Catalan Numbers

Rational Catalan Numbers

Dyck Path

Christoffel words

Well-formed Scales

Narayana Numbers

Block Designs

Johson Works

Catalan Designs

Permutations

Rational Associahedra

Rational Catalan Numbers and Music

Franck Jedrzejewski

Paris-Saclay University - CEA - France

IRMA Strasbourg, Mars 29, 2019

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Charles Eugène Catalan (1814-1894)

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Catalan Numbers

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{(n+1)!n!} = \prod_{k=2}^{n} \frac{n+k}{k}$$

The first Catalan numbers for n = 0, 1, 2, 3, ... are :
1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, 208012, 742900, 2674440, etc.

Recurrence relations :

$$C_0 = 1, \quad C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}$$

$$C_0 = 1, \quad C_{n+1} = \frac{2(2n+1)}{n+2} C_n$$

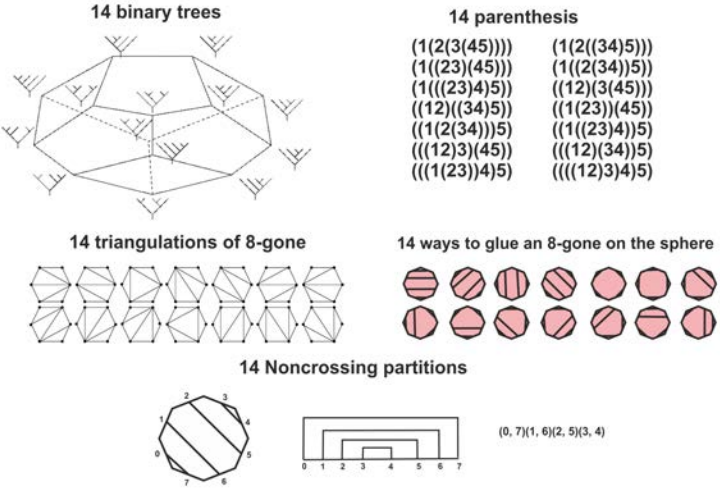
Asymptotic behavior :

$$C_n \sim \frac{4^n}{n^{3/2}\sqrt{\pi}}$$

Integral representation :

$$C_n = \int_0^4 x^n \rho(x) dx, \quad \rho(x) = \frac{1}{2\pi}\sqrt{\frac{4-x}{x}}$$

Catalan Family



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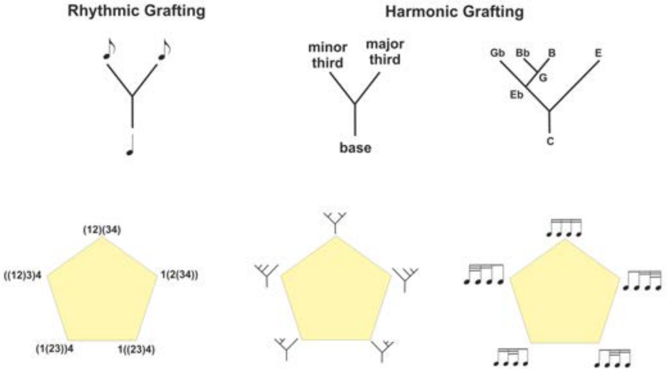
Grafting

Associahedra = representation of the algebra of planar rooted binary trees = dendriform algebra (Jean-Louis Loday)



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Rational Catalan Numbers

Given $x \in \mathbb{Q} \backslash [-1, 0]$, there exist a unique coprime $(a, b) \in \mathbb{N}^2$ such that

$$x = \frac{a}{b-a}$$

The Rational Catalan Number :

$$\mathrm{Cat}(x) = \mathrm{Cat}(a, b) = \frac{1}{a+b}\binom{a+b}{a, b} = \frac{(a+b-1)!}{a!b!}$$

Nikolaus von Fuss (1755-1826)

Special Cases :

1. $a = n,\ b = n+1$ Eugène Charles Catalan (1814-1894)

$$\mathrm{Cat}(n) = \mathrm{Cat}(n, n+1) = \frac{(2n)!}{(n+1)!n!} = C_n$$

2. $a = n,\ b = kn+1$ Nikolaus von Fuss (1755-1826)

$$\mathrm{Cat}(a, b) = \frac{((k+1)n)!}{(kn+1)!n!} = \frac{1}{(k+1)n+1}\binom{(k+1)n+1}{n}$$

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Derived Catalan number

The commutativity $\mathrm{Cat}(a,b) = \mathrm{Cat}(b,a) = \frac{(a+b-1)!}{a!b!}$ implies that the derived Catalan Number satisfies :

$$\mathrm{Cat}'(x) := \mathrm{Cat}\left(\frac{1}{x-1}\right) = \mathrm{Cat}\left(\frac{x}{1-x}\right)$$

Rational Duality :

$$\mathrm{Cat}'\left(\frac{1}{x}\right) = \mathrm{Cat}\left(\frac{1}{1/x-1}\right) = \mathrm{Cat}\left(=\frac{x}{1-x}\right) = \mathrm{Cat}'(x)$$

The process $\mathrm{Cat}(x) \to \mathrm{Cat}'(x) \to \mathrm{Cat}''(x)$... is a categorification of the Euclidean algorithm

Euclidean Algorithm :

$$\begin{aligned} b &= aq_0 + r_0,\ a = q_1 r_0 + r_1, r_0 = q_2 r_1 + r_2, ..., r_n = q_{n+2} r_{n+1} + r_{n+2} \\ g &= \gcd(b,a) = \gcd(a, r_0) = \gcd(r_0, r_1) = \cdots = \gcd(r_n, r_{n+1}) = r_{n+2} \end{aligned}$$

Catalan Algorithm : for the minor third $x = 6/5,\ (a,b) = (5,11)$

$$\begin{aligned} \mathrm{Cat}(5,11) &= 143 \\ \mathrm{Cat}'(5,11) &= \mathrm{Cat}(5,6) = 42 \\ \mathrm{Cat}''(5,11) &= \mathrm{Cat}'(5,6) = \mathrm{Cat}(1,5) = 1 \end{aligned}$$

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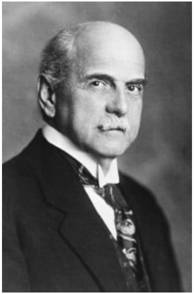
Dyck Words and Dyck Paths

Dyck words ($=$ Well parenthesized words)

alphabet $\Sigma = \{(,)\}$, $\mathrm{imb}(\omega) = |\omega|_(- |\omega|_)$

ω is a Dyck word iff $\mathrm{imb}(\omega) = 0$ and $\mathrm{imb}(u) \geq 0$ for all prefix u of ω

Dyck path from (0,0) to (a, b) $=$ staircase walk that lies below the diagonal (but may touch).



Walther von Dyck (1856-1934)

Theorem (Grossman (1950), Bizley (1954))

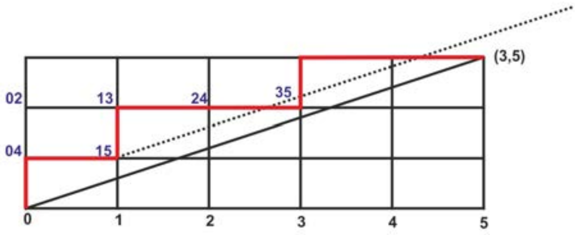
The number of Dyck paths is the Catalan number :

$$|\mathfrak{D}(x)| = \mathrm{Cat}(x)$$

H. D. Grossman. Fun with lattice points : paths in a lattice triangle, *Scripta Math.* 16 (1950) 207–212

M. T. L. Bizley. Derivation of a new formula for the number of minimal lattice paths from (0,0) to (km, kn) having just tcontacts with the line $my = nx$ and having no points above this line ; and a proof of Grossmans formula for the number of paths which may touch but do not rise above this line, *Journal of the Institute of Actuaries.* 80 (1954) 55–62

Rational Dyck Paths



Dyck Paths = Path from (0,0) to (b, a) in the integer lattice $\mathbb{Z}^2$ staying above the diagonal $y = ax/b$.

Bottom of a north step (blue) by laser construction gives the dissection of $\mathbb{P}_{b+1}$

Dyck Path in red : $xyxy^2xy^2$

Number of (a, b)-Dyck Paths $= \mathrm{Cat}(a, b)$.

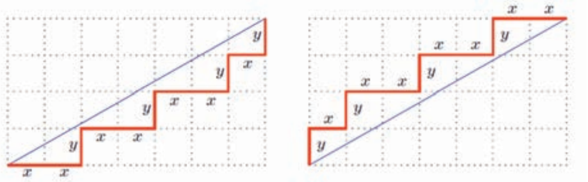
Christoffel words

Definition

The upper (lower) *Christoffel path* of slope b/a is the path from $(0,0)$ to (a,b) in the integer lattice $\mathbb{Z} \times \mathbb{Z}$ that satisfies the following two conditions :
(i) The path lies above (below) the line segment that begins at the origin and ends at (a,b).
(ii) The region in the plane enclosed by the path and the line segment contains no other points of $\mathbb{Z} \times \mathbb{Z}$ besides those of the path.

Definition

Christoffel path of slope b/a determines a word w in the alphabet $\{x,y\}$ by encoding steps of the first type by the letter x and steps of the second type by the letter y.



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Christoffel words

A note by C. Kassel. In Strasbourg, After French-Prussian War in 1870, France lost Alsace-Lorraine to the German Empire. The Prussians created a new university in Strasbourg Christoffel founded the *Mathematisches Institut* in 1872.

Elwin Bruno Christoffel (1829–1900)

Observatio arithmetica, Annali di Matematica Pura ed Applicata, vol. 6 (1875), 148–152.

Exemplum I. Sit $a = 4$, $b = 11$, erit series (r.) notis c, d ornata

r. = 4 8 1 5 9 2 6 10 3 7 0 4
g. = c d c c d c c d c d c

words as g=cdccdccdcdc are called Christoffel words

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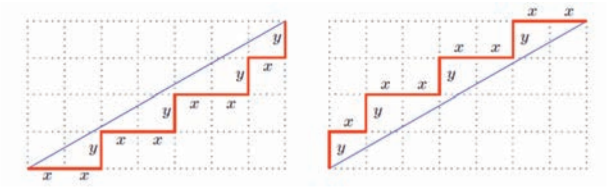
Christoffel Duality



The scale : *fa sol la si do ré mi fa* $\sim \{5, 7, 9, 11, 0, 2, 4, 5\}$
is encoded with $a = tone$, $b = semi - tone$
the Christoffel word : *aaabaab* of slope $5/2$

The same scale *fa do sol ré la mi si* (in the octave *fa* -*fa*)
is encoded with x= fifth up, y = fourth down
the dual Chirstoffel word *xyxyxyy* of slope $4/3$

The dual Christoffel word w of slope a/b is the Christoffel word w^* of slope a^*/b^* with a^* and b^* are multiplicative inverse of a and b in $\mathbb{Z}/(a+b)\mathbb{Z}$.

Example. The multiplicative inverse of 2 is 4 in $\mathbb{Z}_7$, and the inverse of 5 is 3, since $5 \times 3 = 1 \mod 7$ and $2 \times 4 = 1 \mod 7$

Well-formed Scales

Palindromic decomposition (See Kassel, Reteneuauer)

- The lydian word *aaabaab* has a decompostion $w = aub$ with $u = aabaa$ palindromic
- And u has a decomposition $u = rabs$ with $r = a$ and $s = aa$ palindromic.
- The dual word $w^* = xyxyxyy$ has the same decomposition

The scale is well-formed (modulo 12) : 5-generated

$$5 \xrightarrow{5} 0 \xrightarrow{5} 7 \xrightarrow{5} 2 \xrightarrow{5} 9 \xrightarrow{5} 4 \xrightarrow{5} 8$$

step = 3 :

5 *7 9 11* **0** *2 4 5* **7** *9 11 0* **2** *4 5 7* **9** *11 0 2* **4** *5 7 9* **11** *0 2 4* **5**

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Maximally Even Sets

Definition

A *maximally even scale* is a scale in which every generic interval has either one or two consecutive (adjacent) specific intervals—in other words a scale that is "spread out as much as possible."

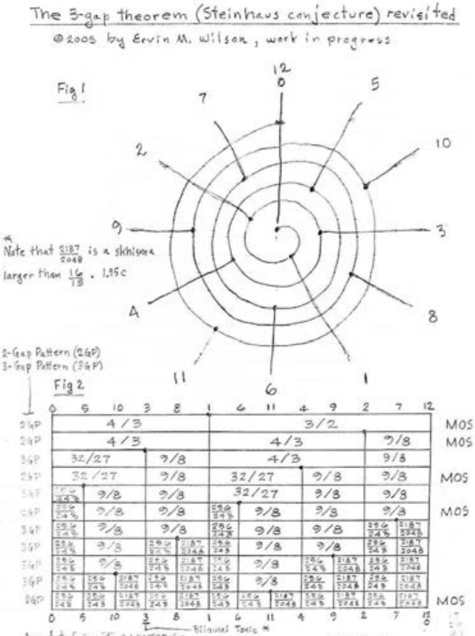
Example. The diatonic scale has interval structure 2212221. The sums of k consecutive intervals has always one or two specific intervals

k	*Partials sums*	*Specific int.*
1	2212221	{1,2}
2	4334433	{3,4}
3	5556555	{5,6}
	...	
7	12	{12}

(Steinhaus Conjecture, Three gaps theorem)

Let N points be placed consecutively around the circle by an angle of α. Then for all irrational α and natural N, the points partition the circle into gaps of at most three different lengths.

Scales Construction



Narayana Numbers

Narayana Numbers

h-vector $= (h_{-1}, h_0, ..., h_{a-2})$ of $\mathrm{Ass}(a, b)$ with

$$h_{i-2} = \mathrm{Nar}(a, b, i) = \frac{1}{a}\binom{a}{i}\binom{b-1}{i-1}$$

$\mathrm{Nar}(a, b, i)$ = Number of (a, b)-Dyck Paths with i non trivial vertices runs.

Kreweras Numbers

Number of (a, b)-Dyck Paths with r_j vertices runs of length j

$$\mathrm{Krew}(a, b, \mathbf{r}) = \frac{(b-1)!}{r_0!r_1!...r_a!}$$

Kirkman Numbers

f-vector $= (f_{-1}, f_0, ..., f_{a-2})$ of $\mathrm{Ass}(a, b)$ with $f_{-1} = 1$, f_i = Number of i-dimensional faces $0 \leq i \leq a-2$

$$f_{i-2} = \mathrm{Kir}(a, b, i) = \frac{1}{a}\binom{a}{i}\binom{b+i-1}{i-1}$$

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Reduced Euler Characteristic

Relations

$$\sum_{i=-1}^{a-2} f_i (t-1)^{a-2-i} = \sum_{i=-1}^{a-2} h_i t^{a-2-i}$$

Reduced Euler Characteristic

$$\chi = \sum_{i=-1}^{a-2} (-1)^i f_i = (-1)^a \mathrm{Cat}'(a,b)$$

Example : Ass(3,5). h-vector $= (1,\ 4,\ 2)$. **f-vector** $= (1,\ 6,\ 7)$
Relations

$$\begin{aligned} \sum_{i=-1}^{1} f_i (t-1)^{1-i} &= (t-1)^2 + 6(t-1) + 7 \\ &= t^2 + 4t + 2 \\ &= \sum_{i=-1}^{1} h_i t^{1-i} \end{aligned}$$

Reduced Euler Characteristic

$$\chi = \sum_{i=-1}^{a-2} (-1)^i f_i = -1 + 6 - 7 = -2$$

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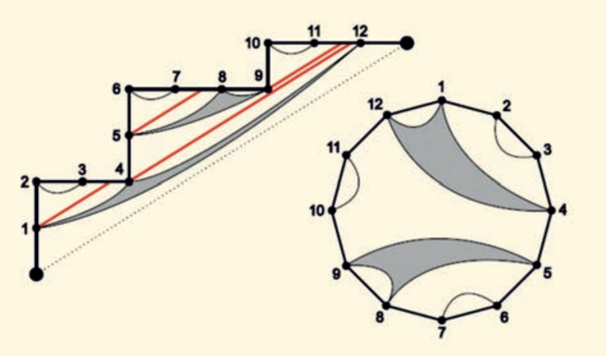
Noncrossing partition

Drew Armstrong How to create a noncrossing partition from a Dyck Path ?

- Start with a Dyck path. Here $(a, b) = (5, 8)$.
- Label the internal vertices by $\{1, 2, \ldots, a + b\}$
- Shoot lasers from the bottom left with slope a/b
- Who can see each other ?

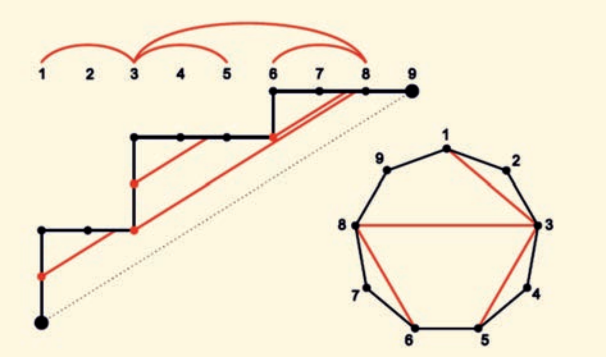


from Rational Catalan Combinatorics (Type A), Drew Armstrong (2012)

Polygon Dissection

Drew Armstrong How to create a polygon dissection from a Dyck Path ?

- Start with a Dyck path. Here $(a, b) = (5, 8)$.
- Label the columns by $\{1, 2, \ldots, b+1\}$
- Shoot some lasers from the bottom left with slope a/b.
- Lift the lasers up.



from Rational Catalan Combinatorics (Type A), Drew Armstrong (2012)

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Combinatorial t-Designs

Is there a relation between associahedron and combinatorial designs ?
What is a combinatorial design ? It has been used by Tom Johnson since 2003.

Definition

A *t-design* $t-(v,k,\lambda)$ is a pair $D=(X,\mathcal{B})$ where X is a v-set ($X=\mathbb{Z}_v$) and $\mathcal{B}$ a collection of k-subsets of X called blocks such that every t-subset of X is contained in exactly λ blocks. D is simple if it has no repeated block.

Examples

$2-(v,k,\lambda)=$ Balanced Incomplete Block Design (BIBD)
$t-(v,k,1)=$ Steiner Systems
$t-(v,3,1)=$ Triple Systems (TS)
$2-(v,3,1)=$ Steiner Triple Systems (STS)
$2-(v,4,1)=$ Steiner Quadruple System (SQS).

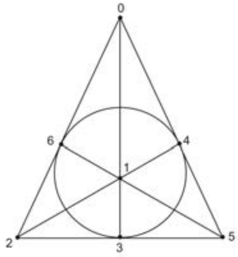
There are no known examples of non trivial t-designs with $t\geq 6$.

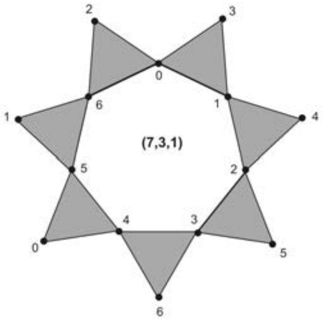
Example : $5-(24,8,1)$ is a Steiner System.

Definition

Two t-designs $(X_1,\mathcal{B}_1)$ and $(X_2,\mathcal{B}_2)$ are *isomorphic* if there is a bijection $\varphi : X_1\to X_2$ such that $\varphi(\mathcal{B}_1)=\mathcal{B}_2$.

Example : Fano Plane (7, 3, 1)





0	0	0	1	1	2	3
1	2	4	2	5	3	4
3	6	5	4	6	5	6

- The complementary of $(7, 3, 1)$ is $(7, 4, 2)$ with blocks $\{0, 1, 2\}^c = \{3, 4, 5, 6\}$, etc.
- Is t-design always represented by base blocks (0,1,3) and transformations (Here $T_1(x) = x + 1 \mod 7$), i.e. generators and relations ?
- How to draw a t-design using n-gones and common subsets ?

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Number of blocks of a t-Design

Number of blocks of a t-Design

$$b = \lambda \frac{v!}{(v-t)!} \frac{(k-t)!}{k!}$$

Number of blocks that contain any i-element set of points

$$b_i = \lambda \binom{v-i}{t-i} / \binom{k-i}{t-i}, \quad i = 0, 1, ..., t$$

If we set

$$r = \lambda \frac{(v-1)!}{(v-t)!} \frac{(k-t)!}{(k-1)!}$$

we get the famous relation

$$bk = vr$$

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Complement of a t-Designs

The complement of $D = (X, \mathcal{B})$, $t - (v, k, \lambda)$ is $D^c = (X, X \backslash \mathcal{B})$ of parameters $t - (v, v - k, \mu)$ with

$$\mu = \lambda \binom{v-t}{k} / \binom{v-t}{k-t} = \lambda \frac{(v-k)!}{(v-t-k)!} \frac{(k-t)!}{k!}$$

D and D^c have the same number of blocks.

For $t = 2$, the block design D with b blocks

$$b = \frac{v(v-1)\lambda}{k(k-1)}, \quad r = \lambda \frac{(v-1)}{(k-1)}, \quad bk = vr$$

has a complement D^c with b blocks and $(v, v-k, b-2r+\lambda)$.

A **symmetric design** is a BIBD (v, k, λ) with $b = v$.

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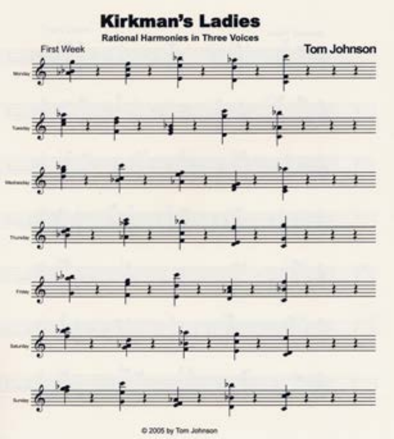
Tom Johnson' Works

- **Block Design for piano** : 4-(12, 6,10) built on 30 base blocks and the automorphism $\sigma = (0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10)(11)$
- **Kirkman's ladies** : (15, 3, 1) with 35 blocks
- **Vermont Rhythms** : 42$\times$11 rhythms based on (11,6,3)



Resolvable Designs

Definition

A *parallel class* in a design is a set of blocks that partition the point set.

Definition

A design (v, k, λ) is resolvable if its blocks can be partitioned into parallel classes

Examples

(9,3,1) is resolvable

(0,1,2)	(0,3,6)	(0,4,8)	(0,5,7)
(3,4,5)	(1,4,7)	(1,5,6)	(1,3,8)
(6,7,8)	(2,5,8)	(2,3,7)	(2,4,6)

Kirkman problem : (15, 3, 1)

Kirkman's Ladies

Thomas Penyngton Kirkman (1806-1895) posed the so-called schoolgirls problem in 1850 *Fifteen young ladies in a school walk out abreast for seven days in succession : it is required to arrange them daily, so that no two walk twice abreast.*

A Kirkman Triple System (KTS) is a resolvable STS.

Theorem

KTS(v) exists if and only if $v \equiv 3 \pmod 6$

There are 7 solutions for $v = 15$. A solution is :

Monday	(0,1,2)	(3,9,11)	(4,7,13)	(5,8,14)	(6,10,12)
Tuesday	(0,3,4)	(1,8,10)	(2,10,14)	(5,7,11)	(6,9,13)
Wednesday	(0,5,6)	(1,7,9)	(2,11,13)	(3,12,14)	(4,8,10)
Thursday	(1,3,5)	(0,10,13)	(2,7,12)	(4,9,14)	(6,8,11)
Friday	(1,4,6)	(0,11,14)	(2,8,9)	(3,7,10)	(5,12,13)
Saturday	(2,3,6)	(0,7,8)	(1,13,14)	(4,11,12)	(5,9,10)
Sunday	(2,4,5)	(0,9,12)	(1,10,11)	(3,8,13)	(6,7,14)

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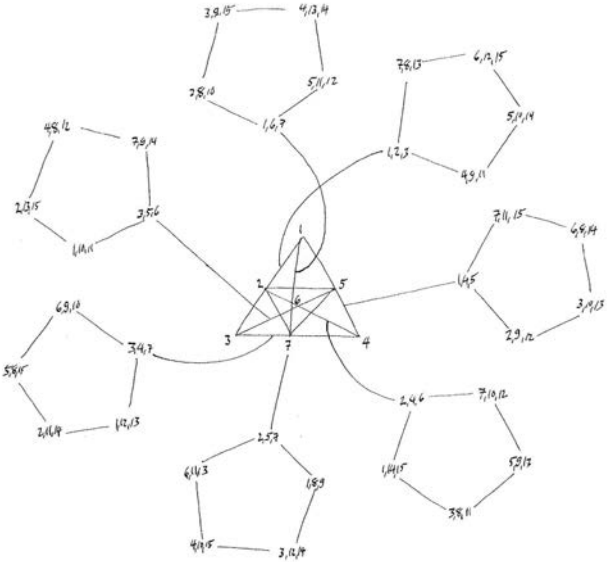
Kirkman's Ladies : (15,3,1)

The parallel classes of (15,3,1) showing its relation with the Fano plane.



How to draw a t-design ?

With genrators (cyclic representations)

- Blocks are constructed from generators $\mathcal{B} = \left\langle B \mid T_1^v(B) \equiv 1 \right\rangle$ with action of the cyclic group. (p prime power)
- Projective geometry, $PG(m-1, p)$

$$2 - \left(\frac{p^m - 1}{p-1}, \frac{p^{m-1}-1}{p-1}, \frac{p^{m-1}-1}{p-1} \right)$$

(7,3,1)	PG(2,2)	(0,1,3)
(13,4,1)	PG(2,3)	(0,1,3,9)
(21,5,1)	PG(2,4)	(0,1,4,14,16)
(31,6,1)	PG(2,5)	(0,1,3,8,12,18)
(57,8,1)	PG(2,7)	(0,1,3,13,32,36,43,52)
(73,9,1)	PG(2,8)	(0,1,3,7,15,31,36,54,63)
(91,10,1)	PG(2,9)	(0,1,3,9,27,49,56,61,77,81)

Theorem (Netto, 1893)

Let p prime, $n \geq 1$, $p^n \equiv 1 \pmod 6$. Let $\mathbb{F}_{p^n}$ be a finite field on X of size $p^n = 6t + 1$ with 0 as its zero element and α a primitive root of unity. The sets

$$B_i = \{\alpha^i, \alpha^{i+2t}, \alpha^{i+4t}\} \qquad \mathrm{mod}\ \ p^n$$

for $i = 1, 2, ..., t-1$ are generators ($T_j(B) = j + B$ mod p^n) of the set blocks of an $STS(p^n)$ on X.

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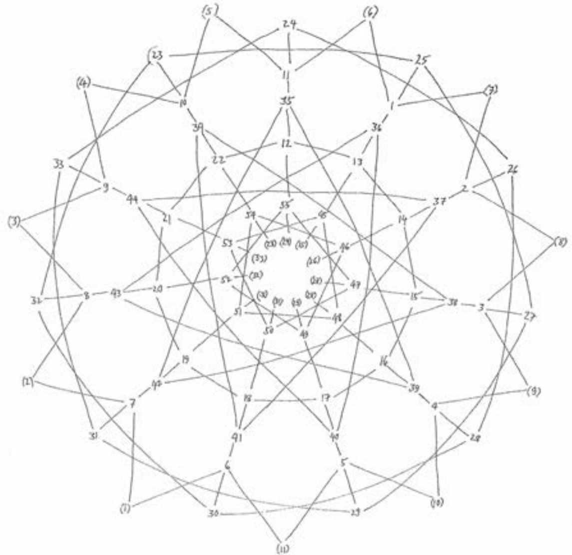
55 Chords for organ : $(11, 4, 6)$

How to draw a t-design ?
Example : *55 Chords* (2009) pour orgue. 23 minutes of organ music all derived from an (11,4,6) block design.

1	{2,3,10,11}	20	{1,4,6,10}	39	{1,7,9,10}
2	{1,3,4,11}	21	{2,5,7,11}	40	{2,8,10,11}
3	{1,2,4,5}	22	{1,3,6,8}	41	{1,3,9,11}
4	{2,3,5,6}	23	{2,3,6,7}	42	{1,2,4,10}
5	{3,4,6,7}	24	{3,4,7,8}	43	{2,3,5,11}
6	{4,5,7,8}	25	{4,5,8,9}	44	{1,3,4,6}
7	{5,6,8,9}	26	{5,6,9,10}	45	{2,6,7,11}
8	{6,7,9,10}	27	{6,7,10,11}	46	{1,3,7,8}
9	{7,8,10,11}	28	{1,7,8,11}	47	{2,4,8,9}
10	{1,8,9,11}	29	{1,2,8,9}	48	{3,5,9,10}
11	{1,2,9,10}	30	{2,3,9,10}	49	{4,6,10,11}
12	{2,4,7,9}	31	{3,4,10,11}	50	{1,5,7,11}
13	{3,5,8,10}	32	{1,4,5,11}	51	{1,2,6,8}
14	{4,6,9,11}	33	{1,2,5,6}	52	{2,3,7,9}
15	{1,5,7,10}	34	{2,4,5,7}	53	{3,4,8,10}
16	{2,6,8,11}	35	{3,5,6,8}	54	{4,5,9,11}
17	{1,3,7,9}	36	{4,6,7,9}	55	{1,5,6,10}
18	{2,4,8,10}	37	{5,7,8,10}		
19	{3,5,9,11}	38	{6,8,9,11}		

55 Chords for organ : $(11, 4, 6)$

Cosmological view : Every single chord has no notes in common with exactly four chords Number 1 (2,3,10,11) has no not in common with Numbers 6, 7, 25 and 36



55 Chords for organ : $(11, 4, 6)$

Pentagonal view : Each chord has one pair of notes in common with one chord, the other pair in common with one other chord, and no notes in common with the adjacent chords.

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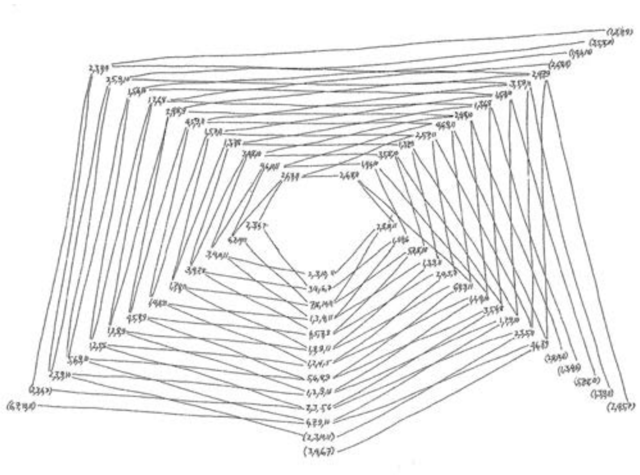
55 Chords for organ : $(11, 4, 6)$

Spider web view : Linking chords with 3 notes in common



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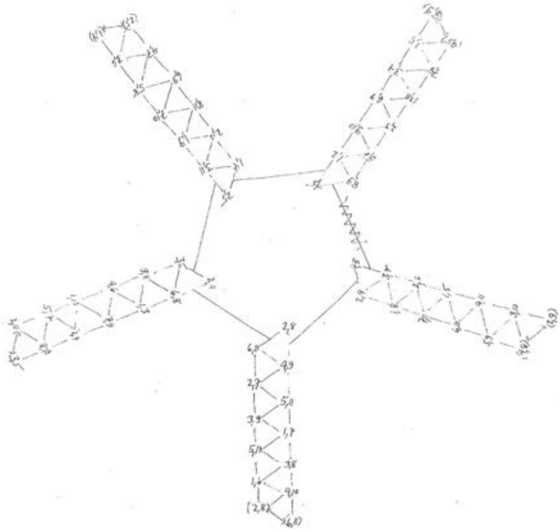
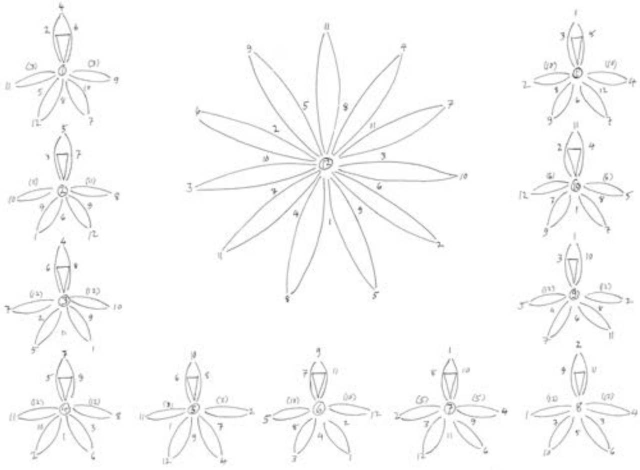
55 Chords for organ : $(11, 4, 6)$

Startfish view : three pairs of notes combine to form 3 chords
Two notes change and two notes continue with each move.



Clarinet Trio – Block design (12,3,2)

- *Clarinet Trio* (2012). Seven kinds of music derived from seven drawings all based on a (12,3,2) combinatorial design.



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Catalan Designs

Is there a t-(v, k, λ) design such that the number of blocks b is a Catalan number of order n ?

$$b = \lambda \frac{v!}{(v-t)!} \frac{(k-t)!}{k!} = \frac{(2n)!}{(n+1)!n!}$$

Catalan numbers = 1, 2, 5, 14, 42, 132, etc.

$b = 14$	(7,3,2), (8,4,3)
$b = 42$	(7,3,6), (8,4,9), (15,5,4), (21,5,2), (21,6,3) (21,10,9), (22,11,10), (28,10,5), (36,6,1), 3-(8,4,3)
$b = 132$	(33,8,7), (33,9,9), (121,11,1), 4-(11,5,2), 4-(12,6,4), 5-(12,6,1)
$b = 429$	(66,6,3), (286,20,2)

Are Catalan designs nicely representable by associahedra ?

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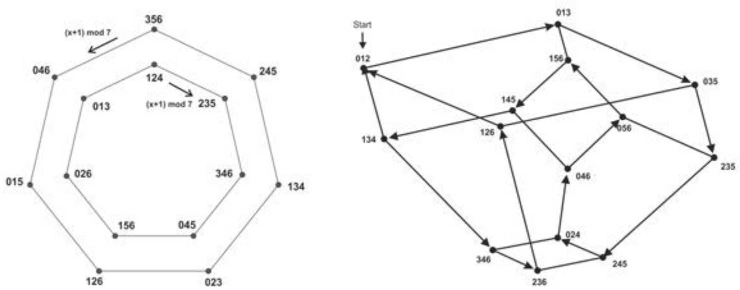
Design (7,3,2)

The design (7,3,2) has b=14 blocks.



Left : Cyclic representation

Right : Hamiltonian cycle through (7,3,2)

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Design 3-(8,4,1)

Construction of the 3-(8,4,1) design :

Add the number 7 to the design (7,3,1).

0	1	2	3	0	1	0
1	2	3	4	4	5	2
3	4	5	6	5	6	6
7	7	7	7	7	7	7

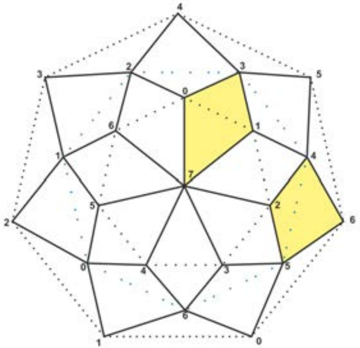
For each bloc add the supplementary block (example 0137 gives 2456, etc...). This leads to the 3-(8,4,1) design. Each pair of notes appears three times.

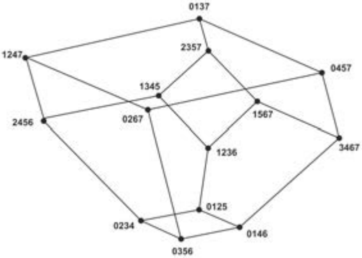
0	1	2	3	0	1	0	2	0	0	0	1	0	1
1	2	3	4	4	5	2	4	3	1	1	2	2	3
3	4	5	6	5	6	6	5	5	4	2	3	3	4
7	7	7	7	7	7	7	6	6	6	5	6	4	5

3-(8,4,1) is a Steiner system.

Design 3-(8,4,1)

The two yellow blocks have no point in common





On the associahedron, connected blocks have 2 points in common

Design (21,6,3)

The design (21,6,3) has two generators

$$u = (0, 1, 3, 11, 16, 20), \qquad v = (0, 1, 7, 12, 15, 19)$$

Consider now sum modulo 21.

1. If n is even, let $a = 3n/2$ and consider the blocks :

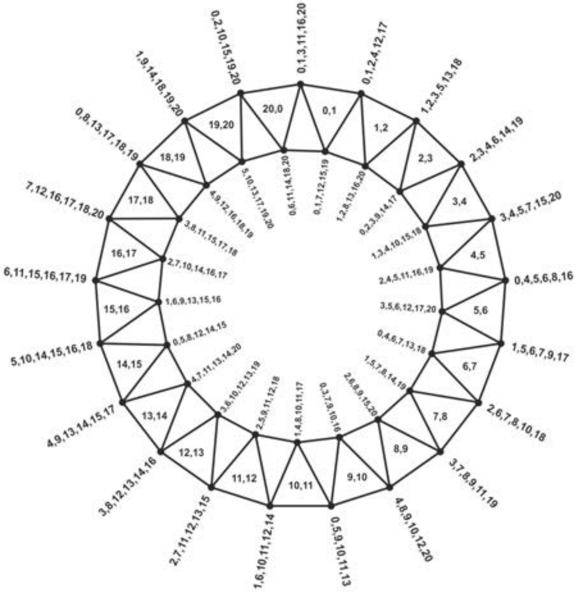
$$\begin{aligned} (a, a+1, a+3, a-1, a+11, a+16) &= a+u \\ (a+1, a+2, a+4, a, a+12, a+17) &= a+u+1 \\ (a, a+1, a+7, a+12, a+15, a+19) &= a+v \end{aligned}$$

2. If n is odd, let $a = (3n+1)/2$ and consider the blocks :

$$\begin{aligned} (a, a+1, a+3, a-1, a+11, a+16) &= a+u \\ (a-1, a, a+6, a+11, a+14, a+18) &= a+v-1 \\ (a, a+1, a+7, a+12, a+15, a+19) &= a+v \end{aligned}$$

All these blocks form the (21,6,3) design. Each block has 6 elements choose on an alphabet of 21 symbols. Each pair appear in exactly 3 blocs has shown on the following figure.

Design (21,6,3)


0,1,3,11,16,20
0,1,2,4,12,17
1,2,3,5,13,18
2,3,4,6,14,19
3,4,5,7,15,20
0,4,5,6,8,16
1,5,6,7,9,17
2,6,7,8,10,18
3,7,8,9,11,19
4,8,9,10,12,20
0,5,9,10,11,13
1,6,10,11,12,14
2,7,11,12,13,15
3,8,12,13,14,16
4,9,13,14,15,17
5,10,14,15,16,18
6,11,15,16,17,19
7,12,16,17,18,20
0,8,13,17,18,19
1,9,14,18,19,20
0,2,10,15,19,20
20,0
0,1
1,2
2,3
3,4
4,5
5,6
6,7
7,8
8,9
9,10
10,11
11,12
12,13
13,14
14,15
15,16
16,17
17,18
18,19
19,20
5,10,13,17,19,20
0,6,11,14,18,20
0,1,7,12,15,19
1,2,8,13,16,20
0,2,3,9,14,17
1,3,4,10,15,18
2,4,5,11,16,19
3,5,6,12,17,20
0,4,6,7,13,18
1,5,7,8,14,19
2,6,8,9,15,20
0,3,7,9,10,16
1,4,8,10,11,17
2,5,9,11,12,18
3,6,10,12,13,19
4,7,11,13,14,20
0,5,8,12,14,15
1,6,9,13,15,16
2,7,10,14,16,17
3,8,11,15,17,18
4,9,12,16,18,19

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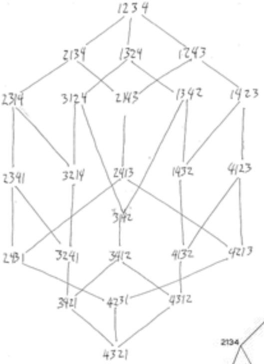
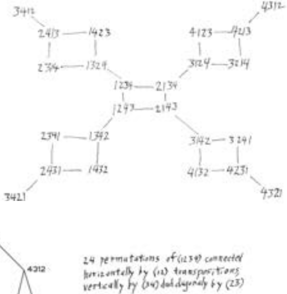
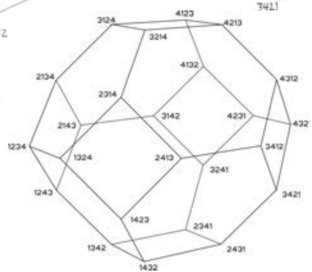
Permutations

Tom Johnson is an American minimalist composer, a former student of Allen Forte and Morton Feldman.
The 24 permutations of (1,2,3,4) arranged in different ways.

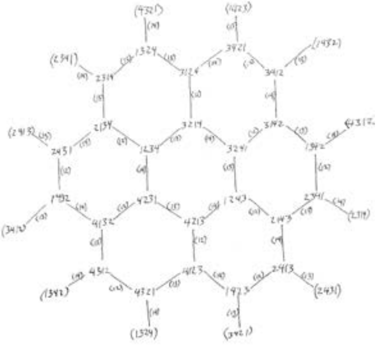


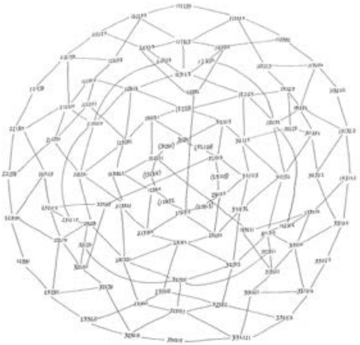




Permutations

Left : Permutations of (1,2,3,4) connected by transpositions (12), (13) and (14)

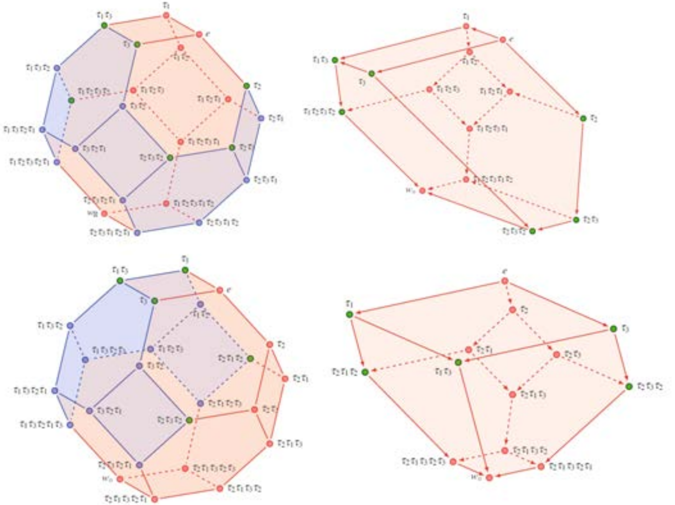




Right : Permutations of 112233

Permutohedra and Associahedra

Some permutations lead to the permutohedron (left) Stasheff polytope or associahedron (right). Two realisations : Loday-Shnider-Sternberg (top) Chapoton-Fomin-Zelevinsky (bottom) © Christian Hohlweg



Rational Associahedra

- Defined by Drew Armstrong. Rational associahedra and noncrossing partitions (2013).
- $\mathrm{Ass}(n, n+1) = \mathrm{Ass}(n)$ is the good old associahedron.
- $\mathrm{Ass}(a, b) =$ simplicial complex consists of all noncrossing dissection of $\mathbb{P}_{b+1}$.
- **Facets** : Collection F(D) of diagonals corresponding to the given Dyck path D. All facets have same cardinality.They are defined by laser construction from bottom of a north step.
- $\mathrm{Ass}(x)$ has $\mathrm{Cat}(x)$ facets, and Euler characteristic $\mathrm{Cat}'(x)$.
- **Vertices :** A diagonal of $\mathbb{P}_{b+1}$ which separates i vertices from $b-i-1$ vertices appears as a vertex of $\mathrm{Ass}(a, b)$ if and only if $i \in S(a, b)$

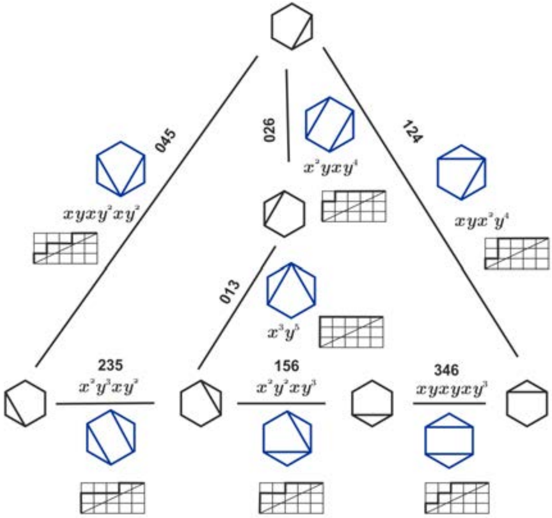
$$S(a,b) = \left\{ \left\lfloor \frac{ib}{a} \right\rfloor, 1 \leq i < a \right\}$$

where $\lfloor x \rfloor = \mathrm{floor}(x) =$ greatest integer $\leq x$. (Well formed scales)

Example :

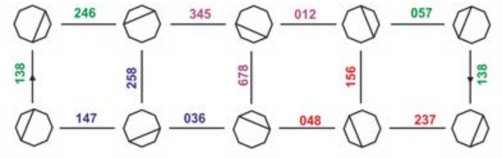
- $S(3,5) = \{1,3\} \Longrightarrow \mathrm{Ass}(3,5)$ has 6 vertices.
- $\mathrm{Cat}(3,5) = 7$ Dyck Paths $\Longrightarrow \mathrm{Ass}(3,5)$ has 7 facets.

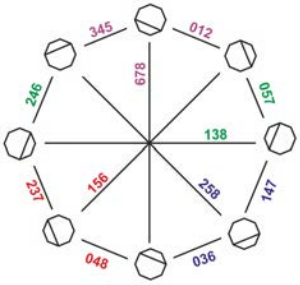
Rational Associahedron Ass(3,5)



Block Design (9,3,1)

Design (9,3,1) has 4 parallel classes (partition of $\mathbb{Z}_9$, 4 colors)
Number of blocks $=12=$ Cat(3,7). $\mathrm{Ass}(3,7)$ has 8 vertices, 12 facets
$S(3,7)=\{2,4\}$. On $\mathbb{P}_8$, each vertex i separates 2 vertices from 4 vertices.
Dick paths lead to 12 facets. Möbius strip (glue the ribbon with respect to the arrows)





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The end

Thank You For Your Attention