 3 4 4 3 3 2 3 3 4 4 3)
```

$$\mathbf{IFUNC}(A,B)(3) \quad = \#\{(0,9),(1,10),(2,11),(6,3)\}$$

Z-relation, homométrie et reconstruction de la phase

- Contenu intervallaire et Fonction de Patterson

$$IC_A(k) = (1_A \star 1_{-A})(k)$$



Expression polynomiale :

$$D(x) = 1 + x + x^2 + x^5$$

Fonction de Patterson (ou d'autocorrelation):

$$D(x)D(x^{-1}) =$$

$$= (1 + x + x^2 + x^5)(1 + x^{11} + x^{10} + x^7)$$

$$= 1 + x + x^2 + x^5 + x^7 + x^8 + x^9 + x^{10} + 2x^{11} + 3x^{12} + x^{13} + x^{15} + x^{16}$$

$$= 4 + 2x + x^2 + x^3 + x^4 + x^5 + x^7 + x^8 + x^9 + x^{10} + 2x^{11}$$

IFUNC(D, D) = IC_D = [4, 2, 1, 1, 1, 1, 1, 1, 1, 1, 2]

• Daniele Ghisi, "From Z-relation to homometry: an introduction and some musical examples", MaMuX, 10/2/2010

Relation Z, DFT et théorie de l'homométrie

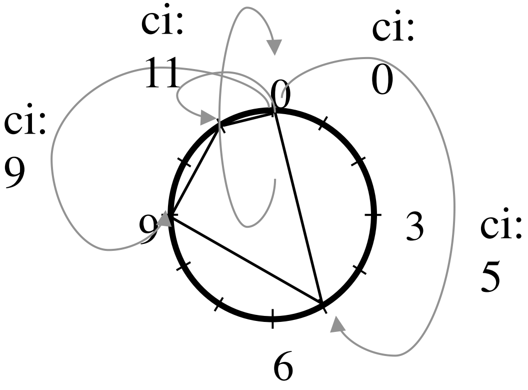
- Le contenu intervallique est équivalent à un produit de convolution de fonctions caractéristiques (Lewin, 1958)

$$\mathcal{F}_A : t \mapsto \sum_{k\in A} e^{-2i\pi kt/c}$$

$$IC_A(k) = (1_A \star 1_{-A})(k)$$

$$1_A * \tilde{1}_B(k) = \sum_i 1_A(i) \times 1_B(i-k) = \sum_{\substack{i\in A\\ i-k\in B}} 1$$

$$\mathcal{F}(1_A * \tilde{1}_B) = \mathcal{F}(1_A) \times \mathcal{F}(\tilde{1}_B)$$

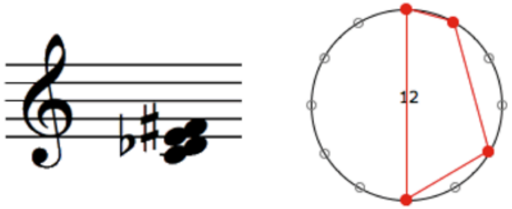


$A = \{0, 5, 9, 11\} \Rightarrow IC_A(k) = 1\ \forall\ k$

$$\mathcal{F}(IC_A) = \mathcal{F}_A \times \mathcal{F}_{-A} = |\mathcal{F}_A|^2$$

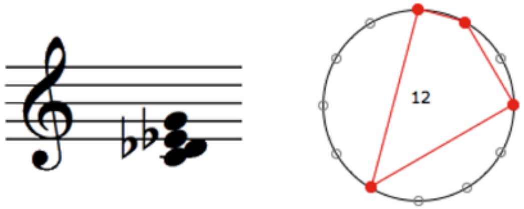
➔ **Relation Z**

$\{0, 1, 4, 6\}_{12}$



iv= [4, 1, 1, 1, 1, 1, 2, 1, 1, 1, 1, 1]

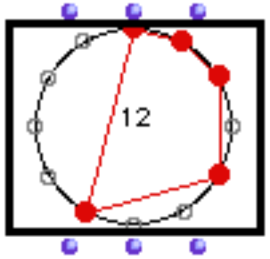
$\{0, 1, 3, 7\}_{12}$



- D. Ghisi, "From Z-relation to homometry: an introduction and some musical examples", Sémnaire MaMuX, 10 décembre 2010
- P. Beauguitte, *Transformée de Fourier discrète et structures musicales*, Master ATIAM, 2011

Une équivalence qui n’est pas induite par une action

La relation Z

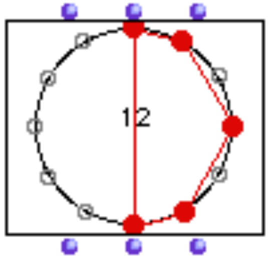


5-30	0,1,4,6,8	121321
5-31	0,1,3,6,9	114112
5-32	0,1,4,6,9	113221
5-33(12)	0,2,4,6,8	040402
5-34(12)	0,2,4,6,9	032221
5-35(12)	0,2,4,7,9	032140
5-Z36	0,1,2,4,7	222121
5-Z37(12)	0,3,4,5,8	212320
5-Z38	0,1,2,5,8	212221
6-1(12)	0,1,2,3,4,5	543210
6-2	0,1,2,3,4,6	443211
6-Z4(12)	0,1,2,4,5,6	432321
6-5	0,1,2,3,6,7	422232
6-Z6(12)	0,1,2,5,6,7	421242
6-7(6)	0,1,2,6,7,8	420243
6-8(12)	0,2,3,4,5,7	343230
6-9	0,1,2,3,5,7	342231
6-Z10	0,1,3,4,5,7	333321
6-Z11	0,1,2,4,5,7	333231
6-Z12	0,1,2,4,6,7	332232
6-Z13(12)	0,1,3,4,6,7	324222

5-Z36 0,1,2,4,7 222121

7-30	0,1,2,4,6,8,9	343542
7-31	0,1,3,4,6,7,9	336333
7-32	0,1,3,4,6,8,9	335442
7-33	0,1,2,4,6,8,10	262623
7-34	0,1,3,4,6,8,10	254442
7-35	0,1,3,5,6,8,10	254361
7-Z36	0,1,2,3,5,6,8	444342
7-Z37	0,1,3,4,5,7,8	434541
7-Z38	0,1,2,4,5,7,8	434442

6-Z36	0,1,2,3,4,7
6-Z37(12)	0,1,2,3,4,8
6-Z38(12)	0,1,2,3,7,8
6-Z39	0,2,3,4,5,8
6-Z40	0,1,2,3,5,8
6-Z41	0,1,2,3,6,8
6-Z42(12)	0,1,2,3,6,9



5-Z12

Ensembles homométriques

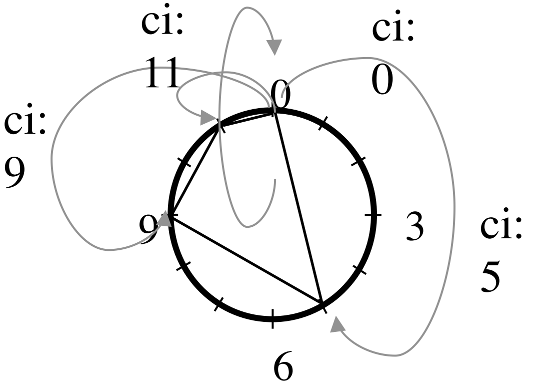
- Le contenu intervallique est équivalent à un produit de convolution de fonctions caractéristiques (Lewin, 1958)

$$\mathcal{F}_A : t \mapsto \sum_{k \in A} e^{-2i\pi kt/c}$$

$$IC_A(k) = (1_A \star 1_{-A})(k)$$

$$1_A * \tilde{1}_B(k) = \sum_i 1_A(i) \times 1_B(i-k) = \sum_{\substack{i \in A \\ i-k \in B}} 1$$

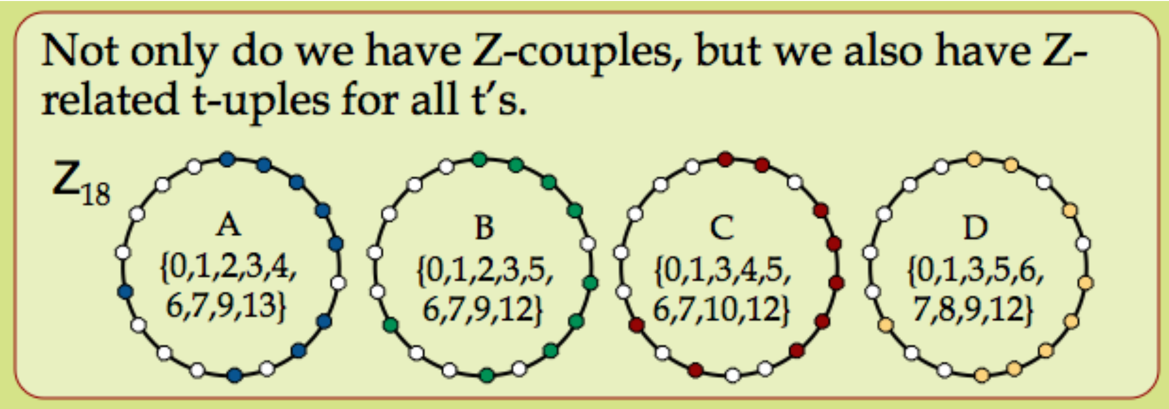
$$\mathcal{F}(1_A * \tilde{1}_B) = \mathcal{F}(1_A) \times \mathcal{F}(\tilde{1}_B)$$


ci: 11
ci: 0
ci: 9
ci: 5
0
3
6
9

$A = \{0, 5, 9, 11\} \Rightarrow IC_A(k) = 1 \ \forall\, k$

$$\mathcal{F}(IC_A) = \mathcal{F}_A \times \mathcal{F}_{-A} = |\mathcal{F}_A|^2$$

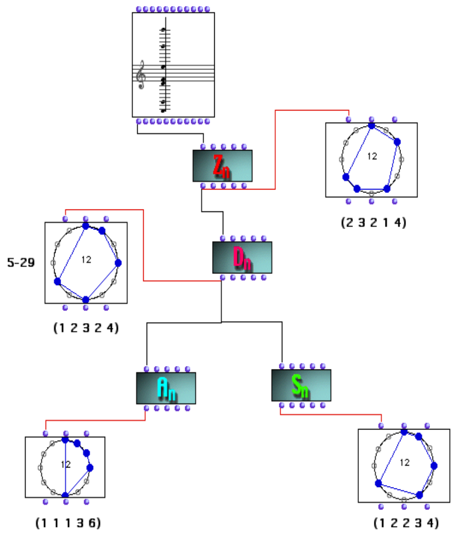
➔ **Relation Z**


Not only do we have Z-couples, but we also have Z-related t-uples for all t's.

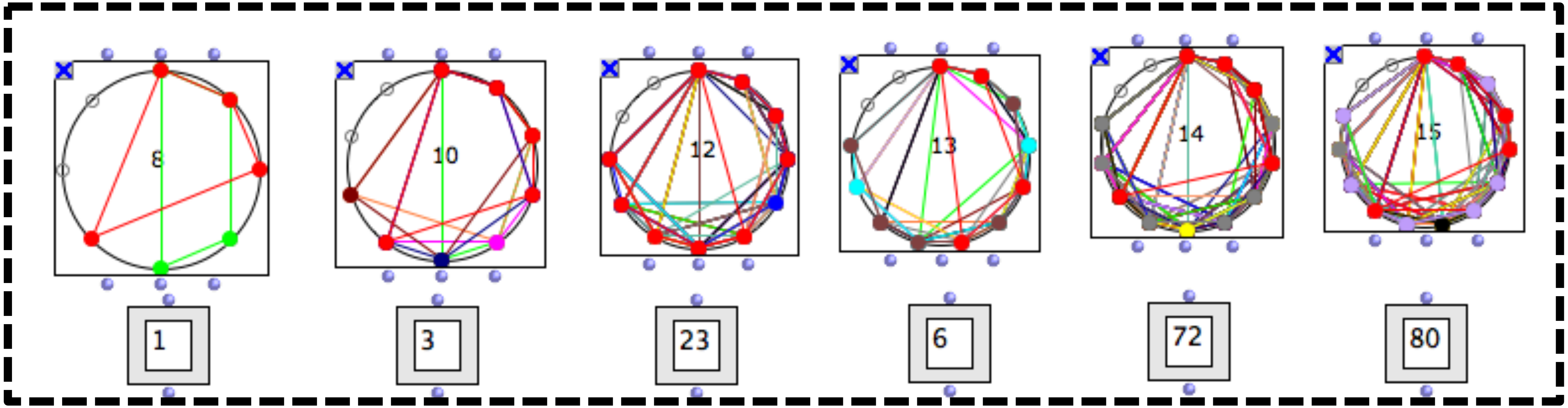
Z_{18}
A
{0,1,2,3,4, 6,7,9,13}
B
{0,1,2,3,5, 6,7,9,12}
C
{0,1,3,4,5, 6,7,10,12}
D
{0,1,3,5,6, 7,8,9,12}

- D. Ghisi, "From Z-relation to homometry: an introduction and some musical examples", Sémnaire MaMuX, 10 décembre 2010
- P. Beauguitte, *Transformée de Fourier discrète et structures musicales*, Master ATIAM, 2011

Classification des ensembles en Z-relations


Zn
(2 3 2 1 4)
5-29
Dn
(1 2 3 2 4)
An
Sn
(1 1 1 3 6)
(1 2 2 3 4)
12

- *Is there a (non-trivial) group action whose orbits are the equivalence classes of homometric sets?*
- *Is there an enumeration formula for homometric sets?*


8
10
12
13
14
15
1
3
23
6
72
80

- John Mandereau, *Étude des ensembles homométriques et leur application en théorie mathématique de la musique et en composition assistée par ordinateur*, Master Thesis, ATIAM, Ircam/Université Paris 6, juin 2009

➔

11-catalogues-z-relation

Relation Z en musique et théorie de l' homométrie

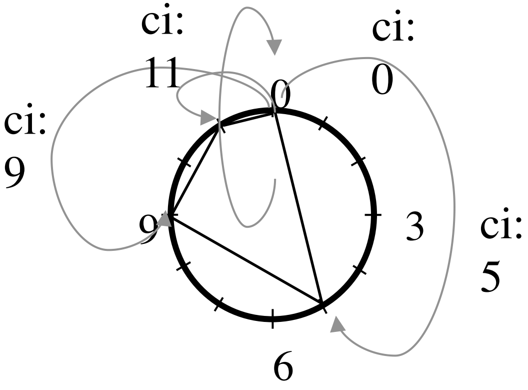
- Le contenu intervallique est équivalent à un produit de convolution de fonctions caractéristiques (Lewin, 1958)

$$\mathcal{F}_A : t \mapsto \sum_{k\in A} e^{-2i\pi kt/c}$$

$$IC_A(k) = (1_A \star 1_{-A})(k)$$

$$1_A * \tilde{1}_B(k) = \sum_i 1_A(i) \times 1_B(i-k) = \sum_{\substack{i\in A\\ i-k\in B}} 1$$

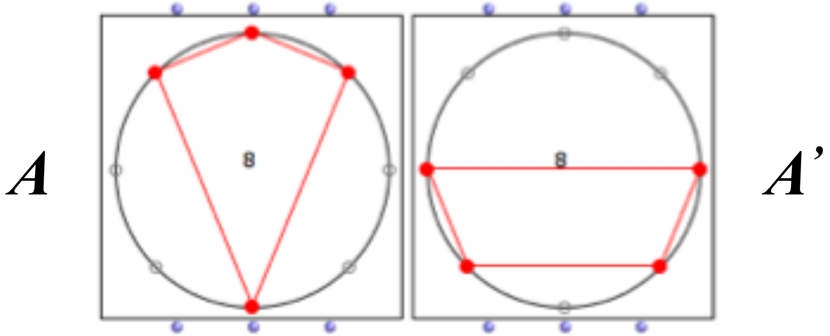
$$\mathcal{F}(1_A * \tilde{1}_B) = \mathcal{F}(1_A) \times \mathcal{F}(\tilde{1}_B)$$



A = {0, 5, 9, 11} ⇒ $IC_A(k) = 1 \ \forall k$

$$\mathcal{F}(IC_A) = \mathcal{F}_A \times \mathcal{F}_{-A} = |\mathcal{F}_A|^2$$

➔ **Relation Z**



➔ $IC_A = IC_{A'}$

FIGURE 1.4 – Plus petite Z-relation non triviale

- P. Beauguitte, *Transformée de Fourier discrète et structures musicales*, Master ATIAM, 2011

Relation Z en musique et théorie de l' homométrie

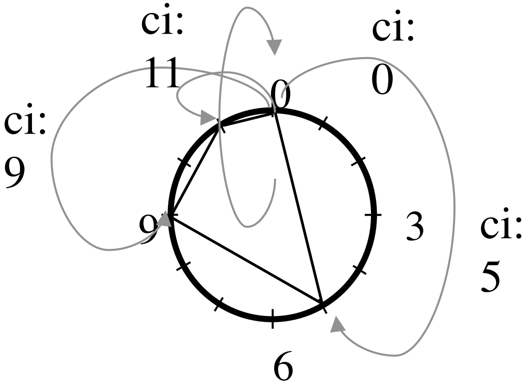
- Le contenu intervallique est équivalent à un produit de convolution de fonctions caractéristiques (Lewin, 1958)

$$\mathcal{F}_A : t \mapsto \sum_{k\in A} e^{-2i\pi kt/c}$$

$$IC_A(k) = (1_A \star 1_{-A})(k)$$

$$1_A * \tilde{1}_B(k) = \sum_i 1_A(i) \times 1_B(i-k) = \sum_{\substack{i\in A\\ i-k\in B}} 1$$

$$\mathcal{F}(1_A * \tilde{1}_B) = \mathcal{F}(1_A) \times \mathcal{F}(\tilde{1}_B)$$

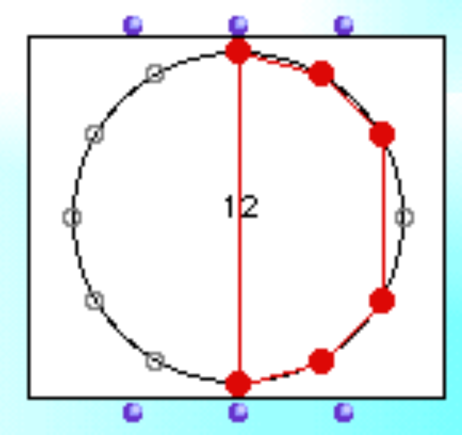


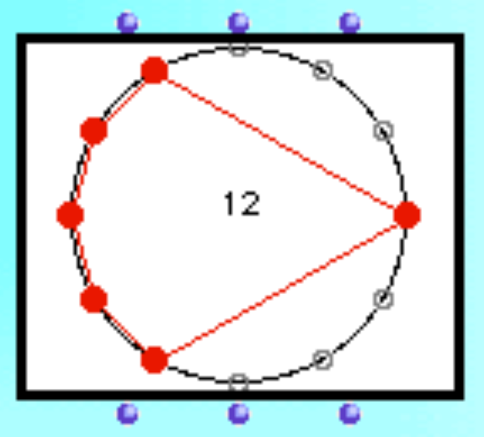
A = {0, 5, 9, 11} ⇒ $IC_A(k) = 1 \ \forall\, k$

$$\mathcal{F}(IC_A) = \mathcal{F}_A \times \mathcal{F}_{-A} = |\mathcal{F}_A|^2$$

➔ **Relation Z**

A





A'

➔ $IC_A = IC_{A'}$

$A' = A^c$

Relation Z en musique et théorie de l' homométrie

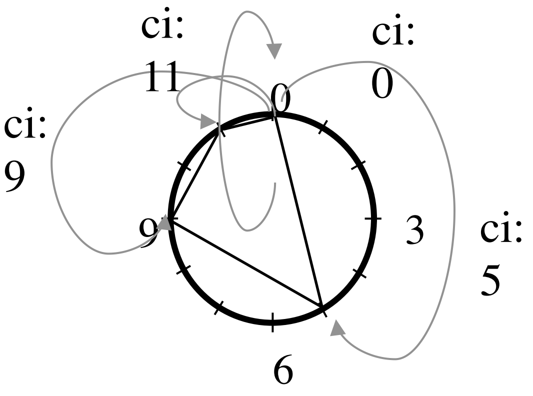
- Le contenu intervallique est équivalent à un produit de convolution de fonctions caractéristiques (Lewin, 1958)

$$\mathcal{F}_A : t \mapsto \sum_{k\in A} e^{-2i\pi kt/c}$$

$$IC_A(k) = (1_A \star 1_{-A})(k)$$

$$1_A * \tilde{1}_B(k) = \sum_{i} 1_A(i) \times 1_B(i-k) = \sum_{\substack{i\in A \\ i-k\in B}} 1$$

$$\mathcal{F}(1_A * \tilde{1}_B) = \mathcal{F}(1_A) \times \mathcal{F}(\tilde{1}_B)$$



A = {0, 5, 9, 11} ⇒ $IC_A(k) = 1\ \forall\ k$

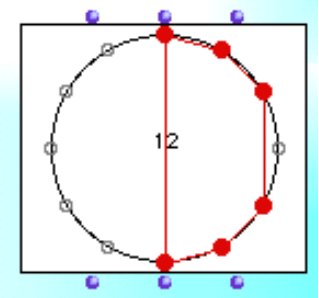
$$\mathcal{F}(IC_A) = \mathcal{F}_A \times \mathcal{F}_{-A} = |\mathcal{F}_A|^2$$

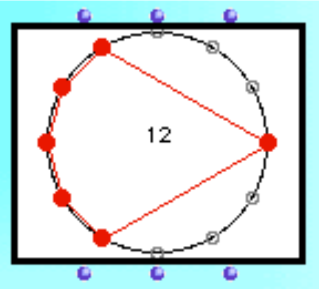
→ **Relation Z**

$$\forall \mathrm{k}\ \mathcal{F}(\mathrm{IC}_{\mathbb{Z}_c\setminus A})(\mathrm{k}) = \mathcal{F}(\mathrm{IC}_A)(\mathrm{k})$$

→ **Théorème de l'hexacorde**

A

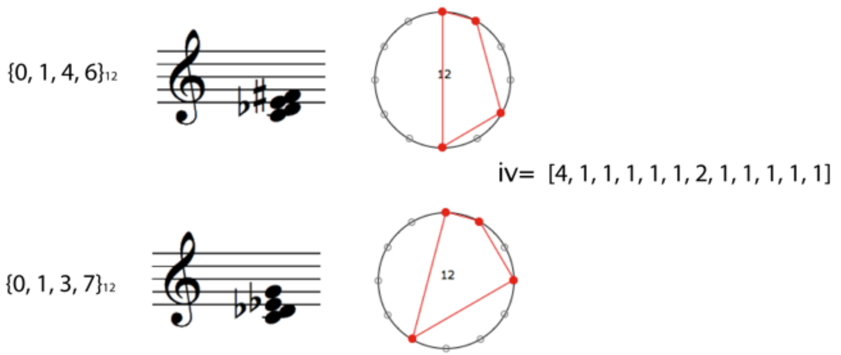




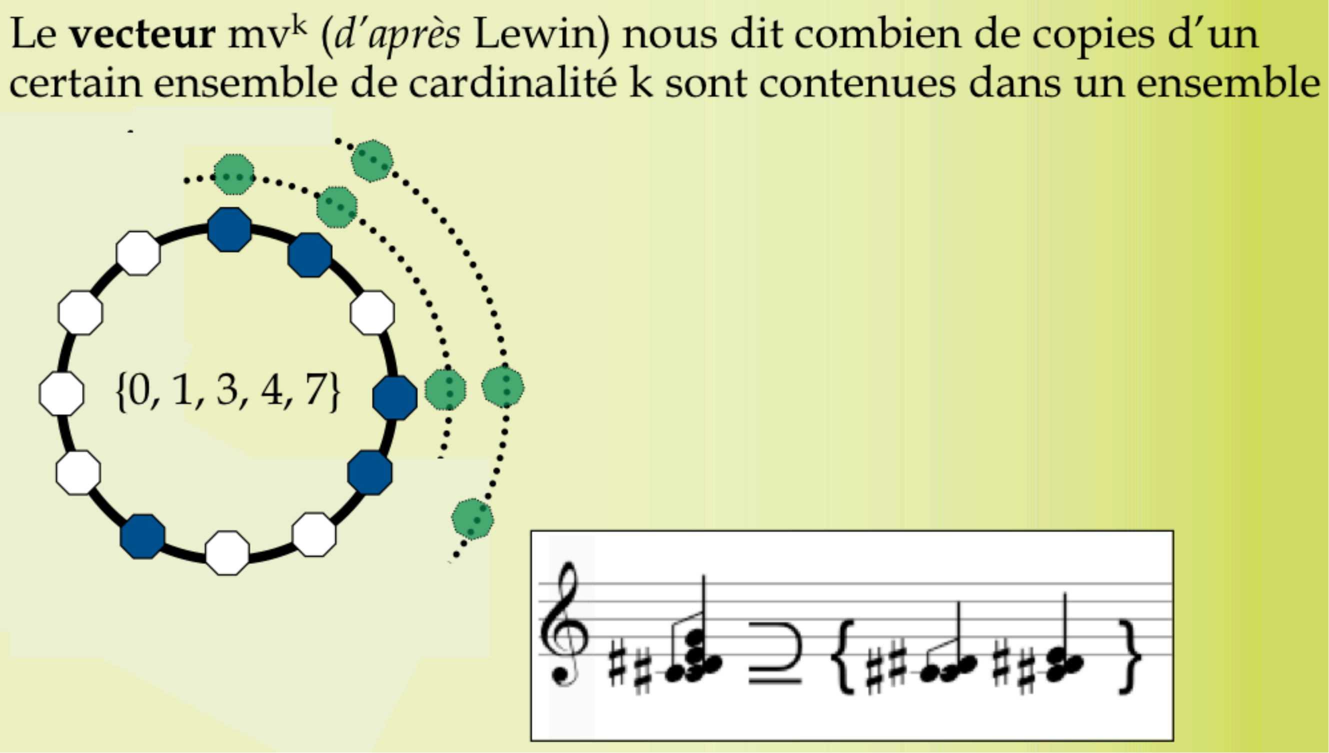
A'

→ $IC_A = IC_{A'}$

Relation Z d’ordre supérieur (vecteur mv^k de Lewin)



➔ **2-homométrie**

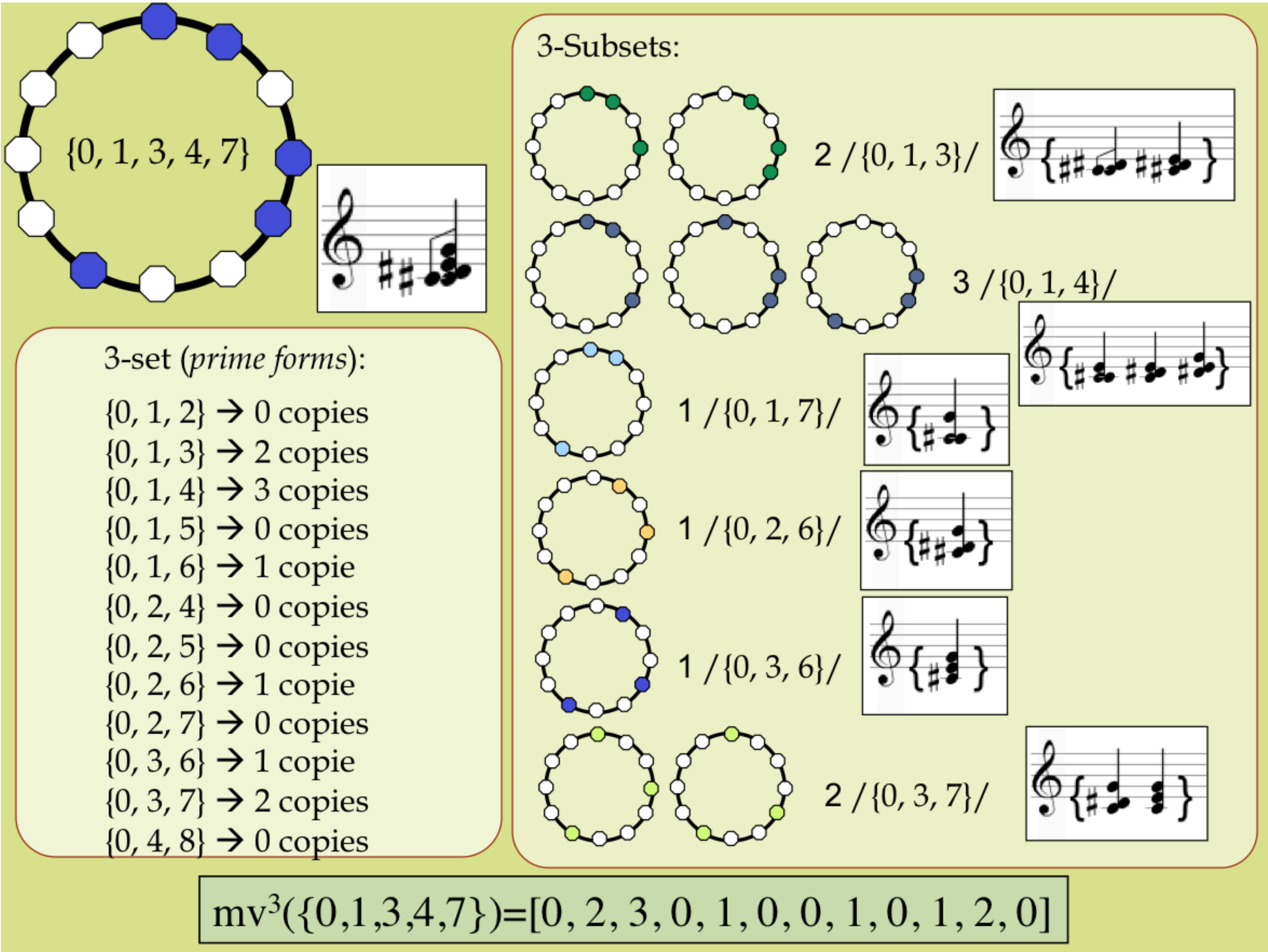


Le **vecteur** mv^k (*d’après* Lewin) nous dit combien de copies d’un certain ensemble de cardinalité k sont contenues dans un ensemble

➔k-homométrie
(k-deck ou k-Deck)

• Daniele Ghisi, "From Z-relation to homometry: an introduction and some musical examples", Sémnaire MaMuX, 10 décembre 2010 [http://repmus.ircam.fr/mamux/saisons/saison10-2010-2011/2010-12-10]

High-order ‘interval’ content: Lewin’s mv^k vector



3-set (*prime forms*):

{0, 1, 2} → 0 copies
{0, 1, 3} → 2 copies
{0, 1, 4} → 3 copies
{0, 1, 5} → 0 copies
{0, 1, 6} → 1 copie
{0, 2, 4} → 0 copies
{0, 2, 5} → 0 copies
{0, 2, 6} → 1 copie
{0, 2, 7} → 0 copies
{0, 3, 6} → 1 copie
{0, 3, 7} → 2 copies
{0, 4, 8} → 0 copies

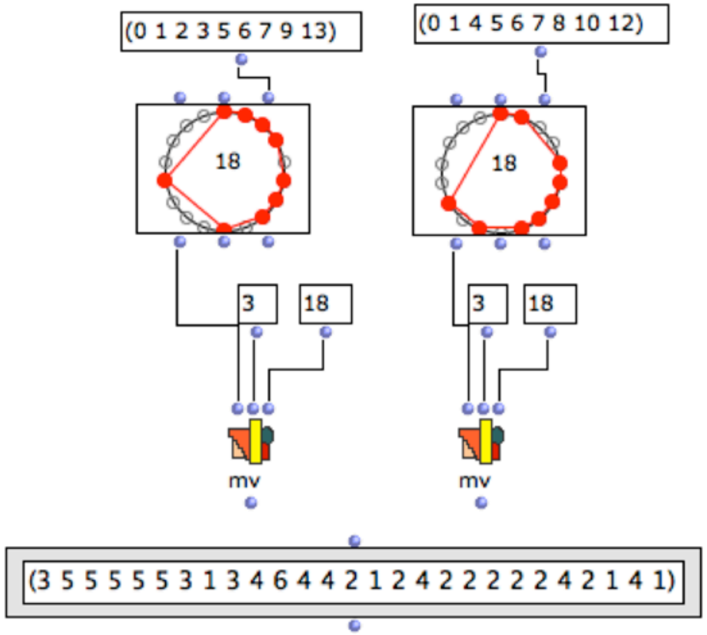
$$mv^3(\{0,1,3,4,7\})=[0, 2, 3, 0, 1, 0, 0, 1, 0, 1, 2, 0]$$

- Daniele Ghisi, "From Z-relation to homometry: an introduction and some musical examples", Séminaire MaMuX, 10 décembre 2010 [http://repmus.ircam.fr/mamux/saisons/saison10-2010-2011/2010-12-10]

➔

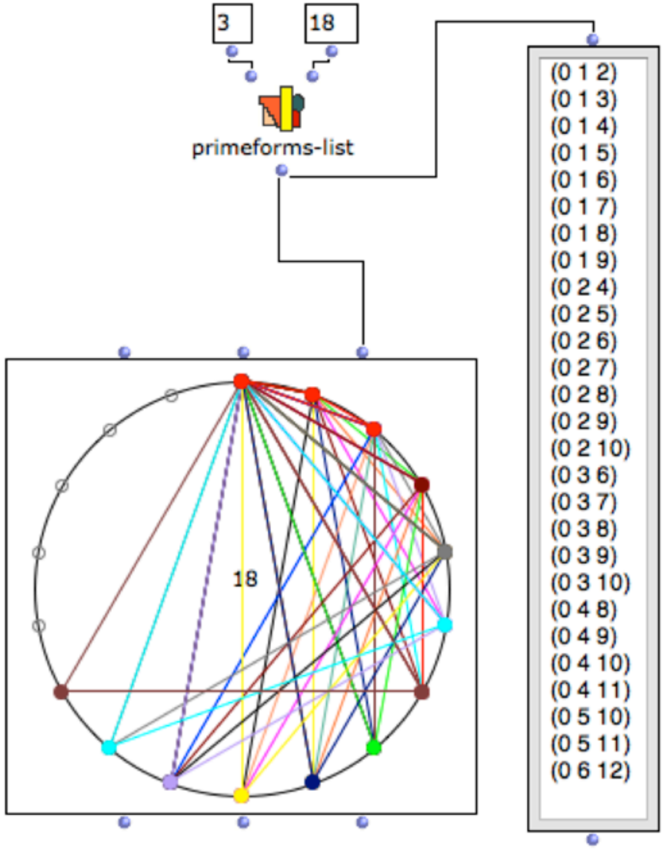
18-mv3-exemple

High-order *Z*-relation and *k*-homometric nesting



```
(0 1 2 3 5 6 7 9 13)
(0 1 4 5 6 7 8 10 12)
(3 5 5 5 5 5 3 1 3 4 6 4 4 2 1 2 4 2 2 2 2 2 4 2 1 4 1)
```

18
18
3
18
3
18
mv
mv



```
(0 1 2)
(0 1 3)
(0 1 4)
(0 1 5)
(0 1 6)
(0 1 7)
(0 1 8)
(0 1 9)
(0 2 4)
(0 2 5)
(0 2 6)
(0 2 7)
(0 2 8)
(0 2 9)
(0 2 10)
(0 3 6)
(0 3 7)
(0 3 8)
(0 3 9)
(0 3 10)
(0 4 8)
(0 4 9)
(0 4 10)
(0 4 11)
(0 5 10)
(0 5 11)
(0 6 12)
```

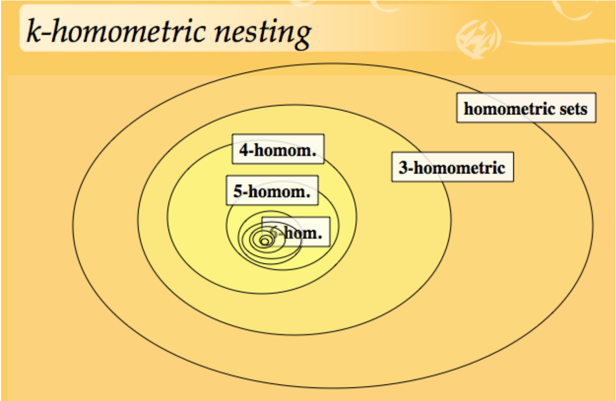
3
18
primeforms-list
18

➔

19-mv3-and-zm-partners

$\mathbf{Z}_{18}$ A = {0, 1, 2, 3, 5, 6, 7, 9, 13}
B = {0, 1, 4, 5, 6, 7, 8, 10, 12}

$mv^3(A) = mv^3(B)$ ➔ 3-homometry (➔ 2-homometry)

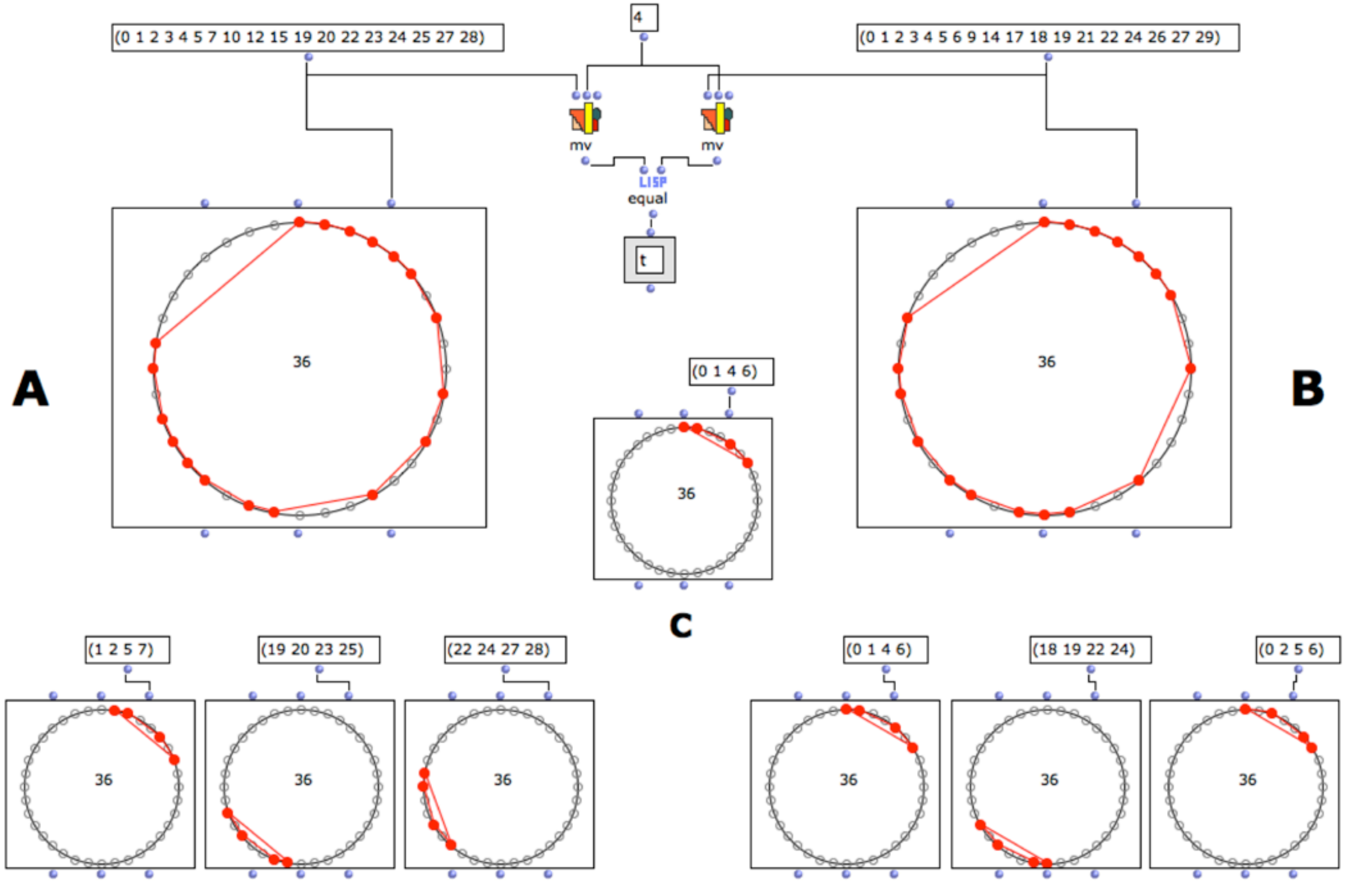


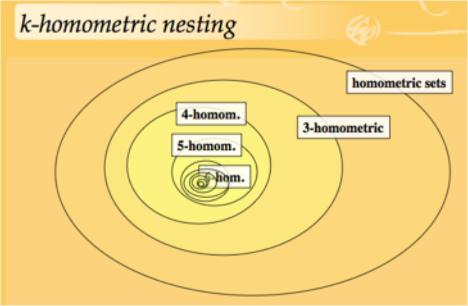
k-homometric nesting
homometric sets
3-homometric
4-homom.
5-homom.
hom.

➔ **Where does this nesting stop?**

Premiers résultats sur la *k*-homométrie

- Mandereau J., D. Ghisi, E. Amiot, M. Andreatta, C. Agon, (2011a), « Z-relation and homometry in musical distributions », *JMM*, vol. 5, n° 2, p. 83-98.
- Mandereau J, D. Ghisi, E. Amiot, M. Andreatta, C. Agon (2011b), « Discrete phase retrieval in musical structures », *JMM*, vol. 5, n° 2, p. 99-116.


(0 1 2 3 4 5 7 10 12 15 19 20 22 23 24 25 27 28)
4
(0 1 2 3 4 5 6 9 14 17 18 19 21 22 24 26 27 29)
mv
mv
LISP
equal
t
A
36
(0 1 4 6)
36
36
B
C
(1 2 5 7)
(19 20 23 25)
(22 24 27 28)
(0 1 4 6)
(18 19 22 24)
(0 2 5 6)
36
36
36
36
36
36


k-homometric nesting
homometric sets
4-homom.
3-homometric
5-homom.
hom.

$\mathbb{Z}_{36}$ A = {0, 1, 2, 3, 4, 5, 7, 10, 12, 15, 19, 20, 22, 23, 24, 25, 27, 28}
B = {0, 1, 2, 3, 4, 5, 6, 9, 14, 17, 18, 19, 21, 22, 24, 26, 27, 29}

$mv^4(A)= mv^4(B)$ ➔ 4-homométrie