Journal of Mathematics and Music, 2018
Vol. 12, No. 1, 35–55, https://doi.org/10.1080/17459737.2017.1406011

Relational poly-Klumpenhouwer networks for transformational and voice-leading analysis

Alexandre Popoff[a], Moreno Andreatta[b]*, and Andrée Ehresmann[c]

[a] *119 Rue de Montreuil, 75011 Paris;*
[b] *IRCAM-CNRS-UPMC, Paris, and IRMA-CNRS-GREAM, Université de Strasbourg, Strasbourg, France;*
[c] *LAMFA, Université de Picardie, Amiens, France*

(*Received 21 December 2016; accepted 12 October 2017*)

In the field of transformational music theory, which emphasizes the possible transformations between musical objects, Klumpenhouwer networks (K-nets) constitute a useful framework with connections in both group theory and graph theory. Recent attempts at formalizing K-nets in their most general form have evidenced a deeper connection with category theory. These formalizations use diagrams in sets, i.e. functors $\mathbf{C} \rightarrow \mathbf{Sets}$ where $\mathbf{C}$ is often a small category, providing a general framework for the known group or monoid actions on musical objects. However, following the work of Douthett–Steinbach and Cohn, transformational music theory has also relied on the use of relations between sets of the musical elements. Thus, K-net formalizations should be extended further to take this aspect into account. The present article proposes a new framework called *relational PK-nets*, an extension of our previous work on poly-Klumpenhouwer networks (PK-nets), in which we consider diagrams in **Rel** rather than **Sets**. We illustrate the potential of relational PK-nets with selected examples, by analyzing pop music and revisiting the work of Douthett–Steinbach and Cohn.

Keywords: transformational analysis; category theory; Klumpenhouwer network; PK-net; relation; parsimonious voice leading

2010 Mathematics Subject Classification: 00A65; 18B10

1. Introduction: from K-nets and PK-nets to relational PK-nets

We begin this section by recalling the definition of a poly-Klumpenhouwer network (PK-net), and then discuss its limitations, as a motivation for introducing relational PK-nets.

1.1. *The categorical formalization of poly-Klumpenhouwer networks (PK-nets)*

Following the work of Lewin (1982, 1987), transformational music theory has progressively shifted the music-theoretical and analytical focus from the "object-oriented" musical content to the operational musical process. As such, transformations between musical elements are emphasized. In the original framework of Lewin, the set of transformations often forms a group, with a corresponding group action on the set of musical objects. Within this framework, Klumpenhouwer networks (henceforth K-nets) have stressed the deep synergy between set-theoretical

*Corresponding author. Email: Moreno.Andreatta@ircam.fr

and transformational approaches thanks to their anchoring in both group theory and graph theory, as observed by many scholars (Nolan 2007). We recall that a K-net is informally defined as a labelled graph, wherein the labels of the vertices belong to the set of pitch classes, and each arrow is labelled with a transformation that maps the pitch class at the source vertex to the pitch class at the target vertex. An example of a K-net is given in Figure 1. Klumpenhouwer networks allow one conveniently to visualize at once the musical elements and selected transformations between them.

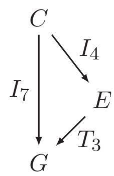

Figure 1. A Klumpenhouwer network (K-net) describing a major triad. The arrows are labelled with specific transformations in the T/I group relating the pitch classes in their domain and codomain.

Following David Lewin's (1990) and Henry Klumpenhouwer's (1991) original group-theoretical description, theoretical studies of K-nets have mostly focused until now on the underlying algebraic methods related to the group of automorphisms of the T/I group or of the more general T/M affine group (Lewin 1990; Klumpenhouwer 1998). Following the very first attempt by Mazzola and Andreatta at formalizing K-nets in a more general categorical setting as limits of diagrams within the framework of denotators (Mazzola and Andreatta 2006), a categorical construction called poly-Klumpenhouwer networks (henceforth PK-nets) has recently been proposed, which generalizes the notion of K-nets in various ways (Popoff, Andreatta, and Ehresmann 2015; Popoff et al. 2016).

We begin by recalling the definition of a PK-net, introduced first by Popoff, Andreatta, and Ehresmann (2015).

Definition 1.1 Let $\mathbf{C}$ be a category, and S a functor from $\mathbf{C}$ to the category **Sets** of (small) sets. Let Δ be a small category and R a functor from Δ to **Sets** with non-empty values. A *PK-net of form R and of support S* is a 4-tuple (R, S, F, ϕ), in which F is a functor from Δ to $\mathbf{C}$, and ϕ is a natural transformation from R to SF.

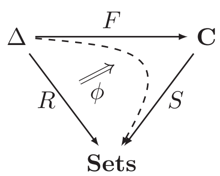

Figure 2. Diagrammatic representation of a PK-net (R, S, F, ϕ).

A PK-net can be represented by the diagram of Figure 2. We now detail below the importance of each element in this definition with respect to transformational music analysis. The category $\mathbf{C}$ and the functor $S : \mathbf{C} \to \mathbf{Sets}$ represent the context of the analysis. Traditional transformational music theory commonly relies on a group acting on a given set of objects: the most well-known examples are the T/I group acting on the set of the twelve pitch classes, the same T/I group acting simply transitively on the set of the 24 major and minor triads, or the PLR group acting simply transitively on the same set, to name a few examples. From a categorical point of view, the data of a group and its action on a set is equivalent to the data of a functor from a single-object category with invertible morphisms to the category of sets. However, this situation can be further generalized by considering any category $\mathbf{C}$ along with a functor $S : \mathbf{C} \to \mathbf{Sets}$. The morphisms of the category **C** are therefore the musical transformations of interest. For example, Noll has studied previously a monoid of eight elements and its action on the set of the twelve pitch classes (Noll 2005; Fiore and Noll 2011): this monoid can be considered as a single-object category **C** with eight non-invertible morphisms along with its corresponding functor $S : \mathbf{C} \to \mathbf{Sets}$, where the image of the only object of **C** is the set of the twelve pitch classes.

The category Δ serves as the abstract skeleton of the PK-net: as such, its objects and morphisms are abstract entities, which are labelled by the functor F from Δ to the category **C**. By explicitly separating the categories Δ and **C**, we allow for the same PK-net skeleton to be interpreted in different contexts. For example, a given category **C** may describe the relationships between pitch classes, while another category $\mathbf{C}'$ may describe the relationships between time spans (Lewin 1987). Different functors $F : \Delta \to \mathbf{C}$ or $F' : \Delta \to \mathbf{C}'$ will then label the arrows of Δ differently with transformations from **C** or $\mathbf{C}'$, depending on whether the PK-net describes pitch classes or time spans. Two PK-nets may actually be related by different kinds of *morphisms of PK-nets*, which have been described previously (Popoff, Andreatta, and Ehresmann 2015; Popoff et al. 2016).

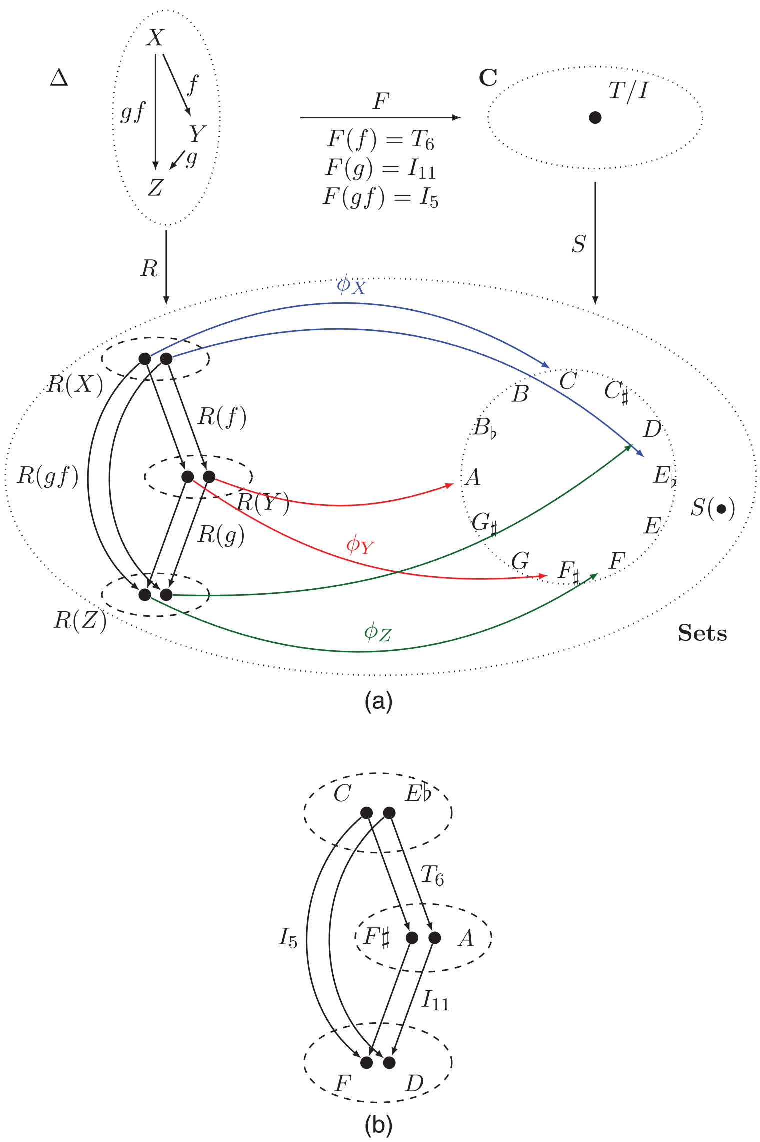

Figure 3. (a) Diagram showing the constitutive elements of an example PK-net (R, S, F, ϕ). (b) A simplified representation of the same PK-net, wherein the arrows are directly labelled by their F-images, and the elements of the R-images of the objects of Δ are directly labelled by their ϕ-component-images.

Note that the objects of Δ do not represent the actual musical elements of a PK-net: these are introduced by the functor R from Δ to **Sets**. This functor sends each object of Δ to an actual set, which may contain more than a single element, and whose elements abstractly represent the musical objects of study. However, these elements are not yet labelled. In the same way the morphisms of Δ represent abstract relationships which are given a concrete meaning by the functor F, the elements in the images of R are given a label in the images of S through the natural transformation ϕ. The elements in the image of S represent musical entities on which the category **C** acts, and one would therefore need a way to connect the elements in the image of R with those in the image of S. However, one cannot simply consider a collection of functions between the images of R and the images of S in order to label the musical objects in the PK-net. Indeed, one must make sure that two elements in the images of R which are related by a function $R(f)$ (with f being a morphism of Δ) actually correspond to two elements in the images of S related by the function $SF(f)$. The purpose of the natural transformation ϕ is thus to ensure the coherence of the whole diagram. The diagram of Figure 3(a) on the previous page sums up the constitutive elements of a simple PK-net (R, S, F, ϕ) where the category **C** is the T/I group considered as a single-object category, the image of this object by the functor S being the set $\mathbb{Z}_{12}$ of the twelve pitch classes. This PK-net can be represented in the simplified form of Figure 3(b) wherein the functors R, S, F, and the natural transformation ϕ are implicit.

While PK-nets have been so far defined in **Sets**, there is *a priori* no restriction on the category that should be used as the codomain of the functors R and S. For example, the approach of Mazzola and Andreatta (2006) uses modules and categories of module-valued presheaves. As noticed by Popoff, Andreatta, and Ehresmann (2015), PK-nets could also be defined in such categories. Alternatively, the purpose of the present article is to consider *relational PK-nets* in which the category **Sets** of sets and functions between them is replaced by the category **Rel** of sets and binary relations between them. The reasons for doing so are detailed in the next sections.

1.2. *Limitations of PK-nets*

The definition of PK-nets introduced above leads to musical networks of greater generality than traditional Klumpenhouwer networks, as was shown previously by Popoff, Andreatta, and Ehresmann (2015) and Popoff et al. (2016). In particular, it allows one to study networks of sets of different cardinalities, not necessarily limited to singletons. We recall here a prototypical example showing how a dominant seventh chord may be obtained from the transformation of an underlying major chord, with an added seventh.

Example 1.2 Let **C** be the T/I group, considered as a single-object category, and consider its natural action on the set $\mathbb{Z}_{12}$ of the twelve pitch classes (with the usual semitone encoding), which defines a functor $S : T/I \to$ **Sets**. Let Δ be the poset of the ordinal number **2**, i.e. the category with two objects X and Y and precisely one morphism $f : X \to Y$, and consider the functor $F : \Delta \to T/I$ which sends f to T_4.

Consider now a functor $R : \Delta \to$ **Sets** such that $R(X) = \{x_1, x_2, x_3\}$ and $R(Y) = \{y_1, y_2, y_3, y_4\}$, and such that $R(f)(x_i) = y_i$, for $1 \leq i \leq 3$. Consider the natural transformation ϕ such that $\phi_X(x_1) = C$, $\phi_X(x_2) = E$, $\phi_X(x_3) = G$, and $\phi_Y(y_1) = E$, $\phi_Y(y_2) = G\sharp$, $\phi_Y(y_3) = B$, and $\phi_Y(y_4) = D$. Then (R, S, F, ϕ) is a PK-net of form R and support S which describes the transposition of the C major triad to the E major triad subset of the dominant seventh E^7 chord. The constitutive elements of this PK-net $(R, S, F\phi)$ are summed up in the diagram of Figure 4.

However, one specific limitation of this approach appears quickly: whereas transformations between sets of increasing cardinalities can easily be modelled in this framework, transformations between sets of decreasing cardinalities sometimes cannot. Consider for example

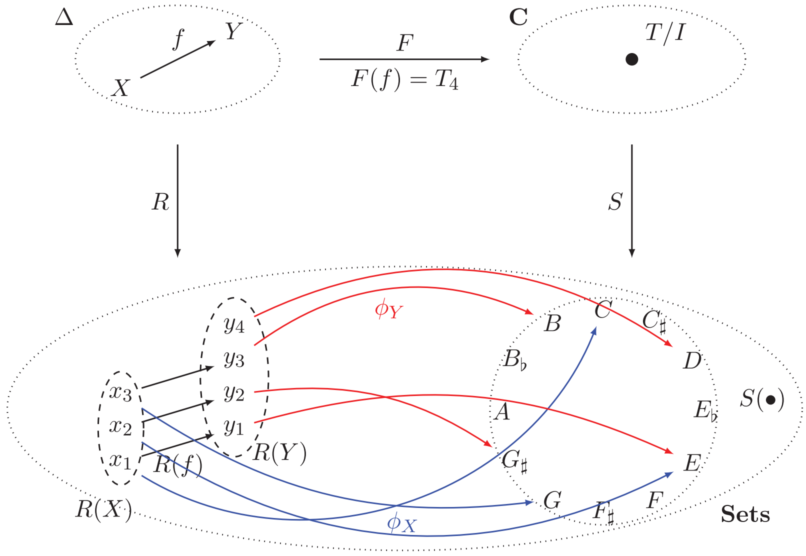

Figure 4. Diagram showing the constitutive elements of the simple PK-net (R, S, F, ϕ) of Example 1.2, which describes the transposition of the C major triad to the E major triad subset of the dominant seventh E^7 chord. The functor F sends the objects X and Y to the group T/I considered as a single-object category, whose image $S(\bullet)$ by the functor S is the set $\mathbb{Z}_{12}$ of the twelve pitch classes.

a PK-net (R, S, F, ϕ) with a category Δ with at least two objects X and Y and a morphism $f : X \to Y$ between them, a functor $R : \mathbf{C} \to \mathbf{Sets}$ such that $|R(X)| > 1$ and $|R(Y)| > 1$, and a group $\mathbf{C}$ with a functor $S : \mathbf{C} \to \mathbf{Sets}$. Consider two elements x_1 and x_2 of $R(X)$, an element y of $R(Y)$, and assume that we have $R(f)(x_1) = R(f)(x_2) = y$. By definition of the PK-net, we must have $SF(f)(\phi_X(x_1)) = SF(f)(\phi_X(x_2)) = \phi_Y(y)$, and since $\mathbf{C}$ is a group this imposes $\phi_X(x_1) = \phi_X(x_2)$, i.e. the two musical objects x_1 and x_2 must have the same labels.

As an example, there is no possibility to define a PK-net showing the inverse transformation from a dominant seventh E^7 chord to a C major triad. If one tries to define a new PK-net (R', S, F', ϕ') such that $F'(f) = T_8$, $R'(X) = \{x_1, x_2, x_3, x_4\}$, $R'(Y) = \{y_1, y_2, y_3\}$, and with a natural transformation ϕ' such that $\phi'_X(x_1) = E$, $\phi'_X(x_2) = G\sharp$, $\phi'_X(x_3) = B$, and $\phi'_Y(y_1) = C$, $\phi_Y(y_2) = E$, $\phi_Y(y_3) = G$, then one quickly sees that no function $R'(f)$ can exist which would satisfy the requirement that ϕ' is a natural transformation from R to SF', since all the elements constituting the seventh chord have different labels in $\mathbb{Z}_{12}$.

In a possible way to resolve this problem, and from the point of view that the E^7 dominant seventh chord consists of an E major chord with an added D note, we would intuitively like to "forget" about the D and consider only the transformation of x_1, x_2, and x_3 in $R(X)$ to y_1, y_2, and y_3 respectively in $R'(Y)$. In other words, we would like to consider *partial functions* between sets, instead of ordinary ones. In order to do so, one must abandon the category **Sets** and choose a category which makes it possible to consider such morphisms. This simple example motivates the introduction in this paper of *relational PK-nets* valued in the category **Rel** of sets and binary relations between them. Although there also exists a category **Par** whose objects are sets and whose morphisms are partial functions between them, the use of **Rel** includes the case of partial functions as well as even more general applications, as will be seen in the next sections.

1.3. *The use of relations in transformational music theory*

The use of relations between musical objects figures prominently in the recent literature on music theory. Perhaps one of the most compelling examples is the work of Douthett–Steinbach and

Cohn on parsimonious voice leading and its subsequent formalization in the form of parsimonious graphs (Douthett and Steinbach 1998; Cohn 2012). In order to formalize the notion of parsimony, Douthett and Steinbach (1998) introduced the $\mathcal{P}_{m,n}$ relation between two pc-sets, the definition of which we recall here. We recall that a *pitch-class set (pc-set)* is a set in $\mathbb{Z}_{12}$ (where $\mathbb{Z}_{12}$ encodes the twelve pitch classes with the usual semitone encoding).

Definition 1.3 Let O and O' be two pitch-class sets of equal cardinality. We say that O and O' are $\mathcal{P}_{m,n}$*-related* if there exists a partition of $O \backslash O'$ into two disjoint subsets O_1 and O_2 of cardinality m and n respectively, along with injective functions $\tau_1 : O_1 \to O' \backslash O$ and $\tau_2 : O_2 \to O' \backslash O$ such that

- the image sets $\tau_1(O_1)$ and $\tau_2(O_2)$ are disjoint, and
- $\tau_1(x) = x \pm 1$, and $\tau_2(x) = x \pm 2$.

In other words, if O' is $\mathcal{P}_{m,n}$-related to O, m pitch classes in O move by a semitone, while n pitch classes move by a whole tone, the rest of the pitch classes being identical. For example, the set $\{C, E, G\}$ representing the C major chord is $\mathcal{P}_{1,0}$-related to the set $\{C, E\flat, G\}$ representing the C minor chord, since they only differ by elements of the set $\{E, E\flat\}$ which are a semitone apart. Note that $\mathcal{P}_{m,n}$ is a symmetric relation. From this definition, Douthett and Steinbach define a *parsimonious graph for a* $\mathcal{P}_{m,n}$ *relation* on a set H of pc-sets as the graph whose set of vertices is H and whose set of edges is the set

$$\{(O, O') \mid O \in H,\ O' \in H,\ \text{and}\ O\mathcal{P}_{m,n}O'\}.$$

A notable example is the "Cube Dance" graph, which is the parsimonious graph for the $\mathcal{P}_{1,0}$ relation (i.e. the voice-leading relation between chords resulting from the ascending or descending movement of a single pitch class by a semitone) on the set H of 28 elements containing the 24 major and minor triads as well as the four augmented triads. This graph is reproduced in Figure 5. One should note that this graph contains the subgraphs defined on the set H by the neo-Riemannian operations L and P viewed as relations. Indeed, given O and O' in H such that $O' = L(O)$ or $O' = P(O)$, one can immediately verify by definition of these neo-Riemannian operations that we have $O\mathcal{P}_{1,0}O'$. This subgraph is called "HexaCycles" by Douthett and Steinbach. The "Cube Dance" adds to "HexaCycles" the possible relations between the augmented triads and the hexatonic cycles, which, as commented by Douthett and Steinbach, "*serve as the couplings between hexatonic cycles and function nicely as a way of modulating between hexatonic sets*."

Douthett and Steinbach also introduced the "Weitzmann's Waltz" graph, which corresponds to the parsimonious graph for the $\mathcal{P}_{2,0}$ relation (i.e. the voice-leading relation between chords resulting from the ascending or descending movement of two pitch classes by a semitone) on the set H of 28 elements containing the 24 major and minor triads as well as the four augmented triads. This graph is reproduced in Figure 6.

Whereas relations can be described as graphs, which can then be used for musical applications, *transformational* analysis using relations is however trickier to define than in the case of groups and group actions. In the framework of Lewin, the image of a given x by the group action of an element g of a group G is determined unambiguously. Hence, one can speak about "applying the operation g to the musical element x," or about "the image of the musical element x under the operation g." Assume instead that, given a relation $\mathcal{R}$ between two sets X and Y and an element x of X, there exist multiple y 's in Y such that we have $x\mathcal{R}y$. How then can one define "the image of the musical element x under the relation $\mathcal{R}$"? This question motivates the use of the more general framework of *relational PK-nets*, which we define in the next section.

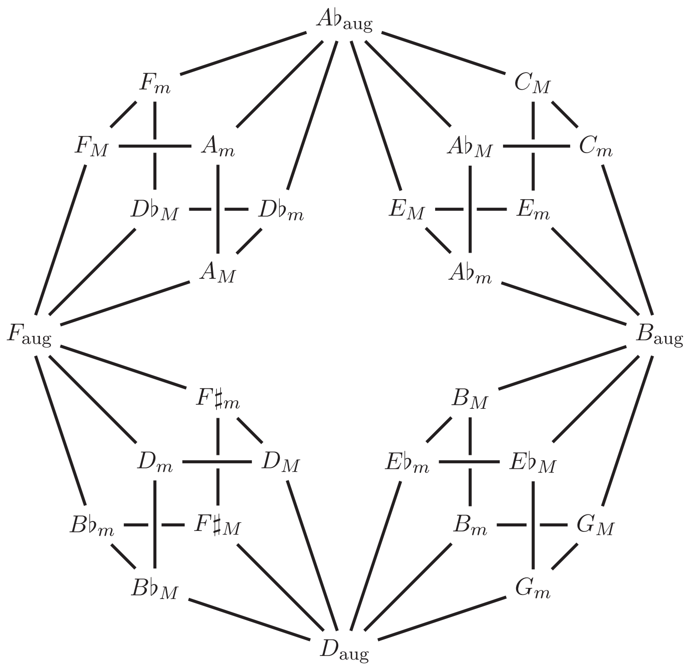

Figure 5. Douthett and Steinbach's "Cube Dance" graph as the parsimonious graph for the $\mathcal{P}_{1,0}$ relation on the set of the 24 major and minor triads and the four augmented triads (Douthett and Steinbach 1998, page 254, Figure 9). The subscripts *M*, *m*, and aug refer to major, minor, and augmented triads, respectively.

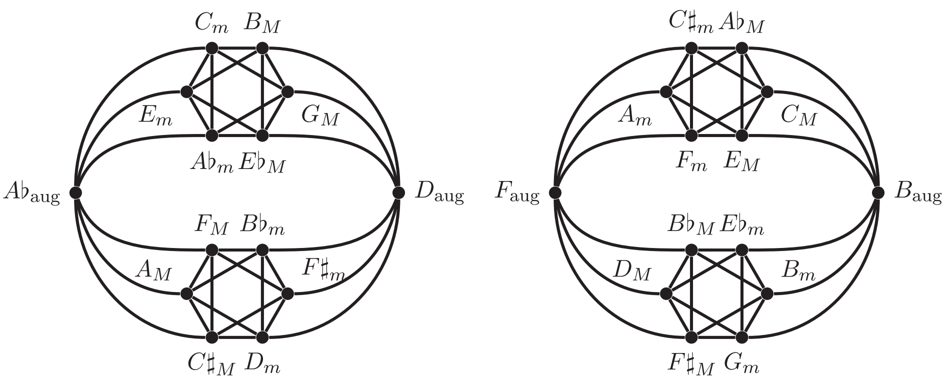

Figure 6. Douthett and Steinbach's "Weitzmann's Waltz" graph as the parsimonious graph for the $\mathcal{P}_{2,0}$ relation on the set of the 24 major and minor triads and the four augmented triads (Douthett and Steinbach 1998, page 260, Figure 12).

2. Defining relational PK-nets

Before introducing the definition of relational PK-nets, we recall basic facts about relations and the associated category **Rel**.

2.1. *The 2-category Rel*

We first recall some basic definitions about relations.

Definition 2.1 Let X and Y be two sets. A *binary relation* $\mathcal{R}$ *between* X *and* Y is a subset of the cartesian product $X \times Y$. We say that $y \in Y$ is related to $x \in X$ by $\mathcal{R}$, which is notated as $x\mathcal{R}y$, if $(x, y) \in \mathcal{R}$.

Definition 2.2 Let $\mathcal{R}$ be a relation between two sets X and Y. We say that $\mathcal{R}$ is *left-total* if, for each $x \in X$, there exists at least one $y \in Y$ such that we have $x\mathcal{R}y$.

Definition 2.3 Let X and Y be two sets, and $\mathcal{R}$ and $\mathcal{R}'$ be two relations between them. The relation $\mathcal{R}$ is said to be *included in* $\mathcal{R}'$ if $x\mathcal{R}y$ implies $x\mathcal{R}'y$, for all pairs $(x, y) \in X \times Y$.

Relations can be composed according to the definition below.

Definition 2.4 Let X, Y, and Z be sets, $\mathcal{R}$ a relation between X and Y, and $\mathcal{R}'$ be a relation between Y and Z. The *composition of* $\mathcal{R}'$ *and* $\mathcal{R}$ is the relation $\mathcal{R}'' = \mathcal{R}' \circ \mathcal{R}$ defined by the pairs (x, z) with $x \in X$ and $z \in Z$ such that there exists at least one $y \in Y$ with $x\mathcal{R}y$ and $y\mathcal{R}'z$.

Sets and binary relations between them form a 2-category **Rel** defined as follows.

Definition 2.5 The 2-category **Rel** is the category which has sets as objects, binary relations as 1-morphisms between them, and inclusion of relations as 2-morphisms between relations.

Notice that the definition of relations includes the special case of functions between sets: if $\mathcal{R}$ is a relation between two sets X and Y, then $\mathcal{R}$ is a function if, given an element $x \in X$, there exists exactly one $y \in Y$ such that we have $x\mathcal{R}y$. As a consequence, the category **Sets** of sets and functions between them is a subcategory of **Rel**. Notice also that the definition of relations also includes the case of partial functions between sets: if $\mathcal{R}$ is a relation between two sets X and Y, it may be possible that, given an element $x \in X$, there exists no $y \in Y$ such that we have $x\mathcal{R}y$. Hence, the category **Par** of sets and partial functions between them is also a subcategory of **Rel**.

Since **Rel** is a 2-category, the notion of a *lax functor* from a 1-category to **Rel** should be recalled to account for the possible 2-morphisms between relations. In its generality, a lax functor has a comparison natural 2-cell for identity morphisms and a comparison natural 2-cell for compositions, and these comparison natural 2-cells are required to satisfy three coherence diagrams. Since there is at most one 2-cell between any two morphisms in **Rel**, all diagrams of 2-cells in **Rel** commute. It follows that the definition of lax functor into **Rel** can be simplified, since the requirements on the coherence diagrams are automatically satisfied. Moreover, all the lax functors we consider in this paper strictly preserve identities, so we are interested in *normal* lax functors. The notion of a normal lax functor from a 1-category to **Rel** can thus be defined more precisely as follows.

Definition 2.6 Let **C** be a 1-category. A *normal lax functor F* from **C** to **Rel** is the data of a map

- which sends each object X of **C** to an object $F(X)$ of **Rel**, and the identity morphism id_X of X to the identity morphism $\mathrm{id}_{F(X)}$ of $F(X)$, and
- which sends each morphism $f : X \to Y$ of **C** to a relation $F(f) : F(X) \to F(Y)$ of **Rel**, such that for each pair (f, g) of composable morphisms $f : X \to Y$ and $g : Y \to Z$ the image relation $F(g)F(f)$ is included in $F(gf)$.

Remark 2.7 All the examples of lax functors considered in this paper are normal, so in all of the following, we use the expression "lax functor" to mean "*normal* lax functor."

A lax functor will be called a 1-functor (coinciding with the usual notion of functor between 1-categories) when $F(g)F(f) = F(gf)$.

Given two lax functors F and G to **Rel**, the notion of a *lax natural transformation* η between F and G should be recalled to account similarly for the possible 2-morphisms between lax functors.

Since the necessary coherence diagrams are automatically satisfied in **Rel**, this notion is defined as follows.

Definition 2.8 Let **C** be a 1-category, and let F and G be two lax functors from **C** to **Rel**. A *lax natural transformation* η between F and G is the data of a collection of relations $\{\eta_X : F(X) \to G(X)\}$ for all objects X of **C**, such that, for any morphism $f : X \to Y$, the relation $\eta_Y F(f)$ is included in the relation $G(f)\eta_X$.

Finally, there exists a notion of inclusion of lax natural transformations between functors going to **Rel**, which we define precisely below.

Definition 2.9 Let **C** be a 1-category, let F and G be two lax functors from **C** to **Rel**, and let η and η' be two lax natural transformations between F and G. We say that η is *included* in η' if, for any object X of **C**, the component η_X is included in the component η'_X.

The required equality axiom of a modification in the sense of 2-category theory is again automatically satisfied because of the specific structure of **Rel**.

2.2. *Relational PK-nets*

With the previous definitions in mind, we now give the formal definition of a relational PK-net.

Definition 2.10 Let **C** be a small 1-category, and S a lax functor from **C** to the category **Rel**. Let Δ be a small 1-category and R a lax functor from Δ to **Rel** with non-empty values. A *relational PK-net of form R and of support S* is a 4-tuple (R, S, F, ϕ), in which F is a functor from Δ to **C**, and ϕ is a lax natural transformation from R to SF, such that, for any object X of Δ, the component ϕ_X is left-total.

A relational PK-net can be represented by a diagram similar to the one represented in Figure 2, with **Sets** replaced by **Rel**. The definition given above is almost similar to that of a PK-net in **Sets**, but the specifics of the 2-category **Rel** impose slight adjustments. The first one is the requirement that the functor R shall be a lax functor to **Rel** instead of a 1-functor. To see why this is the case, let X, Y, and Z be objects of Δ, and let $f : X \to Y$, $g : Y \to Z$, and $h : X \to Z$ be morphisms between them, with $h = gf$. Relations are more general than functions: given the relation $R(f)$ between the sets $R(X)$ and $R(Y)$, it is possible that, for a given element $x \in R(X)$ there exist multiple elements $y \in R(Y)$ such that we have $xR(f)y$, or even none at all. To be consistent, we require that, given $x \in R(X)$ and $z \in R(Z)$, we have that $xR(g) \circ R(f)z$ implies $xR(h)z$. However, we do not require the strict equality of relations, as it gives more flexibility over the possible relations with the elements of $R(Y)$. The first example below will clarify this notion in the case of the analysis of sets of varying cardinalities, in particular for triads and seventh chords. Note that the same logic requires that S shall be a lax functor to **Rel** as well.

The second adjustment corresponds to the requirement that ϕ shall be a lax natural transformation from R to SF instead of an ordinary one. Let us recall the role of the functor S and the natural tranformation ϕ in the case of PK-nets in **Sets**. The lax functor S defines the context of the analysis: for any objects e and e' of **C** and a morphism $f : e \to e'$ between them, the sets $S(e)$ and $S(e')$ represent all the musical entities of interest and the function $S(f)$ represents a transformation between them. Given a set of unnamed musical objects $R(X)$ (with X being an object of the category Δ), the component ϕ_X of the natural transformation ϕ is a function which "names" these objects by their images in $SF(X)$. In the case of relational PK-nets, since $S(f)$ is a relation instead of a function, it is possible that an element of $S(e)$ may be related to more than one element of $S(e')$, or even none. However, the relations between the actual musical objects under study in the images of the lax functor R may not cover the range of possibilities offered by the relations in the image of SF. Thus, we use a lax natural transformation ϕ instead of an ordinary one, such that the images by ϕ of the relations given through R be included in those given through SF. Informally speaking, the lax natural transformation ϕ "selects" a restricted range of related elements among the possibilities given by the functor SF. In addition, we require that each component of ϕ be left-total, so that all musical objects are "named" in the images of the functor SF. All the constitutive elements of a relational PK-net $(R,S,F\phi)$ are summed-up in the diagram of Figure 7. In addition, various examples in Section 3 of this paper will clarify this notion.

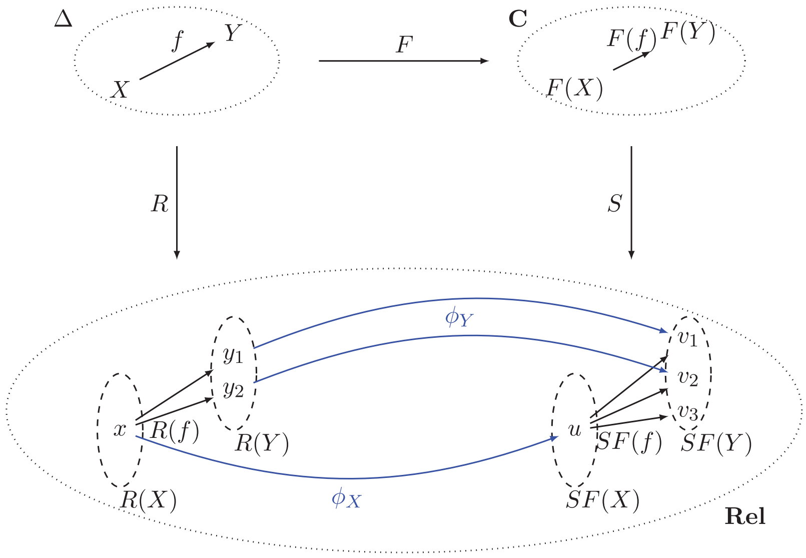

Figure 7. Diagram showing the constitutive elements of a simple relational PK-net (R,S,F,ϕ).

We now give two simple examples to illustrate the advantages of relational PK-nets. The first example deals with sets of varying cardinalities.

Example 2.11 Let **C** be the T/I group, considered as a single-object category, and consider its natural action on the set $\mathbb{Z}_{12}$ of the twelve pitch classes (with the usual semitone encoding), which defines a functor $S : T/I \to \mathbf{Rel}$. Let Δ be the category with three objects X, Y, and Z and precisely three morphisms $f : X \to Y$, $g : Y \to Z$, and $h : X \to Z$, with $h = gf$. Consider the functor $F : \Delta \to T/I$ which sends f to I_3, g to I_5, and h to T_2.
Consider now a lax functor $R : \Delta \to \mathbf{Rel}$ such that we have

- $R(X) = \{x_1,x_2,x_3,x_4\}$, $R(Y) = \{y_1,y_2,y_3\}$, and $R(Z) = \{z_1,z_2,z_3,z_4\}$, and
- the relation $R(f)$ is such that $x_iR(f)y_i$, for $1 \leq i \leq 3$, the relation $R(g)$ is such that $y_iR(g)z_i$, for $1 \leq i \leq 3$, and the relation $R(h)$ is such that $x_iR(h)z_i$, for $1 \leq i \leq 4$.

Consider the left-total lax natural transformation ϕ such that

- $\phi_X(x_1) = C$, $\phi_X(x_2) = E$, $\phi_X(x_3) = G$, $\phi_X(x_4) = B$, and
- $\phi_Y(y_1) = E\flat$, $\phi_Y(y_2) = B$, $\phi_Y(y_3) = G\sharp$, and
- $\phi_Z(z_1) = D$, $\phi_Z(z_2) = F\sharp$, $\phi_Z(z_3) = A$, $\phi_Z(z_4) = C$.

Then (R,S,F,ϕ) is a relational PK-net of form R and support S which describes the T_2 transposition of the dominant seventh C^7 chord to the dominant seventh D^7 chord and the successive I_3 and I_5 inversions of its underlying C major triad. A simplified representation of this relational PK-net is given in Figure 8.

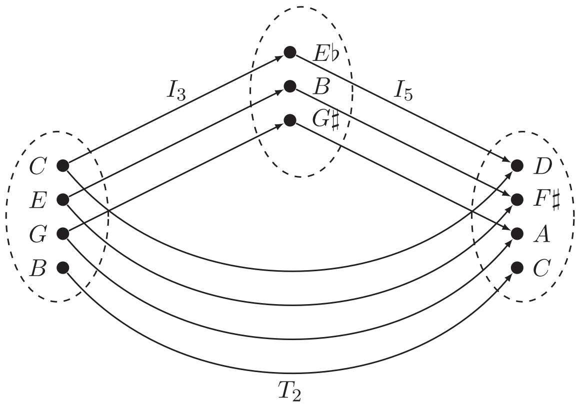

Figure 8. Simplified representation of the relational PK-net (R, S, F, ϕ) of Example 2.11 which describes the T_2 transposition of the dominant seventh C^7 chord to the dominant seventh D^7 chord and the successive I_3 and I_5 inversions of its underlying C major triad.

This simple example is of particular interest as it shows the advantage of relational PK-nets over the usual PK-nets in **Sets** to describe transformations between sets of decreasing cardinalities. Here, the relation $R(f)$ is a partial function from $R(X)$ to $R(Y)$ which "forgets" the element x_4, allowing us to describe the transformation of the major triad on which the initial seventh chord is built. The requirement that R be a lax functor appears clearly: the composite relation $R(g) \circ R(f)$ only relates the first three elements x_1, x_2, and x_3 of $R(X)$ to the first three elements z_1, z_2, and z_3 of $R(Z)$ (i.e. the underlying major triads), whereas the relation $R(h) = R(gf)$ relates all four elements, i.e. it describes the full transformation of the seventh chord by the T_2 transposition. We thus require that the relation $R(g) \circ R(f)$ be included in $R(h)$.

The second example shows how relational PK-nets can model the *split* and *fuse* transformations of (Callender 1998).

Example 2.12 Figure 9(a) shows the essential voice leading from the right hand of the first bar of Scriabin, op. 65 no. 3, reducing the involved pitch components to two distinct sets $\{B\flat, F, E\flat\}$ and $\{A, B, E\}$. Callender (1998) proposes a possible voice-leading relation between them through the simultaneous splitting of E into $E\flat$ and F and the fusing of A and B into $B\flat$. This relation is illustrated in Figure 9(b). We can model this voice-leading relation by the following relational PK-net.

Let $\mathbf{C} = M$ be the monoid, considered as a single-object category, with presentation

$$M = \langle r \mid r^3 = r \rangle.$$

Let S be the functor from $\mathbf{C}$ to **Rel** such that

- the image of the single object of M by S is the set $\mathbb{Z}_{12}$ of the twelve pitch classes, and
- the image of the generator r of M by S is the symmetric relation defined by the subset

$$S(r) = \{(E, E\flat), (E\flat, E), (E, F), (F, E), (B, A), (A, B), (B\flat, B), (B, B\flat)\}$$

of $\mathbb{Z}_{12} \times \mathbb{Z}_{12}$.

Let Δ be the category with two objects X, Y, and precisely four non-identity morphisms $f : X \to Y$, $g : Y \to X$, $gf : X \to X$, and $fg : Y \to Y$. Consider the functor $F : \Delta \to M$ which sends both f and g to r.

Consider now a lax functor $R : \Delta \to$ **Rel** such that we have

- $R(X) = \{x_1, x_2, x_3\}$, $R(Y) = \{y_1, y_2, y_3\}$, and

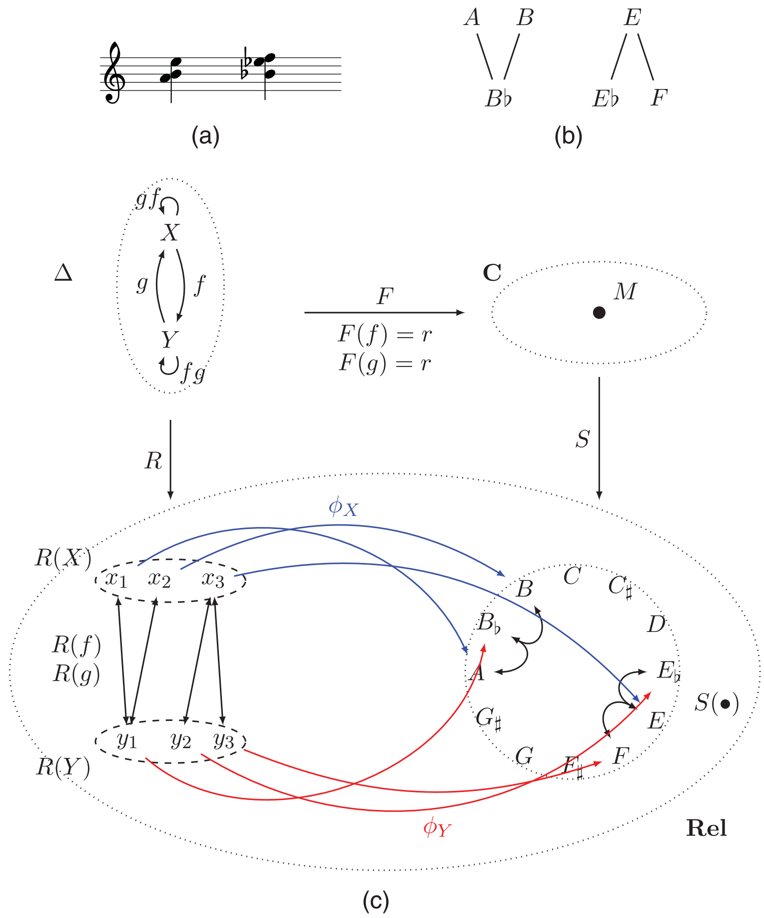

Figure 9. (a), (b) The right hand of the first bar of Scriabin, op. 65 no. 3, wherein the involved pitch components have been reduced as two distinct sets $\{B\flat, F, E\flat\}$ and $\{A, B, E\}$, and a possible voice-leading relation by simultaneous splitting and fusing of the individual pitch classes (Callender 1998, page 224, Figure 5). (c) Diagram showing the constitutive elements of the relational PK-net (R, S, F, ϕ) of Example 2.12 which describes how the set $\{B\flat, F, E\flat\}$ is related to the set $\{A, B, E\}$ by the simultaneous splitting of E and the fusing of A and B.

- the relation $R(f)$ is such that $x_1R(f)y_1$, $x_2R(f)y_1$, $x_3R(f)y_2$, $x_3R(f)y_3$, and
- the relation $R(g)$ is such that $y_1R(g)x_1$, $y_1R(g)x_2$, $y_2R(g)x_3$, $y_3R(g)x_3$.

Consider the left-total lax natural transformation ϕ such that

- $\phi_X(x_1) = A$, $\phi_X(x_2) = B$, $\phi_X(x_3) = E$, and
- $\phi_Y(y_1) = B\flat$, $\phi_Y(y_2) = F$, $\phi_Y(y_3) = E\flat$.

Then (R, S, F, ϕ) is a relational PK-net of form R and support S which describes how the set $\{B\flat, F, E\flat\}$ is related to the set $\{A, B, E\}$ by the simultaneous splitting of E and the fusing of A and B. This relational PK-net is represented in Figure 9(c).

Relational PK-nets of form R can be transformed by PK-homographies, which are defined as follows.

Definition 2.13 A *PK-homography* $(N, \nu) : K \to K'$ from a relational PK-net $K = (R, S, F, \phi)$ to a second relational PK-net $K' = (R, S', F', \phi')$ consists of a functor $N : \mathbf{C} \to \mathbf{C}'$ and a left-total lax natural transformation $\nu : SF \to S'F'$ such that $F' = NF$ and the composite lax natural transformation $\nu \circ \phi$ is included in ϕ'. A PK-homography is called a *PK-isography* if N is an isomorphism and ν is an equivalence.

There exists a notion of inclusion of PK-homographies, defined as follows.

Definition 2.14 Let $(N,\nu) : K \to K'$ and $(N',\nu') : K \to K'$ be PK-homographies from a relational PK-net $K = (R, S, F, \phi)$ to a second relational PK-net $K' = (R, S', F', \phi')$. We say that (N,ν) is *included* in (N',ν') if ν is included in ν'.

For a given lax functor $R : \Delta \to \mathbf{Rel}$, relational PK-nets with fixed functor R form a 2-category $\mathbf{RelPKN_R}$, which is defined as follows.

Definition 2.15 For a given lax functor $R : \Delta \to \mathbf{Rel}$, the 2-category $\mathbf{RelPKN_R}$ has the relational PK-nets of form R as objects, PK-homographies (N,ν) between them as 1-morphisms, and inclusion of PK-homographies as 2-morphisms.

3. Relational PK-nets in monoids of parsimonious relations

In this section, we revisit the work of Douthett and Steinbach about parsimonious relations, and show how we can define proper relational PK-nets for transformational music analysis on major, minor, and augmented triads. We begin by defining a new monoid of parsimonious relations originating from the “Cube Dance”.

3.1. *The monoid $M_{\mathcal{UPL}}$*

As discussed in Section 1.3, the “Cube Dance” graph represented in Figure 5 results from the HexaCycles graph to which the vertices and edges corresponding to the four augmented triads and their $\mathcal{P}_{1,0}$ relation to major and minor triads have been added. The HexaCycles graph itself results from the neo-Riemannian operations L and P viewed as relations on the set of the 24 major and minor triads. We formalize the construction of the “Cube Dance” graph by defining three different relations on the 28-elements set of the 24 major and minor triads and the four augmented triads. We adopt here the usual semitone encoding of pitch classes with $C=0$, and we notate a major chord by n_M and a minor chord by n_m, where n is the root pitch class of the chord, with $0 \leq n \leq 11$. For the four augmented triads, we adopt the following notation: $A\flat_{\text{aug}} = 0_{\text{aug}}$, $F_{\text{aug}} = 1_{\text{aug}}$, $D_{\text{aug}} = 2_{\text{aug}}$, and $B_{\text{aug}} = 3_{\text{aug}}$. All arithmetic operations are understood modulo 12, unless otherwise indicated.

Definition 3.1 Let H be the set of the 24 major and minor triads and the four augmented triads, i.e. $H = \{n_M \mid 0 \leq n \leq 11\} \cup \{n_m \mid 0 \leq n \leq 11\} \cup \{n_{\text{aug}} \mid 0 \leq n \leq 3\}$. We define the following relations on H.

- The relation $\mathcal{P}$ is the symmetric relation such that we have $n_M \mathcal{P} n_m$ for $0 \leq n \leq 11$, and $n_{\text{aug}} \mathcal{P} n_{\text{aug}}$ for $0 \leq n \leq 3$. This is the relational analogue of the neo-Riemannian P operation. The relation $\mathcal{P}$ defined here should not be confused with Douthett and Steinbach’s $\mathcal{P}_{m,n}$ relations.
- The relation $\mathcal{L}$ is the symmetric relation such that we have $n_M \mathcal{L} (n+4)_m$ for $0 \leq n \leq 11$, and $n_{\text{aug}} \mathcal{L} n_{\text{aug}}$ for $0 \leq n \leq 3$. This is the relational analogue of the neo-Riemannian L operation.
- The relation $\mathcal{U}$ is the symmetric relation such that we have $n_M \mathcal{U} (n \pmod 4)_{\text{aug}}$ for $0 \leq n \leq 11$, and $n_m \mathcal{U} ((n+3) \pmod 4)_{\text{aug}}$ for $0 \leq n \leq 11$.

The $\mathcal{P}$ and $\mathcal{L}$ relations come from the neo-Riemannian P and L functions. The relation $\mathcal{U}$ is introduced to couple the hexatonic cycles, by relating any augmented chord with the major or minor chords sharing two tones in common. As such, the relation $\mathcal{U}$ is genuinely not a function since there are always three major chords and three minor chords related by $\mathcal{U}$ to a given augmented chord.

We are now interested in the monoid $M_{\mathcal{UPL}}$ generated by the relations $\mathcal{U}$, $\mathcal{P}$, and $\mathcal{L}$ under the composition of relations introduced in Section 2.1. The structure of this monoid can be determined by hand through an exhaustive enumeration, or more simply with any computational algebra software, such as GAP (2016). The following proposition gives a presentation of the monoid $M_{\mathcal{UPL}}$.

Proposition 3.2 *The monoid $M_{\mathcal{UPL}}$ generated by the relations $\mathcal{U}$, $\mathcal{P}$, and $\mathcal{L}$ has the following presentation:*

$$\begin{aligned} M_{\mathcal{UPL}} = \langle \mathcal{U},\mathcal{P},\mathcal{L} \mid\ & \mathcal{P}^2=\mathcal{L}^2=e,\ \ \mathcal{LPL}=\mathcal{PLP},\ \ \mathcal{U}^3=\mathcal{U},\\ & \mathcal{UP}=\mathcal{UL},\ \ \mathcal{PU}=\mathcal{LU},\ \ \mathcal{U}^2\mathcal{PU}^2=\mathcal{PU}^2\mathcal{PU}^2\mathcal{P},\\ & (\mathcal{UP})^2\mathcal{U}^2=\mathcal{P}(\mathcal{UP})^2\mathcal{U}^2\mathcal{P},\ \ \mathcal{U}^2(\mathcal{PU})^2=\mathcal{PU}^2(\mathcal{PU})^2\mathcal{P}\rangle. \end{aligned}$$

The monoid $M_{\mathcal{UPL}}$ contains 40 elements.

The Cayley graph of this monoid is represented in Figure 10. The only invertible elements of the monoid $M_{\mathcal{UPL}}$ are the set $\{e,\mathcal{L},\mathcal{P},\mathcal{LP},\mathcal{PL},\mathcal{LPL}\}$, which forms a subgroup in $M_{\mathcal{UPL}}$ isomorphic to the dihedral group D_6 generated by the neo-Riemannian operations P and L.

In view of building PK-isographies between relational PK-nets on $M_{\mathcal{UPL}}$, the next proposition establishes the structure of the automorphism group of the monoid $M_{\mathcal{UPL}}$.

Proposition 3.3 *The automorphism group of the monoid $M_{\mathcal{UPL}}$ is isomorphic to the group $D_6 \times \mathbb{Z}_2$.*

Proof Any automorphism N of the monoid $M_{\mathcal{UPL}}$ is entirely determined by its image of generators. Since $\mathcal{L}$ and $\mathcal{P}$ are the only invertible generators, their images belong to the D_6 subgroup $\{e,\mathcal{L},\mathcal{P},\mathcal{LP},\mathcal{PL},\mathcal{LPL}\}$ and induce an isomorphism of this subgroup. From known results about dihedral groups, we have $\mathrm{Aut}(D_6) \simeq D_6$, and the images are given by $N(\mathcal{P}) = (\mathcal{PL})^{m+n}\mathcal{L}$, and $N(\mathcal{L}) = (\mathcal{PL})^n\mathcal{L}$, with $m \in \{-1,1\}$ and $n \in \{0,1,2\}$.

This defines a homomorphism Φ from $\mathrm{Aut}(M_{\mathcal{UPL}})$ to D_6 which associates to any automorphism N of $M_{\mathcal{UPL}}$ the automorphism of D_6 induced by the images of $\mathcal{L}$ and $\mathcal{P}$.

The kernel of Φ consists of the subgroup of $\mathrm{Aut}(M_{\mathcal{UPL}})$ formed by the automorphisms N such that $N(\mathcal{L}) = \mathcal{L}$ and $N(\mathcal{P}) = \mathcal{P}$. It is thus uniquely determined by the possible images of the remaining generator $\mathcal{U}$ by N. An exhaustive computer search shows that only $N(\mathcal{U}) = \mathcal{U}$ and $N(\mathcal{U}) = \mathcal{PUP}$ yield valid automorphisms, i.e. $N(\mathcal{U}) = \mathcal{P}^k\mathcal{UP}^k$, with $k \in \{0,1\}$ considered as the additive cyclic group $\mathbb{Z}_2$. Thus $\mathrm{Aut}(M_{\mathcal{UPL}})$ is an extension of $\mathbb{Z}_2$ by $\mathbb{D}_6$, and any automorphism N is uniquely determined by the pair (g,k) where g is an element of $\mathrm{Aut}(D_6)$, and k is an element of $\mathbb{Z}_2$.

Let $N_1 = (g_1,k_1)$ and $N_2 = (g_2,k_2)$ be two automorphisms of the $M_{\mathcal{UPL}}$ monoid and consider $N = N_2N_1 = (g,k)$. From the discussion above, we have $g = g_2g_1$. The image of the generator $\mathcal{U}$ by N is

$$N(\mathcal{U}) = N_2(\mathcal{P}^{k_1}\mathcal{UP}^{k_1}),$$

which is equal to

$$N(\mathcal{U}) = ((\mathcal{PL})^{m_2+n_2}\mathcal{L})^{k_1}\mathcal{P}^{k_2}\mathcal{UP}^{k_2}((\mathcal{PL})^{m_2+n_2}\mathcal{L})^{k_1}.$$

Since we have $\mathcal{PU} = \mathcal{LU}$ and $\mathcal{UP} = \mathcal{UL}$, all the terms $\mathcal{P}$ in this last equation can be replaced by $\mathcal{L}$, and since $\mathcal{L}^2 = e$, this yields

$$N(\mathcal{U}) = \mathcal{L}^{k_1+k_2}\mathcal{UL}^{k_1+k_2}.$$

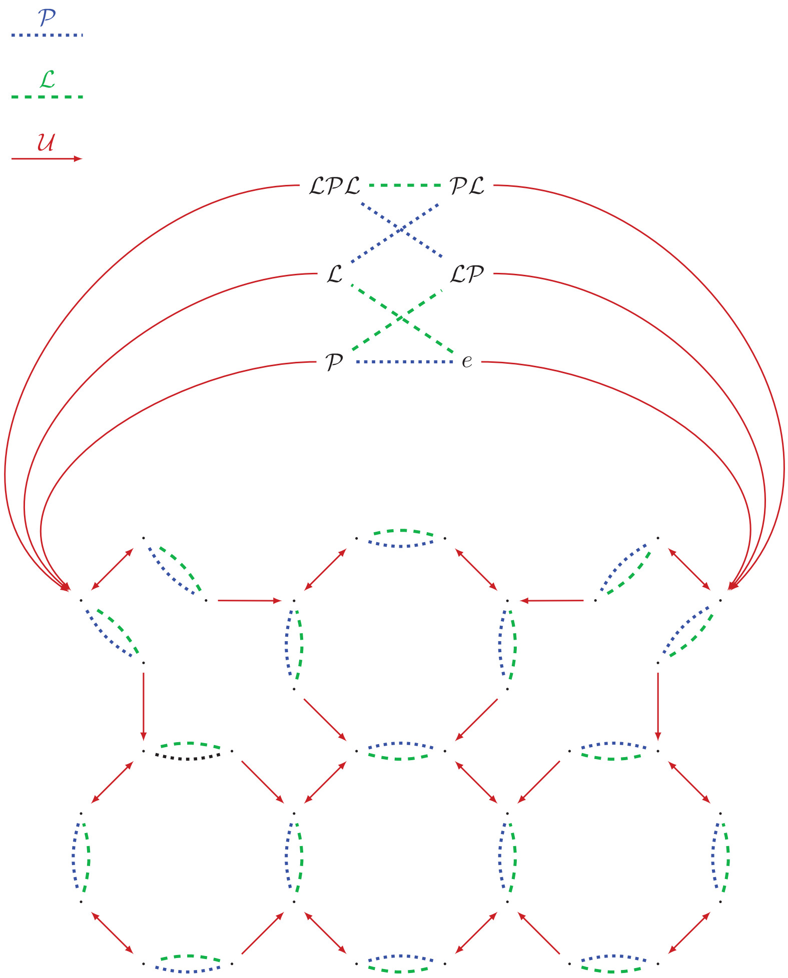

Figure 10. The Cayley graph of the monoid $M_{\mathcal{UPL}}$ generated by the relations $\mathcal{U}$, $\mathcal{P}$, and $\mathcal{L}$. The relations $\mathcal{L}$ and $\mathcal{P}$ are involutions, and are represented as arrowless dashed and dotted lines.

Hence, we have $N = N_2N_1 = (g, k) = (g_2g_1, k_2 + k_1)$, thus proving that $\mathrm{Aut}(M_{\mathcal{UPL}})$ is isomorphic to $D_6 \times \mathbb{Z}_2$. ■

We now illustrate the possibilities offered by the $M_{\mathcal{UPL}}$ monoid for transformational analysis using relational PK-nets. Figure 11(a) shows a reduction of the opening chord progression of the song *Take a Bow* by the English rock band Muse. This progression proceeds by semitone changes from a major chord to an augmented chord to a minor chord. At this point, the minor chord evolves to a major chord on the same root, i.e. the two chords are related by the neo-Riemannian operation P. The same process is then applied five times till the middle of the song (only the first twelve chords are represented in Figure 11(a)).

A first transformational analysis of this progression can be realized as follows. We focus here on the first four chords, since the progression proceeds further identically. The data of the monoid $M_{\mathcal{UPL}}$ and the relations over H of its elements defines a functor $S : M_{\mathcal{UPL}} \to$ **Rel**. Let Δ be the poset of the ordinal number **4** (whose objects are labelled X_i, with $0 \leq i \leq 3$), and let R be the functor from Δ to **Rel** which sends the objects X_i of Δ to singletons $\{x_i\}$. Let F be the functor from Δ to $M_{\mathcal{UPL}}$ which sends the non-trivial morphisms $f_{0,1} : X_0 \to X_1$ and $f_{1,2} : X_1 \to X_2$ of Δ to $\mathcal{U}$ in $M_{\mathcal{UPL}}$, and the non-trivial morphism $f_{2,3} : X_2 \to X_3$ of Δ to $\mathcal{P}$ in $M_{\mathcal{UPL}}$. Finally, let ϕ be the left-total lax natural transformation which sends x_0 to D_M, x_1 to D_{aug}, x_2 to G_m, and x_3 to G_M. Then (R, S, F, ϕ) is a relational PK-net describing the opening progression of Figure 11(a). A simplified representation of this relational PK-net (and its extension to the remaining chords) is

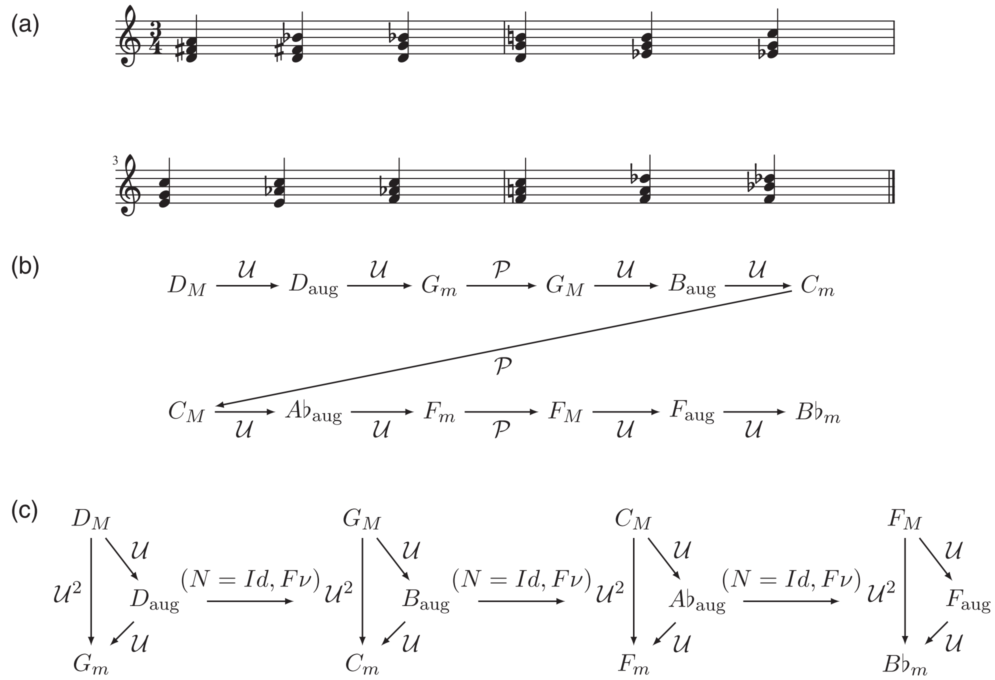

Figure 11. (a) Reduction of the opening progression of *Take a Bow* from Muse (the first twelve chords are represented here). (b) First transformational analysis in the $M_{\mathcal{UPL}}$ monoid showing the sequential regularity of the progression. (c) Second transformational analysis in the $M_{\mathcal{UPL}}$ monoid showing the successive transformations of the initial three-chord cell by the homography $(N = Id, F\nu)$ with $\nu(n_M) = (n+5)_M$, $\nu(n_m) = (n+5)_m$, and $\nu(n_{\text{aug}}) = (n+1 \pmod 4)_{\text{aug}}$.

shown in Figure 11(b), by directly labelling the arrows between the chords with the elements of $M_{\mathcal{UPL}}$. Given two chords on the left and right side of a labelled arrow between them, one should remain aware that this does not mean that the chord on the right is the unique image of the chord by the given element of $M_{\mathcal{UPL}}$. From a relational point of view, as discussed in Section 2.2, one should consider instead the chord on the right to be *related* to the one on the left by the element of $M_{\mathcal{UPL}}$, among the possibly larger range of possibilities given by the functor $S : M_{\mathcal{UPL}} \to \mathbf{Rel}$. For example, given the chord D_{aug}, there exist six chords y (namely D_M, $F\sharp_M$, $B\flat_M$, $E\flat_m$, B_m, and G_m) such that we have $D_{\text{aug}}\mathcal{U}y$. Here, the left-total lax natural transformation ϕ allows us to select precisely one chord, G_m, to explain the given chord progression.

This relational PK-net shows the regularity of the chord progression, but does not clearly evidence the progression by fourths of the initial three-chord cell. We propose now a second transformational analysis based on a specific PK-net isography and its iterated application on a relational PK-net describing this initial cell.

Let Δ be the poset of the ordinal number **3** (whose objects are labelled X_i, with $0 \leq i \leq 2$), and let R be the functor from Δ to **Rel** which sends the objects X_i of Δ to singletons $\{x_i\}$. Let F be the functor from Δ to $M_{\mathcal{UPL}}$ which sends the non-trivial morphisms $f_{0,1} : X_0 \to X_1$ and $f_{1,2} : X_1 \to X_2$ of Δ to $\mathcal{U}$ in $M_{\mathcal{UPL}}$. Finally, let ϕ be the left-total lax natural transformation which sends x_0 to D_M, x_1 to D_{aug}, and x_2 to G_m. Then (R,S,F,ϕ) is a relational PK-net describing the initial three-chord cell of Figure 11(a). Consider now the identity functor $N = Id$ on $M_{\mathcal{UPL}}$, along with the left-total lax natural transformation $\nu : S \to S$ defined on the set of the major, minor, and augmented triads by $\nu(n_M) = (n+5)_M$, $\nu(n_m) = (n+5)_m$, and $\nu(n_{\text{aug}}) = (n+1 \pmod 4)_{\text{aug}}$. By applying the PK-isography $(N, F\nu)$ on (R,S,F,ϕ), one obtains a second relational PK-net (R,S,F,ϕ') such that ϕ' sends x_0 to G_M, x_1 to B_{aug}, and x_2 to C_m, thus describing the progression by fourths of the initial cell. The successive chords are given by the iterated application of the same PK-isography $(N, F\nu)$, as shown in Figure 11(c).

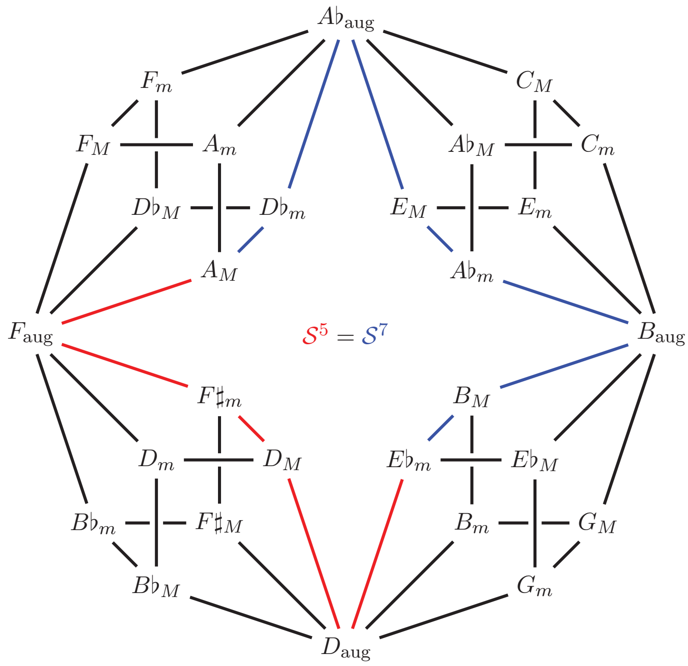

Figure 12. A graphical illustration on Douthett and Steinbach's "Cube Dance" graph of the identity $\mathcal{S}^7 = \mathcal{S}^5$ in the monoid $M_{\mathcal{S}}$. The chord A_M is related to $E\flat_m$ by the relation $\mathcal{S}^5$, which can be viewed as the result of taking two distinct paths of length five or seven in the graph.

3.2. *The monoids $M_{\mathcal{S}}$, $M_{\mathcal{T}}$, and $M_{\mathcal{ST}}$*

In the previous monoid of parsimonious relations, we distinguished the subrelations $\mathcal{U}$, $\mathcal{P}$, and $\mathcal{L}$ included in the $\mathcal{P}_{1,0}$ relation in order to differentiate the contributions of the neo-Riemannian operations P and L, and the relation $\mathcal{U}$ which bridges the hexatonic system *via* the augmented triads. However, we could also consider Douthett and Steinbach's $\mathcal{P}_{1,0}$ relation as a whole, to focus on the parsimonious voice leading between chords by any single semitone displacement.

For clarity of notation, we rename the $\mathcal{P}_{1,0}$ relation to $\mathcal{S}$ and we recall its definition on the set of major, minor, and augmented triads.

Definition 3.4 Let H be the set of the 24 major and minor triads and the 4 augmented triads, i.e. $H = \{n_M \mid 0 \leq n \leq 11\} \cup \{n_m \mid 0 \leq n \leq 11\} \cup \{n_{aug} \mid 0 \leq n \leq 3\}$. The $\mathcal{S}$ relation on H is defined as the symmetric relation such that we have $n_M \mathcal{S} n_m$, $n_M \mathcal{S} (n+4)_m$, $n_M \mathcal{S} (n \pmod 4)_{aug}$, and $n_m \mathcal{S} ((n+3) \pmod 4)_{aug}$ for $0 \leq n \leq 11$.

We are now interested in the monoid generated by the relation $\mathcal{S}$, under the composition of relations introduced in Section 2.1. The structure of this monoid can easily be determined by hand or with a computer, and is given in the following proposition.

Proposition 3.5 *The monoid $M_{\mathcal{S}}$ generated by the relation $\mathcal{S}$ has the following presentation:*

$$M_{\mathcal{S}} = \langle \mathcal{S} \mid \mathcal{S}^7 = \mathcal{S}^5 \rangle.$$

The identity $\mathcal{S}^7 = \mathcal{S}^5$ can be proved rigorously by writing down explicitly the subset of $H \times H$ to which it corresponds. Alternatively, this identity can be directly visualized on Douthett and Steinbach's "Cube Dance" graph, as shown in Figure 12. As a quick application of this monoid, consider the following example. The data of the monoid $M_{\mathcal{S}}$ and the relations over H of its elements define a functor $S : M_{\mathcal{S}} \rightarrow \mathbf{Rel}$. Let Δ be the poset of the ordinal number $\mathbf{2}$, i.e. the

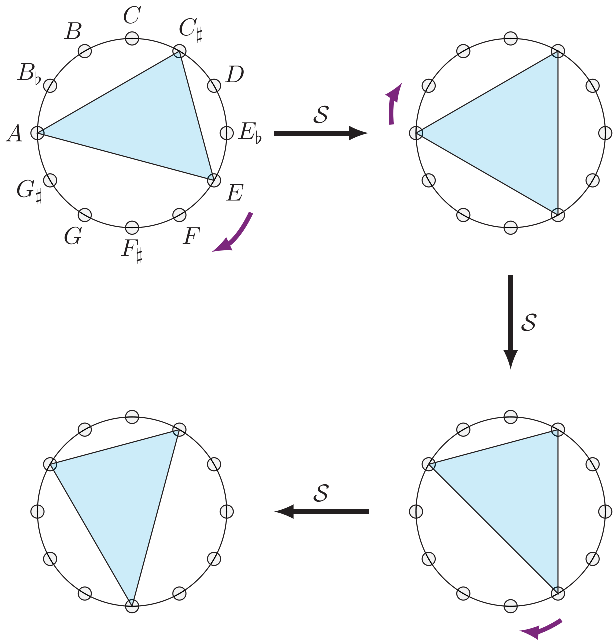

Figure 13. A possible path for the progression from A major to $F\sharp$ major by successive semitone displacement. The labels above the arrows indicate that the chord on the right is related to the chord on the left by the $\mathcal{S}$ relation. The corresponding semitone displacements are indicated by the curved violet arrows.

category with only two objects X and Y and only one non-trivial morphism $f : X \to Y$ between them, and let R be the functor from Δ to **Rel** which sends the objects X and Y of Δ to the singletons $\{x\}$ and $\{y\}$. Let $\mathbf{C}$ be the $M_{\mathcal{P}_{1,0}}$-monoid with the above-defined functor $S : M_{\mathcal{S}} \to \mathbf{Rel}$. Finally, let ϕ be the left-total lax natural transformation which sends x to A_M, and y to $F\sharp_M$. We are interested in the possible functors F such that (R,S,F,ϕ) is a relational PK-net describing the relation between A_M and $F\sharp_M$. A rapid verification through the elements of $M_{\mathcal{S}}$ shows that only $F(f) = \mathcal{S}^3$ and $F(f) = \mathcal{S}^5$ yield valid choices for F. Observe that these relations correspond to the first two shortest distances between A_M and $F\sharp_M$ in the "Cube Dance" of Figure 5. A possible path for the progression from A_M to $F\sharp_M$ by three successive semitone displacements is given in Figure 13, which shows the successive relations $A_M\mathcal{S}F_{\text{aug}}$, $F_{\text{aug}}\mathcal{S}B\flat_m$, and $B\flat_m\mathcal{S}F\sharp_m$.

As introduced in Section 1.3, Douthett and Steinbach also studied the parsimonious graph induced by the $\mathcal{P}_{2,0}$ relation on the set of major, minor, and augmented triads (the "Weitzmann's Waltz" graph shown in Figure 6). The $\mathcal{P}_{2,0}$ relation relates two pitch-class sets if one can be obtained from the other by the displacement of two pitch classes by a semitone each, which includes both the case of the parallel displacement of these pitch classes, as well as their contrary movement.

As before, we rename the $\mathcal{P}_{2,0}$ relation as $\mathcal{T}$ for clarity of notation, and we recall its specific definition on the set of major, minor, and augmented triads.

Definition 3.6 Let H be the set of the 24 major and minor triads and the four augmented triads, i.e. $H = \{n_M \mid 0 \le n \le 11\} \cup \{n_m \mid 0 \le n \le 11\} \cup \{n_{\text{aug}} \mid 0 \le n \le 3\}$. The $\mathcal{T}$ relation over H is defined as the symmetric relation such that we have

$$\begin{array}{r} n_M\mathcal{T}(n+4)_M \\ n_M\mathcal{T}(n+8)_M \\ n_m\mathcal{T}(n+4)_m \\ n_m\mathcal{T}(n+8)_m \end{array}, \quad \begin{array}{c} n_M\mathcal{T}(n+1)_m \\ n_M\mathcal{T}(n+5)_m \end{array}, \quad \text{and} \quad \begin{array}{r} n_M\mathcal{T}((n+3) \pmod 4)_{\text{aug}} \\ n_m\mathcal{T}(n \pmod 4)_{\text{aug}} \end{array}$$

for $0 \leq n \leq 11$.

As before, the structure of the monoid $M_{\mathcal{T}}$ generated by the relation $\mathcal{T}$ can easily be determined, and is given in the following proposition.

Proposition 3.7 *The monoid $M_{\mathcal{T}}$ generated by the relation $\mathcal{T}$ has the following presentation:*

$$M_{\mathcal{T}} = \langle \mathcal{T} \mid \mathcal{T}^4 = \mathcal{T}^3 \rangle.$$

Transformational analysis using relational PK-nets can be performed in the context of the monoid $M_{\mathcal{T}}$, but it should be noticed that given two chords x and y there may not always exist a relation $\mathcal{R}$ in $M_{\mathcal{T}}$ such that $x\mathcal{R}y$. Therefore, it may be interesting to combine both the $\mathcal{S}$ and the $\mathcal{T}$ relations in order to describe the relations between chords by a series of one or two pitch-class movements by semitones. We are thus interested in the structure of the monoid $M_{\mathcal{ST}}$ generated by the relations $\mathcal{S}$ and $\mathcal{T}$, which is given in the following proposition.

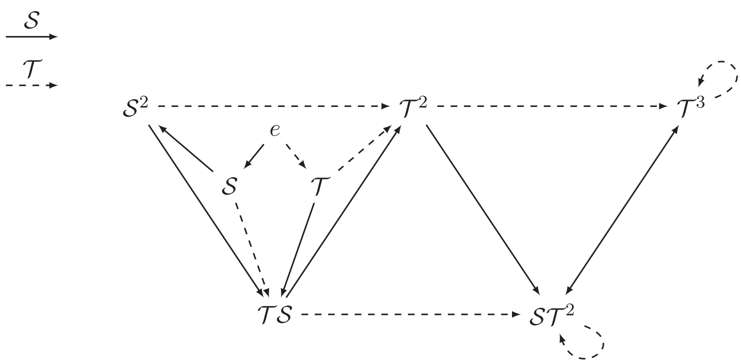

Figure 14. The Cayley graph of the monoid generated by the relations $\mathcal{S}$ and $\mathcal{T}$.

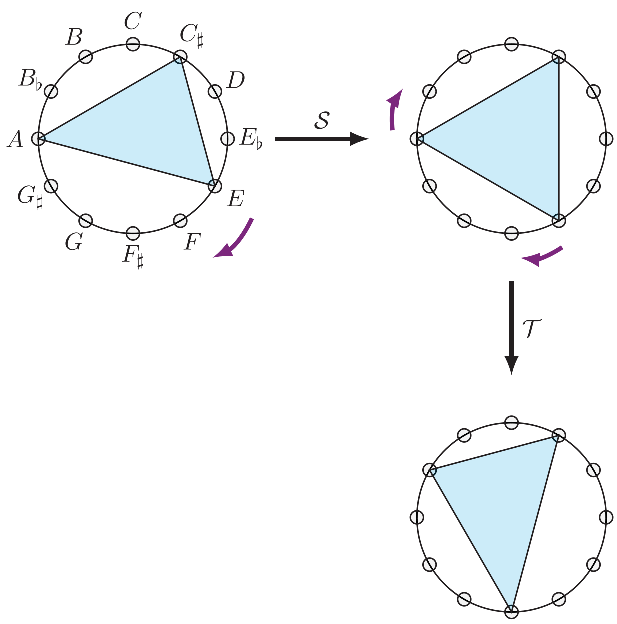

Figure 15. Transformational analysis of the progression A major to $F\sharp$ major in the context of the $M_{\mathcal{ST}}$ monoid. The semitone changes are indicated by the violet arrows.

Proposition 3.8 *The monoid $M_{\mathcal{S}\mathcal{T}}$ generated by the relations $\mathcal{S}$ and $\mathcal{T}$ has the following presentation:*

$$M_{\mathcal{S}\mathcal{T}} = \langle \mathcal{S}, \mathcal{T} \mid \mathcal{T}\mathcal{S} = \mathcal{S}\mathcal{T},\ \ \mathcal{S}^3 = \mathcal{S}\mathcal{T},\ \ \mathcal{T}^4 = \mathcal{T}^3,\ \ \mathcal{T}\mathcal{S}^2 = \mathcal{T}^2,\ \ \mathcal{S}\mathcal{T}^3 = \mathcal{S}\mathcal{T}^2 \rangle.$$

The monoid $M_{\mathcal{S}\mathcal{T}}$ contains eight elements.

The Cayley graph of this monoid is represented in Figure 14. As a quick application of this monoid, consider the aforementioned example for the relation between A_M and $F\sharp_M$ and let **C** be the new monoid $M_{\mathcal{S}\mathcal{T}}$. An enumeration of the elements of this monoid shows that only $F(f) = \mathcal{T}\mathcal{S}$ and $F(f) = \mathcal{S}\mathcal{T}^2$ yield valid choices for F. A possible path for the progression from A_M to $F\sharp_M$ is shown in Figure 15, which shows the successive relations $A_M \mathcal{S} F_{\text{aug}}$ and $F_{\text{aug}} \mathcal{T} F\sharp_m$.

4. Conclusions

We have presented in this work a new framework, called relational PK-nets, in which we consider diagrams in **Rel** rather than **Sets** as an extension of our previous work on PK-nets. We have shown how relational PK-nets capture both the group-theoretical approach and the relational approach of transformational music theory. In particular, we have revisited Douthett and Steinbach's parsimonious relations $\mathcal{P}_{m,n}$ by studying the structure of monoids based on the $\mathcal{P}_{1,0}$ or $\mathcal{P}_{2,0}$ relations (or subrelations of these), and their corresponding functors to **Rel** relating major, minor, and augmented triads. Further perspectives of relational PK-nets include their integration for computational music theory, providing a way for the systematic analysis of music scores.

Acknowledgments

We thank the anonymous referee and Co-Editor-in-Chief Thomas Fiore for valuable comments that lead to the improvement of the paper. We also thank Thomas Fiore for corrections to the galley proof.

Disclosure statement

No potential conflict of interest was reported by the authors.

References

Clifton, Callender. 1998. "Voice-Leading Parsimony in the Music of Alexander Scriabin." *Journal of Music Theory* 42 (2): 219–233.

Cohn, Richard. 2012. *Audacious Euphony: Chromaticism and the Triad's Second Nature*. New York: Oxford University Press.

Douthett, Jack, and Peter Steinbach. 1998. "Parsimonious Graphs: A Study in Parsimony, Contextual Transformations, and Modes of Limited Transposition." *Journal of Music Theory* 42 (2): 241–263.

Fiore, Thomas M., and Thomas Noll. 2011. *Commuting Groups and the Topos of Triads*, 69–83. Vol. 6726 of *Lecture Notes in Artificial Intelligence*, edited by Carlos Agon, Moreno Andreatta, Gérard Assayag, Emmanuel Amiot, Jean Bresson, and John Mandereau. Berlin: Springer-Verlag.

GAP. 2016. *GAP – Groups, Algorithms, and Programming, Version 4.8.6*, The GAP Group. http://www.gap-system.org.

Klumpenhouwer, Henry. 1991. "A Generalized Model of Voice-Leading for Atonal Music." PhD thesis, Harvard University.

Klumpenhouwer, Henry. 1998. "The Inner and Outer Automorphisms of Pitch-Class Inversion and Transposition: Some Implications for Analysis with Klumpenhouwer Networks." *Intégral* 12, 81–93.

Lewin, David. 1982. "Transformational Techniques in Atonal and Other Music Theories." *Perspectives of New Music* 21 (1-2): 312–381.

Lewin, David. 1987. *Generalized Music Intervals and Transformations*. New Haven, CT: Yale University Press.

Lewin, David. 1990. "Klumpenhouwer Networks and Some Isographies that Involve Them." *Music Theory Spectrum* 12 (1): 83–120.

Mazzola, Guerino, and Moreno Andreatta. 2006. "From a Categorical Point of View: K-Nets as Limit Denotators." *Perspectives of New Music* 44 (2): 88–113.

Nolan, Catherine. 2007. "Thoughts on Klumpenhouwer Networks and Mathematical Models: The Synergy of Sets and Graphs." *Music Theory Online* 13 (3): http://www.mtosmt.org/issues/mto.07.13.3/mto.07.13.3.nolan.pdf.

Noll, Thomas. 2005. "The Topos of Triads." In *Proceedings of the Colloquium on Mathematical Music Theory*, 103–135. Vol. 347 of *Grazer mathematische Berichte*, edited by Harald Fripertinger and Ludwig Reich, Karl-Franzens-Universität, Graz, Austria.

Popoff, Alexandre, Carlos Agon, Moreno Andreatta, and Andrée Ehresmann. 2016. "From K-Nets to PK-Nets: A Categorical Approach." *Perspectives of New Music* 54 (2): 5–63.

Popoff, Alexandre, Moreno Andreatta, and Andrée Ehresmann. 2015. "A Categorical Generalization of Klumpenhouwer Networks." In *Proceedings of the 5th International Conference on Mathematics and Computation in Music (MCM 2015)*, London, UK, 22–25 June 2015, 303–314, Vol. 9110 of *Lecture Notes in Computer Science*, edited by Tom Collins, David Meredith, and Anja Volk. Cham, Switzerland: Springer International Publishing.