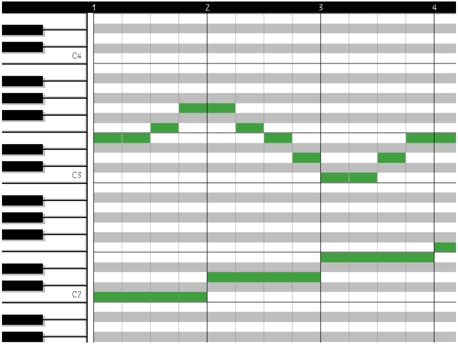
Mathematical Morphology and Musical Representations

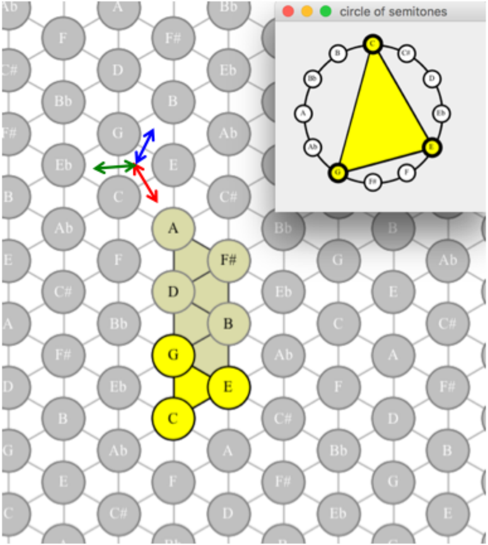
Carlos Agon, Moreno Andreatta, Jamal Atif, Isabelle Bloch

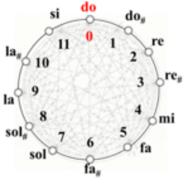
2019

Spatial representations of music

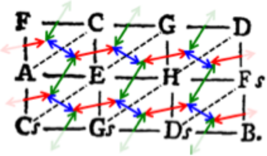

1
2
3
4
C4
C3
C2

Spatial representations and structure





Marin Mersenne



Leonhard Euler's *Speculum Musicum*

➔http://www.lacl.fr/~lbigo/hexachord

Mathematical morphology in a nutshell

Dilation: operation in complete lattices that commutes with the supremum.
Erosion: operation in complete lattices that commutes with the infimum.

$\Rightarrow$ applications on sets, fuzzy sets, functions, logical formulas, graphs, etc.

Using a structuring element:

- dilation as a degree of conjunction: $\delta_B(X) = \{x \in \mathcal{S} \mid B_x \cap X \neq \emptyset\}$,
- erosion as a degree of implication: $\varepsilon_B(X) = \{x \in \mathcal{S} \mid B_x \subseteq X\}$.

A lot of other operations...

Demonstrations...

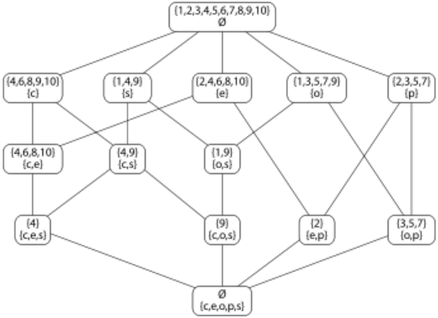
Formal framework: complete lattices

- Lattice: $(\mathcal{T}, \leq)$ ($\leq$ partial ordering) such that $\forall (x, y) \in \mathcal{T}, \exists x \vee y$ and $\exists x \wedge y$.
- Complete lattice: every family of elements (finite or not) has a smallest upper bound and a largest lower bound.
- Examples of complete lattices:
  - $(\mathcal{P}(E), \subseteq)$: complete lattice, Boolean (complemented and distributive)
  - $(\mathcal{F}(\mathbb{R}^d), \subseteq)$
  - functions of $\mathbb{R}^n$ in $\overline{\mathbb{R}}$ for the partial ordering $\leq$:
    $f \leq g \Leftrightarrow \forall x \in \mathbb{R}^n, \;\; f(x) \leq g(x)$
  - partitions
  - logics (propositional logics, modal logics...)
  - fuzzy sets, bipolar fuzzy sets
  - rough sets and fuzzy rough sets
  - formal concepts
  - ...

FCA (Ganter et al.)

- G = set of objects.
- M = set of attributes or properties.
- $I \subseteq G \times M$.
- (X, Y) $(X \in \mathcal{P}(G), Y \in \mathcal{P}(M))$ = formal concept if (X, Y) is maximal for $X \times Y \subseteq I$.
- Partial ordering: $(X_1, Y_1) \preceq (X_2, Y_2) \Leftrightarrow X_1 \subseteq X_2 (\Leftrightarrow Y_2 \subseteq Y_1)$.
- $\Rightarrow$ Lattice structure $\mathbb{C}$ and
  $\bigwedge_{t \in T}(X_t, Y_t) = \left(\bigcap_{t \in T} X_t, \alpha(\beta(\bigcup_{t \in T} Y_t))\right)$,
  $\bigvee_{t \in T}(X_t, Y_t) = \left(\beta(\alpha(\bigcup_{t \in T} X_t)), \bigcap_{t \in T} Y_t\right)$.
- Derivation operators:
  $\alpha(X) = \{m \in M \mid \forall g \in X, (g, m) \in I\}$,
  $\beta(Y) = \{g \in G \mid \forall m \in Y, (g, m) \in I\}$.
- (X, Y) formal concept $\Leftrightarrow \alpha(X) = Y$ and $\beta(Y) = X$.

$\mathbb{K}$	composite	even	odd	prime	square
1			×		×
2		×		×	
3			×	×	
4	×	×			×
5			×	×	
6	×	×			
7			×	×	
8	×	×			
9	×		×		×
10	×	×			



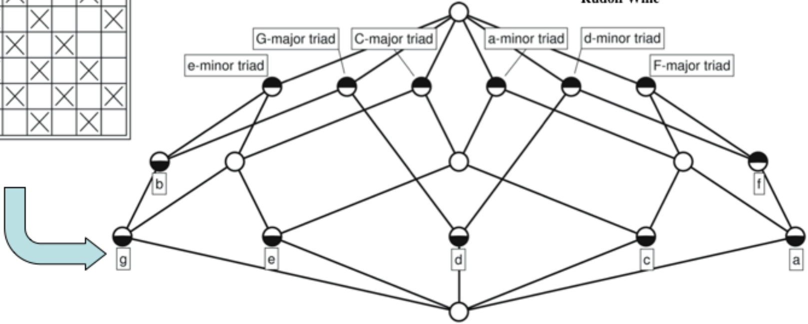
Objects: integers from 1 to 10.

Attributes: composite (c), even (e), odd (o), prime (p) and square (s).

A concept lattice for musical structures

	C-major triad	d-minor triad	e-minor triad	F-major triad	G-major triad	a-minor triad
c	X			X		X
d		X			X	
e	X		X			X
f		X		X		
g	X		X		X	
a		X		X		X
b			X		X	

Rudolf Wille



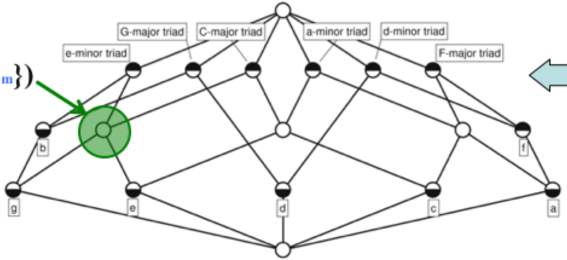
[R. Wille & R. Wille-Henning, « Towards a Semantology of Music », ICCS 2007, Springer, 2007]

A concept lattice for the diatonic scale

	C-major triad	d-minor triad	e-minor triad	F-major triad	G-major triad	a-minor triad
c	X			X		X
d		X			X	
e	X		X			X
f		X		X		
g	X		X		X	
a		X		X		X
b			X		X	

	C_M	E_m				
mi	X	X				
sol	X	X				

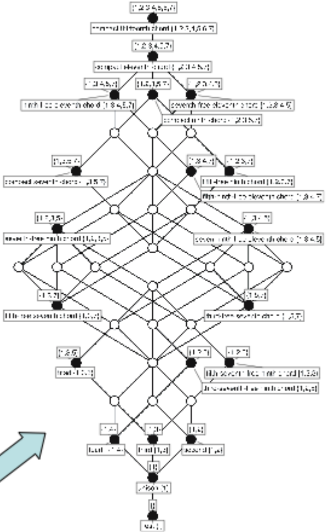
({*mi, sol*},{C_M, E_m})



[R. Wille & R. Wille-Henning, « Towards a Semantology of Music », ICCS 2007, Springer, 2007]

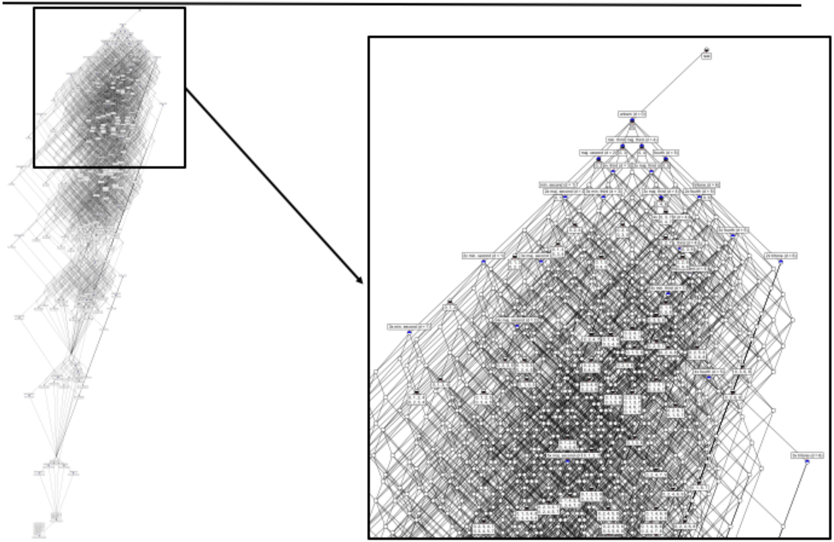
A different concept lattice for the diatonic scale

	{}	{1}	{1,2}	{1,3}	{1,4}	{1,2,3}	{1,3,7}	{1,2,5}	{1,5,7}	{1,3,5}	{1,3,5,7}	{1,2,3,5}	{1,3,4,7}	{1,3,4,5}	{1,2,3,7}	{1,3,4,5,7}	{1,2,3,5,7}	{1,2,3,4,5}	{1,2,3,4,5,7}	{1,2,3,4,5,6,7}
rest { }	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X
unison {1}		X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X
second {1,2}			X			X	X	X	X		X	X	X	X	X	X	X	X	X	X
third {1,3}				X		X	X		X	X	X	X	X	X	X	X	X	X	X	X
fourth {1,4}					X		X	X	X	X	X	X	X	X	X	X	X	X	X	X
fifth-seventh-free ninth chord {1,2,3}						X						X		X	X	X	X	X	X	X
fifth-free seventh chord {1,3,7}							X				X	X	X		X	X	X	X	X	X
third-seventh-free ninth chord {1,2,5}								X				X	X	X		X	X	X	X	X
third-free seventh chord {1,5,7}									X		X		X	X	X	X	X	X	X	X
triad {1,3,5}										X	X	X		X		X	X	X	X	X
compact seventh chord {1,3,5,7}											X					X	X		X	X
seventh-free ninth chord {1,2,3,5}												X				X	X	X	X	X
fifth-ninth-free eleventh chord {1,3,4,7}													X			X		X	X	X
seventh-ninth-free eleventh chord {1,3,4,5}														X		X	X	X	X	X
fifth-free ninth chord {1,2,3,7}															X		X	X	X	X
ninth-free eleventh chord {1,3,4,5,7}																X			X	X
compact ninth chord {1,2,3,5,7}																	X		X	X
seventh-free eleventh chord {1,2,3,4,5}																		X	X	X
compact eleventh chord {1,2,3,4,5,7}																			X	X
compact thirteenth chord {1,2,3,4,5,6,7}																				X



[R. Wille & R. Wille-Henning, « Towards a Semantology of Music », ICCS 2007, Springer, 2007]

How to reduce the combinatorial explosion?



• T. Schlemmer, S. E. Schmidt, « A formal concept analysis of harmonic forms and interval structures », *Annals of Mathematics and Artificial Intelligence* 59(2), 241– 256 (2010)

A lattice structure on intervals

Core idea:

- Harmonic forms = objects
- Intervals = attributes

Harmonic system: $\mathbb{T} = (T, \Delta, \mathrm{I})$, with T = set of tones, I = musical intervals, and $\Delta : T \times T \to I$ s.t.

$$\forall (t_1, t_2, t_3), \Delta(t_1, t_2) + \Delta(t_2, t_3) = \Delta(t_1, t_3) \text{ and } \Delta(t_1, t_2) = 0 \text{ iff } t_1 = t_2$$

Here: $\mathbb{T}_n = (\mathbb{Z}_n, \Delta_n, \mathbb{Z}_n)$, where $n \in \mathbb{Z}_+$ represents an octave, $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$, and Δ_n is the difference modulo n.

Harmonic forms $\mathcal{H}(\mathbb{T}_n)$: equivalence classes of Ψ:

$$\forall H_1 \subseteq \mathbb{Z}_n, \forall H_2 \subseteq \mathbb{Z}_n,\ H_1 \Psi H_2 \text{ iff } \exists i \text{ s.t. } H_1 = H_2 + i$$

where $H + i = \{t + i \mid t \in H\}$ if $t + i$ exists for all $t \in H$.

Musical context $\mathbb{K} = (\mathcal{H}(\mathbb{T}_n), \mathbb{Z}_n, R)$:

- $G = \mathcal{H}(\mathbb{T}_n) =$ objects $=$ harmonic forms,
- $M = \mathbb{Z}_n =$ attributes $=$ intervals,
- $R =$ occurrence of an interval in an harmonic form.

Example: 7-tet $\mathbb{T}_7$ (C, D, E, F, G, A, B)
Intervals $=$ unison (0), second (1), third (2), fourth (3).

Adjunction and Galois connection

Equivalent concepts by reversing the order on one space.

$\delta : A \to B$, $\varepsilon : B \to A$	$\alpha : B \to A, \beta : A \to B$
$\delta(a) \leq_B b \Leftrightarrow a \leq_A \varepsilon(b)$	$a \leq_A \alpha(b) \Leftrightarrow b \leq_B \beta(a)$
	$(\Leftrightarrow \beta(a) \leq'_B b$ with $\leq'_B \equiv \geq_B)$
increasing operators	decreasing operators
$\varepsilon\delta\varepsilon = \varepsilon, \delta\varepsilon\delta = \delta$	$\alpha\beta\alpha = \alpha, \beta\alpha\beta = \beta$
$\varepsilon\delta =$ closing, $\delta\varepsilon =$ opening	$\alpha\beta$ and $\beta\alpha =$ closings
$Inv(\varepsilon\delta) = \varepsilon(B)$, $Inv(\delta\varepsilon) = \delta(A)$	$Inv(\alpha\beta) = \alpha(B)$, $Inv(\beta\alpha) = \beta(A)$
$\varepsilon(B) =$ Moore family	$\alpha(B)$ and $\beta(A) =$ Moore families
$\delta(A) =$ dual Moore family	
$\delta =$ dilation: $\delta(\vee_A a_i) = \vee_B(\delta(a_i))$	$\alpha(\vee_B b_i) = \wedge_A \alpha(b_i)$
$\varepsilon =$ erosion: $\varepsilon(\wedge_B b_i) = \wedge_A(\varepsilon(b_i))$	$\beta(\vee_A a_i) = \wedge_B \beta(a_i)$ (anti-dilation)

Dilation and erosion in a concept lattice

- From join and meet irreducible elements (and for any element by disjunction/conjunction of irreducible elements).
- From valuations and derived distances (and structuring element = ball of the chosen distance).

(Atif et al. ICFCA 2013, LFA 2015, IJUFKS 2016, Agon et al., ICCS 2018)

Reducing a concept lattice using congruences

Congruence: equivalence relation θ on a lattice $\mathcal{L}$, compatible with join and meet, i.e. $(\theta(a, b)$ and $\theta(c, d)) \Rightarrow (\theta(a \vee c, b \vee d)$ and $\theta(a \wedge c, b \wedge d))$, for all $a, b, c, d \in \mathcal{L}$.

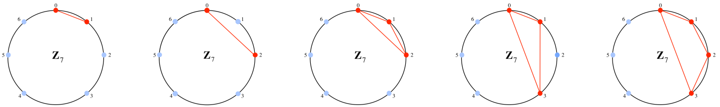
Quotient lattice: $\mathcal{L}/\theta$

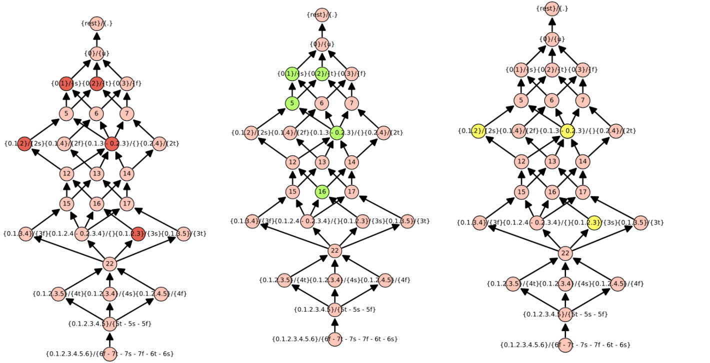
Example: congruence grouping the most common harmonic forms in a same equivalence class.

Harmonico-morphological descriptors:

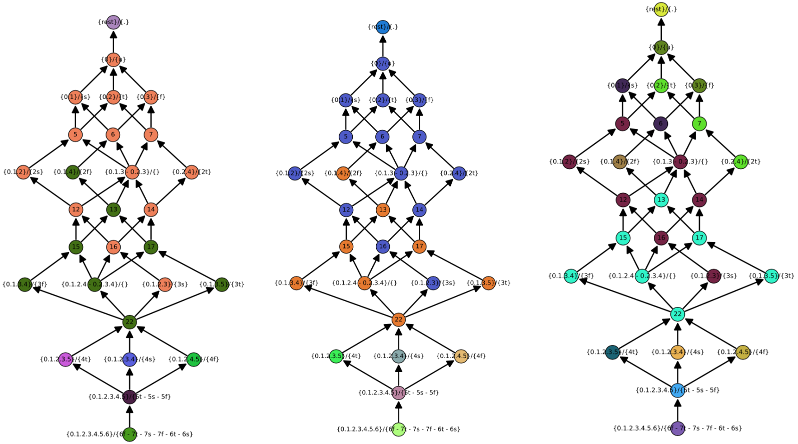
- Musical piece $\mathcal{M}$, harmonic system $\mathbb{T}_{\mathcal{M}}$, concept lattice $\mathbb{C}(\mathcal{M})$
- $H_{\mathbb{C}}^{\mathcal{M}}$: formal concepts corresponding to the harmonic forms in $\mathcal{M}$
  - θ grouping all formal concepts in $H_{\mathbb{C}}^{\mathcal{M}}$ into one same class;
  - θ_δ grouping all formal concepts in $\delta(H_{\mathbb{C}}^{\mathcal{M}})$ into one same class;
  - θ_ε grouping all formal concepts in $\varepsilon(H_{\mathbb{C}}^{\mathcal{M}})$ into one same class.
- Proposed harmonic descriptors: quotient lattices $\mathbb{C}(\mathcal{M})/\theta$, $\mathbb{C}(\mathcal{M})/\theta_\delta$, and $\mathbb{C}(\mathcal{M})/\theta_\varepsilon$.

Example: Ligeti's String Quartet No. 2 (M2 Pierre Mascarade)

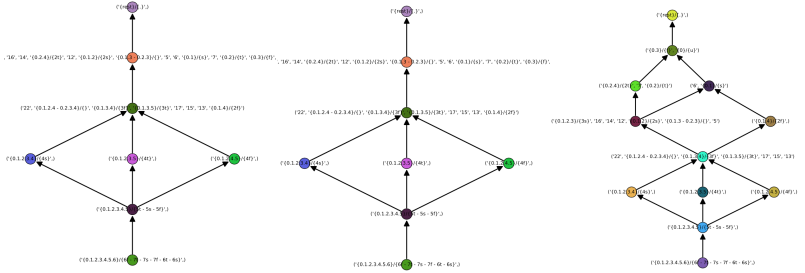




Formal concepts associated with the harmonic forms found in $H^{\mathcal{M}}$: $H^{\mathcal{M}}_{\mathbb{C}}$ (red), dilation $\delta(H^{\mathcal{M}}_{\mathbb{C}})$ (green), and erosion $\varepsilon(H^{\mathcal{M}}_{\mathbb{C}})$ (yellow).



Congruence relations θ, θ_δ, and θ_ε on $\mathbb{C}(\mathcal{M})$ (7-tet) generated by: $H_{\mathbb{C}}^{\mathcal{M}}$, $\delta(H_{\mathbb{C}}^{\mathcal{M}})$, and $\varepsilon(H_{\mathbb{C}}^{\mathcal{M}})$.

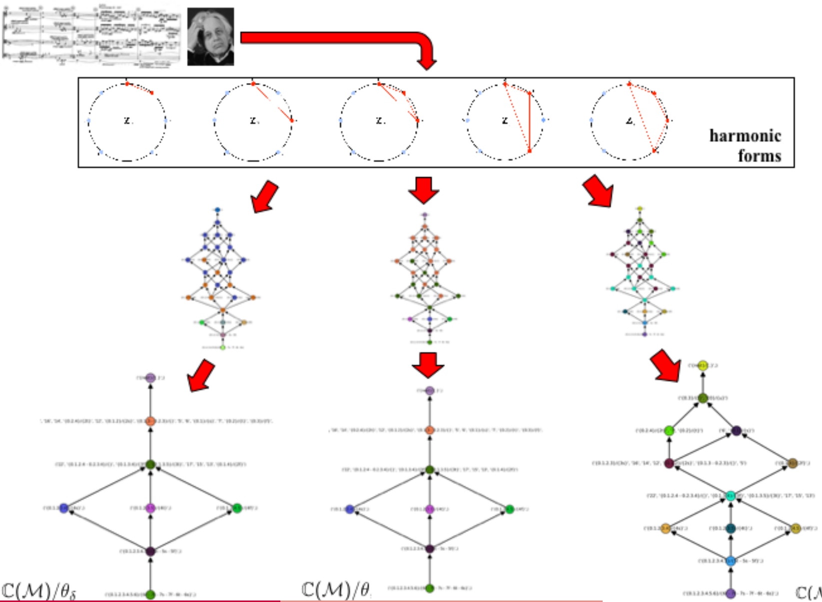


Quotient lattices: $\mathbb{C}(\mathcal{M})/\theta$, $\mathbb{C}(\mathcal{M})/\theta_\delta$, and $\mathbb{C}(\mathcal{M})/\theta_\varepsilon$.

Interpretation:

- Dilations and erosions of the set of formal concepts provide upper and lower bounds of the description.
- Congruences provide a structural summary of the harmonic forms.
- Proposed descriptors = good representative of $\mathcal{M}$, since they preserve the intervallic structures and provide compact summaries, which would allow for comparison between musical pieces.

Summary of the lattice-reduction process

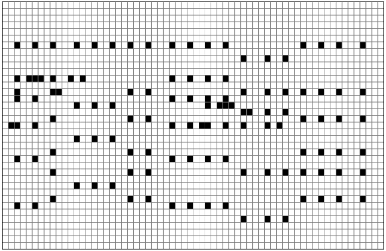


Mathematical morphology on various spatial representations

- Rythms:
  - Object = rhythm
  - Structuring element = rhythm
  - Dilation via time translation
  - Concatenation
- Melodies
- Piano roll
- Perspective: Tonnetz, ...

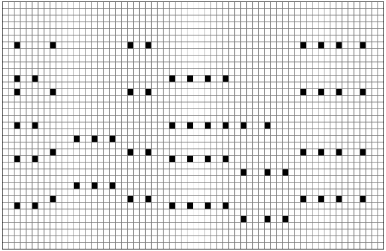
Examples on piano roll representations (M2 Paul Lascabettes)

Pyramid Song (Radiohead)



Play

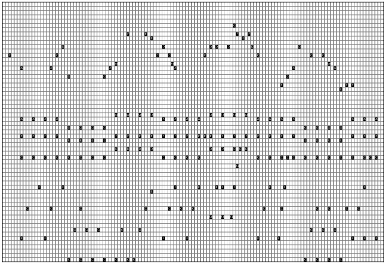
Opening



Play

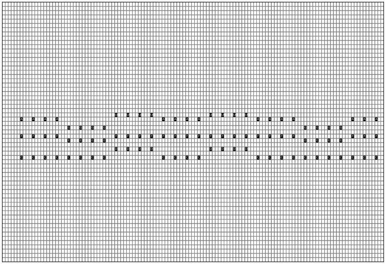
Structuring element (fifth): 

Hey Jude (Beatles)



Play

Opening



Play

Structuring elements (triad):