From K-Nets to PK-Nets: A Categorical Approach

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1. Introduction

Transformational approaches have a long tradition in formalized music analysis in the American as well as in the European tradition. Since the publication of pioneering work by David Lewin and Guerino Mazzola, this paradigm has become an autonomous field of study making use of more and more sophisticated mathematical tools, ranging from group theory to categorical methods via graph-theoretical constructions (Nolan 2007). Within the transformational approach, Klumpenhouwer networks (henceforth K-nets) are prototypical examples of music-theoretical constructions unifying the three domains we just mentioned, since they provide a description of the inner structure of chords by focusing on the transformations between their elements rather than on the elements themselves. For this reason, K-nets represent a complementary approach to the traditional set-theoretical one with respect to the problem of chord enumeration and classification. One of the main interests for the "working mathemusician" lies in their deep connections with some common constructions used in category theory. In fact, following Mazzola's original intuition on the relevance of the categorical approach to the formalization of musical structures and process, Klumpenhouwer networks seem suitable for music-theoretical investigations making use of category theory, since they are based on concepts (such as isographies) and principles (such as the recursive network construction) which are naturally grasped by the functorial approach. However, as we have suggested elsewhere (Popoff et al. 2015), although K-nets and, more generally, group action-based theoretical constructions, such as Lewin's "Generalized Interval Systems" (GIS), are naturally described in terms of categories and functors, the categorical approach to transformational theory remains relatively marginal with respect to the major trend in the math-music community (Mazzola 2002; Lavelle, unpublished paper; Fiore 2011; Popoff, submitted paper). Following Lewin's (1990) and Klumpenhouwer's (1991) original group-theoretical descriptions, theoretical studies have mostly focused until now on the automorphisms of the T/I group or of the more general T/M affine group (Lewin 1990; Klumpenhouwer 1998). This enables one to define the main notions of positive and negative isographies, notions which can easily be extended by taking into account the affine group on $\mathbb{Z}_{12}$, together with high-order isographies. Since a prominent feature of K-nets is their ability to instantiate an in-depth multi-level model of musical structure, category theory seems nowdays the most suitable mathematical framework to capture this recursive potentiality of the graph-theoretical construction (Mazzola et al. 2006).

From a graph-theoretical perspective, a K-net is a directed graph (also called digraph) where the vertices (or nodes) consist of pitch-classes (or pitch-class sets) and the edges (or arrows) are elements of the T/I group (or, in a more general setting, of the T/M affine group). As an example, we represent two K-nets in Example 1. In addition, these K-nets are $\langle T_2 \rangle$-isographic, in the sense that every edge of the form T_p is sent to T_p and every edge of the form I_p is sent to I_{p+2}. In the general case, what music theorists call the "path consistency condition" (Hook 2007) is nothing else than the composition and associativity law of morphisms, together with the definition of functors in the categorical framework. Moreover, all K-nets correspond to commutative diagrams in category theory, where the term "diagram" is taken here in a naive sense (the technical concept of a diagram will be introduced in Section 5.1.)

$$
\begin{tikzcd}
G_\sharp \arrow[r, "T_2"] \arrow[d, "I_1"'] & B_\flat \arrow[d, "I_7"] \\
F \arrow[r, "T_4"'] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
B_\flat \arrow[r, "T_2"] \arrow[d, "I_3"'] & C \arrow[d, "I_9"] \\
F \arrow[r, "T_4"'] & A
\end{tikzcd}
$$

EXAMPLE 1: TWO K-NETS

Commutative diagrams, together with the notion of isography as an algebraic relation between K-nets which is independent of the content of the nodes, clearly suggests that the categorical approach is the natural one for the study of any kind of networks. Moreover, category theory shows how to go beyond the K-nets commonly used in transformational analysis by defining diagrams which do not necessarily have this property and by extending the isographic relation to different levels, by considering transformations between networks of networks and so on.

Following the very first attempt at formalizing K-nets in terms of limits of diagrams within the framework of denotators (Mazzola et al. 2006), we have recently proposed a categorical construction, called poly-K-nets (henceforth PK-nets) and taking values in **Sets** (Popoff et al. 2015). This construction generalizes the notion of K-nets in various ways by solving the sensitivity problem of the K-nets to the label of their arrows. The concept of isography is in fact highly dependent on the selection of specific transformations, meaning that two isographic K-nets loose this isomorphic relation by eventually changing the musical transformations between their nodes. To see this, we consider the two K-nets of Example 1, but with a different labelling of the edges (Example 2). One can quickly check that these K-nets are not

$$
\begin{tikzcd}
G_\sharp \arrow[r, "T_2"] \arrow[d, "T_9"'] & B_\flat \arrow[d, "T_{11}"] \\
F \arrow[r, "T_4"'] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
B_\flat \arrow[r, "T_2"] \arrow[d, "T_7"'] & C \arrow[d, "T_9"] \\
F \arrow[r, "T_4"'] & A
\end{tikzcd}
$$

EXAMPLE 2: TWO K-NETS, RELABELED

isographic anymore. This asks for a more general setting in which isographic networks remain isographic when the nodes are preserved and the family of transformations between the nodes is changed (which leads to the notion of *complete isography*), but with the possibility of recovering the standard case in traditional K-nets (via the notion of *local isography*).

PK-nets have been introduced as natural extensions of K-nets, also enabling the analyst to compare in a categorical framework digraphs with different cardinalities. This fact may eventually occur in musically interesting analytical situations and is outside of the scope of classical K-nets theory. PK-nets also realize Lewin's intuition that transformational networks do not necessarily have groups as support spaces, since one can define PK-nets in any category. Morphisms of PK-nets clearly show the structural role of natural transformations by which one can generalize the case of isographic K-nets. In particular they enabled us to define K-nets which remain isographic for any choice of transformations between the original pitch-classes (or pitch-class sets). After introducing some preliminaries, we will define PK-nets and provide the first elementary examples of this theoretical construction.

Formally a PK-net K with values in **Sets** consists of a functor R from a category $\mathbf{\Delta}$ to **Sets** defining its form, a functor S from a small category C to **Sets** representing its support in the musical context, a functor F from Δ to C and a natural transformation φ from R to SF showing how the category actions of Δ and C (via R and S) are related. K-nets correspond to the case where R takes its values in singletons. In In a previous work, we have defined and studied the category $\mathbf{PKN_R}$ of PK-nets of form R whose morphisms measure the changes between musical contexts (Popoff et al. 2015); its isomorphisms are related to the notion of complete isography between PK-nets.

Here we introduce other morphisms between PK-nets, called PK-homographies. In particular, two different kinds of PK-homographies are distinguished: *complete homographies*, and *local homographies* between PK-nets with the same form and the same support. These different kinds of homographies, as well as their corresponding PK-isographies, are compared and illustrated by many musical examples, often in the frame of K-nets. We study the category $\mathbf{HoPKN_R}$ of PK-nets and their PK-homographies, as well as its sub-category of complete PK-homographies (which is a quotient of $\mathbf{PKN_R}$); e.g., constructing limits and connected colimits.

Brief indications are given on the construction of higher level PK-nets, for instance by taking squares in $\mathbf{HoPKN_R}$ or defining PK-nets taking their values in categories other than **Sets**, in particular in the category Diag(**Sets**) (which allows re-iteration of the *Diag* operation).

2. Preliminaries

The reader is supposed to be familiar with standard arithmetics in the cyclic group $\mathbb{Z}_{12}$. By an abuse of notation this group will also represent the set of the twelve pitch-classes as well as the twelve interval-classes, with the usual semitone encoding. All arithmetic in $\mathbb{Z}_{12}$ is supposed to be performed modulo 12. Some elementary notions of category theory (functors, natural transformations, limit and colimit constructions [Kan 1958]) are also supposed known. A glossary containing the definitions of the category theory terms used throughout this article is included in Annex 3. In contrast to Mazzola's category-based mathematical music theory (2002), all functors considered in the paper are covariant.

The group T/I is the group of transpositions and inversions of the twelve-pitch-classes. It is generated by the transposition T, whose action on an element x of $\mathbb{Z}_{12}$ is $T\cdot x = x+1$, and by the inversion I, whose action on an element x of $\mathbb{Z}_{12}$ is $I\cdot x = -x$. The T/I group has for presentation $T/I = \langle T,\ I \mid T^{12} = I^2 = ITIT = 1\rangle$, and is thus isomorphic to the dihedral group D_{24} of order 24. The transpositions T^n are usually notated T_n, while the inversions T^nI are usually notated I_n.

One can also consider the larger group of transformations T/M acting on $\mathbb{Z}_{12}$, by adding the transformation M, whose action on an element x of $\mathbb{Z}_{12}$ is $M\cdot x = 5\mathrm{x}$. Similarly, we will notate the elements T^nM as M_n. The group generated by T, I, and M is isomorphic to $\mathbb{Z}_{12} \rtimes \mathrm{Aut}(\mathbb{Z}_{12})$; i.e., the holomorph of $\mathbb{Z}_{12}$. Notice that the T/M group can also be described as the group of affine transformations of $\mathbb{Z}_{12}$, whose elements are functions $f(x) = kx + l$, $k \in \{1, 5, 7, 11\}$, and $l \in \mathbb{Z}_{12}$.

It is a well-known result that the automorphism group of T/I is isomorphic to $\mathbb{Z}_{12} \rtimes \mathrm{Aut}(\mathbb{Z}_{12})$. Notice that $\mathrm{Aut}(T/I)$ is isomorphic to the T/M group. In the most general form, its elements will be notated $\langle k_l \rangle$ in this paper, with $k \in \{1, 5, 7, 11\}$, and $l \in \mathbb{Z}_{12}$, by analogy with the more traditional notation for positive and negative isographies, namely $\langle T_l \rangle = \langle 1_l \rangle$, and $\langle I_l \rangle = \langle 11_l \rangle$, which we will also use. The action of these automorphisms on elements of the T/I group is given by

- $\langle k_l \rangle (T_p) = T_{kp}$, and
- $\langle k_l \rangle (I_p) = I_{kp+1}$.

The automorphism group of T/M is isomorphic to:

$$\mathbb{Z}_2 \times (\mathbb{Z}_{12} \rtimes \mathrm{Aut}(\mathbb{Z}_{12})).$$

Its elements are notated $(u\,\langle k_l \rangle)$, with $u \in \mathbb{Z}_2 = \{1_{\mathbb{Z}_2}, z\}$, $k \in \{1, 5, 7, 11\}$, and $l \in \mathbb{Z}_{12}$. Their action on elements of the T/M group is given by:

- $(1_{\mathbb{Z}_2}, \langle k_l \rangle)(T_p) = T_{kp}$,
- $(1_{\mathbb{Z}_2}, \langle k_l \rangle)(I_p) = T_{kp+l}$,
- $(1_{\mathbb{Z}_2}, \langle k_l \rangle)(M_p) = M_{kp+4l}$,
- $(z, \langle 1_0 \rangle)(T_p) = T_p$,
- $(z, \langle 1_0 \rangle)(I_p) = I_p$,
- $(z, \langle 1_0 \rangle)(M_p) = M_{p+6}$.

3. Defining PK-Nets

We begin by defining a Poly-Klumpenhouwer network (PK-net) as introduced in Popoff et al. (2015).

Definition 1. *Let* **C** *be a category, and S a functor from* **C** *to the category* **Sets**. *Let* Δ *be a small category and R a functor from* Δ *to* **Sets** *with non-void values. A PK-net of form R and of support S is a* 4*-tuple* (R, S, F, φ), *in which*

- *F is a functor from* Δ to **C**,
- *and* φ *is a natural transformation from R to SF.*

The definition of a PK-net is illustrated by the diagram in Example 3. The category Δ serves as the abstract skeleton of the PK-net: as such, its objects and morphisms are abstract entities. Their labelling by musical entities is performed by the functor F from Δ to the category **C**. Traditional transformational music theory commonly relies on a group acting on a given set of objects: the T/I group acting on the set of the twelve pitch-classes, the same T/I group acting simply transitively on

$$\begin{tikzcd}
\Delta \arrow[rr, "F"] \arrow[dr, "R"'] \arrow[rr, phantom, "\phi"', shift right=5] & & \mathbf{C} \arrow[dl, "S"] \\
& \mathbf{Sets} &
\end{tikzcd}$$

Example 3: PK-net definition summary

the set of the 24 major and minor triads, the PLR group acting simply transitively on the same set, etc. From a categorical point of view, the data of a group and its action on a set is equivalent to the data of a functor from a single-object category with invertible morphisms to the category of sets. However, this situation can be further generalized by considering any category $\mathbf{C}$ along with a functor S: $\mathbf{C} \rightarrow \mathbf{Sets}$. This is the point of view we include in the definition of a PK-net. Note that the functor S corresponds to an action of the category $\mathbf{C}$ on the disjoint union of the sets $S(c)$ for the objects $c \in \mathbf{C}$ as shown by Charles Ehresmann (1957). Recent examples in transformational music theory have taken advantage of this more general definition. For example, Noll (2005) considers the action of an eight-element monoid on the set of the twelve pitch-classes, which can be considered as a single-object category $\mathbf{C}$ with eight non-invertible morphisms along with its corresponding functor S: $\mathbf{C} \rightarrow \mathbf{Sets}$, where the image of the only object of $\mathbf{C}$ is the set of the twelve pitch-classes.

The functor F then allows to label each object of Δ by an object of $\mathbf{C}$, and each morphism of by a morphism of $\mathbf{C}$. By explicitly separating the categories Δ and $\mathbf{C}$, we allow for a same PK-net skeleton to be interpreted in different contexts. For example, a given category $\mathbf{C}$ may describe the relationships between pitch-classes, while another category $\mathbf{C}'$ may describe the relationships between time-spans (Lewin 1987). Different functors F: $\mathbf{\Delta} \rightarrow C$ and F': $\mathbf{\Delta} \rightarrow \mathbf{C}'$ will then label the arrows of Δ differently, depending on whether the PK-net describes pitch-classes or time-spans. Two PK-nets may actually be related by different kinds of *morphisms* of PK-nets, some of which have already been described previously (Popoff et al. 2015), while the others will formally be introduced in the next sections.

The objects of Δ do not represent the actual musical elements of a PK-net, which are introduced by the functor R: $\Delta \rightarrow \mathbf{Sets}$. This functor sends each object of Δ to an actual set, which may contain more than a single element, and whose elements abstractly represent the musical objects of study. However, these elements are not yet labelled. In the same way the morphisms of $\mathbf{\Delta}$ represent abstract relationships which are given a concrete meaning by the functor F, these elements are labelled by the natural transformation φ. The elements in the image of S represent musical entities on which the category $\mathbf{C}$ acts, and one would therefore need a way to connect the elements in the image of R with those in the image of S. However, one cannot simply consider a collection of functions between the images of R and the images of S in order to label the musical objects in the PK-net. Indeed, one must make sure that two elements in the images of R which are related by a function $R(f)$, f being a morphism of Δ, actually correspond to two elements in the image of S related by the function $SF(f)$. The purpose of the natural transformation is thus to ensure the coherence of the whole diagram.

Note that the definition of PK-nets encompasses the usual case of K-nets, for which

- the category $\mathbf{C}$ corresponds to the group T/I of transpositions and inversions, considered as a single-object category,
- the group T/I acts on the set $\mathbb{Z}_{12}$ of the twelve pitch-classes in the usual way, which defines a functor S: $T/I \rightarrow$ **Sets**, and
- the functor R is such that for any object $X \in \mathbf{\Delta}$, the set $R(X)$ is a singleton.

In a previous issue of *Perspectives of New Music*, Mazzola and Moreno Andreatta have proposed a categorical formalization of K-nets as elements of the limit of a diagram of sets or of modules (Mazzola et al. 2006). We can compare the notion of PK-net with this notion of K-net (Popoff et al. 2015). A PK-net does not represent a unique K-net, but the set of K-nets associated to SF and a way to "name" them (via $\lim_\varphi$) by the K-sets of R. The PK-net "reduces" to a K-net if $R(X)$ is a singleton for each object X of Δ. The difference between the notion of PK-net exposed here and the formalization of Mazzola and Andreatta lies mainly in the presence of the functor R and the associated natural transformation φ: $R \rightarrow SF$. Their set-oriented formalization introduces a direct functor F: $\mathbf{\Delta} \rightarrow$ **Sets**, wherein each object of $\mathbf{\Delta}$ is mapped to a copy of $\mathbb{Z}_{12}$, and where each arrow is mapped to an affine function from $\mathbb{Z}_{12}$ to $\mathbb{Z}_{12}$. As explained above, by introducing the functor R, we separate the actual musical objects (the elements of the images by R) from their interpretation in the context S. This allows us to change the context through morphisms of PK-nets (not necessarily confining ourselves to the usual T/I group), a possibility which was not described in the work of Mazzola and Andreatta.

Notice that the framework of PK-nets allows much more general networks to be defined, by replacing the category **Sets** by any category **H** to obtain the notion of a P(oly-)K-net in **H**. For example, one could consider the category of ordered sets, or the category of sets and partial functions between them. Some other interesting situations include the case where **H** is a category of presheaves: the networks considered by Mazzola et al. (2006), correspond to PK-nets in

Mod@$\mathbb{Z}$ of the form (R, S, F, φ) where $\mathbf{C} = T/I$ (p. 104) and to PK-nets in a category of presheaves, with F an identity, which are called networks of networks (106–107). In addition, the authors show how to define iterated networks using powerset constructions. In another case, $\mathbf{H}$ could be the category Diag($\mathbf{C}$) of diagrams in a category $\mathbf{C}$, which could be used to define a hierarchy of PK-nets of increasing orders (without recourse to powerset constructions as in Mazzola et al., [2006]).

We now detail several examples showing the advantages of our more general definition of PK-nets

DEMONSTRATION 1. *The functor* R *allows one to consider sets* $R(X)$, $X \in \mathbf{\Delta}$, *whose cardinality* $|R(X)|$ *is greater than* 1.

For example, let $\mathbf{C}$ *be the group T/I, considered as a single-object category, and consider its natural action on the set* $\mathbb{Z}_{12}$ *of the twelve pitch-classes (with the usual semi-tone encoding), which defines a functor* S: $T/I \to$ **Sets**. *Let* Δ *be the interval category (i.e., the category with two objects* X *and* Υ *and precisely one morphism* f: $X \to (\Upsilon)$, *and consider the functor* F: $\mathbf{\Delta} \to T/I$ *which sends* f *to* T_4.

Consider now a functor R: $\mathbf{\Delta} \to$ **Sets** *such that*

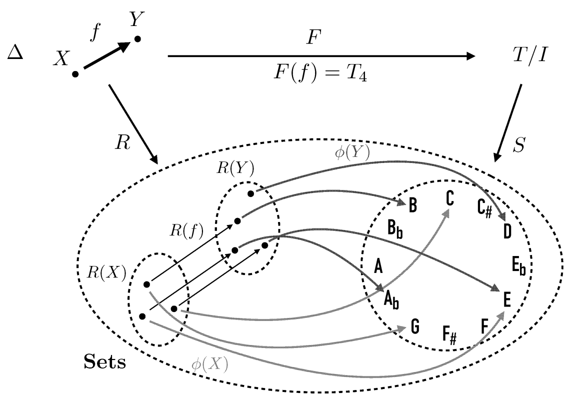
$$R(X) = \{x_1, x_3, x_3,\} \text{ and } R(\Upsilon) = \{y_1, y_2, y_3, y_4,\}, \text{ and such that}$$

$$R(f)(x_i) = y_i, \text{ for } 1 \leq i \leq 3.$$

Consider the natural transformation φ *such that*

$\varphi(x_1) = C$, $\varphi(x_2) = E$, $\varphi(x_3) = G$, *and* $\varphi(y_1) = E$, $\varphi(y_2) = G\sharp$, $\varphi(y_3) = B$, *and* $\varphi(y_4) = D$. *Then* (R, S, F, φ) *is a PK-net of form* R *and support* S *which describes the transposition of the C-major triad to the E-major triad subset of the dominant seventh* E^7 *chord. This functorial construction is shown in Example 4.*

This example should make clear to the reader the main differences between a PK-net and a traditional K-net. Here, the vertices of the PK-net are not restricted to singletons, and may be sets of arbitrary cardinality. In addition, this allows us to use an injective function from one set to the other, thus formalizing the dominant seventh E^7 chord as being built upon a triad (the E-major triad transposed by four semitones) with an added note. As the next examples show, PK-nets have additional advantages.



EXAMPLE 4: DIAGRAM SHOWING THE FUNCTORIAL CONSTRUCTION UNDERLYING THE DEFINITION OF PK-NETS AS APPLIED TO DEMONSTRATION 1

DEMONSTRATION 2. *The definition of PK-nets allows one to consider networks of greater generality than the usual K-nets.*

Consider the category $\mathbf{C} = T/I$ *and the functor* S: $T/I \to$ **Sets** *as in the previous example, and consider the category* Δ *with one single-object* X *and one non-trivial morphism* f: $X \to X$ *such that* $f_2 = id_X$. *Consider now the functor* F: $\mathbf{\Delta} \to T/I$ *which sends* f *to* $I_1 \in T/I$.

If we restrict ourselves to functors R: $\mathbf{\Delta} \to$ **Sets** *such that* $R(X)$ *is a singleton, then there exists no natural transformation* φ: $R \to SF$, *since the equation* $\varphi(x) = 1 - \varphi(x)$ *has no solution in* $\mathbb{Z}_{12}$. *However, it is possible to consider a functor* R *such that* $R(X) = \{x_1, x_2\}$ *with* $R(f)(x_1) = x_2$ *and vice-versa, and a natural transformation* φ *which sends* x_1 *to* 0 *and* x_2 *to* 1. *Then* (R, S, F, φ) *is a valid PK-net of form* R *and support* S.

DEMONSTRATION 3. *In addition to groups, the definition of PK-nets allows one the use of any category* $\mathbf{C}$. *Thus, PK-nets can describe networks of musical objects being transformed by the image morphisms of* $\mathbf{C}$ *through* S.

Consider, for example, the set of the twelve pitch-classes $\mathbb{Z}_{12}$ *(with the usual semi-tone encoding), and the two functions* m, n: $\mathbb{Z}_{12} \to \mathbb{Z}_{12}$ *such that* $m(x) = 3x + 7$ *and* $n(x) = 8x + 4$. *These functions are the generators of the 8-element monoid studied by Noll (2005).*

$$M = \langle m, n \mid m^3 = m, n^3 = n, m^2 n = nm^2, mn^2 = mn, nm^2 = nm \rangle$$

Let Δ be a small category with three elements X, Y, and Z, and only three non-trivial morphisms $f\colon X \to Y$, $g\colon Y \to Z$, and $gf\colon X \to Z$. Let F be the functor from Δ to M such that $F(f) = F(g) = n$ (and $F(gf) = n^2$). Let R be the functor from Δ to **Sets**, *such that $R(X)$, $R(Y)$, and $R(Z)$ are singletons, and let φ be the natural transformation which maps these singletons to $\{D\}$, $\{G\sharp\}$, and $\{G\sharp\}$ in $\mathbb{Z}_{12}$, respectively. Then (R, S, F, φ) is a PK-net of form R and support S, which expresses the successive transformations of the pitch-class D by the monoid element m, giving us an example of a PK-net in a monoid which can be represented in a succinct way with the diagram in Example 5.*

$$D \xrightarrow{\;n\;} G_\sharp \xrightarrow{\;n\;} G_\sharp$$

EXAMPLE 5

4. Homographies of PK-Nets

As stated in the previous section, the category **C** along with the functor $S\colon \mathbf{C} \to \mathbf{Sets}$ can be interpreted as the context in which the PK-net analysis is performed. These contexts can be quite varied (pitch-class transformations, time-span transformations, etc.), and one therefore needs a way to pass from one context to another. We will therefore define in this section the notion of *PK-net homography*. This will allow us to define different categories of PK-nets. From now on, we will restrict ourselves to PK-nets of a given form R. Studying morphisms of PK-nets where the form is allowed to vary could be considered in a future, more general, work.

The first notion we define is that of a lax PK-net homography. Let $K = (R, S, F, \varphi)$ be a PK-net. We define a functor $SF_{|\varphi}\colon \mathbf{\Delta} \to \mathbf{Sets}$ as follows. It associates to an object X of Δ the subset $\mathrm{Im}_\varphi(X)$ of $SF(X)$ image of $\varphi(X)$:

$$SF_{|\varphi}(X) = \mathrm{Im}_\varphi(X) = \{\varphi(X)(a) \mid a \in R(X)\}$$

and associates to $d\colon X \to Y$ in the map $SF_{|\varphi}(d)\colon \mathrm{Im}_\varphi(X) \to \mathrm{Im}_\varphi(Y)$ such that $SF_{|\varphi}(d)(a) = SF(d)(a)$. There is a natural transformation $\varphi_|\colon R \to SF_{|\varphi}$ associating to X the map $\varphi_|(X)\colon R(X) \to SF_{|\varphi}(\mathrm{X})$, which is the restriction of $\varphi(X)$. We then have the following definition of a lax PK-net homography.

DEFINITION 2. *A* **lax PK-homography** (N, ν): $K \rightarrow K'$ *from* K *to* $K' = (R, S', F', \varphi')$ *consists of a functor* N: $\mathbf{C} \rightarrow \mathbf{C'}$ *and a natural transformation* ν: $SF_{|\varphi} \rightarrow S'F'_{|\varphi'}$ *such that* $F' = NF$ *and* $\varphi'_{|} = \nu \circ \varphi_{|}$. *It is called a* **lax PK-isography** *if* N *is an isomorphism and* ν *is an equivalence.*

This allows us to define a first category $\mathbf{LaxHoPKN_R}$ of PK-nets and lax homographies between them.

DEFINITION 3. *For a given functor* R: $\mathbf{\Delta} \rightarrow \mathbf{Sets}$, *the category* $\mathbf{LaxHoPKN_R}$ *has for objects the PK-nets of form* R, *and for morphisms the lax PK-homographies between them.*

The situation simplifies in the case of K-nets, as shown by the following proposition.

PROPOSITION 1. *Let* K *and* K' *be two K-nets such that there exists a functor* N *with* $F' = NF$. *Then there exists a lax PK-homography* (N, ν): $K \rightarrow K'$. *It is a lax PK-isography if* N *is an isomorphism.*

PROOF. *For each object* X *of* Δ, $R(X)$ *is a singleton, and so are* $Im_{\varphi}(X)$ *and* $Im_{\varphi'}(X)$, *so that the natural transformation* ν *is defined canonically.*

There exists a stronger notion of homography, which we simply refer to as a *PK-net homography*, and which is defined as follows.

DEFINITION 4. *A* **PK-homography** (N, ν): $K \rightarrow K'$ *from* K *to* $K' = (R, S, F, \varphi)$ *consists of a functor* N: $\mathbf{C} \rightarrow \mathbf{C'}$ *and a natural transformation* ν: $SF \rightarrow S'F'$ *such that* $F' = NF$ *and* $\varphi' = \nu \circ \varphi$. *A PK-homography is called a* **PK-isography** *if* N *is an isomorphism and* ν *is an equivalence.*

This new definition allows us to define a second category $\mathbf{HoPKN_R}$ of PK-nets and homographies between them.

DEFINITION 5. *For a given functor* R: $\mathbf{\Delta} \rightarrow \mathbf{Sets}$, *the category* $\mathbf{HoPKN_R}$ *has for objects the PK-nets of form* R, *and for morphisms the PK-homographies between them. The composition of two PK-homographies* (N, ν) *and* (N', ν') *is given by* $(N', \nu')(N, \nu) = (N'N, \nu' \circ \nu)$.

As a remark, we can notice that $\mathbf{HoPKN_R}$ has "many" isomorphisms as the following example shows. Let K be a PK-net with the following property: for an object U of $\mathbf{C}$ which is not of the form $F(X)$ for some X in Δ, there is no arrow from U to an object in $F(\Delta)$. For such a PK-net, in which one finds objects U of $\mathbf{C}$ not belonging to $F(\Delta)$, we have an isomorphism $(\mathrm{Id}_{\mathbf{C}}, \mathrm{Id})\colon K \to K'$ where $K' = (R, S, F, \varphi)$ such that S and S' have the same restriction on $F(\Delta)$, and such that $S'(U)$ is reduced to a singleton for the objects U. In this paper, we will often consider PK-nets where the functor F is surjective on objects. Note that this is the usual case in transformational music theory, in particular if $\mathbf{C}$ is a group.

The distinction between PK-homographies and lax PK-homographies is a subtle one, and may not be apparent at first look. The following example will help clarify this distinction.

DEMONSTRATION 4. *Let K and K' be two K-nets with the same functors F and S such that:*

- *the category Δ has two objects X and Y and a unique morphism $d\colon X \to Y$,*
- $\mathbf{C}$ *is the 3-element monoid generated by a single element t and such that $t^3 = t^2$,*
- *the functor F maps d onto t,*
- *the functor* $S\colon \mathbf{C} \to \mathbf{Sets}$ *is associated to the action of* $\mathbf{C}$ *on the set $\{A, B, 0\}$ such that $t(A) = B$, $t(B) = 0$, $t(0) = 0$,*
- *for the K-Net K, $Im_{\varphi}(X) = \{A\}$, $Im_{\varphi}(Y) = \{B\}$, whereas for the K-Net K', $Im_{\varphi'}(X) = \{B\}$, $Im_{\varphi'}(Y) = \{0\}$.*

Then

- *The PK-nets K and K' are lax PK-isographs but are not PK-isographs.*
- *There is a PK-homography $(Id, \nu)\colon K \to K'$, where $\nu(X)(A) = B$, $\nu(X)(B) = 0$, $\nu(X)(0) = 0$, and $\nu(Y)$ is the constant function on 0.*
- *There is no PK-homography from K' to K.*

When dealing with small musical networks, the categories Δ are usually *finite posets*: i.e., small categories such that for any objects X and Y in Δ there is at most one morphism d: $X \to Y$ or d: $Y \to X$ between them. As an illustration, Example 6 presents three categories Δ_2, Δ_3, and Δ_4, frequently used for the construction of networks. Observe that Δ_2 and Δ_3 were used in Demonstrations 2 and 3 in the previous section. If Δ is a poset with a bottom or top element (which is the case for Δ_3 and Δ_4), and if $\mathbf{C}$ is a group, then the distinction between lax PK-isographies and PK-isographies vanishes, as proposition 2 shows.

$$\begin{tikzcd} X \arrow[d, "f_{XY}"] \\ Y \end{tikzcd}$$

(A)

$$\begin{tikzcd} X \arrow[dd, "f_{XZ}"'] \arrow[dr, "f_{XY}"] & \\ & Y \arrow[dl, "f_{YZ}"] \\ Z & \end{tikzcd}$$

(B)

$$\begin{tikzcd} W \arrow[r, "f_{WX}"] \arrow[d, "f_{WY}"'] \arrow[dr, "f_{WZ}"] & X \arrow[d, "f_{XZ}"] \\ Y \arrow[r, "f_{YZ}"'] & Z \end{tikzcd}$$

(C)

EXAMPLE 6: THREE CATEGORIES (A) $\Delta 2$, (B) $\Delta 3$, AND (C) $\Delta 4$ FREQUENTLY USED FOR THE CONSTRUCTION OF PK-NETS IN MUSICAL ANALYSIS

PROPOSITION 2. *Let Δ be a poset with a bottom (or top) element X, and let $\mathbf{C}$ and $\mathbf{C}'$ be two groups with units o and o'. Let F: $\mathbf{\Delta} \to \mathbf{C}$, F': $\mathbf{\Delta} \to \mathbf{C}'$, S: $\mathbf{C} \to$ **Sets**, and S': $\mathbf{C}' \to$ **Sets** be functors. Then*

1. *Given a map μ_X: $S(o) \to S'(o')$, there is one and only one natural transformation μ: $SF \to S'F'$ extending μ_X.*

2. *Let $K = (R, S, F, \varphi)$ and $K' = (R, S', F', \varphi')$ be PK-nets. If (N, ν) is a lax PK-homography from K to K', then there are PK-homographies (N, μ): $K \to K'$ extending (N, ν). If K and K' are lax PK-isographs and if $S(o)$ and $S'(o')$ are isomorphic, then K and K' are also PK-isographs.*

PROOF.

1. *If μ: $SF \to S'F'$ is a natural transformation, it must satisfy the following equation for each object Y of Δ, where f_{XY}: $X \to Y$ is the unique arrow in Δ between them:*

$$\mu_Y = S'F'(f_{XY})\mu_X SF(f_{XY})^{-1}$$

Since **C** *and* **C'** *are groups,* $SF(f_{XY})$ *and* $S'F'(f_{XY})$ *are bijections, and thus the data of the map* μ_X *determines uniquely the natural transformation* μ *given by the above equation.*

2. *Let* (N, ν) *be a lax PK-homography. Then we have the map* ν_X: $\mathrm{Im}(\varphi) \to \mathrm{Im}(\varphi')$. *We can extend it to a map* μ_X: $S(o) \to S'(o')$ *by associating to each element in* $S(o)$ *not in* $\mathrm{Im}(\varphi)$ *an element of* $S'(o')$. *Then we extend* μ_X *into a natural transformation:* $SF \to S'F'$ *by the previous result. As* φ *and* φ' *take their values in their images, the equality* $\varphi'| = \nu \circ \varphi'|$ *extends to the equality* $\varphi' = \mu \circ \varphi'$, *so that* (N, μ) *is a PK-homography from K to K'. If* (N, ν) *is a lax PK-isography, and if* $S(o)$ *is isomorphic to* $S'(o')$, then *the bijection* ν_X *can be extended into a bijection* μ_X: $S(o) \to S'(o')$. *Then, from the previous result, the different* μ_Y *are also bijections, and* μ *is an equivalence. Thus* (N, μ) *is a PK-isography.*

Since the networks considered in transformational music theory usually satisfy the conditions of Propositions 1 and 2, we will therefore only consider the cases of PK-homographies and PK-isographies in the rest of this paper.

In order to give a musical example of a PK-homography which allows a change of context for musical analysis, we consider the following cases.

DEMONSTRATION 5. *Let the category* **C** *be the cyclic group* $G = \mathbb{Z}_{12}$, *generated by an element t of order* 12. *Consider the action of t on the set* $\mathbb{Z}_{12}$ *of the twelve pitch-classes given by* $t \cdot x = x + 1, \forall\, x \in \mathbb{Z}_{12}$. *This defines a functor S:* $G \to$ **Sets**, *which corresponds to the traditional action of* $\mathbb{Z}_{12}$ *by transpositions by semitones. Consider now the action of t on the set* $\mathbb{Z}_{12}$ *of the twelve pitch-classes given by* $t \cdot x = x + 5, \forall\, x \in \mathbb{Z}_{12}$. *This defines another functor S':* $G \to$ **Sets**, *which corresponds to the action of* $\mathbb{Z}_{12}$ *by transpositions by fourths.*

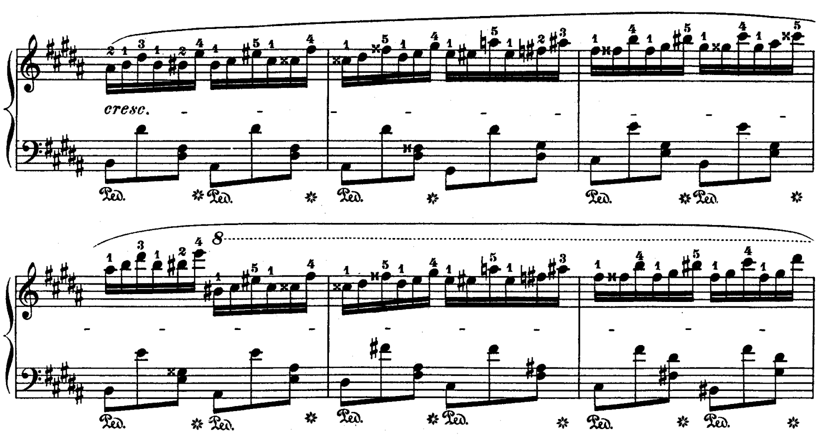
Let N be the automorphism of G which sends $t^p \in G$ *to* t^{5p} *in G,* $\forall\, p \in \{1, \ldots, 12\}$, *and let* ν *be the identity function on the set* $\mathbb{Z}_{12}$. *It is easily checked that* ν *is a natural transformation from S to S'N, which thus extends to a natural transformation from SF to S'F'.*

Let (R, S, F, φ) *be the PK-net built on* Δ_2 *(see Example 6a), where F is the functor from* Δ_2 *to G which sends the non-trivial morphism f:* $X \to Y$ *of* Δ_2 *to* t^{10} *in G, R is the functor from* Δ_2 *to* **Sets** *which* *sends the objects of* Δ_2 *to singletons, and* φ *is the natural transformation which sends* $R(X)$ *to* $\{0\} \subset \mathbb{Z}_{12}$ *and* $R(\Upsilon)$ *to* $\{10\} \subset \mathbb{Z}_{12}$*. This PK-net describes the transformation of* C *to* B♭ *by a transposition of ten semitones.*

By the morphism of PK-nets (N, ν) *introduced above, one obtains a new PK-net* (R', S', F', φ')*, wherein the functor* $F' = NF$ *sends* $f \in \Delta_2$ *to* $t^2 \in G$*, and the natural transformation* $\varphi' = (F\nu) \circ \varphi$ *sends* $R(X)$ *to* $\{0\} \subset \mathbb{Z}_{12}$ *and* $R(\Upsilon)$ *to* $\{10\} \subset \mathbb{Z}_{12}$*. This new PK-net describes the transformation* C *to* B♭ *by a transposition of two fourths.*

DEMONSTRATION 6. *We here give an example of a morphism between a PK-net of beats and a PK-net of pitches. Example 7 shows a passage from the final movement of Chopin's Piano Sonata No. 3, op. 58 in B minor, wherein the initial six-note motive is raised by two semitones every half-bar.*

Let the category **C** *be the infinite cyclic group* $G = \mathbb{Z}$*, generated by an element* t*. Let* $\mathbb{Z}$ *be the set of equidistant beats of a given duration and consider the action of* t *on this set given by* $t \cdot x = x + 1$*,* $\forall x \in \mathbb{Z}$*. This action defines a functor* S*:* $G \rightarrow$ **Sets**. *Let the category* **C'** *be the cyclic group* $G' = \mathbb{Z}_{12}$*, generated by an element* t' *of order* 12. *Consider the set* $U = \{u_i, \mathrm{i} \in \mathbb{Z}_{12}\}$ *of the twelve successive transpositions of the pitch class set* $u_0 = \{10, 11, 0, 3, 4\}$*, and consider the action of* t' *on* U *given by* $t' \cdot u_i = u_{i+1} \pmod{12}$*,* $\forall i \in \mathbb{Z}_{12}$*. This defines a functor* S'*:* $G' \rightarrow$ **Sets**.



EXAMPLE 7: A PASSAGE FROM THE FINAL MOVEMENT OF CHOPIN'S PIANO SONATA NO. 3, OP. 58 IN B MINOR

Let (R, S, F, φ) *be the PK-net wherein:*

- Δ *defines the order of the ordinal number* **4** *(whose objects are labelled* X_i*),*
- F *is the functor from* Δ *to* G *which sends the non-trivial morphisms* $f_{i,\, i+1}\colon X_i \to X_{i+1}$ *of* Δ *to* t *in* G*,*
- R *is the functor from* Δ *to* **Sets** *which sends the objects* X_i *of* Δ *to singletons* $\{x_i\}$*, and*
- φ *is the natural transformation sending* $R(X_i)$ *to* $\{i\} \subset \mathbb{Z}$*.*

This PK-net describes the successive transformations of the initial set by translation of one half-bar in time.

Let (R', S', F', φ') *be the PK-net wherein:*

- F' *is the functor from to* Δ G' *sending the non-trivial morphisms* $f_{i,\, i+1}\colon X_i \to X_{i+1}$ *of* Δ *to* t'^2 *in* G'*,*
- φ' *is the natural transformation which sends* $R(X_i)$ *to* $\{u_{2i}\} \subset U$*.*

This PK-net describes the successive transformations of the initial set $\{10, 11, 0, 3, 4\}$ *by transpositions of two semitones.*

Consider the functor $N\colon G \to G'$ *which sends* t *to* t'^2*, together with the natural transformation* $\nu\colon \mathbb{Z} \to U$ *given by* $\nu(x) = u_{2x\,(\mathrm{mod}\ 12)}$*. This natural transformation extends to a natural transformation from* SF *to* $S'F'$*. The morphism of PK-nets* (N, ν) *thus describes the relation between the translation in time and the transposition in pitch.*

The next two sections are devoted to two specific forms of homographies which are particularly relevant for musical analysis.

5. Complete Homographies

Demonstrations 5 and 6 introduced a particular class of PK-homographies (or PK-isographies) in which the natural transformation $\nu\colon SF \to S'F'$ is of the form $\tilde{\nu}F$, where $\tilde{\nu}$ is a natural transformation from S to $S'N$.

DEFINITION 6. *A PK-homography* (N, v) *between two PK-nets K and K' is called a* **complete homography** *if the natural transformation v can be expressed as* $v = \tilde{v}F$, *where* $\tilde{v}$ *is a natural transformation from S to S'N.*

We note that $\tilde{v}$ is not always unique if F is not surjective on objects. The reason these homographies are called complete is because they do not depend of the choice of functor F. Indeed, given two objects X and Y in **C**, and two elements $x \in S(X)$, $y \in S(Y)$ such that $y = S(f)(x)$ for some morphism $f \in$ **C**, we have

$$\tilde{v}(y) = \tilde{v}(S(f)(x)) = S'N(f)(\tilde{v}(x)),$$

by definition of the natural transformation $\tilde{v}$. In other words, whatever the transformation f in **C** which relates the elements x and y, their images by $\tilde{v}$ are related by the image transformation $N(f)$.

PK-nets and the complete homographies between them form a category, hereby called **CompHoPKN$_\mathbf{R}$**, which is a subcategory of **HoPKN$_\mathbf{R}$**.

DEFINITION 7. *For a given functor* R: $\mathbf{\Delta} \to$ ***Sets****, the category* **CompHoPKN$_\mathbf{R}$** *has the PK-nets of form R for objects, and complete PK-homographies as morphisms between them.*

The following proposition describes the structure of the categories **HoPKN$_\mathbf{R}$** and **CompHoPKN$_\mathbf{R}$**. The proof is available in Annex 1. Note that the construction of limits or colimits of PK-nets represents a universal method for grouping interacting PK-nets into a single network.

PROPOSITION 3. *We have the following results regarding the categories* **HoPKN$_\mathbf{R}$** *and* **CompHoPKN$_\mathbf{R}$**.

1. **CompHoPKN$_\mathbf{R}$** *has all small limits and all small connected colimits, and they are preserved by the insertion functor from the sub-category* **CompHoPKN$_\mathbf{R}$** *to* **HoPKN$_\mathbf{R}$**.

2. **HoPKN$_\mathbf{R}$** *has all small limits and all small connected colimits relative to PK-nets in which the functor F is surjective on objects.*

Note that studying the natural transformations $\tilde{v}$ from S to $S'N$ amounts in fact to studying the morphisms having the functor S as their domain in the category Diag(**Sets**) of diagrams in **Sets**. Let us recall some results about this category.

5.1 CATEGORIES OF DIAGRAMS IN SETS

We begin by recalling the definition of the category Diag(**Sets**).

DEFINITION 8. *The category* Diag(**Sets**) *has:*

- *pairs* $(\mathbf{C}, S)$ *as objects, where* $\mathbf{C}$ *is a small category, and* S *is a functor from* C *to* **Sets**, *and*
- *pairs* $(N, \tilde{v})$ *as morphisms between two objects* $(\mathbf{C}_1, S_1)$ *and* $(\mathbf{C}_2, S_2)$, *where* N *is a functor from* $\mathbf{C}_1$ *to* $\mathbf{C}_2$, *and* $\tilde{v}$ *is a natural transformation from* S_1 *to* S_2N.

This is a complete and co-complete category, which includes a (complete and co-complete) subcategory GrDiag(**Sets**) wherein the categories $\mathbf{C}$ are groups.

DEFINITION 9. *The category* GrDiag(**Sets**) *has*

- *pairs* (G, S) *as objects, where* G *is a group, and* S *is a functor from* G *to* **Sets**, *and*
- *pairs* $(N, \tilde{v})$ *as morphisms between objects* (G_1,S_1) *and* (G_2,S_2), *where* N *is a functor from* G_1 *to* G_2 *(i.e., a group homomorphism), and* $\tilde{v}$ *is a natural transformation from* S_1 *to* S_1N.

In turn, GrDiag(**Sets**) includes a subcategory, which is neither complete nor co-complete, wherein the functors S are representable. Observe that, from a result of Vuza (1988) and Kolman (2004), a GIS is known to be equivalent to a simply transitive group action on a set, which is in turn equivalent to a representable functor from the group to **Sets**. The subcategory of GrDiag(**Sets**) where the functors S are representable is therefore the category of GIS and GIS morphisms studied by Noll and Fiore (2011).

Since we are not limited to GIS, we can work more generally in GrDiag(**Sets**), or even Diag(**Sets**). The definition above allows us to define the automorphism group of an object $(\mathbf{C}, S)$ of this category.

DEFINITION 10. *Let* $(\mathbf{C}, S)$ *be an object of* Diag(**Sets**). *The automorphism group of* $(\mathbf{C}, S)$ *is the set* $Aut((\mathbf{C}, S)) = \{(N, \tilde{v})\}$ *of morphisms from* $(\mathbf{C}, S)$ *to* $(\mathbf{C}, S)$, *where* N *is an isomorphism and* $\tilde{v}$ *an equivalence, equipped with the product given by*

$$(N_2, \tilde{v}_2)(N_1, \tilde{v}_1) = (N_2 N_1, (\tilde{v}_2\ N_1) \circ \tilde{v}_1).$$

In the case of PK-nets having a functor F which is surjective on objects, a complete homography is in one-to-one correspondence to a morphism in Diag(**Sets**), and a complete isography to an isomorphism of Diag(**Sets**). In the particular context of isographies, the question remains to determine the automorphism groups of objects of Diag(**Sets**), or in other words, of functors S: $\mathbf{C} \to \mathbf{Sets}$.

Before analyzing these groups in detail, it should be remarked that topoi and their characteristic morphisms form a great source of PK-net morphisms. It is a well-known result that for any small category $\mathbf{C}$, the category of functors $\mathbf{Sets}^{\mathbf{C}}$ is a topos. The category $\mathbf{Sets}^{\mathbf{C}}$ therefore has a subobject classifier Ω, and for any subobject $A \in \mathbf{Sets}^{\mathbf{C}}$ of an object $B \in \mathbf{Sets}^{\mathbf{C}}$, there exists a *characteristic map* χ_A: $B \to \Omega$. Topoi have found applications in music theory; for example, in the work of Mazzola (2002) and more recently in the work of Noll and Fiore (2011) and Fiore et al. (2013). In the context of PK-nets, the characteristic map can be considered as a morphism of PK-nets. Let (R, S, F, φ) be a PK-net of form R and of support $S \in \mathbf{Sets}^{\mathbf{C}}$. Let A be a subobject of S: this defines a characteristic map χ_A: $S \to \Omega$ which is equivalent to a morphism of PK-nets $(\mathrm{id}_{\mathbf{C}}, \chi_A)$. This morphism thus defines a new PK-net (R, Ω, F, φ').

5.2 The Automorphism Groups of Representable Functors

Given the knowledge of two functors S: $\mathbf{C} \to \mathbf{Sets}$ and S': $\mathbf{C'} \to \mathbf{Sets}$, and a functor N: $\mathbf{C} \to \mathbf{C'}$, there may not always exist a natural transformation $\tilde{\nu}$: $S \to S'N$. The following Proposition gives a sufficient condition on S for the existence of the natural transformation $\tilde{\nu}$. The proof can be found in Popoff et al. (2015).

Proposition 4. *Let S: $\mathbf{C} \to \mathbf{Sets}$, S': $\mathbf{C'} \to \mathbf{Sets}$, and N: $\mathbf{C} \to \mathbf{C'}$ be three functors, where S' has non-empty values. If S is a representable functor, then there exists at least one natural transformation $\tilde{\nu}$: $S \to S'N$.*

An immediate corollary of this result is that, given a PK-net (R, S, F, φ) where S is a Generalized Interval System (GIS), a functor S': $\mathbf{C'} \to \mathbf{Sets}$ (which may not necessarily be representable), and a functor N: $\mathbf{C} \to \mathbf{C'}$, one can always form a new PK-net $(R', S', F' = NF, \varphi')$. Indeed, a GIS is known to be equivalent to a representable functor from the group (as a single-object category) to **Sets**. The previous Proposition can then be used to form the new PK-net. In fact, if $\mathbf{C} = G$ is a group and S is representable, the following proposition allows us to describe exactly the structure of the automorphism group $\mathrm{Aut}((G, S))$.

PROPOSITION 5. *Let (G, S) be an object of GrDiag(**Sets**), where S is a representable functor from G to **Sets**. The automorphism group of (G, S) is isomorphic to $G \rtimes Aut(G)$; i.e., the holomorph of G.*

PROOF. *Let o be the single object of G. Let N be an automorphism of G. We want to determine the natural transformation $\tilde{v}: S \to SN$. Since S is representable, there is a natural isomorphism with a particular Hom functor H, which defines a bijection $X: G \to S(o)$ between the elements of G and the elements of the set. By the Yoneda lemma, the natural transformations from H to SN are in bijection with the elements of the set $S(o)$; i.e., with elements of G (through X). Given a particular element g_0 of G, the natural transformation $\tilde{v}: S \to SN$ is then given by*

$$\tilde{v}(x) = X(M(X^{-1}(x))g_0).$$

Thus the set of morphisms $(N, \tilde{v})$ from (G, S) to (G, S) is in bijection with pairs (N, G), where N is an automorphism of G and g is an element of G. Given two morphisms $(N_1, \tilde{v}_1)$, and $(N_2, \tilde{v}_2)$ corresponding to pairs (N_1, g_1), and (N_2, g_2), the composition of the natural transformations is given by

$$\tilde{v}_2(\tilde{v}_1(x)) = X(N_2(X^{-1}(X(N_1(X^{-1}(x))\,g_1)))\,g_2),$$

which is equal to

$$\tilde{v}_2(\tilde{v}_1(x)) = X(N_2(N_1(X^{-1}(x)g_1))g_2),$$

in turn equal to

$$\tilde{v}_2(\tilde{v}_1(x)) = X((N_2 \circ N_1)(X^{-1}(x))N_2(g_1)g_2),$$

which corresponds to the natural transformation from (G, S) to (G, S) corresponding to the pair $(N_2N_1, N_2(g_1)g_2)$. This last expression is that of a semidirect product of G by Aut(G).

Notice that in the case where $G = T/I$, we obtain the affine maps first discussed by Kolman, and later by Noll and Fiore (2011). As a corollary, observe that G is a normal subgroup in Aut((G, S)). It is the subgroup of Aut((G, S)) corresponding to identity isomorphisms of G. In other words, given two elements $x, y \in S(o)$, which are connected by a transformation g of G, their images $\tilde{v}(x)$ and $\tilde{v}(y)$ are also connected by the same transformation g; i.e., we have the diagram shown in Example 8.

$$\begin{tikzcd} x \arrow[r, "g"] \arrow[d, "\tilde{\nu}"'] & y \arrow[d, "\tilde{\nu}"] \\ \tilde{\nu}(x) \arrow[r, "g"'] & \tilde{\nu}(y) \end{tikzcd}$$

EXAMPLE 8

In the particular case of $G = T/I$ acting on the set of the 24 major and minor triads, this result is better known in terms of dual groups (in the sense of Lewin). In this case, the action of the normal subgroup of Aut((T/I, S)) on the set of triads coincides with that of the PRL group. In the general case, Aut((T/I, S)) is the group of order 1152 isomorphic to $T/I \rtimes T/M$. Given an element $(\langle kl \rangle, g)$ of Aut((T/I, S)), we can calculate explicitly the corresponding natural transformation. This result is given in Annex 2.

Of course, one could consider G to be the PLR group, acting on the set of the 24 major and minor triads, in which case the action of the normal subgroup of the automorphism group Aut((G, S)) would coincide with that of the T/I group. We give an application of this case in the following demonstration.

DEMONSTRATION 7. *Example 9 is a passage from Gesualdo's five voice motet* Deus refugium et virtu. *We are particularly interested in the chord progressions in the four repeated "Pietatis." The first one (A) goes from A major to* C *minor and then to G major. The analysis in the context of the PLR group yields the K-net shown in Example 10a. The second one (B) goes from E major to* C *minor and then to D major. The analysis in the context of the PLR group yields the K-net shown in Example 10b. The third one (C) goes from E major to G minor and then to D major. The analysis in the context of the PLR group yields the K-net shown in Example 10c. The last one (D) goes from B major to G minor and then to A minor. This last chord stands out against the rest of the progressions: one would have expected a A major, had everything been symmetric. The analysis in the context of the PLR group yields the K-net shown in Example 10d.*

The K-nets (A) and (C) are clearly isographic, the isomorphism being the identity on the PLR group, and so are the K-nets (B) and

(D) (omitting the second arrow). As per the discussion above, the natural transformation of the morphism from (A) to (C) corresponds to an element of the normal subgroup of Aut((PLR, S)). In other words, it corresponds to an element of the T/I group, which is here the T_7 transposition.

Moreover, the K-nets (A) and (B) are also isographic, and so are the K-nets (C) and (D). Indeed, consider the automorphism N which sends the generators L and R to R and RLR respectively. Then $N((RL)^6R) = (RL)^7R$ and $N((RL)^4R) = (RL)^5R$. The natural transformation $\tilde{v}\colon S \to SN$ associated with the automorphism N which describes the transformation from (A) to (C) and from (B) to (D) is the bijective function on the set of 24 major and minor triads which sends a major triad n_{Maj} to $\tilde{v}(n_{Maj}) = (n+7)_{Maj}$, and a minor triad n_{Min} to $\tilde{v}(n_{Min}) = n_{Min}$. We thus obtain the diagram in Example 11 of PK-nets and isographies between them.

EXAMPLE 9: A PASSAGE FROM GESUALDO'S FIVE-VOICE MOTET *DEUS REFUGIUM ET VIRTU*

$$A_{Maj} \xrightarrow{\quad PRP=(RL)^6R \quad} C_{Min} \xrightarrow{\quad PLR=(RL)^4R \quad} G_{Maj}$$

EXAMPLE 10A

$$E_{Maj} \xrightarrow{\quad PLP=(RL)^7R \quad} C_{Min} \xrightarrow{\quad P(LR)^2=(RL)^5R \quad} D_{Maj}$$

EXAMPLE 10B

$$E_{Maj} \xrightarrow{\quad PRP=(RL)^6R \quad} G_{Min} \xrightarrow{\quad PLR=(RL)^4R \quad} D_{Maj}$$

EXAMPLE 10C

$$B_{Maj} \xrightarrow{\quad PLP=(RL)^7R \quad} G_{Min} \dashrightarrow (A_{Min})$$

EXAMPLE 10D

$$\begin{array}{ccc}
\begin{array}{l} A_{Maj} \\ \quad\searrow PRP=(RL)^6R \\ \quad C_{Min} \\ \qquad\searrow RLP=(RL)^4R \\ \qquad G_{Maj} \end{array}
& \begin{array}{c} \xrightarrow{\quad N=Id \quad} \\ \left\{\begin{array}{l} \tilde{\nu}(n_{Maj})=(n+7)_{Maj} \\ \tilde{\nu}(n_{Min})=(n+7)_{Min} \end{array}\right. \end{array}
& \begin{array}{l} E_{Maj} \\ \quad\searrow PRP=(RL)^6R \\ \quad G_{Min} \\ \qquad\searrow RLP=(RL)^4R \\ \qquad D_{Maj} \end{array} \\
\begin{array}{r} N(L)=R \\ N(R)=RLR \end{array} \Bigg\downarrow \left\{\begin{array}{l} \tilde{\nu}(n_{Maj})=(n+7)_{Maj} \\ \tilde{\nu}(n_{Min})=n_{Min} \end{array}\right.
& &
\begin{array}{r} N(L)=R \\ N(R)=RLR \end{array} \Bigg\downarrow \left\{\begin{array}{l} \tilde{\nu}(n_{Maj})=(n+7)_{Maj} \\ \tilde{\nu}(n_{Min})=n_{Min} \end{array}\right. \\
\begin{array}{l} E_{Maj} \\ \quad\searrow PLP=(RL)^7R \\ \quad C_{Min} \\ \qquad\searrow P(LR)^2=(RL)^5R \\ \qquad D_{Maj} \end{array}
& \begin{array}{c} \xrightarrow{\quad N=Id \quad} \\ \left\{\begin{array}{l} \tilde{\nu}(n_{Maj})=(n+7)_{Maj} \\ \tilde{\nu}(n_{Min})=(n+7)_{Min} \end{array}\right. \end{array}
& \begin{array}{l} B_{Maj} \\ \quad\searrow PLP=(RL)^7R \\ \quad G_{Min} \\ \qquad\searrow \\ \qquad (A_{Min}) \end{array}
\end{array}$$

EXAMPLE 11

5.3 The automorphism groups of non-representable functors

If the functor S: $\mathbf{C} \to \mathbf{Sets}$ is not representable, the situation is a bit more complex, and there is no answer to the existence of natural transformations $S \to SL$ in the general case. We can, however, work it out in some special cases.

In a first part, we study the automorphism group of the functor $S: T/I \to \mathbf{Sets}$ given by the standard action of the T/I group on the set of pitch-classes $\mathbb{Z}_{12}$. This result has been discussed in Popoff et al. (2015).

Proposition 6. *The automorphism group of $(T/I, S)$, where S is given by the standard action of the T/I group on the set of pitch-classes $\mathbb{Z}_{12}$, is isomorphic to $Aut(T/I) \simeq T/M$.*

Proof. *The proof has been given in Popoff et al. (2015) in the general case of dihedral groups. Let L be an automorphism of T/I: i.e., an element $\langle k_l \rangle \in Aut(T/I)$. Then, if l is odd, there exists no natural transformation v: $S \to SL$. If l is even, there exist exactly two natural transformations: $S \to SL$, given by $v(x) = kx + l/2 + 6$. The set of all natural transformations equipped with the composition given above is isomorphic to the T/M group, which (as seen above), is isomorphic to $Aut(T/I)$.*

Let $Z = \{0, 6\}$ be the additive subgroup of order 2 of the cyclic group $\mathbb{Z}_{12}$. The elements of Aut($(T/I, S)$) can be bijectively identified with pairs $(\langle k_l \rangle, z)$, where k $\in \{1, 5, 7, 11\}$, l is even, and $z \in Z$. Thus Aut($(T/I$, S)) can be described as an extension of $\mathbb{Z}_2$ by $\mathbb{Z}_6 \rtimes D_4$, which involves a non-trivial 2-cocycle η from $(\mathbb{Z}_6 \rtimes D_4) \times (\mathbb{Z}_6 \rtimes D_4)$ to $\mathbb{Z}_2$. More precisely, given two elements $\langle k_l \rangle$ and $\langle k'_{l'} \rangle$ of $\mathbb{Z}_6 \rtimes$ D4, we have

$$\eta(\langle k'_{l'} \rangle, \langle k_l \rangle) = (k'(l/2) + l'/2) - (k'l + l')/2.$$

We now give an application to isographic networks. Consider the following K-net (R, S, F, φ), along with the isomorphism $N = \langle T_2 \rangle$; i.e., the isomorphism N: $T/I \to T/I$ such that $N(T_p) = T_p$, and $N(I_p) = I_{p+2}$. (See Example 12.)

By Proposition 6, there exists two natural transformations from S to SN, given by the functions $\tilde{v}_1(x) = x + 1$ and $\tilde{v}_2(x) = x + 7$, for $x \in \mathbb{Z}_{12}$. By the PK-net morphisms $(N, \tilde{v}_1)$ and $(N, \tilde{v}_2)$ applied to (R, S, F, φ), two new PK-nets are obtained, which are represented in Example 13.

$$
\begin{tikzcd}
G_\sharp \arrow[r, "T_2"] \arrow[d, "I_1"'] & B_\flat \arrow[d, "I_7"] \\
F \arrow[r, "T_4"'] & A
\end{tikzcd}
$$

EXAMPLE 12

$$
\begin{tikzcd}
E_\flat \arrow[r, "T_2"] \arrow[d, "I_3"'] & F \arrow[d, "I_9"] & {} & {} \arrow[l, "N = \langle T_2 \rangle"', "\tilde{\nu}_2(x) = x + 7"] & G_\sharp \arrow[r, "T_2"] \arrow[d, "I_1"'] & B_\flat \arrow[d, "I_7"] & {} \arrow[r, "N = \langle T_2 \rangle", "\tilde{\nu}_1(x) = x + 1"'] & {} & A \arrow[r, "T_2"] \arrow[d, "I_3"'] & B \arrow[d, "I_9"] \\
C \arrow[r, "T_4"'] & E & & & F \arrow[r, "T_4"'] & A & & & F_\sharp \arrow[r, "T_4"'] & B_\flat
\end{tikzcd}
$$

EXAMPLE 13

As $(N, \tilde{\nu}_1)$ and $(N, \tilde{\nu}_2)$ are complete isographies, for any other choice of transformations between the original pitch-classes, these PK-nets remain isographic to the initial one. For example, the pitch-class G♯ can also be transformed into F by the T_9 transformation, which is the case for the pitch-classes A and F♯, or E♭ and C.

In a more general case, we could also consider the automorphism group of the functor S: $T/M \to$ **Sets** given by the standard action of the T/M group on the set of pitch-classes $\mathbb{Z}_{12}$.

PROPOSITION 7. *The automorphism group $(T/M, S)$, where S is given by the standard action of the T/M group on the set of pitch-classes $\mathbb{Z}_{12}$, is isomorphic to T/M.*

PROOF. *The automorphism group of the group T/M is isomorphic to $\mathbb{Z}_2 \times (\mathbb{Z}_{12} \rtimes Aut(\mathbb{Z}_{12}))$, with elements $(z, \langle k_l \rangle)$ (not to be confused with elements of $Aut(T/I, S)$). Let L be the element $(1_{\mathbb{Z}_2}, \langle k_l \rangle)$. For a natural transformation:* S $\to$ *SL, we must check the following conditions, for all $p \in \mathbb{Z}_{12}$:*

- $T_{kp} \cdot \nu(x) = \nu(T_p \cdot x)$,
- $I_{kp+l} \cdot \nu(x) = \nu(I_p \cdot x)$,
- $M_{kp+4l} \cdot \nu(x) = \nu(M_p \cdot x)$,

The first condition imposes $v(x) = kx + v(0)$. *The second condition imposes* $v(0) = l - v(0)$, *while the third one imposes* $v(0) = 4l + 5v(0)$. *One notes that* $v(0) = l - v(0)$ *is sufficient to have* $v(0) = 4l + 5v(0)$. *Therefore, the natural transformation exists iff* $v(0) = l/2$ *or* $v(0) = l/2 + 6$. *The natural transformations associated with* $(1_{\mathbb{Z}_2}, \langle k_l \rangle)$ *are therefore of the form* $v(x) = kx + l/2$ *or* $v(x) = kx + l/2 + 6$. *The subgroup of* $Aut((T/M, S))$ *consisting of elements* $((1_{\mathbb{Z}_2}, \langle k_l \rangle), v)$ *is thus isomorphic to* T/M.

Let L be the element $(z, \langle 1_0 \rangle)$. *For a natural transformation* $v\colon S \rightarrow SL$, *we must check the following conditions, for all* $p \in \mathbb{Z}_{12}$:

- $T_p \cdot v(x) = v(T_p \cdot x)$,
- $I_p \cdot v(x) = v(I_p \cdot x)$,
- $M_{p+6} \cdot v(x) = v(M_p \cdot x)$,

The first condition imposes $v(x) = x + v(0)$. *The second condition imposes* $v(0) = -v(0)$, *while the third condition imposes* $4v(0) + 6 = 0$. *This has no solution in* $\mathbb{Z}_{12}$, *and therefore there exists no natural transformation associated with the automorphism* $(z, \langle 1_0 \rangle)$ *of* T/M.

Thus, $Aut((T/M, S))$ *is isomorphic to* T/M, *and its elements consist of pairs* $((1_{\mathbb{Z}_2}, \langle k_l \rangle), v)$, *with* $v(x) = kx + l/2$ *or* $v(x) = kx + l/2 + 6$.

6 Local Homographies

However, complete isographies do not cover all cases of isographies. Consider for example the two K-nets of Example 14. Clearly, they are isographic by the $\langle T_1 \rangle$ isography. As complete isographies only cover the isographies $\langle k_l \rangle$ where l is even, there is no complete isography which can describe a transformation of the first K-net into the second.

$$\begin{tikzcd}
G_\sharp \arrow[r, "T_2"] \arrow[d, "I_1"'] & B_\flat \arrow[d, "I_7"] \\
F \arrow[r, "T_4"'] & A
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
B \arrow[r, "T_2"] \arrow[d, "I_2"'] & C_\sharp \arrow[d, "I_8"] \\
E_\flat \arrow[r, "T_4"'] & G
\end{tikzcd}$$

EXAMPLE 14

Such a transformation can yet be described by a *local isography*. This notion, along with the more general notion of *local homography*, is defined below.

DEFINITION 11. *A PK-homography* (N, ν) *between two PK-nets K and K′ of identical form R, is called a* ***local PK-homography*** *if they have the same support S and if there is a natural transformation* $\hat{\nu}\colon F \to F' = NF$ *such that* $\nu = S\hat{\nu}$. *It is a* ***local PK-isography*** *if N is an isomorphism and ν is an equivalence.*

A local homography attributes to each object X of Δ a morphism of $\mathbf{C} = \mathbf{C}'$ which describes the local transformation $\hat{\nu}_X$ at this node. Notice that there can be several natural transformations $\hat{\nu}$ giving the same $\hat{\nu} = S\hat{\nu}$: for instance if $\mathbf{C}$ is a group and its action via S is not free.[1] For a given form R, we can define a subcategory of $\mathbf{HoPKN_R}$ of PK-nets and local homographies between them; its isomorphisms are the local PK-isographies.

DEFINITION 12. *For a given functor* $R\colon \mathbf{\Delta} \to \mathbf{Sets}$, *the category* $\mathbf{LocHoPKN_R}$ *has for objects the PK-nets of form R, and for morphisms the local PK-homographies between them.*

We illustrate more clearly the concept of local isography on the the two PK-nets of Example 14. These PK-nets are built on the category $\mathbf{\Delta_4}$, along with the functor R which sends each object of $\mathbf{\Delta_4}$ to a singleton, and the functor $S\colon T/I \to \mathbf{Sets}$. The functor F is such that $F(f_{WX}) = T_2$, $F(f_{XZ}) = I_7$, $F(f_{WY}) = I_1$, and $F(f_{YZ}) = T_4$, while the functor F' is such that $F'(f_{WX}) = T_2$, $F'(f_{XZ}) = I_8$, $F'(f_{WY}) = I_2$, and $F'(f_{YZ}) = T_4$. Let $N = \langle T_1 \rangle$ be the isomorphism of the T/I group which sends T_p to T_p, and I_p to I_{p+1}. We are looking for a natural transformation $\hat{\nu}$ from F to $F' = NF$. Since $\mathbf{C}$ is a group, for any object U of Δ, the component $\hat{\nu}_U$ of $\hat{\nu}$ is an element of T/I. Consider the natural transformation $\hat{\nu}$ such that $\hat{\nu}_W = T_3$, $\hat{\nu}_X = T_3$, $\hat{\nu}_Y = T_{10}$, and $\hat{\nu}_Z = T_{10}$. One can quickly verify that it is indeed a natural transformation: for instance, Example 15, which amounts, in more simple terms, to the equation $2 - (x + 3) = 11 - x = 10 + (1 - x)$, for all x in $\mathbb{Z}_{12}$. Then $(N, S\hat{\nu})$ is the PK-isography, illustrated in Example 16, that describes the transformation of the first network into the second.

$$
\begin{tikzcd}
\mathrm{F(W)} \arrow[r, "I_1"] \arrow[d, "\hat{\nu}_W = T_3"'] & \mathrm{F(Y)} \arrow[d, "\hat{\nu}_Y = T_{10}"] \\
\mathrm{F'(W)} \arrow[r, "I_2"'] & \mathrm{F'(Y)}
\end{tikzcd}
$$

EXAMPLE 15

$$
\begin{tikzcd}
G_\sharp \arrow[r, "T_2"] \arrow[d, "I_1"'] \arrow[rrr, bend left, dashed, red, "T_3"] & B_\flat \arrow[d, "I_7"] \arrow[rrr, bend left, dashed, red, "T_3"] & & B \arrow[r, "T_2"] \arrow[d, "I_2"'] & C_\sharp \arrow[d, "I_8"] \\
F \arrow[r, "T_4"'] \arrow[rrr, bend right, dashed, red, "T_{10}"'] & A \arrow[rrr, bend right, dashed, red, "T_{10}"'] & & E_\flat \arrow[r, "T_4"'] & G
\end{tikzcd}
$$

EXAMPLE 16

The next two propositions study the particular case where Δ is a poset with a bottom element and $\mathbf{C}$ is a group acting freely via S.

PROPOSITION 8. *Let $K = (R, S, F, \varphi)$ be a PK-net where $\boldsymbol{\Delta}$ is a poset with a bottom element X and $\mathbf{C}$ is a group acting freely via S. Then:*

1. *Given a group homomorphism $N\colon \mathbf{C} \to \mathbf{C}$, there is an injection from the set H_N of local PK-homographies $(N, \nu)\colon K \to K' = (R, S, NF, \varphi')$ to $\mathbf{C}$, mapping (N, ν) on $\hat{\nu}_X$.*

2. *If $F' = F$, then the set H of local PK-isographies $(Id_{\mathbf{C}}, \nu)$ with domain K is in bijection with G.*

PROOF.

1. *Let* (N, ν) *be a PK-homography, and let* o *be the single object of* $\mathbf{C}$*. From Proposition 2, it follows that* ν *is uniquely determined by its restriction* ν_X: $S(o) \to S(o)$, *where* X *is the bottom element of* $\mathbf{\Delta}$*. Since the action of* $\mathbf{C}$ *is free, there is at most one element* g *in* $\mathbf{C}$ *such that* $S(g) = \nu_X$, *hence a unique natural transformation* $\hat{\nu}$: $F \to NF$ *such that* $\hat{\nu}_X = G$*. Thus by mapping* (N, ν) *on the element* g *of the group* $\mathbf{C}$*, we define an injective map* $HN \to \mathbf{C}$.

2. *Let us suppose that* $F' = F$*. For each* g *of* $\mathbf{C}$*, we can define a natural transformation* $\hat{\nu}(g)$: $F \to F$ *by* $\hat{\nu}(g)_X = g$ *and* $\hat{\nu}(g)_Y = F(f)gF(f)^{-1}$ *for each object* Y *of* $\mathbf{\Delta}$ *with* f: $X \to Y$*. Then* $(Id_{\mathbf{C}}, S\hat{\nu})$ *is a local PK-homography from* K *to* $K' = (R, S, F, \varphi')$*, where* $\varphi' = S\hat{\nu} \circ \varphi$*. The injective map introduced above, which is associated to* H_N *is a bijection in the case of* $N = Id_{\mathbf{C}}$.

PROPOSITION 9. *Let* $K = (R, S, F, \varphi)$ *be a PK-net where* $\mathbf{\Delta}$ *is a poset with a bottom element* X *and* C *is a group acting freely via* S*. The group* $\mathbf{C}$ *operates on the set* H_N *of local PK-homographies* (N, ν): $K \to K' = (R, S, NF, \varphi')$ *by* $g(N, \nu) = (N, \mu_g)$*, where* $\mu_g = S\hat{\mu}$*, and* $\hat{\mu}_X = g\hat{\nu}_X$.

PROOF. *Let* (N, ν): $K \to K'$ *be a local PK-homography and* $\hat{\nu}$: $F \to F' = NF$ *be the natural transformation such that* $\nu = S\hat{\nu}$*. For each* g *in* $\mathbf{C}$ *we can define another natural transformation* $\hat{\mu}$ *from* F *to* F' *by* $\hat{\mu}_X = g\hat{\nu}_X$*, and* $\hat{\mu}_Y = NF(f)g\hat{\nu}_XF(f)^{-1}$ *for any morphism* f: $X \to Y$ *in* $\mathbf{\Delta}$*. Then we have a local PK-homography* (N, μ_g): $K \to K' = (R, S, F', \varphi')$ *by taking* $\mu_g = S\hat{\mu}$ *and* $\varphi' = \mu_g \circ \varphi$*. The map* $(g, (N, \nu)) \mapsto (N, \mu_g)$ *defines an action of the group* $\mathbf{C}$ *on* H_N.

The above results are valid for any group. We are now going to study some applications in the specific case where G is the T/I group.

6.1 $\langle T_0 \rangle$- AND $\langle T_1 \rangle$-LOCAL ISOGRAPHIES

Among all possible local isographies, those corresponding to $N = \langle T_0 \rangle$ and $N = \langle T_1 \rangle$ are of particular importance. Indeed, from the results of Section 5.3, we have seen that complete isographies with $G = T/I$ are only possible if the isomorphism $N = \langle k_l \rangle$ is such that l is even. If one now considers isomorphisms of the form $N = \langle k_{2p+1} \rangle$, then there exists no automorphism of the functor S: $T/I \to$ **Sets** which would yield a complete isography. However, we have

$$N = \langle k_{2p+1} \rangle = N_2 N_1 = \langle 1_1 \rangle \langle k_{2p} \rangle.$$

Thus, we may consider the successive isographic transformations of a PK-net by a complete isography corresponding to N_1, followed by a local $\langle T_0 \rangle$ isography corresponding to N_2. The way these isographies may be composed is summed-up in Example 17.[2]

If one considers isomorphisms of the form $N = \langle k_{2p} \rangle$, we have seen that automorphisms of the functor S: $T/I \to$ **Sets** exist. The associated natural transformation is a linear function of the form $\hat{v}(x) = kx + p$ or $\hat{v}(x) = kx + p + 6$. However, not all PK-net isographies can be described with such a linear mapping, as we saw Example 14. Nevertheless, as in the previous case, we may decompose an isomorphism $N = \langle k_{2p} \rangle$ as

$$N = \langle k_{2p} \rangle = N_2 N_1 = \langle 1_0 \rangle \langle k_{2p} \rangle.$$

Then, we may consider the successive transformations of a PK-net by a complete isography corresponding to N_1, followed by a local $\langle T_0 \rangle$ isography corresponding to N_2. Consider for example the two K-nets illustrated in Example 18.

These K-nets are $\langle T_2 \rangle$-isographic, but it can quickly be checked that none of the complete isographies presented in the previous section can describe the transformation of the first K-net into the second. However, the desired transformation may be described as the composition of a complete isography, followed by a local one, as shown in Example 19.

$$\begin{tikzcd}
\Delta \arrow[rr, "F"] \arrow[d, dash, dashed] & & T/I \arrow[rr, "S"] \arrow[d, dash, dashed] \arrow[dr, phantom, "\Downarrow \tilde{\nu}_1"] & & \mathbf{Sets} \arrow[d, dash, dashed] \\
\Delta \arrow[rr, "F"] \arrow[d, dash, dashed] \arrow[drr, phantom, "\Downarrow \hat{\nu}_2"] & & T/I \arrow[r, "N_1"] & T/I \arrow[r, "S"] \arrow[d, dash, dashed] & \mathbf{Sets} \arrow[d, dash, dashed] \\
\Delta \arrow[r, "F"] & T/I \arrow[r, "N_1"] & T/I \arrow[r, "N_2"] & T/I \arrow[r, "S"] & \mathbf{Sets}
\end{tikzcd}$$

EXAMPLE 17

$$\begin{tikzcd} G_\sharp \arrow[r, "T_2"] \arrow[d, "I_1"'] & B_\flat \arrow[d, "I_7"] \\ F \arrow[r, "T_4"'] & A \end{tikzcd} \qquad\qquad \begin{tikzcd} B \arrow[r, "T_2"] \arrow[d, "I_3"'] & C_\sharp \arrow[d, "I_9"] \\ E \arrow[r, "T_4"'] & G_\sharp \end{tikzcd}$$

EXAMPLE 18

$$\begin{tikzcd} G_\sharp \arrow[r, "T_2"] \arrow[d, "I_1"'] & B_\flat \arrow[d, "I_7"] \\ F \arrow[r, "T_4"'] & A \end{tikzcd} \quad \xrightarrow[\tilde{\nu}_1(x) = x + 7]{N_1 = \langle T_2 \rangle} \quad \begin{tikzcd} E_\flat \arrow[r, "T_2"] \arrow[d, "I_3"'] \arrow[rr, bend left, dashed, red, "T_8"] & F \arrow[d, "I_9"] \arrow[rr, bend left, dashed, red, "T_8"] \arrow[dr, phantom, "\xrightarrow{N_2 = \langle T_0 \rangle}"] & B \arrow[r, "T_2"] \arrow[d, "I_3"'] & C_\sharp \arrow[d, "I_9"] \\ C \arrow[r, "T_4"'] \arrow[rr, bend right, dashed, red, "T_4"'] & E \arrow[rr, bend right, dashed, red, "T_4"'] & E \arrow[r, "T_4"'] & G_\sharp \end{tikzcd}$$

EXAMPLE 19

Obviously, one can also transform Example 18's first PK-net directly by using an appropriate local $\langle T_2 \rangle$-isography. The choice of one transformation over the other to describe the transformation of PK-nets depends on the musical context of the analysis.

In conclusion, local isogaphies corresponding to the isomorphisms $\langle T_0 \rangle$ and $\langle T_1 \rangle$ are of particular interest to us. We now give a few applications to post-tonal music.

DEMONSTRATION 8. *A particularly symmetric example of* $\langle T_0 \rangle$ *transformations can be found in Webern's* Three Little Pieces for Cello and Piano*, Op. 11/2, at bars 4–5, represented in Example 20. Each three-note segment can be considered as a K-net, according to the diagrams shown in Example 21.*

These K-nets are built on the category $\mathbf{\Delta_3}$*, and since they are all* $\langle T_0 \rangle$*-isographic, the functor* F *is identical for all three K-nets, with* $F(f_{XY}) = I_8$, $F(f_{YZ}) = I_9$, *and* $F(f_{XZ}) = T_1$. *We consider the natural transformation* $\hat{\nu}$: $F \rightarrow F$ *such that* $\hat{\nu}_X = T_4$, $\hat{\nu}_Y = T_8$, *and* $\hat{\nu}_Z = T_4$. *Applying repeatedly the local isography* $(Id_{T/I}, S\hat{\nu})$ *on the first PK-net, we thus get the next* $\langle T_0 \rangle$*-isographic networks, as shown in Example 22.*

EXAMPLE 20: WEBERN, OP. 11/2, BARS 4–5

$$
\begin{tikzcd}
A \arrow[dd, "T_1"'] \arrow[dr, "I_8"] & \\
& B \arrow[dl, "I_9"] \\
B_\flat &
\end{tikzcd}
\qquad
\begin{tikzcd}
C_\sharp \arrow[dd, "T_1"'] \arrow[dr, "I_8"] & \\
& G \arrow[dl, "I_9"] \\
D &
\end{tikzcd}
\qquad
\begin{tikzcd}
F \arrow[dd, "T_1"'] \arrow[dr, "I_8"] & \\
& E_\flat \arrow[dl, "I_9"] \\
F_\sharp &
\end{tikzcd}
$$

EXAMPLE 21

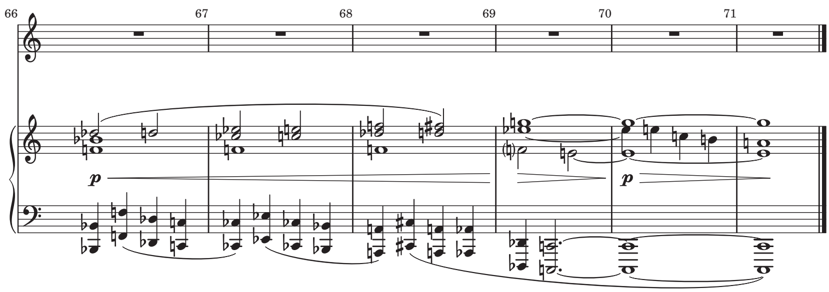
$$
\begin{tikzcd}
A \arrow[dd, "T_1"'] \arrow[dr, "I_8"] \arrow[rr, bend left, dashed, red, "T_4"] & & C_\sharp \arrow[dd, "T_1"'] \arrow[dr, "I_8"] \arrow[rr, bend left, dashed, red, "T_4"] & & F \arrow[dd, "T_1"'] \arrow[dr, "I_8"] & \\
& B \arrow[dl, "I_9"] \arrow[rr, bend right, dashed, red, "T_8"'] & & G \arrow[dl, "I_9"] \arrow[rr, bend right, dashed, red, "T_8"'] & & E_\flat \arrow[dl, "I_9"] \\
B_\flat \arrow[rr, bend right, dashed, red, "T_4"'] & & D \arrow[rr, bend right, dashed, red, "T_4"'] & & F_\sharp &
\end{tikzcd}
$$

EXAMPLE 22

As the chords {A, B, B♭}, {C♯, G, D}, and {F, E♭, F♯} do not belong to the same set class, Demonstration 8 demonstrates the advantage of local isographies for describing the transformations of these networks, over more traditional analyses of pitch class sets using the T/I or the PLR group, which are restricted to members of the same set class.

DEMONSTRATION 9. *An example of $\langle T_1 \rangle$-isographic networks is given at the end of Schoenberg's Op. 14/2. The chords on the right hand in bars 66 to 68 (see Example 23) are described by the $\langle T_1 \rangle$-isographic networks shown in Example 24.*

We consider the same category $\mathbf{\Delta_3}$ as in the previous Demonstration 8, and the natural transformation $\hat{v}$: $F \rightarrow NF$, with $N = \langle T_1 \rangle$, such that $\hat{v}_X = T_1$, $\hat{v}_Y = T_0$, and $\hat{v}_Z = T_1$. Then, by applying repeatedly the local isography $(N, S\hat{v})$ on the first PK-net, we get the next $\langle T_1 \rangle$-isographic networks, as shown in Example 25.



EXAMPLE 23: SCHOENBERG, OP. 14/2, BARS 66–71

$$\begin{tikzcd} D \arrow[dd, "T_8"'] \arrow[dr, "I_7"] & \\ & F \arrow[dl, "I_3"] \\ B_\flat & \end{tikzcd} \quad \begin{tikzcd} E_\flat \arrow[dd, "T_8"'] \arrow[dr, "I_8"] & \\ & F \arrow[dl, "I_4"] \\ B & \end{tikzcd} \quad \begin{tikzcd} E \arrow[dd, "T_8"'] \arrow[dr, "I_9"] & \\ & F \arrow[dl, "I_5"] \\ C & \end{tikzcd} \quad \begin{tikzcd} F \arrow[dd, "T_8"'] \arrow[dr, "I_{10}"] & \\ & F \arrow[dl, "I_6"] \\ C_\sharp & \end{tikzcd} \quad \begin{tikzcd} F_\sharp \arrow[dd, "T_8"'] \arrow[dr, "I_{11}"] & \\ & F \arrow[dl, "I_7"] \\ D & \end{tikzcd} \quad \begin{tikzcd} G \arrow[dd, "T_8"'] \arrow[dr, "I_0"] & \\ & F \arrow[dl, "I_8"] \\ E_\flat & \end{tikzcd}$$

EXAMPLE 24

$$\begin{tikzcd}
D \arrow[dd, "T_8"'] \arrow[dr, "I_7"] \arrow[rr, dashed, red, bend left, "T_1"] & & E_\flat \arrow[dd, "T_8"'] \arrow[dr, "I_8"] \arrow[rr, dashed, red, bend left, "T_1"] & & E \arrow[dd, "T_8"'] \arrow[dr, "I_9"] \arrow[rr, dashed, red, bend left, "T_1"] & & F \arrow[dd, "T_8"'] \arrow[dr, "I_{10}"] \arrow[rr, dashed, red, bend left, "T_1"] & & F_\sharp \arrow[dd, "T_8"'] \arrow[dr, "I_{11}"] \arrow[rr, dashed, red, bend left, "T_1"] & & G \arrow[dd, "T_8"'] \arrow[dr, "I_0"] & \\
& F \arrow[dl, "I_3"] \arrow[rr, dashed, red, bend left, "T_0"] & & F \arrow[dl, "I_4"] \arrow[rr, dashed, red, bend left, "T_0"] & & F \arrow[dl, "I_5"] \arrow[rr, dashed, red, bend left, "T_0"] & & F \arrow[dl, "I_6"] \arrow[rr, dashed, red, bend left, "T_0"] & & F \arrow[dl, "I_7"] \arrow[rr, dashed, red, bend left, "T_0"] & & F \arrow[dl, "I_8"] \\
B_\flat \arrow[rr, dashed, red, bend right, "T_1"'] & & B \arrow[rr, dashed, red, bend right, "T_1"'] & & C \arrow[rr, dashed, red, bend right, "T_1"'] & & C_\sharp \arrow[rr, dashed, red, bend right, "T_1"'] & & D \arrow[rr, dashed, red, bend right, "T_1"'] & & E_\flat &
\end{tikzcd}$$

EXAMPLE 25

7 Construction of Higher Order PK-Homographies and PK-Nets

7.1 Transformations of PK-Net Transformations

Following Lewin (2004), hyper-isographies have also become standard objects in K-net theory, enabling the music analyst to describe transformations of networks of K-nets at different levels of recursion. Informally speaking, these hyper-isographies may be viewed in categorical terms as morphisms in a higher category; i.e., morphisms of morphisms. We present here a framework for hyper-homographies based on the notion of double categories, the definition of which we recall below.

DEFINITION 13. *A* **double category** *is a triple* $(D, \circ, \star)$ *satisfying*

- *D is a set, whose elements are called squares,*
- *the pair $(D, \circ)$ is a category, the law $\circ$ being called the horizontal composition,*
- *the pair $(D, \star)$ is a category, the law $\star$ being called the vertical composition,*
- *the two laws $\circ$ and $\star$ satisfy the distributivity axiom*

$$(\delta \circ \gamma) \star (\beta \circ \alpha) = (\delta \star \beta) \circ (\gamma \star \alpha),$$

corresponding to the diagram shown in Example 26, iff the four composites $(\delta \circ \gamma)$, $(\beta \circ \alpha)$, $(\delta \star \beta)$, and $(\gamma \star \alpha)$ are defined.

$$\begin{tikzcd}
{} \arrow[r] & {} \arrow[r] & {} \\
{} \arrow[u] \arrow[r] \arrow[ur, phantom, "\delta"] & {} \arrow[u] \arrow[r] \arrow[ur, phantom, "\gamma"] & {} \arrow[u] \\
{} \arrow[u] \arrow[r] \arrow[ur, phantom, "\beta"] & {} \arrow[u] \arrow[r] \arrow[ur, phantom, "\alpha"] & {} \arrow[u]
\end{tikzcd}$$

EXAMPLE 26

It is a straightforward result that, for a given 1-category **C**, we obtain a double category **Sq**(**C**) whose squares are the commutative squares in **C**. Applying this result to our case, the 1-category $\mathbf{HoPKN_R}$ gives rise to a double category $\mathbf{Sq}(\mathbf{HoPKN_R})$ as described in Definition 14.

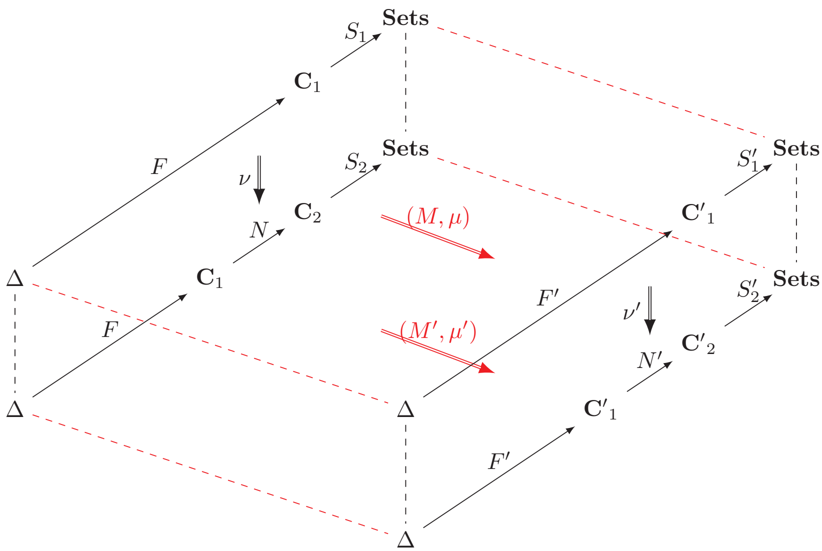
DEFINITION 14. *The double category* $\mathbf{Sq}(\mathbf{HoPKN_R})$ *has its squares defined by four PK-homographies,*

- $(N, \nu)\colon K_1 \to K_2$,
- $(N', \nu')\colon K'_1 \to K'_2$,
- $(M, \mu)\colon K_1 \to K'_1$,
- $(M', \mu')\colon K_2 \to K'_2$,

such that we have the commutative diagram shown in Example 27; i.e., we have $(M', \mu')(N, \nu) = (N', \nu')(M, \mu)$.

$$\begin{tikzcd}
K_1 \arrow[r, "{(M,\mu)}"] \arrow[d, "{(N,\nu)}"'] & K'_1 \arrow[d, "{(N',\nu')}"] \\
K_2 \arrow[r, "{(M',\mu')}"'] & K'_2
\end{tikzcd}$$

EXAMPLE 27



EXAMPLE 28: A SQUARE IN THE DOUBLE CATEGORY $\mathbf{SQ}(\mathbf{HoPKN_R})$

Example 27 is presented more explicitly in Example 28. It is easy to see that the Gesualdo example (Demonstration 7) can be understood as a special square in which we have $M = M'$ and $\mu = \mu'$.

The vertical 1-category $\mathbf{Sq}(\mathbf{C})$ can also be viewed as the category of presheaves $\mathbf{C}^{\Delta_2}$: its objects encode the data of two objects of $\mathbf{C}$ and a morphism between them, and the morphisms of $\mathbf{C}^{\Delta_2}$ identify to the commutative squares of $\mathbf{C}$. The composition is the vertical composition of squares. Observe that an object of $\mathbf{C}^{\Delta_2}$ is a special case of a *diagram* in the category $\mathbf{C}$. Thus the objects of $\mathbf{HoPKN_R}^{\Delta_2}$ are simply the data of two PK-nets of form R and a morphism between them, and a morphism of $\mathbf{HoPKN_R}^{\Delta_2}$ is a commutative square as defined above. This construction may be further generalized by replacing Δ_2 with any small category Γ and forming the category of presheaves $\mathbf{HoPKNR}^{\Gamma}$, or in other words the category of Γ-diagrams in $\mathbf{HoPKN_R}$ and morphisms between them.

The square construction can also be iterated, leading to n-fold categories of hyper-homographies.

7.2 Recurrent Construction of PK-nets of Increasing Orders

In Section 3 we alluded to the notion of a PK-net (R, S, F, φ) in any category $\mathbf{H}$, meaning that the functors R and S take their values in $\mathbf{H}$ rather than in **Sets**. A K-net in $\mathbf{H}$ is a PK-net in which R is a constant functor on an object of $\mathbf{H}$.

Proposition 10. *To a PK-net in* $\mathbf{H}$ *is associated a functor* Φ *from* $\mathbf{\Delta}$ *to* $\mathbf{Sq}(\mathbf{H})$ *whose restriction to objects reduces to the natural transformation* φ.

Proof. *Given the PK-net* (R, S, F, φ) *in* $\mathbf{H}$, *we construct a functor* Φ *from* $\mathbf{\Delta}$ *to the (vertical)* 1*-category* $\mathbf{Sq}(\mathbf{H})$ *as follows: it maps the object* X *of* $\mathbf{\Delta}$ *on* $\varphi(X)$ *and a morphism* $d\colon X \to Y$ *on the commutative square* $(R(d), \varphi(X), \varphi(Y), SF(d))$ *such that* $SF(d)\varphi(X) = \varphi(Y)R(d)$.

This result can be used in the construction of "higher order" PK-nets, or PK^{n+1}-nets, taking their values in categories deduced from **Sets** by n iterations of the Diag operation. Let us describe the first step of the construction, namely what is a PK-net in Diag(**Sets**), which we call PK^2-net, or PK-net of PK-nets. This last name is justified by its following description. Let $K = (R, S, F, \varphi)$ be a PK^2-net, so that R and S are functors to Diag(**Sets**). Such a functor $S\colon \mathbf{C} \to$ Diag(**Sets**) has been called a distructure presheaf on $\mathbf{C}$ (Ehresmann 2011). It associates to each morphism c of $\mathbf{C}$ a morphism of Diag(**Sets**), hence a PK-net (in **Sets**) $S(c) = (R_c, S_c, F_c, \varphi_c)$. Applying the Proposition 10, we see that a PK^2-net corresponds to the following data:

- a functor $F\colon \mathbf{\Delta} \to \mathbf{C}$,
- a distructure R on $\mathbf{\Delta}$, and a distructure S on $\mathbf{C}$,
- a functor from to $\mathbf{Sq}$(Diag(**Sets**)) which associates to a morphism $d\colon X \to Y$ of $\mathbf{\Delta}$ a commutative square of PK-nets, connecting the PK-nets $R(X)$ to $SF(Y)$.

8 Computational Aspects

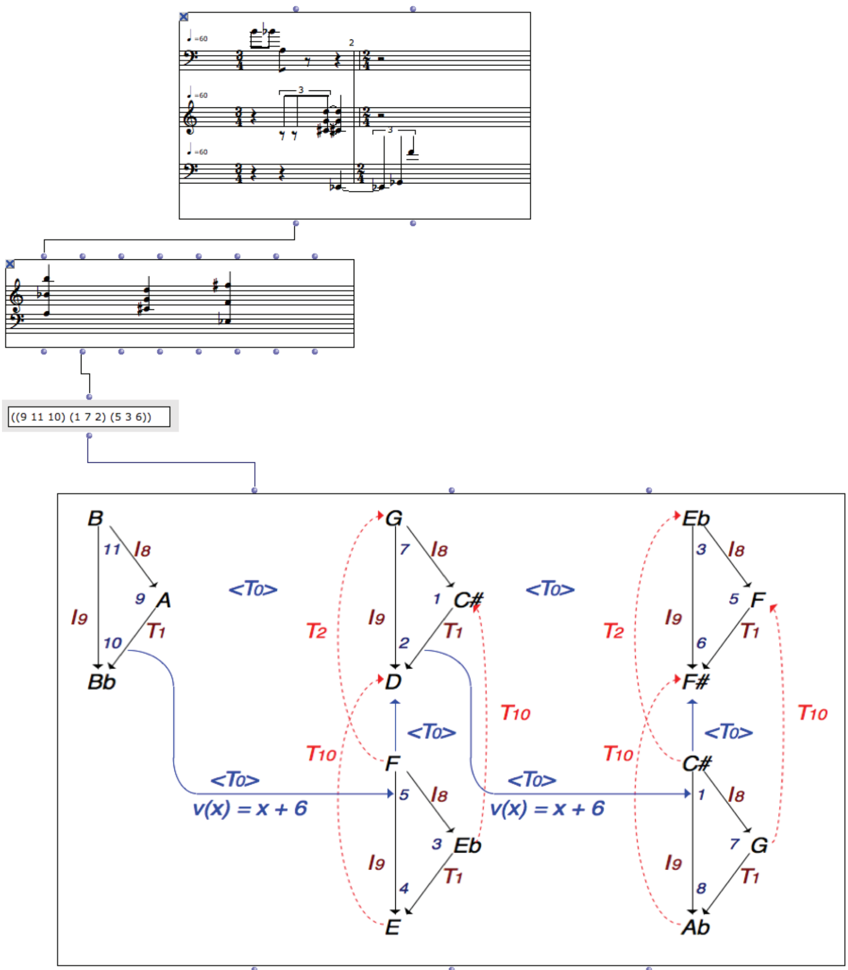
Thanks to their strong computational character, the concepts we have introduced and discussed in this paper are suitable for being integrated into some programming languages devoted to computer-aided music analysis. In a previous issue of *Perspectives of New Music*, we have presented the main functions of the Math Tool environment in OpenMusic visual programming language, with focus on tiling canons constructions and the underlying algebraic combinatorics (Agon et al. 2011). K-net and PK-net theory represents an alternative approach to classical group-based paradigmatic classification of musical chords, the notion of isography being weaker than the action-based equivalence of the different set-theoretical catalogues. In order to illustrate the implementation of K-nets and PK-nets in OpenMusic, let us firstly come back to Webern's *Three Little Pieces for Cello and Piano*, Op. 11/2 discussed previously. The following figure (Example 29) shows the three trichords in bars 4–5 and the associated computer-aided analysis of the PK-nets. The three K-nets described previously, together with the $\langle T_0 \rangle$ isographic relations are one of the possible ways of representing the combinatorial potential of these musical structures. There are in fact many other possible configurations with associated isographic relations, and the music analyst can now choose which solution is the more suitable for a given music-analytical purpose. Example 29 shows a different $\langle T_0 \rangle$ isographic relation among the same three trichords. The $\langle T_0 \rangle$ isography is factorized as a product of a complete $\langle T_0 \rangle$ isography followed by a local isography.

We now consider the special case of the PLR group acting on major and minor triads, as introduced in section 5.2. We are interested in exploring some combinatorial aspects of complete and local homographies between various instances of major and minor chords. In particular, let us take a generating cell of the Tonnetz obtained by applying to the C major chord the RLPRLP series of neo-Riemannian transformations, as shown in Example 30.

Is it possible to associate to the P, L, and R operators a series of complete isographies? Example 31 shows that this is actually the case and depicts the unique complete isographies between the chords in this cell.

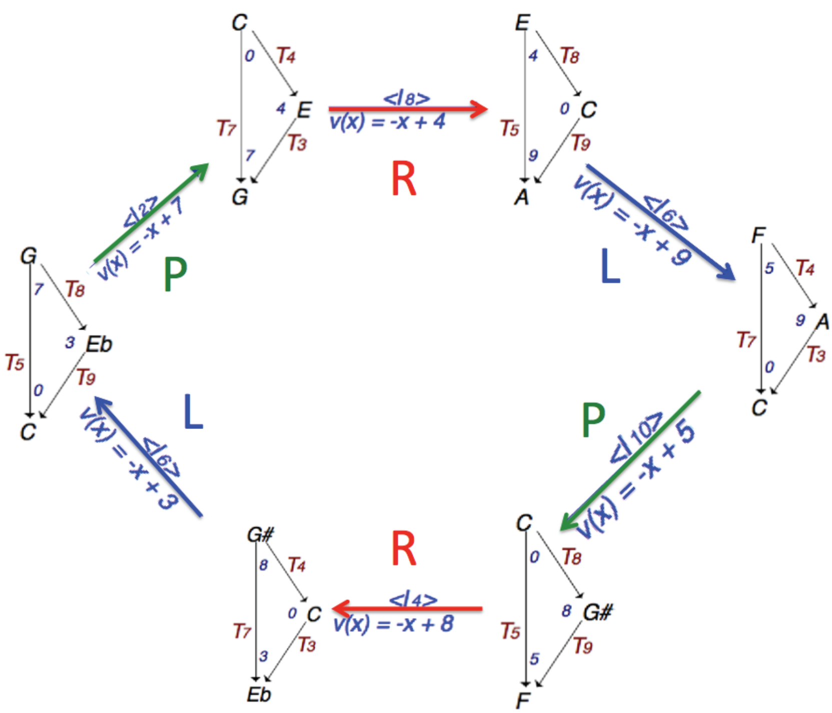
The previous results show that one can associate to the Tonnetz seen as an hexagonal lattice of the plane a second-order PK-net where the nodes are specific instances of K-nets which have been selected within the catalogue of possible K-nets and in such a way that the corresponding isographies are all complete. In other words, we have found a new GIS corresponding to a PK-net-based transitive action on the family of all major and minor chords. OpenMusic enables one to directly visualize these isographies in a score, as shown in Example 32.

Although these complete isographies are unique, since the action is simply transitive, they can still be decomposed as a complete isography followed by a local $\langle T_0 \rangle$ one. Open-Music can readily give the local and complete components of the isography, by selecting the isographies corresponding to the same natural transformation (which is, in this case, $v(x) = -x + 1$). The result is shown in Example 33.

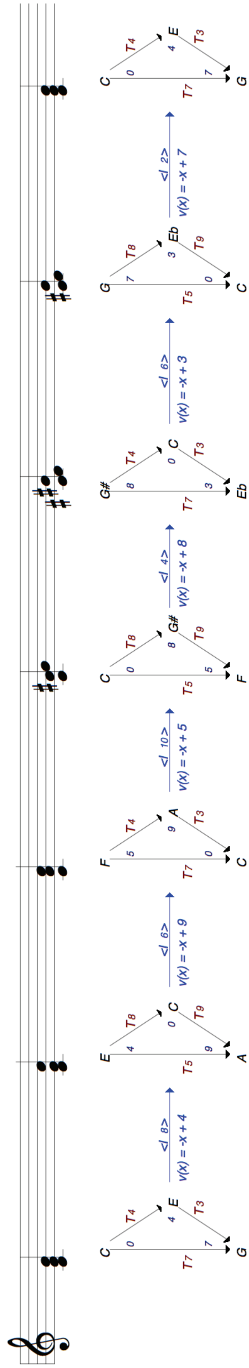


EXAMPLE 29: A COMPUTER-AIDED K-NETS ANALYSIS OF TWO BARS OF WEBERN'S *THREE LITTLE PIECES FOR CELLO AND PIANO*, OP. 11/2, IN OPENMUSIC

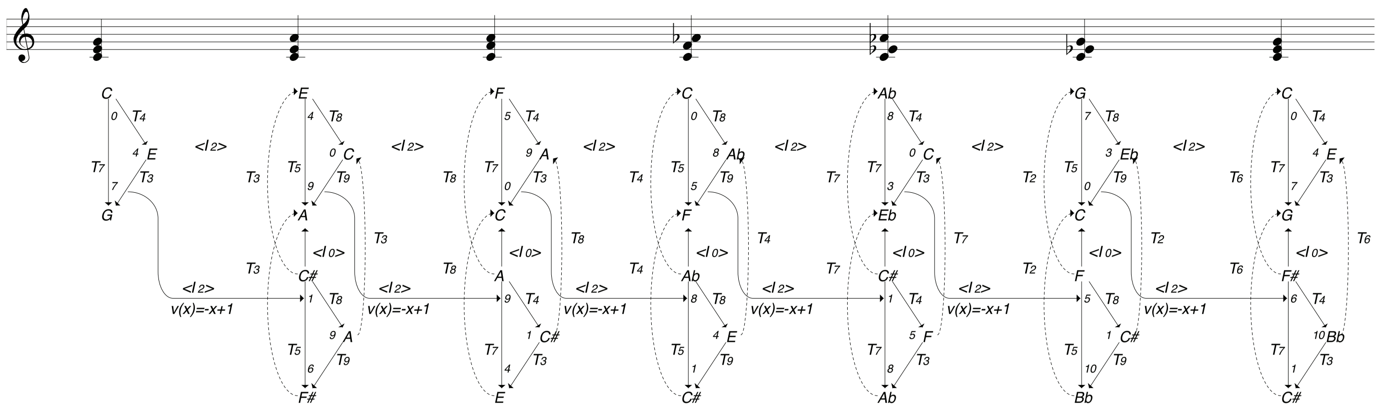
EXAMPLE 30: THE GENERATING CYCLE OF THE TONNETZ



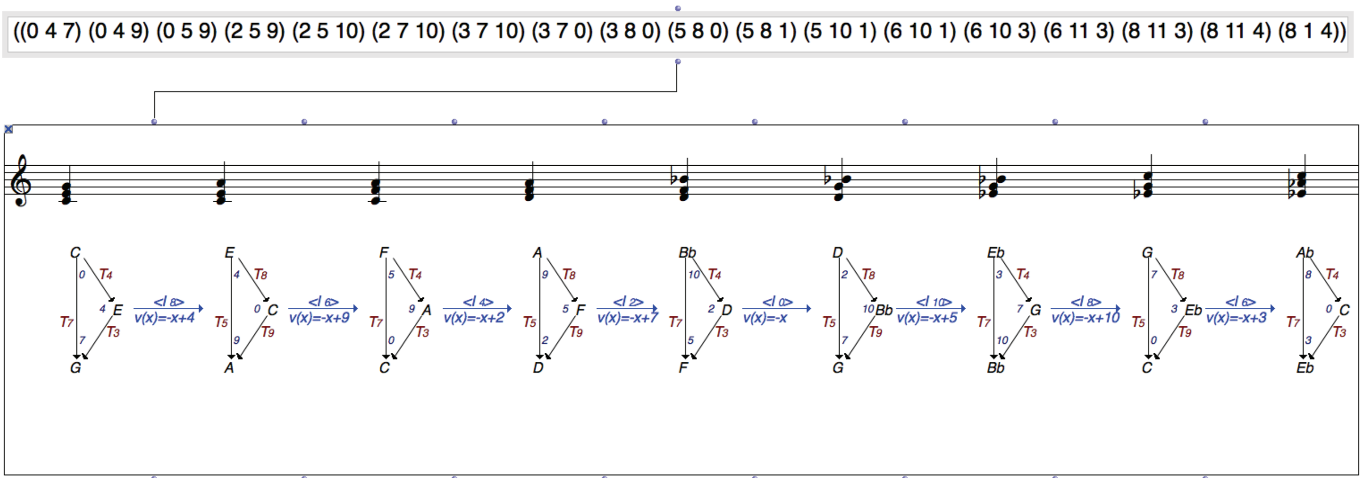
EXAMPLE 31: THE SERIES OF COMPLETE ISOGRAPHIES CORRESPONDING TO THE P, L, AND R OPERATORS USED TO GENERATE THE TONNETZ



EXAMPLE 32: THE SERIES OF COMPLETE ISOGRAPHIES CORRESPONDING TO THE P, L, AND R OPERATORS BETWEEN THE CHORDS OF THE TONNETZ CELL OF EXAMPLE 30



EXAMPLE 33: THE DECOMPOSITION OF THE MAIN ISOGRAPHIES
AS A PRODUCT OF A COMPLETE AND A LOCAL ISOGRAPHY,
WHERE THE LOCAL PART ONLY USES TRANSPOSITION OPERATIONS



EXAMPLE 34: A COMPUTER-AIDED K-NETS ANALYSIS OF A PORTION OF THE TRIADIC CHORD PROGRESSION ALTERNATING R AND L OPERATORS IN THE SECOND MOVEMENT OF BEETHOVEN'S NINTH SYMPHONY

Triadic chord progressions can therefore be associated to a series of complete isographies, as Example 34 shows in the case of the Neo-Riemannian celebrated harmonic progression found in Beethoven's Ninth Symphony (second movement). Note that the symmetric character of the harmonic progression, generated by applying alternatively the R and L operators, is well captured by the regularity of the underlying complete isographies.

9 Conclusions

Following the preliminary results presented in a recent article (Popoff et al. 2015), we have summarized in this study the main theoretical construction underlying our generalized framework of Klumpenhouwer Networks based on category theory. In particular, we have generalized the notion of isography between K-nets thanks to the new concept of homography between PK-nets. Example 35 sums-up the different categories and functors we have studied in this paper, and illustrated by many musical examples. All the functors restrict to the identity on objects.

In Section 7 we have raised the problem of constructing hyper-homographies between PK-nets. Another problem which we intend to study in the future is the study of PK-nets taking their values in a category **H** other than **Sets**; in particular for **H** being the category Diag(**Sets**), we have indicated how the construction leads to "PK-nets of PK-nets," and its iteration would lead to a whole hierarchy of PK-nets of increasing orders.

Starting from some pedagogical examples, we have shown some computational aspects of the interplay between local and complete homographies for future computer-aided PK-nets analysis. These computational models are currently integrated in existing programming languages for computer-aided music theory and analysis, such as the MathTools environment in OpenMusic (Agon et al. 2011). The computational approach enables the music analyst to dramatically reduce the time necessary to calculate the possible isographic relations between different chords in a given musical excerpt. It is hoped that the concepts introduced in this paper, as well as the computational tools which were developed in application, will prove to be useful analytic tools for music theorists in the future.

$$
\begin{tikzcd}
\mathbf{PKN_R} \arrow[d, "{(N,\tilde{\nu}) \to (N,\tilde{\nu}F)}"] & \\
\mathbf{CompHoPKN_R} \arrow[d, hook] & \\
\mathbf{HoPKN_R} \arrow[d, "{(N,\nu) \to (N,\nu_{|})}"] & \mathbf{LocHoPKN_R} \arrow[l, hook'] \\
\mathbf{LaxHoPKN_R} &
\end{tikzcd}
$$

EXAMPLE 35: THE DIFFERENT CATEGORIES AND FUNCTORS STUDIED IN THIS PAPER

Annexes

Annex 1: Limits and Colimits in the Categories and Subcategories of PK-Nets

We first consider the case of the category $\mathbf{PKN_R}$ defined in Popoff et al. (2015). We recall that it has for objects the PK-nets of form R, and for morphisms from $K = (R, S, F, \varphi)$ to $K' = (R', S', F', \varphi')$ the pairs $(N, \tilde{\nu})$, where N is a functor from $\mathbf{C}$ to $\mathbf{C'}$ such that $F' = NF$, and $\tilde{\nu}$ is a natural transformation from S to $S'N$ such that $\varphi' = (\tilde{\nu}, F)$. There is a functor J from $\mathbf{PKN_R}$ to $\mathbf{HoPKN_R}$ which is the identity on objects and which associates to the morphism $(N, \tilde{\nu})$: $K \to K'$ of $\mathbf{PKN_R}$ the complete homography $(N, \tilde{\nu}F)$: $K \to K'$, so that the image of J is the sub-category $\mathbf{CompHoPKN_R}$ of $\mathbf{HoPKN_R}$. The functor J is not injective since two natural transformations $\tilde{\nu}$ and $\tilde{\nu}'$ from S to $S'N$ may have the same composite $\tilde{\nu}F = \tilde{\nu}F$ with F. Thus $\mathbf{CompHoPKN_R}$ is a quotient category of $\mathbf{PKN_R}$ with the same objects. However the restriction of J is injective on Q, where Q is the full subcategory of $\mathbf{PKN_R}$ whose objects are PK-nets in which F is surjective on objects (which is generally the case in musical applications). The following proposition is indicated in Popoff et al. (2015) but without proof.

Proposition 11. *The category* $\mathbf{PKN_R}$ *has all small limits and all small connected colimits.*

- *We first construct explicitly the limits in* $\mathbf{PKN_R}$.

Let P: $I \to \mathbf{PKN_R}$ *be a functor. For an object* i *of* I *we write its image* $P_i = (R_i, S_i, F_i, \varphi_i)$ *and for* h: $i \to j$, *we write* $P(h) = (P_h, p_h)$: $P_i \to P_i$. *We are going to prove that* P *has a limit* $\lim P = (R, S, F, \varphi)$.

Let C *be the limit of the functor* P': $I \to \mathbf{Cat}$ *associating* P_h *to* h, *and let* pr_i: $C \to C_i$ *be the projection of this limit. Thus an object of* C *is a family* $(e_i)_{i\in|I|}$ *of objects of the* C_i *such that* $e_j = P_h(e_i)$ *for each* h: $i \to j$ *in* $\mathbf{\Delta}$, *and the analog for the morphisms* $(f_i)_{i\in|I|}$. *The* F_i *define a cone from* $\mathbf{\Delta}$ *to* P' *which factorizes by* F: $\mathbf{\Delta} \to C$, *so that* $F(X) = (F_i(X))_{i\in|I|}$.

- *We now construct the appropriate functor* S: $C \to$ **Sets**.

Let $e = (e_i)_{i\in|I|}$ *be an object of* C. *We define a functor* V_e: $I \to$ Sets *such that* $V_e(i) = S_i(e_i)$, *and such that for* h: $i \to j$ *we have*

$$V_e(h) = p_h(e_i)\colon S_i(e_i) \to S_j P_h(e_i) = S_j(e_j).$$

*Then $S(e)$ is the limit of V_e. Thus S: $C \to$ **Sets** is defined as follows:*

$$S(e) = lim\ V_e = \left\{ (s_i)_{i \in |I|} \mid s_i \in S_i(e_i),\ s_j = p_h(e_i)(s_i) \right\}_{i \in |I|},$$

and

$$S(f)\colon S(e) \to S(e')\colon (s_i)_{i \in |I|} \mapsto (S_i(f_i)(s_i))_{i \in |I|},$$

for $f = (f_i)_{i|I|}\colon e \to e'$.

- *We now construct the appropriate natural transformation φ by the construction of its components $(\varphi_i)_X$: $R(X) \to SF(X)$ for an object X of $\mathbf{\Delta}$.*

The maps $(\varphi_i)_X$: $R(X) \to S_i F_i(X)$ define a cone from $R(X)$ to $V_{F(X)}$ which factors through the limit $SF(X)$ by

$$\varphi_X\colon R(X) \to SF(X)\colon a \to ((\varphi_i)_X(a))_{i \in |I|}.$$

The i-th projection of lim $P = (R, S, F, \pi_i)$ to P_i is given by (pr_i, π_i) where pr_i: $\mathbf{C} \to C_i$: $(f_i)_{i \in |I|} \mapsto f_i$ and $\pi_i(e)$: $S(e) \to S_i(e_i)$: $(s_i)_{i \in |I|} \mapsto s_i$.

Finally, to verify the universal property, let K' be a PK-net vertex of a cone $((L_i, \lambda_i)\colon K' \to P_i)_{i \in |I|}$ with basis P. We must uniquely factor this cone through K via (L, λ): $K' \to K$. We define L: $C' \to C$: $e' \mapsto (L_i(e'))_{i \in |I|}$ and $\lambda(e')$: $S'(e') \to SL(e')$: $s' \mapsto (\lambda_i(e')(s'))_{i \in |I|}$.

- *We now construct explicitly the connected colimits in* $\mathbf{PKN_R}$.

*Let P: $I \to$ $\mathbf{PKN_R}$ be a functor, where I is connected: i.e., any two objects in I are connected by a zig-zag of morphisms. Let C' be the colimit of the functor P': $I \to$ **Cat**, and let v_i: $C_i \to C'$ be the corresponding injections. The category C' is the quotient of the category sum of the family $(C_i)_{i \in |I|}$ by the equivalence E generated by $e_i \sim e_j$ if there exists h: $C_i \to C_j$ with $P_h(e_i) = e_j$. An object e' of C' will be an equivalence class of an object e_i of C_i for E, and similarly for morphisms.*

Since I is connected, the composites $v_i F_i$: $\mathbf{\Delta} \to C'$ must have the same value F' for each i, and this defines the functor F': $\mathbf{\Delta} \to C'$: $X \mapsto v_i F_i(X)$.

Let e' be an object of C'. Let $I_{e'}$ be the full sub-category of I having for objects the I such that $e' = v_i(e_i)$ for some e_i in C_i. We define a *functor $V'_{e'}$: $I_{e'} \to$ **Sets** associating $S_i(e_i)$ to the object i of $I_{e'}$, and associating $p_h(e_i)$: $S_i(e_i) \to S_j(e_j)$ to h: $i \to j$ in $I_{e'}$. Then $S'(e') =$ colim $V'_{e'}$. Let v'_i be the canonical map from $S_i(e_i)$ to the colimit $S'(e')$. We thus obtain the functor S': $C' \to$ **Sets**.*

Finally, let X be an object of $\boldsymbol{\Delta}$. We define φ'_X:$R(X) \to S'F(X)$ as $\varphi'_X(a) = v'_i((\varphi_i)_X(a))$ for i in $I_{F'(X)}$. This construction does not extend to the case where I is not connected.

Let us consider the above defined functor J from $\mathbf{PKN_R}$ to $\mathbf{HoPKN_R}$ which is the identity on objects, and which associates to a morphism (N, v) the complete homography $(N, \tilde{v}F)$.

PROPOSITION 12. *The functor J preserves the limits and the connected colimits.*

- *We take the notation introduced in Proposition 11 and consider the composite functor JP: $I \to$ $\mathbf{HoPKN_R}$, which is the identity on objects, and maps the morphism (P_h, p_h) on the complete homography (P_h, p_hF): $P_i \to P_j$. We are going to prove that $K = J(K)$ is also the limit of JP in $\mathbf{HoPKN_R}$. The homography projection on P_i will be (pr_i, π_iF): $K \to P_i$. What remains to prove is the universal condition when we give a cone $(L_i, \mu_i)_{i\in|I|}$ from a PK-net K'' to JP, in which the homographies (L_i, μ_i) are not necessarily complete; that is, the natural transformations are of the most general form μ_i: $S''F'' \to S_iF_i$. However, now we only need a homography (L, μ): $K'' \to K$, and it is defined in the same manner as above. More precisely, for an object X of $\boldsymbol{\Delta}$, we define $\mu(X)$: $S''F''(X) \to SF(X)$: $s' \mapsto (\mu_i(X)(s'))_{i\in|I|}$.*

- *To prove that the colimit K' is also colimit of JP in $\mathbf{HoPKN_R}$, we have essentially to prove that the universal property still extends to a cone in $\mathbf{HoPKN_R}$, and the proof is analog to the case of limits.*

We finally address the general question of limits and connected colimits in $\mathbf{CompHoPKN_R}$ and $\mathbf{HoPKN_R}$.

PROPOSITION 13. *The categories $\mathbf{CompHoPKN_R}$ and $\mathbf{HoPKN_R}$ have all small limits and all small connected colimits of PK-nets in which the functor F is surjective on objects.*

- *Let P: $I \to$ $\mathbf{HoPKN_R}$ be a functor: we are going to prove that P has a colimit in $\mathbf{HoPKN_R}$. We keep the same notations as above,* *with the exception that $P(h)$ is now any homography, denoted by (P_h, q_h), where q_h: $S_iF_i \to S_jF_j$. We will construct the limit K of P in* **HoPKN$_\mathbf{R}$**.

The construction of C and F is the same as above. The construction of S must be modified because the natural transformation q_h only gives indications for objects of the form $F_i(X)$, and an object e_i of C_i may be the image by F of several objects X of **Δ**.

Let $e = (e_i)_{i \in |I|}$ be an object of C of the form $e = F(X) = (F_i(X))_{i \in |I|}$ for at least one object X of **Δ**. *We consider the functor V_X: $I \to$* **Sets** *such that $V_X(i) = S_iF_i(X)$, and $V_X(h) = q_h(X)$: $S_iF_i(X) \to S_jF_j(X)$ for h: $i \to j$. If we denote by $S_X(e)$ the limit of V_X, we have*

$$S_X(e) = \{(s_i)_{i \in |I|} \mid s_i \in S_iF_i(X) \text{ and } s_j = q_h(X)(s_I)\} \qquad (1)$$

Now we define $S(e)$ as the union of the sets $S_X(e)$ for the different X of **Δ** *sent on e by F. It consists of the families $(s_i)_{i \in |I|}$ satisfying the condition* (1) *for a X such that $e = F(X)$. We thus define a functor S: $C \to$* **Sets**, *since we have supposed that F is surjective on objects.*

The remainder of the proof is easily adapted from the preceding proposition. Let us note that condition of surjectivity on objects for the P_i is sufficient, but not necessary. For instance, products always exist as well as limits of functors of the form JP', where P' is a functor to **PKN$_\mathbf{R}$**.

- *Let P: $I \to$* **HoPKN$_\mathbf{R}$** *be a functor such that I is connected. We are going to prove that P has a colimit in* **HoPKN$_\mathbf{R}$**. *The construction of the colimit in* **HoPKN$_\mathbf{R}$** *is analog to that given in* **PKN$_\mathbf{R}$** *except for the construction of S'. Indeed, an object e' of C' can be the image by F of several objects X, Y, etc. of* **Δ**. *For each such X we construct a functor V'_X: $I_{e'} \to$* **Sets** *associating $S_iF_i(X)$ to the object i of $I_{e'}$ and associating $q_h(X)$: $S_iF_i(X) \to S_jF_j(X)$ to h: $i \to j$ in $I_{e'}$. Let $S'_X(e') = \mathrm{colim}\ V'_X$. Then $S'(e')$ is the union of the $S'_X(e')$ for the different X with $F(X) = e'$.*

ANNEX 2: AUTOMORPHISMS OF $((T/I, S))$

Section 5 has shown that $\mathrm{Aut}((T/I, S))$ is the group of order 1152 isomorphic to $T/I \rtimes T/M$. For a given automorphism $N = \langle k_l \rangle$ of T/I, we have seen that the elements of $(N, \tilde{v})$ of $\mathrm{Aut}((T/I, S))$ are in bijection with elements $(\langle k_l \rangle, \mathrm{g})$ of $T/I \rtimes T/M$. Starting from an element $(\langle k_l \rangle, g)$ of $T/I \rtimes T/M$, we are thus going to calculate explicitly the corresponding element $(N, \tilde{v})$ of $\mathrm{Aut}((T/I, S))$.

We label the major (resp. minor) chords by n_{Maj} (resp. n_{Min}), where n is the root pitch class of the chord. Recall that the action of the elements T_1 and I_0 of T/I on the chords is given by

- $T_1 \cdot n_{Maj} = (n+1)_{Maj}$, and $T_1 \cdot n_{Min} = (n+1)_{Min}$, and
- $I_0 \cdot n_{Maj} = (5-n)_{Min}$, and $I_0 \cdot n_{Min} = (5-n)_{Maj}$.

This action, which defines the functor S, is simply transitive. By identifying one particular chord to the identity element of the group T/I, it allows us to define a bijection X between the elements of T/I and the elements of the set of major and minor chords. We arbitrarily choose the identity element to be identified with the C major chord, in which case X identifies a major chord n_{Maj} with the element $T_n \in T/I$, and a minor chord n_{Min} with the element $I_{n-5} \in T/I$. Let $N = \langle k_l \rangle$ be an automorphism of T/I, and let g be an element of T/I. We study the natural transformation associated with an element $(\langle k_l \rangle, G)$ of $\mathrm{Aut}((T/I, S))$. Recall that, given an element x from the set of the major and minor triads, we have

$$\tilde{v}(x) = X(N(X^{-1}(x))\,g).$$

Depending on the element g, we distinguish two cases.

- Case $g = T_r$
  - A chord n_{Maj} is uniquely identified with the element $T_n \in T/I$; i.e., we have $X^{-1}(n_{Maj}) = T_n$. We then have $\langle k_l \rangle(T_n) = T_{kn}$, and thus

    $$N(X^{-1}(n_{Maj}))g = T_{kn}T_r = T_{kn+r}.$$

    Since $X(T_{kn+r}) = (kn+r)_{Maj}$, the natural transformation therefore sends the major chord n_{Maj} to the major chord $(kn+r)_{Maj}$.

– A chord n_{Min} is uniquely identified with the element $I_{n-5} \in T/I$; i.e., we have $\mathcal{X}^{-1}(n_{Min}) = I_{n-5}$. We then have $\langle k_l \rangle(I_{n-5}) = I_{kn-5k+l}$, and thus

$$N(\mathcal{X}^{-1}(n_{Min}))g = I_{k(n-5)+l}T_r = I_{k(n-5)+l-r}.$$

Since $\mathcal{X}(I_{k(n-5)+l-r}) = (k(n-5)+l-r+5)_{Min}$, the natural transformation therefore sends the minor chord n_{Min} to the major chord $(k(n-5)+l-r+5)_{Min}$.

- Case $g = I_r$

– Similarly, we have $N(\mathcal{X}^{-1}(n_{Maj}))g = T_{kn}I_r = I_{kn+r}$. Since $\mathcal{X}(I_{kn+r}) = (kn+r+5)_{Min}$, the natural transformation therefore sends the major chord n_{Maj} to the minor chord $(kn+r+5)_{Min}$.

– Likewise, we have $N(\mathcal{X}^{-1}(n_{Min}))g = I_{k(n-5)+l}I_r = T_{k(n-5)+l-r}$. Since $\mathcal{X}(T_{k(n-5)+l-r}) = (k(n-5)+l-r)_{Maj}$, the natural transformation therefore sends the minor chord n_{Min} to the major chord $(k(n-5)+l-r)_{Maj}$.

As we saw in Section 5, the normal subgroup of Aut$((T/I, S))$ consisting of all elements of the form $(\langle T_0 \rangle, G)$ is isomorphic to T/I. Its action is that of the PRL group, where the element $(\langle T_0 \rangle, I_{11})$ corresponds to the L operation, and the element $(\langle T_0 \rangle, I_4)$ corresponds to the R operation.

Annex 3: Glossary of Category Theory Terms

We give below the definition of the main category theory terms used throughout the article.

Automorphism group of an object *X* of a category *K*: it is the sub-group of K consisting of the isomorphisms g: $X \to X$ of K having X for their domain and codomain.

Category: a category K is defined as a graph equipped with an internal (partial) composition law which associates to each pair of two consecutive arrows f: $X \to X'$ and g: $X' \to X''$, a composite $g \circ f$: $X \to X''$. This composition satisfies the following conditions: (i) it is associative: $h \circ (g \circ f) = (h \circ g) \circ f$ if the composites are defined; (ii) each object X has an identity arrow id_X: $X \to X$ such that $f \circ id_X = f$ and $id_{X'} \circ f = f$. The set of objects of K is denoted by $|K|$, and an arrow f: $X \to X'$ is also called a *morphism* of K with domain X and codomain X'. If no confusion is possible, an object X is identified to its identity morphism id_X. For a general theory of categories, we refer to Mac Lane's 1971 book.

Category action: an action of a category K on a set M consists in the following data: (i) a map p: $M \to |K|$; (ii) a composition map k: $K \star M \to M$: $(f, s) \mapsto f(s)$ such that: $K \star M = \{(f, s) \mid p(s) = \mathrm{dom}(f)\}$, $p(f(s)) = \mathrm{codom}(f)$, $(f' \circ f)s = f'(f(s))$ and $p(s)s = s$. This action is equivalent to the data of a functor R: $K \to$ **Sets** which associates to an object X of K the set $R(X) = \{s \mid p(s) = X\}$ and such that $R(f)(s) = f(s)$ for f: $X \to X'$. If the category K is a group, we obtain the usual notion of a group action.

Colimit: let P: $I \to K$ be a functor. A colimit of P, if it exists, is an object cP of K satisfying the following condition: there is a cone (c_i) from P to cP such that each cone (f_i) from P to any object X of K binds into a unique morphism f: $cP \to X$ satisfying the equations $f_i = c_i \circ f$ for each object i of I. The functor P may have no colimit in K; if it has a colimit, this colimit is unique (up to an isomorphism).

Cocomplete category: a category K is said to be cocomplete if each functor from a small category I to K admits a colimit.

Complete category: a category K is said to be complete if each functor from a small category I to K admits a limit.

Complete PK-homography: cf. Section 5, Definition 6.

Cone from P to X in a category K: it is a natural transformation from the functor P: $I \to K$ to the functor from I to K constant on the object X of K. It reduces to a family (f_i) of morphisms f_i: $P(i) \to X$ for each object i of I such that $f_j \circ P(x) = f_i$ for each morphism x: $i \to j$ of I.

Diagram in a category K: it is a functor P: $I \to K$ from a small category I to K.

Form (of a PK-net): cf. Section 3, Definition 1.

Functor: a functor F from a category K to a category K' is a map which associates to each object X of K an object $F(X)$ of K', to each morphism f: $X \to X'$ of K a morphism $F(f)$: $F(X) \to F(X')$ of K, and which preserves composition, that is $F(f' \circ f) = F(f') \circ F(f)$ if f': $X' \to X''$. Such a functor F: $K \to K'$ is also called a covariant functor to distinguish it from a contravariant functor G from K to K' which is defined as a (covariant) functor G from the opposite category K^{op} of K to K'. A contravariant functor is also called a presheaf on K with values in K'. To avoid ambiguities, we only consider covariant functors in this text, contrarily to Mazzola who, using many presheaves, uses both covariant and contravariant functors.

Graph: a graph K (or more precisely a directed multi-graph) consists of a set $O(K)$ of objects X (its vertices), and a set of oriented edges between them, represented by arrows f: $X \to X'$; we call X the source of f, and X' its target. There can be several “parallel” arrows from X to X'.

Groupoid: a groupoid is a category in which each morphism is an isomorphism. A group is a groupoid with only one object. In any category C, the set of its isomorphisms defines a groupoid.

Isomorphism in a category K: a morphism f: $X \to X'$ of K is an isomorphism if there is a morphism f^{-1}: $X' \to X$ in K, called the inverse of f, such that $f \circ f^{-1}$ and $f^{-1} \circ f$ are identities.

Lax PK-homography: cf. Section 4, Definition 2

Limit of a functor P: $I \to K$: it is a colimit of the opposite functor from I^{op} to K^{op}.

Monoid: it is a category with a unique object, namely its unit.

Morphism of a category: cf. the definition of a category above.

Natural transformation: let $S: K \to K'$ and $S': K \to K'$ be functors. A natural transformation from S to S' is a map from $|K|$ to K' which associates to an object X of K a morphism $\varphi(X)$: $S(X) \to S'(X)$ such that we have $\varphi(X')S(f) = S'(f)\varphi(X)$ for each morphism f: $X \to X'$ of K. The natural transformation is an equivalence from K to K' if $\varphi(X)$ is a bijection for each object X of K.

Opposite category: each category K admits an opposite category K^{op} with the same objects and obtained by inverting the direction of its arrows.

PK-homography: cf. Section 4, Definition 4.

PK-net: abbreviation for Poly-Klumpenhouwer-network: cf. Section 3, Definition 1.

Presheaf P of sets on a category C: it is a functor from the opposite category C^{op} of C to the category **Sets** (cf. the definitions of Functor and of Sets).

Representable functor: it is a functor H: $K \to$ **Sets** which is equivalent to a functor Hom(X, –): $K \to$ **Sets** for some object X of K, where Hom(X, X') is the set of morphisms from X to X'.

Restriction of a functor $F: K \to K'$: it is a functor F': $C \to C'$ such that C is a sub-category of K, C' is a sub-category of K', and $F'(c) = F(c)$ for each morphism c of C'.

Sets: it is the category of sets which has for objects the (small) sets E, for morphisms from E to E' the maps from E to E', with the usual composition of maps: i.e., the composite of f: $E \to E'$ with f': $E' \to E''$ is the map $f'f$: $E \to E''$ which sends an element e of E to $f'(f(e))$ in E''. This category raises set theory problems to avoid the paradox of "the set of all sets." One solution is to make a distinction between "proper class" and "set" (as in the Von Neumann–Bernays–Gödel set theory). Another solution, which we adopt, is to work within the Zermelo–Fraenkel set theory with a non-trivial Grothendieck universe U. We take U for the set of objects of **Sets**, and a set belonging to U is called a small set.

Small category: it is a category such that the set of its morphisms is a small set.

Small colimit (**or small limit**): it is the colimit of a functor P: $I \to C$ where I is a small category.

Sub-category of a category K: it is a category C whose objects and morphisms are respectively objects and morphisms of K, and which is closed by composition in K—if morphisms c and c' of C have a composite $c'c$ in K, then $c'c$ is also in C.

Support (**of a PK-net**): cf. Section 3, Definition 1.

Top and bottom elements of posets: in a partially ordered set (or poset), a top element, if it exists, is the greatest element for the order; and the bottom element is the smallest element, if it exists.

Notes

1. Recall that a group action $G \times S \to S$ on a set S is *free* if, given an element x in S and a group element g in G such that $g \cdot x = x$, then g is the identity element.
2. So that this diagram, and the subsequent composition of natural transformations, would be easy to understand, we have represented here the identity functors on Δ, on T/I, and on **Sets** as dashed arrowless lines.

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