Hidden Categories: A New Perspective on Lewin's Generalized Interval Systems and Klumpenhouwer Networks

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Abstract. In this work we provide a categorical formalization of several constructions found in transformational music theory. We first revisit David Lewin's construction of a Generalized Interval System (GIS) to show that even a subset of the GIS conditions already implies a sequence of functors between categories. When all the conditions in Lewin's definition are fullfilled, this sequence involves the category of elements $\int_{\mathbf{G}} S$ for the group action $S\colon \mathbf{G} \rightarrow \mathbf{Sets}$ implied by the GIS structure. By focusing on the role played by categories of elements in such a context, we reformulate previous definitions of transformational networks in a **Cat**-based diagrammatic perspective, and present a new definition of categorical transformational networks, or *CT-Nets*, in more general musical categories. We show how such an approach provides a bridge between algebraic, geometrical, and graph-theoretical approaches in transformational music analysis. We end with a discussion on the new perspectives opened by such a formalization of transformational theory, in particular with respect to **Rel**-based transformational networks which occur in well-known music-theoretical constructions such as Douthett's and Steinbach's Cube Dance.

Keywords: Transformational Music Theory · Generalized Interval System · David Lewin · Category Theory · Transformational Networks · CT-Nets · Cube Dance

1 Introduction

Transformational theory represents a challenging topic in contemporary "mathemusical" research. It not only constitutes a turning point in the field of music analysis but also naturally leads to fundamental questions about the object-based

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vs. operation-based duality in the formalization of musical structures and processes. Starting from the original group-based contributions by David Lewin in the field of transformational music theory and analysis, the role of category theory has been quickly recognized [13–15, 17, 24] as a useful foundation in order to build extended generalizations. Following previous work on the categorical formalization of Lewin's transformational theory and transformational networks in music analysis, this paper has two main goals. In a first part, we show how a restricted form of Lewin's definition of Generalized Interval Systems implicitly defines diagrams of categories. In case all the conditions of Lewin's definition are fullfilled, these diagrams involve the notion of categories of elements. We take this observation as the starting point for further generalization and discuss the importance of such categories for transformational music theory and for graph-based geometrical approach. This shift in focus allows us, in a second part, to revisit Klumpenhouwer networks and to provide a new definition of categorical transformational networks ("CT-Nets"), as well as that of morphisms between them.

2 Revisiting Lewin's Generalized Interval Systems from a Categorical Perspective

In this section, we will show that a Generalized Interval System (GIS) of Lewin [10] inherently defines a category, which coincides with the category of elements for the group action the GIS defines due to the additional conditions imposed by Lewin.

2.1 Lewin's Generalized Interval Systems

We recall here Lewin's definition of a Generalized Interval System (GIS) for transformational music analysis.

Definition 1 (Lewin, 1987). *A Generalized Interval System (GIS) is a triple* $(X, IVLS, int)$ *where*

- X *is a set of musical elements,*
- $IVLS$ *is a group (the group of intervals for the GIS), and*
- int *is a function* $X \times X \to IVLS$

such that

1. *for all* x, y, *and* z *in* X, $int(x, y) * int(y, z) = int(x, z)$, *and*
2. *for all* $x \in X$ *and* $g \in IVLS$, *there exists a unique* $y \in X$ *such that* $int(x, y) = g$.

We know from Vuza [28, p. 270] and Kolman [9, p. 157] that the data of a GIS is equivalent to the data of a simply transitive (right) group action of $IVLS$ on the set X, or equivalently, of a representable functor $\mathbf{IVLS}^{\text{op}} \to \mathbf{Sets}$[1]Note that

[1] By an abuse of notation, for a given group G, we will notate throughout this paper its corresponding single-object category as $\mathbf{G}$. All functors are assumed to be covariant.

Lewin [10, pp 157–158] also discusses how a GIS can be obtained from a simply transitive left action of a group $STRANS$ on a set, by constructing a group $IVLS$ isomorphic to $STRANS^{\mathrm{op}}$. This result was extended to an equivalence of categories between the category of GISs and the category of left simply transitive actions by Fiore [5, p. 10]. Note that if we replace condition (1) by $\mathrm{int}(y, z) * \mathrm{int}(x, y) = \mathrm{int}(x, z)$, we obtain the data of a simply transitive left group action of $IVLS$ on X.

The condition (2) is necessary to construct the group action of $IVLS$ on X from the data of the GIS. Indeed, we can observe that it enforces the surjectivity of the interval function int. Without condition (2), the function $\mathrm{int}\colon X \times X \to IVLS$ may not be surjective, and there would then be elements of $IVLS$ for which we cannot define their action on X. However we show in the following proposition that condition (1) alone in Lewin's definition inherently defines the data of a category $\mathbf{C}$ along with a functor to the group associated to the function int. Note that we use the alternative form of condition (1) to agree with the usual composition of morphisms in a category.

Proposition 1. *We consider a triple* (X, G, int) *where*

- X *is a set of musical elements,*
- G *is a group, and*
- *int is a function* $X \times X \to G$ *such that for all* x, y, *and* z *in* X, $int(y, z) * int(x, y) = int(x, z)$.

This defines the data of a category $\mathbf{C}$ *along with a functor* $\mathbf{C} \to \mathbf{G}$.

Proof. We construct $\mathbf{C}$ as the category having X as its set of objects, such that for all $(x, y) \in X^2$, $\mathrm{Hom}(x, y) = \{(x, y)\}$. This is called the *indiscrete category on* X. Composition of morphisms is straightforward, with $(y, z)(x, y) = (x, z)$. By the properties of the function int, we can immediately define a functor from $\mathbf{C}$ to $\mathbf{G}$. □

We now recall the definition of the category of elements (also called the Grothendieck construction) for a functor $S\colon \mathbf{C} \to \mathbf{Sets}$.

Definition 2. *Let* $S\colon \mathbf{C} \to \mathbf{Sets}$ *be a functor from a category* $\mathbf{C}$ *to* $\mathbf{Sets}$. *The category of elements* $\int_{\mathbf{C}} S$ *is defined as the category having*

- *objects of the form* (X, x) *with* X *an object of* $\mathbf{C}$ *and* x *an element of the set* $S(X)$, *and*
- *morphisms between objects* (X, x) *and* (Y, y) *of the form* (x, f, y) *with* f *being a morphism of* $\mathbf{C}$ *such that* $y = S(f)(x)$.

There is a canonical projection functor $\pi_S\colon \int_{\mathbf{C}} S \to \mathbf{C}$ *sending each object* (X, x) *to* X.

If both condition (1) and (2) are used in Proposition 1, we then obtain the following immediate corollary.

Corollary 1. *Let* (X, G, int) *be a triple as defined in Proposition 1, such that for all* $x \in X$ *and* $g \in G$*, there exists a unique* $y \in X$ *such that* $int(x, y) = g$*. Then, the category* $\mathbf{C}$ *defined in Proposition 1 is the category of elements* $\int_{\mathbf{G}} S$ *for the simply transitive group action* $S\colon \mathbf{G} \to \mathbf{Sets}$ *derived from* (X, G, int)[2].

In the following subsection, we explore how these results provide insights into the relationship between transformational music theory and category theory.

2.2 From Lewin's Generalized Interval Systems to Categories, and to Graphs

As we have seen above, the condition (2) enforces the existence of a simply transitive group action of G on X, i.e. a functor $S\colon \mathbf{G} \to \mathbf{Sets}$ mapping the single object of $\mathbf{G}$ to X and which is representable. This gives rise to a sequence of functors of the form $\int_{\mathbf{G}} S \to \mathbf{G} \to \mathbf{Sets}$. Recent advances in transformational music theory have made use of various groups acting on sets of musical elements. Such groups do not always act in a simply transitive manner: this is the case for example of the T/I group acting on pitch classes, or Hook's UTT group [6] acting on major and minor triads. Despite the absence of a corresponding GIS structure, they nevertheless are useful and widely used tools for transformational analysis [12].

In this regard, dropping condition (2) altogether encourages us to shift our focus to categories rather than group actions, and to consider the functor $\mathbf{C} \to \mathbf{G}$ implied by Proposition 1 as a key notion. We will show in this section how we can generalize this observation by going beyond the definition of the interval function int.

As a first observation, it can be noted that for any group G acting on a set X we can always form the corresponding category of elements $\int_{\mathbf{G}} S$. While properly defining an interval function int might be complicated if the group does not act in a simply transitive way, the definition of $\int_{\mathbf{G}} S$ and the associated functor $\int_{\mathbf{G}} S \to \mathbf{G}$ already contains the necessary properties of composability and associativity.

Recent research has emphasized the significance of extending beyond Lewin's original group-based approach, by, for instance, exploring the use of groupoids or categories in general [21,23]. In this view, the group $\mathbf{G}$ and its action on a set is replaced with a general functor $S\colon \mathbf{C} \to \mathbf{Sets}$, and we can thus consider the corresponding category of elements $\int_{\mathbf{C}} S$ together with the canonical projection functor $\int_{\mathbf{C}} S \to \mathbf{C}$.

This generalization can be extended further by recognizing that the definition of the category of elements expands beyond functors to $\mathbf{Sets}$. In particular, this notion can be defined for functors $S\colon \mathbf{C} \to \mathbf{Rel}$, in which $\mathbf{Rel}$ is the category of finite sets and binary relations between them [27]. Such functors have been shown to occur in transformational music analysis, for example as in the algebraic

[2] In the particular case of a group action, the category of elements $\int_{\mathbf{G}} S$ is also called the *action groupoid*. As discussed in the next section, this paper considers more general cases and we thus retain "category of elements" as a common unifying term.

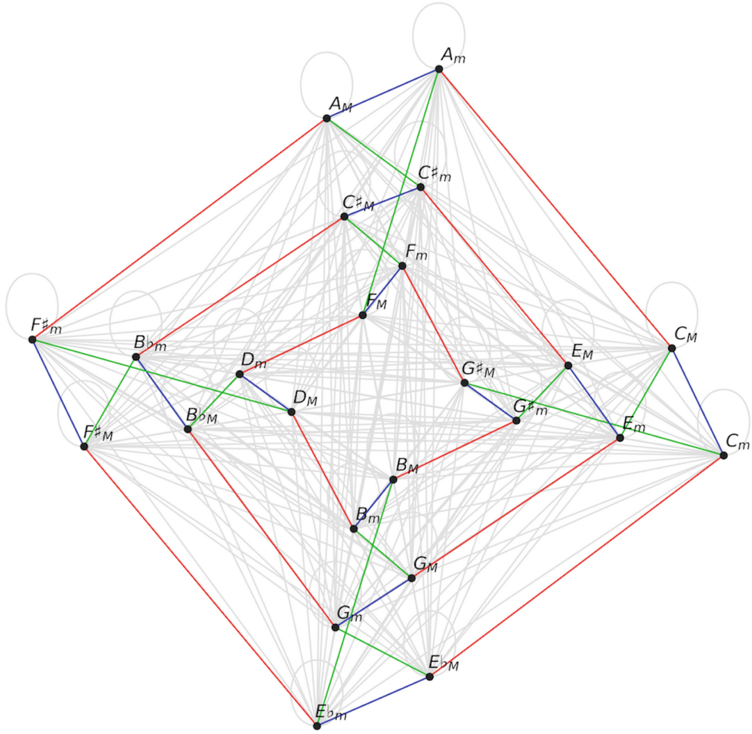


Fig. 1. The quiver corresponding to the category of elements for the simply transitive action of the PLR group on the set of major and minor triads. Edges (corresponding to invertibles morphisms) are represented with undirected edges colored in gray for clarity, except for morphisms projecting on R (red), P (blue), and L (green) in the PLR group. This subquiver corresponds to the "chicken-wire torus" [4]. Best viewed in color. (Color figure online)

formalization of the "Cube Dance" graph [19,22]. One advantage of using the category of elements $\int_{\mathbf{C}} S$ is that it "forgets" the target category of the functor S, whether it is **Rel** or **Sets**, which allows one to consider them on an equal footing.

In passing, it can also be observed that the category of elements is a discrete analogue to the fundamental groupoid of a topological space: objects of the fundamental groupoid are points of the space, and 1-morphisms are paths between these points. This reconciles with Lewin's view of transformations as ways to pass from one object to another[3].

Finally, it can be noted that the notion of category of elements establishes a bridge between algebraic transformational approaches and geometrical

[3] During the redaction of this manuscript, the authors have been made aware by Dmitri Tymoczko of his current work on groupoids. Category of elements are encountered as a common point between his approach, rooted in geometrical and topological considerations, and ours, which stems from algebraic ones.

graph-based approaches which have been studied extensively [2–4,18,25,26]. Indeed, it is well known that there is a forgetful functor from **Cat** to **Quiv**, the category of quivers, i.e. directed multigraphs ("multidigraphs"). The forgetful functor returns the underlying multidigraph of a category and forgets all information about composition of arrows. Certain well-known musical graphs can be obtained from the application of this forgetful functor. For example, the so-called "chicken-wire torus" [4] is a sub-multidigraph of the multidigraph obtained by application of the forgetful functor $\mathbf{Cat} \to \mathbf{Quiv}$ on the category of elements $\int_{PLR} S$ for the usual action S of the PLR group on the set of major and minor triads. This is illustrated in Fig. 1, in which the multidigraph obtained from the category of elements is depicted in gray (since the action is simply transitive, the corresponding quiver is a complete one), with the "chicken-wire torus" highlighted in color.

This colored multidigraph is a special case of the category-based general case, in which we consider the canonical projection functor $\int_{\mathbf{C}} S \to \mathbf{C}$ for a general functor $S\colon \mathbf{C} \to \mathbf{Sets}$. In this context, applying the forgetful functor $\mathbf{Cat} \to \mathbf{Quiv}$ induces additional structure, as shown in the construction below.

Construction 1. *Given a functor* $\mathbf{U} \to \mathbf{V}$*, the forgetful functor* $\mathbf{Cat} \to \mathbf{Quiv}$ *induces a coloring on nodes and edges of the multidigraph underlying* $\mathbf{U}$*. The nodes are colored in the set of nodes of the multidigraph corresponding to* $\mathbf{V}$*, while the edges are colored in the set of edges of same multidigraph.*

In the above example, the category of elements $\int_{\mathbf{PLR}} S$ and the associated functor $\int_{\mathbf{PLR}} S \to \mathbf{PLR}$ for the usual action S of the PLR group on the set of major and minor triads gives an edge-colored multidigraph in which each edge is colored (labelled) by an element of the PLR group. Note that we have colored the edges in Fig. 1 in gray merely for clarity. The nodes have only one color since we are considering a single-object category.

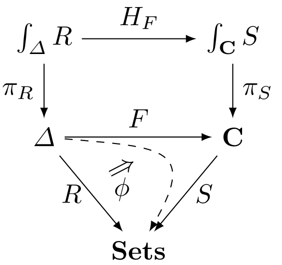
3 A New Definition of Transformational Networks

By following our shift of focus on categories of elements, we consider in this section a new definition of transformational networks which relates musical objects and transformations between them in a consistent diagram. Transformational networks first appeared in Lewin's work [10], and so-called *Klumpenhouwer networks* were developed by Klumpenhouwer [7,8,11] to specifically study pitch classes and their transformations by the T/I group. In recent years, transformational networks have been revisited with new definitions from a categorical perspective [16,20,21,24]. Our starting point is the definition of [21] which we recall below.

Definition 3. *Let* $\mathbf{C}$ *be a category, and* S *a functor from* $\mathbf{C}$ *to the category* $\mathbf{Sets}$ *of (small) sets. Let* Δ *be a small category and* R *a functor from* Δ *to* $\mathbf{Sets}$ *with non-empty values. A PK-net of form* R *and of support* S *is a 4-tuple* (R, S, F, ϕ)*, in which* F *is a functor from* Δ *to* $\mathbf{C}$*, and* ϕ *is a natural transformation from* R *to* SF*.*

As per the discussion above, our objective is to shift the focus from **Sets**-based functors to the corresponding categories of elements, which we do by leveraging the following theorem.

Theorem 2. *Assume we have a PK-net* (R, S, F, ϕ) *of form* R *and of support* S*. Then there exists a functor* H_F *between the category of elements* $\int_\Delta R$ *and* $\int_{\mathbf{C}} S$ *such that the top square in the following diagram commute.*



Proof. We construct the functor H_F as follows.

- For any object (X, x) of $\int_\Delta R$, the functor H_F sends it to $(F(X), \phi_X(x))$.
- For any morphism (x, f, y) between two objects (X, x) and (Y, y) of $\int_\Delta R$, the functor H_F sends this morphism to $(\phi_X(x), F(f), \phi_Y(y))$.

The construction in the first item is consistent, since by the definition of the natural transformation ϕ, $\phi_X(x)$ is indeed an element of $SF(X)$. Similarly, the construction in the second item is also consistent, since for any (x, y) such that $y = R(f)(x)$, we have $\phi_Y(y) = SF(f)(\phi_X(x))$. We have $H_F((y, f', z)(x, f, y)) = H_F((y, f', z))H_F((x, f, y))$ by horizontal composition of the natural transformation. □

The definition of PK-nets allows us to define networks in which nodes are sets, and not necessarily singletons. In this diagram, the role of the functor $\int_\Delta R \to \Delta$ is to remember which objects of $\int_\Delta R$ belong to the same set. The fact that the top square is commutative ensures that objects corresponding to elements of the same set are transformed by the same transformations.

While this possibility gives more generality to transformational networks, the majority of actual transformational analyses in the musicological literature uses networks with only one musical object per node, i.e. singletons. In this case, $\int_\Delta R$ is the same category as Δ, and since the data of the content of sets is no longer relevant, this makes the functor $\int_\Delta R \to \Delta$ somewhat unnecessary. We will thus now consider the specific case of singleton-based networks and drop this functor altogether. We can then rename $\int_\Delta R$ as Δ to indicate its new role as a category defining the skeleton of the transformational network to be considered.

We therefore obtain a sequence of functors of the form $\Delta \xrightarrow{M} \int_{\mathbf{C}} S \xrightarrow{\pi_S} \mathbf{C}$ which encodes transformational networks with a single musical element on each node. The nodes and arrows of the network are given by the objects and morphisms of the category Δ. The functor M maps the objects of Δ to the objects of $\int_{\mathbf{C}} S$ which, as we have seen above, corresponds to musical elements. In other words, the functor M labels the nodes of the network with musical elements. In addition, it maps the arrows of the network to morphisms of $\int_{\mathbf{C}} S$, which are then given a label in $\mathbf{C}$ (for example, group elements if $\mathbf{C}$ is a group) by the functor $\pi_S \colon \int_{\mathbf{C}} S \to \mathbf{C}$. Thus, the functor $\pi_S \circ M$ labels the arrows of Δ with morphisms (transformations) of $\mathbf{C}$.

Example 1. The transformational network represented below

$$\begin{array}{ccc} B & \xrightarrow{T_2} & C_\sharp \\ {\scriptstyle I_3}\downarrow & & \downarrow{\scriptstyle I_9} \\ E & \xrightarrow[T_4]{} & G_\sharp \end{array}$$

can be conceived as a sequence of functors $\Delta \xrightarrow{M} \int_{\mathbf{T/I}} S \xrightarrow{\pi_S} \mathbf{T/I}$, where $S \colon \mathbf{T/I} \to \mathbf{Sets}$ is the usual action of the T/I group on pitch-classes, Δ is the commutative square $\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} \\ Z & \xrightarrow[k]{} & W \end{array}$, and where

$$M(X) = B, M(Y) = C_\sharp, M(Z) = E, M(W) = G_\sharp, \text{and}$$

$$\pi_S \circ M(f) = T_2, \pi_S \circ M(g) = I_9, \pi_S \circ M(h) = I_3, \pi_S \circ M(k) = T_4.$$

In a final generalization, we can observe that, since we do not consider **Sets**-based (or **Rel**-based) functors anymore, we can consider all categories in our definition of a transformational network, and not just category of elements. We thus obtain the following definition.

Definition 4. *A categorical transformational network (CT-Net) is a sequence of functors of the form* $\Delta \xrightarrow{M} \mathbf{C}_{el} \xrightarrow{\pi} \mathbf{C}_T$ *, where*

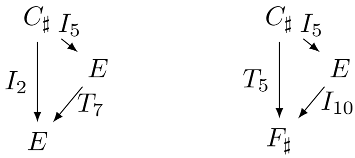
- Δ *is a category representing the skeleton of the network,*
- $\mathbf{C}_{el}$ *is a category whose objects represent musical objects of interest, with morphisms between them, and*
- $\mathbf{C}_T$ *is a category whose objects represent classes of musical objects, and musical transformations between them.*

The nodes of the network are labelled by the images of the objects of Δ *by* M*, and the edges of the network are labelled by the images of the morphisms of* Δ *by* $\pi \circ M$*.*

Note that we do not *a priori* impose conditions on the categories $\mathbf{C}_{\mathrm{el}}$ and $\mathbf{C}_T$, nor on the functor π. In fact, $\mathbf{C}_T$ may well be $\mathbf{C}_{\mathrm{el}}$ itself, in which case transformations are individualized for each musical object.

Since our definition of transformational networks is entirely categorical, all constructions in **Cat** (pullbacks, pushforwards, equalizers, etc.) can in fact be used to construct new CT-Nets, as shown in the following example.

Example 2. Consider the two networks represented below,



viewed as in the previous example as sequences of functors $\Delta \overset{M}{\underset{M'}{\rightrightarrows}} \int_{\mathbf{T/I}} S \longrightarrow \mathbf{T/I}$. Taking the equalizer on the functors M and M' returns a new sequence of functors

$$\Delta' \longrightarrow \Delta \overset{M}{\underset{M'}{\rightrightarrows}} \int_{\mathbf{T/I}} S \longrightarrow \mathbf{T/I}$$

and thus a new CT-Net representing the network each have in common, i.e. $C_\sharp \xrightarrow{I_5} E$. Since such categorical constructions can readily be implemented computationally, and more easily than the previously exposed transformational network frameworks, this opens new perspectives for computational score analysis. Furthermore, as mentioned above, the passage from **Cat** to **Quiv** is functorial which allows to translate these constructions into the corresponding multidigraphs (and morphisms between them).

4 Morphisms of CT-Nets

In this section, we revisit the notions of morphisms of transformational networks and explore their implications for diagrams involving categories of elements. We will build upon the morphisms of PK-Nets defined in previous work, and in this view we will revert to category of elements for functors $\mathbf{C} \rightarrow \mathbf{Sets}$ before giving the final definition in the general case.

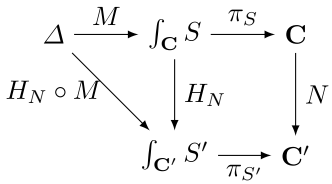
We first consider the notion of *complete homographies* introduced in the formalization of PK-Nets [20]. We recall their definition below.

Definition 5. *Let* (R, S, F, ϕ) *be a PK-Net, and assume we have a functor* $S' \colon \mathbf{C}' \rightarrow \mathbf{Sets}$*. A complete homography is a pair* (N, ν) *with*

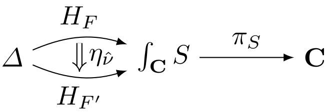
- $N\colon \mathbf{C} \to \mathbf{C}'$, *and*
- $\nu\colon SF \to S'FN$ *such that* $\nu = \tilde{\nu}F$, *where* $\tilde{\nu}$ *is a natural transformation from* S *to* $S'N$

Applying (N, ν) *on the PK-net* (R, S, F, ϕ) *transforms it into the new PK-Net* $(R, S', FN, \nu \circ \phi)$.

When $\mathbf{C}$ and $\mathbf{C}'$ are groups, with simply transitive group actions, this is known as a morphism of GIS [5]. For a given complete homography, we can apply Theorem 2 on the diagram formed by S and $S'N$. This defines a functor H_N from $\int_{\mathbf{C}} S$ to $\int_{\mathbf{C}'} S'$. Starting from a transformational network $\Delta \xrightarrow{M} \int_{\mathbf{C}} S \xrightarrow{\pi_S} \mathbf{C}$, we thus obtain a new transformational network, as shown in the diagram below.

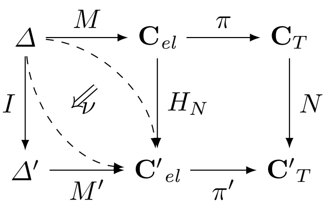


Secondly, we consider *local transformations* [23], which are natural transformations $\hat{\nu}\colon F \to F'$, extended by S such that $\phi' = (S\hat{\nu})\phi$. In such a case, it can readily be seen that the natural transformation $\hat{\nu}\colon F \to F'$ induces a natural transformation $\eta_{\hat{\nu}}\colon H_F \to H_{F'}$ between the corresponding functors $H_F\colon \int_\Delta R \to \int_{\mathbf{C}} S$ and $H_{F'}\colon \int_\Delta R \to \int_{\mathbf{C}} S$. We then obtain the following diagram.



By combining these two notions, and by generalizing it through the possibility of changing the category Δ, we propose a unified notion of morphism of CT-Nets as defined below.

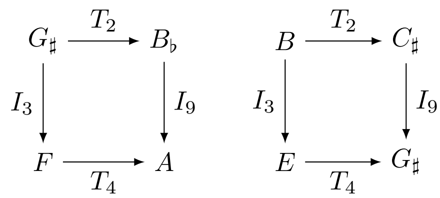
Definition 6. *A morphism between two categorical transformational networks* $\Delta \xrightarrow{M} \mathbf{C}_{el} \xrightarrow{\pi} \mathbf{C}_T$ *and* $\Delta' \xrightarrow{M'} \mathbf{C}'_{el} \xrightarrow{\pi'} \mathbf{C}'_T$ *is defined as a 4-tuple* (I, N, H_N, ν) *such that the following diagram commutes.*



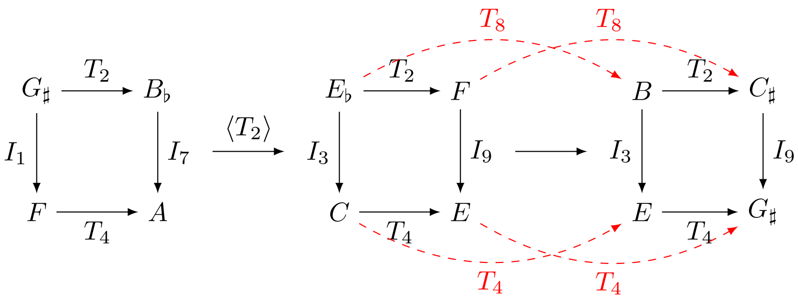
Morphisms of CT-Nets are composable, via the appropriate stacking of the corresponding diagrams.

In the above definition of a morphism of CT-Nets, the rightmost square represents a global transformation of the musical elements and the corresponding transformations between them, independently of the specific network Δ. As detailed above, if $\mathbf{C}_{\mathrm{el}}$ is the category of elements for a given group action, this corresponds to a morphism of a group action. The leftmost square represents a local transformation of the network nodes: the components of the natural transformation ν are morphisms in $\mathbf{C'}_{\mathrm{el}}$, which can be given labels via π' in $\mathbf{C'}_T$. We give below an example of such a combination of global and local transformation.

Example 3. The two transformational networks shown below are $\langle T_2 \rangle$-isographic in the sense of Klumpenhouwer.



However, no morphism of the T/I group action can transform the first into the second. This can nevertheless be achieved with a combination of a global transformation with a local one, as shown below. We refer the reader to [20] for more details about global and local transformations of networks.



5 Conclusions

In this work, we have revisited Lewin's original framework of Generalized Interval Systems from a categorical perspective, by stressing the importance of diagrams $\mathbf{C}_{\mathrm{el}} \to \mathbf{C}_T$, in which $\mathbf{C}_{\mathrm{el}}$ is a category of musical objects which maps onto $\mathbf{C}_T$, a category of musical transformations. We have shown that Lewin's definition of GIS implictly contains the definition of such a diagram, and that the condition for simple transitivity turns $\mathbf{C}_{\mathrm{el}}$ into the category of elements $\int_{\mathbf{G}} S$ for the corresponding group action $S\colon \mathbf{G} \to \mathbf{Sets}$. This consideration extends beyond group and group actions however, and category of elements $\int_{\mathbf{C}} S$ for a given functor $S\colon \mathbf{C} \to \mathbf{Sets}$ or even $S\colon \mathbf{C} \to \mathbf{Rel}$ may be considered. Ultimately, one may consider diagrams $\mathbf{C}_{\mathrm{el}} \rightarrow \mathbf{C}_T$ in which $\mathbf{C}_{\mathrm{el}}$ is not necessarily a category of elements.

Drawing from previous work on the formalization of transformational networks, we have proposed a new categorical definition of transformational networks, hence called CT-Nets, as a sequence of functors $\Delta \rightarrow \mathbf{C}_{\mathrm{el}} \rightarrow \mathbf{C}_T$, in which Δ represents the skeleton of the network. The definition operates directly in **Cat**, and thus unifies the consideration of **Sets**- and **Rel**-based functors at once. The definition of morphisms of CT-Nets results from previous considerations of global and local morphisms, the former corresponding to global transformations of the musical action, independently of the network being considered, the latter corresponding to individual node transformations. As diagrams, CT-Nets can undergo the machinery of all constructions in **Cat** (pullbacks, pushforwards, equalizers, etc.) to give new networks. In addition, the forgetful functor **Cat** $\rightarrow$ **Quiv** readily gives the corresponding multidigraphs (and morphisms between them), bridging algebraic and geometrical graph-based methods in music theory.

This new categorical formalization of transformational networks also opens further research perspectives. For example, while categories of elements can be constructed for functors $\mathbf{C} \rightarrow \mathbf{Rel}$, Theorem 2 cannot be applied to relational PK-nets as introduced in [22] if the natural transformation ϕ is a binary relation and not simply a function. In such a case, the machinery of profunctors [1] may be used between the corresponding category of elements, with new implications regarding the musicological meaning of transformations between musical objects.

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